Thin Coverings of Modules

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Abstract. Thin coverings are a method of constructing graded-simple modules from simple (ungraded) modules. After a general discussion, we classify the thin coverings of (quasifinite) simple modules over associative algebras graded by finite abelian groups. The classification uses the representation theory of cyclotomic quantum tori. We close with an application to representations of multiloop Lie algebras.

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0 Introduction

In a recent series of papers [2, 3, 1], B. Allison, S. Berman, A. Pianzola, and J. Faulkner examined the structure of multiloop algebras. These algebras are formed by a generalization of the twisting process used in Kac's construction [10] of the affine Lie algebras.

More specifically, for any algebra $\mathcal{B}$ over the complex field $\mathbb{C}$ and finite-order commuting automorphisms $\sigma_1, \dots, \sigma_N$ of $\mathcal{B}$ of period $m_1, \dots, m_N$, respectively, the multiloop algebra $L(\mathcal{B}, \sigma_1, \dots, \sigma_N)$ is the following subal-

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gebra of $\mathcal{B} \otimes \mathbb{C}[t_1^{\pm}, \dots, t_N^{\pm}]$:

$$L(\mathcal{B}, \sigma_1, \dots, \sigma_N) = \bigoplus_{(i_1, \dots, i_N) \in \mathbb{Z}^N} \mathcal{B}_{\bar{\imath}_1, \dots, \bar{\imath}_N} \otimes \mathbb{C} t_1^{i_1} \cdots t_N^{i_N},$$

where $\mathcal{B}_{\bar{\imath}_1, \dots, \bar{\imath}_N}$ is the common eigenspace

$$\{b \in \mathcal{B} \mid \sigma_j b = \xi_j^{i_j} b \text{ for } 1 \leq j \leq N\}$$

for fixed primitive m_jth roots of unity ξ_j.

In particular, when $\mathcal{B}$ is a finite-dimensional simple Lie algebra, one can construct realizations for almost all extended affine Lie algebras by taking central extensions of the resulting multiloop algebras and adjoining appropriate derivations [1, 14].

While studying the representation theory of such extended affine Lie algebras [6], we considered methods of "twisting" simple $\mathcal{B}$-modules into graded-simple modules for the multiloop algebras $L(\mathcal{B}, \sigma_1, \dots, \sigma_N)$. As part of our construction, we used structures called thin coverings of modules.

Let $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ be an algebra graded by an abelian group G. A *thin covering* of a left $\mathcal{B}$-module M is a family of subspaces $\{M_g \mid g \in G\}$ with the following properties:

(i) $M = \sum_{g \in G} M_g$,

(ii) $\mathcal{B}_g M_h \subseteq M_{g+h}$ for all $g, h \in G$,

(iii) If $\{N_g \mid g \in G\}$ also satisfies (i) and (ii) and $N_g \subseteq M_g$ for all $g \in G$, then $N_g = M_g$ for all $g \in G$.

We consider finite-dimensional and infinite-dimensional quasifinite modules over a G-graded associative algebra $\mathcal{A}$. Each thin covering of an (ungraded) left $\mathcal{A}$-module M is associated with a graded left $\mathcal{A}$-module $\widetilde{M}$. Although thin coverings are generally not unique, the isomorphism class of M completely determines the graded-isomorphism class of $\widetilde{M}$, under mild conditions. It is trivial to show that $\widetilde{M}$ is graded-simple whenever M is simple.

An $\mathcal{A}$-module may be twisted with an $\mathcal{A}$-automorphism associated with its G-grading. Such twists do not change the isomorphism class of the graded modules coming from the thin covering construction, and they play a vital role in our classification of thin coverings.

After these applications to graded theory, we give an explicit characterization of thin coverings of quasifinite simple modules over arbitrary (unital) associative algebras graded by finite abelian groups. The classification uses the fact that any such module is also a module for a cyclotomic quantum torus coming from isomorphisms between twists of the module.

We conclude the paper with an application to the representation theory of multiloop Lie algebras. Our approach is an alternative to Clifford theory, where graded-simple $\mathcal{A}$-modules are constructed by inducing from simple $\mathcal{A}_0$-modules. (See [12], Thm. 2.7.2 and also [8, 9].)

This method gives an explicit procedure for constructing graded-simple $\mathcal{A}$-modules from (ungraded) simple $\mathcal{A}$-modules via the action of a cyclotomic quantum torus. For the applications we have in mind, it would be difficult to get a classification of simple $\mathcal{A}_0$-modules, whereas ungraded simple $\mathcal{A}$-modules are well-understood.

Note: Although we work over the field $\mathbb{C}$ of complex numbers, all the results of this paper hold over an arbitrary algebraically closed field of characteristic zero. All modules are left modules unless otherwise indicated. Likewise, all hypotheses of simplicity and semisimplicity should be interpreted as left simplicity and left semisimplicity.

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1 Thin Coverings and Graded Modules

Let $\mathcal{A}$ be a (unital) associative algebra graded by an abelian group G. For any module M over $\mathcal{A}$, a set of subspaces $\{M_g \mid g \in G\}$ is a *G-covering* of M if $\sum_{g\in G} M_g = M$ and $\mathcal{A}_g M_h \subseteq M_{g+h}$ for all $g, h \in G$. We will drop the prefix G from "G-covering" when there is no ambiguity. Two coverings $\{M_g \mid g \in G\}$ and $\{M'_g \mid g \in G\}$ are *equivalent*, and we write

$$\{M_g \mid g \in G\} \sim \{M'_g \mid g \in G\}$$

if there is a fixed $h \in G$ such that $M_g = M'_{g+h}$ for all $g \in G$.

The set $\mathfrak{C} := \mathfrak{C}(M, \mathcal{A}, G)$ of coverings of the module M is partially ordered: for any $\{M_g \mid g \in G\}$ and $\{N_g \mid g \in G\}$ in $\mathfrak{C}$, let

$$\{M_g \mid g \in G\} \preceq \{N_g \mid g \in G\}$$

if the covering $\{M_g\}$ is equivalent to a covering $\{M'_g\}$ for which $M'_g \subseteq N_g$ for all $g \in G$.

The minimal coverings in the poset $\mathfrak{C}$ are called *thin coverings*. If M is a finite-dimensional or infinite-dimensional quasifinite module (see Section 4), then it has a thin covering.

Remark 1.1.1 Let $\mathfrak{g}$ be a Lie algebra graded by an abelian group G, and let M be a $\mathfrak{g}$-module. The grading on $\mathfrak{g}$ extends naturally to a G-grading on the universal enveloping algebra $U(\mathfrak{g})$. Note that the coverings of M as a $\mathfrak{g}$-module coincide with its coverings as a $U(\mathfrak{g})$-module. Thus, by studying thin coverings of modules over associative algebras, we automatically obtain the corresponding results for Lie algebras. The same applies to Lie superalgebras as well, where we would need to consider the $\mathbb{Z}_2$-graded version of a thin covering. (See the definition in Section 4.)

The following elementary, but important lemma says that the thin coverings of a simple $\mathcal{A}$-module M are essentially determined by the simple $\mathcal{A}_0$-submodules of M.

Lemma 1.1.2 *Let $\{M_g \mid g \in G\}$ be a covering for a simple module M over an associative algebra $\mathcal{A}$ graded by an abelian group G. This covering is thin if and only if*

$$M_g = \mathcal{A}_{g-h}u,$$

for all $g, h \in G$ and any nonzero $u \in M_h$.

In particular, in a thin covering, every nonzero space M_h is a simple $\mathcal{A}_0$-module.

Proof Since M is a simple $\mathcal{A}$-module, $\sum_{g\in G} \mathcal{A}_{g-h}u = \mathcal{A}u = M$. This gives a covering of M with $\mathcal{A}_{g-h}u \subseteq M_g$ for all $g \in G$. Thus $\{M_g\}$ is a thin covering if and only if $\mathcal{A}_{g-h}u = M_g$ for all g. □

Although the sum $\sum_{g\in G} M_g$ of the spaces M_g in a thin covering need not be direct, the *external* direct sum of these subspaces always has a natural graded module structure. In fact, this construction "preserves simplicity", as described in the following theorem.

Theorem 1.1.3 *Let M be a simple (not necessarily graded) module over an associative algebra $\mathcal{A}$ graded by an abelian group G. Then for any thin covering $\{M_g \mid g \in G\}$ of M, the space*

$$\widetilde{M} := \bigoplus_{g\in G} M_g$$

is a graded-simple $\mathcal{A}$-module.

Proof The space $\widetilde{M}$ has an obvious graded module structure, namely $\widetilde{M}_g := M_g$ and $\mathcal{A}_g : \widetilde{M}_h \to \widetilde{M}_{g+h}$ for all $g, h \in G$.

By Lemma 1.1.2, $\mathcal{A}_{h-g} u = M_h$ for all $g \in G$ and nonzero elements $u \in M_g$. Likewise, $A_{h-g}\widetilde{u} = \widetilde{M}_h$ for every $h \in G$, where $\widetilde{u}$ is the element of $\widetilde{M}$ having the nonzero element $u \in M_g$ in the $\widetilde{M}_g$-component and 0 elsewhere.

Suppose $\widetilde{N} = \bigoplus_{g\in G} \widetilde{N}_g$ is a nonzero G-graded submodule of $\widetilde{M}$. Then for any nonzero component $\widetilde{N}_g$ and $h \in G$, we have $\mathcal{A}_{h-g}\widetilde{N}_g = \widetilde{M}_h$. Hence $\widetilde{N} = \widetilde{M}$. □

In the next theorem, we show that graded-simple $\mathcal{A}$-modules come from thin coverings of simple (non-graded) $\mathcal{A}$-modules. The assumption on the existence of a maximal submodule holds in the finite-dimensional case as well as for the infinite-dimensional quasifinite modules. (See Remark 4.4.2.)

Theorem 1.1.4 *Let $\widetilde{M}$ be a graded-simple module over an associative algebra $\mathcal{A}$ graded by an abelian group G:*

$$\widetilde{M} = \bigoplus_{g\in G} \widetilde{M}_g.$$

Assume that as a non-graded $\mathcal{A}$-module, $\widetilde{M}$ has a maximal (non-graded) submodule and a simple (non-graded) quotient M:

$$\varphi : \widetilde{M} \to M.$$

Let $M_g = \varphi(\widetilde{M}_g)$. Then $\{M_g \mid g \in G\}$ is a thin covering of the module M and $\widetilde{M}$ is graded-isomorphic to the graded $\mathcal{A}$-module associated with this thin covering,

$$\widetilde{M} \cong \bigoplus_{g\in G} M_g,$$

as in Theorem 1.1.3.

Proof Since the map φ is surjective, we get that $\sum_{g\in G} \varphi(\widetilde{M}_g) = \varphi(\widetilde{M}) = M$, and hence $\{\varphi(\widetilde{M}_g) \mid g \in G\}$ is a covering of M.

Moreover, the restrictions of φ to $\widetilde{M}_g$,

$$\varphi_g : \widetilde{M}_g \to M_g,$$

are bijections. If these maps had non-trivial kernels, then $\bigoplus\limits_{g\in G} \ker \varphi_g$ would be a non-trivial graded submodule of $\widetilde{M}$, which contradicts the fact that $\widetilde{M}$ is graded-simple.

We now show that the covering $\{\varphi(\widetilde{M}_g) \mid g \in G\}$ is thin. Suppose $\{N_g \mid g \in G\}$ is another covering of M with $N_g \subseteq M_g$. Let $\widetilde{N}_g \subseteq \widetilde{M}_g$ be the pre-image of N_g under the map φ_g. It is easy to see that

$$\widetilde{N} = \bigoplus_{g\in G} \widetilde{N}_g$$

is a graded $\mathcal{A}$-submodule in $\widetilde{M}$. To verify this, it is enough to check that $\mathcal{A}_h\widetilde{N}_g \subseteq \widetilde{N}_{g+h}$. However, $\varphi(\mathcal{A}_h\widetilde{N}_g) = \mathcal{A}_h N_g \subseteq N_{g+h}$, and $\widetilde{N}_{g+h}$ is the pre-image of N_{g+h} under φ_{g+h}. Thus $\mathcal{A}_h\widetilde{N}_g \subseteq \widetilde{N}_{g+h}$, and $\widetilde{N}$ is a nonzero graded submodule of $\widetilde{M}$. Since $\widetilde{M}$ is graded-simple, we conclude that $\widetilde{N}_g = \widetilde{M}_g$ for all $g \in G$. This implies that $N_g = M_g$ because the map φ_g is a bijection. Hence the covering $\{M_g \mid g \in G\}$ is indeed thin.

Once again using bijectivity of φ_g, we see that the graded $\mathcal{A}$-module associated with this covering is isomorphic to $\widetilde{M}$:

$$\bigoplus_{g\in G} M_g \cong \bigoplus_{g\in G} \widetilde{M}_g.$$

□

Gradings on $\mathcal{A}$ by finite abelian groups can be alternatively described via finite abelian subgroups of the group of automorphisms of $\mathcal{A}$. Suppose $\mathcal{A}$ is graded by a finite abelian group G, and consider the dual group $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$. Then we can interpret an element $\sigma \in \widehat{G}$ as the finite order $\mathcal{A}$-automorphism defined by $\sigma(a) = \sigma(g)a$ for $a \in \mathcal{A}_g$. Although the groups G and $\widehat{G}$ are isomorphic, there is no canonical isomorphism between them.

Conversely, a finite abelian group $\widehat{G}$ of automorphisms of $\mathcal{A}$ defines a grading. The algebra $\mathcal{A}$ is graded by the group $G = \mathrm{Hom}(\widehat{G}, \mathbb{C}^*)$:

$$\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g,$$

where $\mathcal{A}_g = \left\{ a \in \mathcal{A} \;\middle|\; \sigma(a) = g(\sigma)a, \text{ for all } \sigma \in \widehat{G} \right\}$.

To be consistent with the additive notation for grading groups, we will slightly abuse notation and always treat G as an additive group. The sum of two elements $g, h \in G$ is defined by

$$(g+h)(\sigma) = g(\sigma)h(\sigma) \text{ for } \sigma \in \widehat{G}, \tag{1.1.5}$$

and we will denote the identity element of G by 0.

An $\mathcal{A}$-module (M, ρ) can be twisted by an automorphism $\sigma \in \mathrm{Aut}(\mathcal{A})$, and we use the notation M^σ for the module $(M, \rho\sigma)$. That is, the module action in M^σ is given by $\rho\sigma : \mathcal{A} \to \mathrm{End}_{\mathbb{C}} M$.

The following proposition shows that a simple $\mathcal{A}$-module and its twists with automorphisms in $\widehat{G}$ yield isomorphic graded-simple modules.

Proposition 1.1.6 *Let $\mathcal{A}$ be an associative algebra graded by a finite abelian group $\widehat{G}$. Let M be a simple $\mathcal{A}$-module, and let $\sigma \in \widehat{G}$.*

(i) Any thin covering $\{M_g \mid g \in G\}$ of M is also a thin covering of M^σ.

(ii) Let $\{M_g \mid g \in G\}$ be a thin covering of M, and let $\{M_g^\sigma \mid g \in G\}$ be the same covering of M^σ. Then the following graded-simple $\mathcal{A}$-modules are naturally isomorphic

$$\bigoplus_{g \in G} M_g^\sigma \cong \left(\bigoplus_{g \in G} M_g \right)^\sigma .$$

(iii) If M is a G-graded $\mathcal{A}$-module, then $M^\sigma \cong M$ for all $\sigma \in \widehat{G}$.

Proof We first show that a covering of M is also a covering of M^σ. For all $a \in A_g$ and $m \in M_h$, we have $\rho(a)m \in M_{g+h}$, so

$$\rho\sigma(a)m = g(\sigma)\rho(a)m \in M_{g+h}.$$

This gives a bijection between the coverings of M and the coverings of M^σ. Clearly the thin coverings of M correspond to the thin coverings of M^σ.

Part (ii) is obvious.

To establish (iii), we define the isomorphism

$$\theta : M = \bigoplus_{g \in G} M_g \to M^\sigma$$

by $\theta(m) = g(\sigma)m$ for $m \in M_g$. Let us check that θ commutes with the action of $\mathcal{A}$. For $a \in \mathcal{A}_g, m \in M_h$, we have

$$\rho\sigma(a)\theta(m) = g(\sigma)h(\sigma)\rho(a)m = \theta(\rho(a)m).$$

□

Remark 1.1.7 The following partial converse is true: if the modules M and M^σ are isomorphic for some finite order automorphism $\sigma \in \mathrm{Aut}\mathcal{A}$, then M admits a grading by the cyclic group $\langle\sigma\rangle$. (See Example 3.3.25 below.) However, if we replace a cyclic group $\langle\sigma\rangle$ by a finite abelian group G, the analogous statement is, in general, false (cf. Example 3.3.26).

2 Thin coverings of modules over a semisimple algebra

Recall that an associative algebra $\mathcal{A}$ is *semisimple* if each module for $\mathcal{A}$ is completely reducible. In Theorem 2.2.2, we prove that if $\mathcal{A}$ is semisimple, $1 \in \mathcal{A}_g\mathcal{A}_{-g}$ for all g, and each $\mathcal{A}_g$ is an irreducible $\mathcal{A}_0$-module, then the isomorphism class of the graded-simple module $\widetilde{M}$ does not depend on the choice of the thin covering of M. The following lemma is well-known (cf. [12], Cor. 2.74, for instance):

Lemma 2.2.1 *Let $\mathcal{A}$ be a semisimple associative algebra. If $\mathcal{A}$ is graded by an abelian group G, then its subalgebra $\mathcal{A}_0$ is also semisimple.* □

Theorem 2.2.2 *Let $\mathcal{A}$ be a semisimple associative algebra graded by an abelian group G. Assume that for each $g \in G$, the graded component $\mathcal{A}_g$ satisfies the following two properties:*
(i) $1 \in \mathcal{A}_g\mathcal{A}_{-g}$,
(ii) $\mathcal{A}_g$ *is irreducible as a* $\mathcal{A}_0$*-module.*
Let $\{M_g \mid g \in G\}$ and $\{N_g \mid g \in G\}$ be thin coverings for an irreducible $\mathcal{A}$-module M. Then for some $h \in G$,

$$\bigoplus_{g\in G} M_g \cong \bigoplus_{g\in G} N_{g-h}.$$

as graded $\mathcal{A}$-modules.

Proof If $M = 0$, there is nothing to prove. Otherwise, choose $h \in G$ so that $N_h \neq 0$. Note that every N_g is nonzero, since

$$0 \neq N_h \subseteq \mathcal{A}_{h-g}\mathcal{A}_{g-h}N_h \subseteq \mathcal{A}_{h-g}N_g.$$

Likewise, $M_g \neq 0$ for all $g \in G$. By Lemma 2.2.1, N_0 has a direct complement H as an $\mathcal{A}_0$-module. Since $\{M_g \mid g \in G\}$ is a covering, there is an $h \in G$ such that the projection

$$\pi_h : \; M_h \to N_0$$

(given by the splitting $M_h \subseteq M = N_0 \oplus H$) is nonzero. Since π_h is a nonzero homomorphism of simple $\mathcal{A}_0$-modules (using Lemma 1.1.2), $M_h \cong_{\mathcal{A}_0} N_0$ by Schur's Lemma.

Our next task is to show that

$$\bigoplus_{g\in G} M_g \cong \mathcal{A} \otimes_{\mathcal{A}_0} M_h$$

as graded $\mathcal{A}$-modules. We begin by noting that for each $g \in G$, the $\mathcal{A}_0$-module $\mathcal{A}_g \otimes_{\mathcal{A}_0} M_h$ is irreducible.

Let $0 \neq \sum_i a_i \otimes m_i \in \mathcal{A}_g \otimes_{\mathcal{A}_0} M_h$, and let $m \in M_h$ be nonzero. Since M_h is irreducible as an $\mathcal{A}_0$-module (Lemma 1.1.2), there exist $b_i \in \mathcal{A}_0$ such that $b_i m = m_i$ for all i. Then

$$\sum_i a_i \otimes m_i = \left(\sum_i a_i b_i\right) \otimes m.$$

Since $\mathcal{A}_g$ is irreducible as a $\mathcal{A}_0$-module, we have

$$\begin{aligned}
\mathcal{A}_0\left(\left(\sum_i a_i b_i\right) \otimes m\right) &= \mathcal{A}_g \otimes m \\
&\subseteq \mathcal{A}_g \mathcal{A}_0 \otimes m \\
&= \mathcal{A}_g \otimes \mathcal{A}_0 m \\
&= \mathcal{A}_g \otimes_{\mathcal{A}_0} M_h.
\end{aligned}$$

Thus $\mathcal{A}_g \otimes_{\mathcal{A}_0} M_h$ (and $\mathcal{A}_g \otimes_{\mathcal{A}_0} N_0$) is an irreducible $\mathcal{A}_0$-module.

Note that $\mathcal{A} \otimes_{\mathcal{A}_0} M_h = \bigoplus_{g\in G} (\mathcal{A}_g \otimes_{\mathcal{A}_0} M_h)$, and there is a well-defined graded map

$$\phi := \bigoplus_{g\in G} \phi_g : \bigoplus_{g\in G} (\mathcal{A}_g \otimes_{\mathcal{A}_0} M_h) \to \bigoplus_{g\in G} M_g$$

given by $\phi_g(a \otimes m) = am \in M_g$ for all $a \in \mathcal{A}_{g-h}$ and $m \in M_h$. The map ϕ_g is a nonzero homomorphism between simple $\mathcal{A}_0$-modules, so is an $\mathcal{A}_0$-module isomorphism. Thus ϕ is a bijection. Also

$$\begin{aligned}
\phi(ab \otimes m) &= abm \\
&= a\phi(b \otimes m)
\end{aligned}$$

for any homogeneous $a, b \in \mathcal{A}$ and $m \in M_h$, so ϕ is an isomorphism of graded $\mathcal{A}$-modules. Likewise,

$$\mathcal{A} \otimes_{\mathcal{A}_0} N_0 = \bigoplus_{g\in G} \mathcal{A}_g \otimes_{\mathcal{A}_0} N_0 \cong \bigoplus_{g\in G} N_g$$

as graded $\mathcal{A}$-modules.

Since the $\mathcal{A}_0$-modules M_h and N_0 are isomorphic, we get that $\mathcal{A} \otimes_{\mathcal{A}_0} M_h \cong \mathcal{A} \otimes_{\mathcal{A}_0} N_0$, and thus

$$\bigoplus_{g\in G} M_g \cong \bigoplus_{g\in G} N_{g-h}.$$

as graded $\mathcal{A}$-modules. □

While this method of associating graded modules $\widetilde{M}$ with modules M is not functorial (it is not even well-defined if M is not simple), it is surjective in the sense that not only does every graded module come from a thin covering, but (in the context of the previous theorem) every graded-simple module comes from a thin covering of a simple module.

The first assertion ("graded modules come from thin coverings") is trivial. Let $M = \bigoplus_{g\in G} M_g$ be a graded module of a graded associative algebra $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$. Forgetting the graded structure on M gives an (ungraded) $\mathcal{A}$-module M with thin covering $\{M_g \mid g \in G\}$. The second claim ("graded-simples come from simples") is the following theorem. (See also Theorem 1.1.4.)

Theorem 2.2.3 *Let $M = \bigoplus_{g\in G} M_g$ be a graded-simple module for a semisimple associative algebra $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ graded by an abelian group G. Assume that $\mathcal{A}$ satisfies conditions (i) and (ii) of Theorem 2.2.2. Then there is a simple module N for $\mathcal{A}$ so that N has a thin covering $\{N_g \mid g \in G\}$ with*

$$\bigoplus_{g\in G} N_g \cong \bigoplus_{g\in G} M_g$$

as graded $\mathcal{A}$-modules.

Proof Let N be a simple $\mathcal{A}$-submodule of M, and let N' be a simple $\mathcal{A}_0$-submodule of N. Since N is simple,

$$N = \mathcal{A}N' = \sum_{g\in G} \mathcal{A}_g N'.$$

Clearly $\mathcal{A}_g\mathcal{A}_h N' \subseteq \mathcal{A}_{g+h}N'$, so $\{\mathcal{A}_g N' \mid g \in G\}$ is a covering of N. By the argument in the proof of Theorem 2.2.2, the $\mathcal{A}_0$-modules $\mathcal{A}_g N'$ are simple, as are the components N_g in any subcovering $\{N_g \mid g \in G\}$. The N_g are all nonzero by the proof in Theorem 2.2.2. Thus the covering $\{\mathcal{A}_g N' \mid g \in G\}$ is thin.

Since N is nonzero, the projection

$$\pi_h: \ N \to M_h$$

is nonzero for some $h \in G$. Choose $h' \in G$ so that the restriction of π_h to $\mathcal{A}_{h'}N'$ is also nonzero. For each $g \in G$, let

$$C_g = \mathcal{A}_{g-h+h'}N'.$$

Then $\{C_g \mid g \in G\}$ is a thin covering of N with the property that the projection map

$$\pi_h: \ C_h \to M_h$$

is nonzero. Since $C_g = \mathcal{A}_{g-h}C_h$ and $M_g = \mathcal{A}_{g-h}M_h$, we see that the projection $\pi_g: \ C_g \to M_g$ is also nonzero.

Let

$$\pi = \bigoplus_{g\in G} \pi_g : \ \bigoplus_{g\in G} C_g \to \bigoplus_{g\in G} M_g,$$

where the restriction of π to C_g is the projection π_g. Then since each π_g is a nonzero homomorphism between simple $\mathcal{A}_0$-modules C_g and M_g, the maps π_g are $\mathcal{A}_0$-module isomorphisms. Thus π is a bijection.

Finally, π is also a graded homomorphism since

$$\pi(an) = \pi_{g+g'}(an) = a\pi_{g'}(n) = a\pi(n)$$

for all $a \in \mathcal{A}_g$ and $n \in C_{g'}$. Thus

$$\bigoplus_{g\in G} C_g \cong \bigoplus_{g\in G} M_g$$

as graded $\mathcal{A}$-modules. $\square$

3 Classification of Thin Coverings

In this section, we will assume that $\mathcal{A}$ is a (unital) associative algebra over $\mathbb{C}$, and M is a finite-dimensional irreducible $\mathcal{A}$-module whose action is given by the homomorphism $\rho: \ \mathcal{A} \to \mathrm{End}_{\mathbb{C}}M$.

In order to find the thin coverings of a module M, it is important to know for which $\sigma \in \widehat{G}$, the modules M and M^σ are isomorphic.

Lemma 3.3.1 *Let $\widehat{G}$ be a subgroup of* $\operatorname{Aut}\mathcal{A}$ *and let*

$$\widehat{H} = \left\{ \sigma \in \widehat{G} \;\middle|\; M^\sigma \cong M \text{ as } \mathcal{A}\text{-modules} \right\}.$$

Then

(i) $\widehat{H}$ is a subgroup of $\widehat{G}$.

(ii) For $\sigma_1, \sigma_2 \in \widehat{G}$ we have $M^{\sigma_1} \cong M^{\sigma_2}$ if and only if $\sigma_1 \sigma_2^{-1} \in \widehat{H}$.

Proof An isomorphism between $\mathcal{A}$-modules M and M^σ is a map $T_\sigma \in \mathrm{GL}(M)$ satisfying

$$T_\sigma \rho(a) u = \rho(\sigma(a)) T_\sigma u,$$

for all $a \in \mathcal{A}$ and $u \in M$. Equivalently,

$$\rho(\sigma(a)) = T_\sigma \rho(a) T_\sigma^{-1}.$$

The first part of the lemma follows from this relation.

The fact that the modules M^{σ_1} and M^{σ_2} are isomorphic is equivalent to the existence of a $T \in \mathrm{GL}(M)$, such that $\rho(\sigma_1(a)) = T\rho(\sigma_2(a))T^{-1}$. Letting $b = \sigma_2(a)$, we get that $\rho(\sigma_1\sigma_2^{-1}(b)) = T\rho(b)T^{-1}$, which means that $\sigma_1\sigma_2^{-1} \in \widehat{H}$. This establishes the second claim of the lemma. □

Consider the operator T_σ introduced in the above proof. If $\sigma \in \widehat{H}$ has order m, then for all $a \in \mathcal{A}$,

$$\rho(a) = \rho(\sigma^m(a)) = T_\sigma^m \rho(a) T_\sigma^{-m}.$$

By Schur's Lemma, T_σ^m is a scalar operator, and we may normalize T_σ to get $T_\sigma^m = \mathrm{id}$. It is often convenient to use this normalization of T_σ.

Recall that there is an equivalent description of gradings and coverings in terms of the dual groups $H = \mathrm{Hom}(\widehat{H}, \mathbb{C}^*)$ and $G = \mathrm{Hom}(\widehat{G}, \mathbb{C}^*)$ (cf. Section 1). Our strategy will be to describe the thin H-coverings of M, and then use them to determine the thin G-coverings. In order to understand the structure of thin H-coverings, we need to better understand the H-grading on $\mathcal{A}$ and on its image $\rho(\mathcal{A}) = \mathrm{End}_{\mathbb{C}}(M)$. This can be done in two ways. One possibility is to apply the theorem of Bahturin et al. ([4], Thm. 6) on gradings of a matrix algebra. A second, equivalent approach, is to use the classification of cyclotomic quantum tori [13, 1]. We will follow the second approach here, since it can also be adapted for the infinite-dimensional (quasifinite) set-up.

Let us recall the definition of a quantum torus. A *quantum torus* is the unital associative algebra $\mathcal{R}_q = \mathcal{R}_q \left\langle t_1^{\pm}, \dots, t_r^{\pm} \right\rangle$, whose generators t_i are subject to the defining relations

$$t_i t_j = q_{ij} t_j t_i, \quad i, j = 1, \dots, r,$$

where $q = (q_{ij})$ is a complex $r \times r$ matrix satisfying $q_{ii} = 1$, $q_{ij} = q_{ji}^{-1}$.

We say that the quantum torus $\mathcal{R}_q$ is *cyclotomic* if all q_{ij} are complex roots of 1.

Lemma 3.3.2 *Let $\eta_1, \dots, \eta_r$ be a set of generators of the group $\widehat{H}$ of orders $s_1, \dots, s_r$, respectively, and let $T_1, \dots, T_r$ be invertible operators on M satisfying*

$$\rho(\eta_j(a)) = T_j \rho(a) T_j^{-1}, \quad \text{for all} \quad a \in \mathcal{A}, \ j = 1, \dots, r. \tag{3.3.3}$$

Then the map $t_i \mapsto T_i$ defines a representation of a cyclotomic quantum torus $\mathcal{R}_q \left\langle t_1^{\pm}, \dots, t_r^{\pm} \right\rangle$, with

$$q_{ij}^{\gcd(s_i, s_j)} = 1.$$

Proof The automorphisms η_i and η_j commute. Thus for all $a \in \mathcal{A}$,

$$T_i T_j \rho(a) T_j^{-1} T_i^{-1} = T_j T_i \rho(a) T_i^{-1} T_j^{-1},$$

or equivalently,

$$T_i^{-1} T_j^{-1} T_i T_j \rho(a) = \rho(a) T_i^{-1} T_j^{-1} T_i T_j.$$

However, by Schur's Lemma, the centralizer of a finite-dimensional simple module is a multiple of the identity map, so

$$T_i T_j = q_{ij} T_j T_i$$

for some nonzero $q_{ij} \in \mathbb{C}$.

Moreover, the relation $\eta_i^{s_i} = 1$ implies that $T_i^{s_i}$ is a multiple of the identity. Since $T_i^{s_i} T_j = q_{ij}^{s_i} T_j T_i^{s_i}$, we conclude that

$$q_{ij}^{s_i} = 1.$$

Likewise, $q_{ij}^{s_j} = 1$, so $q_{ij}^{\gcd(s_i, s_j)} = 1$. □

We will need elements of the representation theory of cyclotomic quantum tori. We call an $\mathcal{R}_q$-module U *diagonalizable* if every generator t_i is diagonalizable on U. Whenever $\mathcal{R}_q$ is noncommutative, these operators clearly cannot be diagonal with respect to the same basis. Let $\mathcal{C}$ be the category of finite-dimensional diagonalizable $\mathcal{R}_q$-modules. Since $T_i^{s_i}$ is a multiple of the identity on M, we see that M belongs to Category $\mathcal{C}$.

Lemma 3.3.4 *Let $\mathcal{R}_q$ be a cyclotomic quantum torus, and let U be a module in Category $\mathcal{C}$. Then for any $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$, the monomial $\mathbf{t}^{\mathbf{m}} = t_1^{m_1} \ldots t_r^{m_r}$ is diagonalizable on U.*

Proof Choose a positive integer b such that $q_{ij}^b = 1$ for all $i, j = 1, \ldots, r$. Then the elements $t_i^{\pm b}$ belong to the centre of $\mathcal{R}_q$. Since each of them is diagonalizable on U, they are simultaneously diagonalizable. Moreover, $(\mathbf{t}^{\mathbf{m}})^b$ belongs to the subalgebra generated by $t_1^{\pm b}, \ldots, t_r^{\pm b}$, and hence diagonalizable. This implies that $\mathbf{t}^{\mathbf{m}}$ is diagonalizable as well. $\square$

Proposition 3.3.5 *Every module in Category $\mathcal{C}$ for the cyclotomic quantum torus $\mathcal{R}_q = \mathcal{R}_q \langle t_1^{\pm}, \ldots, t_r^{\pm} \rangle$ is completely reducible, and all simple $\mathcal{R}_q$-modules in Category $\mathcal{C}$ have the same dimension. More explicitly, fix an arbitrary simple $\mathcal{R}_q$-module N in $\mathcal{C}$. Then all other simple $\mathcal{R}_q$-modules in $\mathcal{C}$ can be obtained from N as twists by the rescaling automorphisms of $\mathcal{R}_q$:*

$$t_j \mapsto \alpha_j t_j, \quad j = 1, \ldots, r,$$

where $\alpha_1, \ldots, \alpha_r \in \mathbb{C}^*$.

Proof To prove this proposition, we will put $\mathcal{R}_q$ in a normal form, using the classification of cyclotomic quantum tori [13, 1]. If we make a change of variables in a quantum torus using a matrix $(a_{ij}) \in \mathrm{GL}_r(\mathbb{Z})$:

$$\bar{t}_i = t_1^{a_{i1}} t_2^{a_{i2}} \ldots t_r^{a_{ir}},$$

we will get another quantum torus with a new matrix $\bar{q}_{ij}$. The classification theorem states that by means of such a change of variables, a cyclotomic quantum torus can be brought to the normal form

$$\mathcal{R}_q \langle t_1^{\pm}, \ldots, t_r^{\pm} \rangle \cong \mathcal{R}_{\zeta_1} \langle x_1^{\pm}, y_1^{\pm} \rangle \otimes \ldots \otimes \mathcal{R}_{\zeta_\ell} \langle x_\ell^{\pm}, y_\ell^{\pm} \rangle \otimes \mathbb{C}[z_{2\ell+1}^{\pm}, \ldots, z_r^{\pm}], \tag{3.3.6}$$

where the first ℓ tensor factors are rank two cyclotomic quantum tori, and the last tensor factor is a commutative algebra of Laurent polynomials. By

Lemma 3.3.4, an $\mathcal{R}_q$-module's membership in Category $\mathcal{C}$ is independent of the choice of the generators $t_i^{\pm}$ of $\mathcal{R}_q$.

The rank two quantum tori $\mathcal{R}_\zeta = \mathcal{R}_\zeta \langle x^{\pm}, y^{\pm} \rangle$ that appear in (3.3.6) have the following defining relation

$$xy = \zeta yx,$$

where ζ is a primitive complex dth root of unity. The centre of $\mathcal{R}_\zeta$ is generated by $x^{\pm d}$ and $y^{\pm d}$. Hence the centre of $\mathcal{R}_q$ is the Laurent polynomial algebra:

$$\mathcal{Z}(\mathcal{R}_q) = \mathbb{C}[x_1^{\pm d_1}, y_1^{\pm d_1}, \dots, x_\ell^{\pm d_\ell}, y_\ell^{\pm d_\ell}, z_{2\ell+1}^{\pm}, \dots, z_r^{\pm}].$$

Let U be a finite-dimensional diagonalizable $\mathcal{R}_q$-module. Since the action of $\mathcal{Z}(\mathcal{R}_q)$ on U is diagonalizable, we may decompose U into the direct sum of $\mathcal{R}_q$-submodules corresponding to various central characters $\chi : \mathcal{Z}(\mathcal{R}_q) \to \mathbb{C}$:

$$U = \bigoplus_{\chi} U^{\chi}.$$

Note that a central character χ is determined by its (nonzero) values $\chi(x_i^{d_i})$, $\chi(y_i^{d_i})$, and $\chi(z_i)$. Hence, by means of the rescaling automorphisms of $\mathcal{R}_q$, we may twist a module with central character χ into a module with any other central character.

We will show that, up to isomorphism, there exists a unique finite-dimensional simple $\mathcal{R}_q$-module for each central character.

In order to prove that the module U is is completely reducible, it is enough to consider its single component U^{χ}. This component is a module for the quotient $\mathcal{R}_q^{\chi}$ of $\mathcal{R}_q$ by the central ideal:

$$\mathcal{R}_q^{\chi} = \mathcal{R}_q / \big\langle z - \chi(z)1 \mid z \in \mathcal{Z}(\mathcal{R}_q) \big\rangle.$$

The algebra $\mathcal{R}_q^{\chi}$ decomposes into the tensor product:

$$\mathcal{R}_q^{\chi} \cong \mathcal{R}_{\zeta_1}^{\chi} \langle x_1^{\pm}, y_1^{\pm} \rangle \otimes \ldots \otimes \mathcal{R}_{\zeta_\ell}^{\chi} \langle x_\ell^{\pm}, y_\ell^{\pm} \rangle .$$

The rank two factors here have the following structure:

$$\mathcal{R}_\zeta^{\chi} = \mathcal{R}_\zeta \langle x^{\pm}, y^{\pm} \rangle \Big/ \Big\langle x^d - \chi(x^d)1,\, y^d - \chi(y^d)1 \Big\rangle .$$

In fact, $\mathcal{R}_\zeta^{\chi}$ is isomorphic to the matrix algebra $M_d(\mathbb{C})$ under the following isomorphism:

$$x \longmapsto \big(\chi(x^d)\big)^{1/d} \begin{pmatrix} 1 & 0 & \cdot & \cdot & 0 \\ 0 & \zeta & \cdot & \cdot & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdot & \cdot & \zeta^{d-1} \end{pmatrix}$$

$$\text{and} \quad y \longmapsto \begin{pmatrix} 0 & 0 & \cdot & 0 & \chi(y^d) \\ 1 & 0 & \cdot & 0 & 0 \\ 0 & 1 & \cdot & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdot & 1 & 0 \end{pmatrix}. \tag{3.3.7}$$

This implies that the algebra $\mathcal{R}_q^\chi$ is isomorphic to the matrix algebra of rank $d_1 \cdots d_\ell$. It is well-known that the matrix algebra is semisimple and has a unique, up to isomorphism, finite-dimensional simple module of dimension equal to the rank of the matrix algebra. □

The following corollary is an immediate consequence of Proposition 3.3.5.

Corollary 3.3.8 *Let N be a simple module in Category $\mathcal{C}$ for the cyclotomic quantum torus $\mathcal{R}_q = \mathcal{R}_q \left\langle t_1^\pm, \dots, t_r^\pm \right\rangle$. Then every module M in Category $\mathcal{C}$ may be written as a finite sum*

$$M \cong N \otimes \left(\bigoplus_{\alpha \in (\mathbb{C}^*)^r} V^\alpha \right), \tag{3.3.9}$$

where the action of $\mathcal{R}_q$ on $N \otimes V^\alpha$ is given by the formula

$$t_i(u \otimes v^\alpha) = t_i(u) \otimes \alpha_i v^\alpha, \quad \textit{for} \quad u \in N, v^\alpha \in V^\alpha. \tag{3.3.10}$$

Proof If we take a one-dimensional space V^α, then $N \otimes V^\alpha$ is a simple $\mathcal{R}_q$-module, which is a twist of N by the rescaling automorphism of $\mathcal{R}_q$ given by $\alpha \in (\mathbb{C}^*)^r$. Thus the decomposition (3.3.9) is just an isotypic decomposition of the module M. □

We now look at the representation of $\mathcal{R}_q$ described in Lemma 3.3.2. We show that in this module, the elements α appearing in the isotypic decomposition of M are not arbitrary, but belong to the group H.

Lemma 3.3.11 *Let M be a finite-dimensional simple module for an H-graded algebra $\mathcal{A}$. Suppose that $T_1, \dots, T_r$ are operators on M that satisfy (3.3.3) and define the action of a cyclotomic quantum torus $\mathcal{R}_q$ on M. Let N be a simple $\mathcal{R}_q$-submodule of M. Then the isotypic decomposition (3.3.9) for M can be written as*

$$M \cong N \otimes \left(\bigoplus_{h \in H} V^h \right), \tag{3.3.12}$$

where the action of $\mathcal{R}_q$ on $N \otimes V^h$ is given by the formula

$$t_i(u \otimes v^h) = t_i(u) \otimes h(\eta_i) v^h, \quad \textit{for} \quad u \in N, v^h \in V^h. \tag{3.3.13}$$

Proof The H-grading of the algebra $\mathcal{A}$ yields an H-grading of the matrix algebra

$$\mathrm{End}_{\mathbb{C}}(M) = \rho(\mathcal{A}) = \bigoplus_{h \in H} \rho(\mathcal{A}_h), \tag{3.3.14}$$

where the endomorphisms in $\rho(\mathcal{A}_h)$ satisfy

$$T_j \rho(a) T_j^{-1} = \rho(\eta_j(a)) = h(\eta_j) \rho(a). \tag{3.3.15}$$

This can be viewed as an eigenspace decomposition of $\mathrm{End}_{\mathbb{C}}(M)$ with respect to conjugations by the T_j's.

On the other hand, we can construct such an eigenspace decomposition using the isotypic decomposition (3.3.9). In particular, let $N \otimes V^0$ be the isotypic component of N in M, and let $N \otimes V^\alpha$ be another nontrivial component in (3.3.9). We claim that $\mathrm{id}_N \otimes \mathrm{Hom}_{\mathbb{C}}(V^0, V^\alpha)$ can be viewed as the eigenspace (with eigenvalue α_j) for the conjugation action of T_j on $\mathrm{End}_{\mathbb{C}}(M)$. Indeed, extend $S \in \mathrm{Hom}_{\mathbb{C}}(V^0, V^\alpha)$ to $\mathrm{End}_{\mathbb{C}}(\bigoplus_\beta V^\beta)$ by setting $S\left(V^\beta\right) = 0$ when $\beta \neq 0$. Then for $n \in N$ and $v \in V^0$, we have

$$\begin{aligned} T_j(\mathrm{id}_N \otimes S) T_j^{-1}(n \otimes v) &= T_j(\mathrm{id}_N \otimes S)(T_j^{-1}(n) \otimes v) \\ &= T_j(T_j^{-1}(n) \otimes Sv) \\ &= \alpha_j n \otimes Sv. \end{aligned}$$

Thus $T_j(\mathrm{id}_N \otimes S) T_j^{-1} = \alpha_j(\mathrm{id}_N \otimes S)$. By (3.3.14) and (3.3.15), all eigenvalues of the conjugation action of T_j on $\mathrm{End}_{\mathbb{C}}(M)$ correspond to the elements of H, so there exists $h \in H$ such that $\alpha_j = h(\eta_j)$ for all $j = 1, \dots, r$. $\square$

Next we explicitly describe the eigenspace decomposition of $\operatorname{End}_{\mathbb{C}}(M)$. In order to do this, we introduce the group homomorphism

$$\gamma : \mathbb{Z}^r \to H,$$

defined by the formula

$$\gamma(\mathbf{a})(\eta^{\mathbf{b}}) = \prod_{i,j=1}^{r} q_{ij}^{a_i b_j}, \quad \text{for } \mathbf{a}, \mathbf{b} \in \mathbb{Z}^r.$$

Here we use the multi-index notation $\eta^{\mathbf{b}} = \eta_1^{b_1} \dots \eta_r^{b_r}$ and $\mathbf{T}^{\mathbf{b}} = T_1^{b_1} \dots T_r^{b_r}$. The fact that $\gamma(\mathbf{a}' + \mathbf{a}'') = \gamma(\mathbf{a}') + \gamma(\mathbf{a}'')$ is obvious (see (1.1.5) for the definition of the additive notation), so all we need to verify is that this map is well-defined. That is, we should check that $\gamma(\mathbf{a})(\eta^{\mathbf{b}'}) = \gamma(\mathbf{a})(\eta^{\mathbf{b}''})$ whenever $\eta^{\mathbf{b}'} = \eta^{\mathbf{b}''}$ in H.

Taking $\mathbf{b} = \mathbf{b}' - \mathbf{b}''$, it is enough to show that $\gamma(\mathbf{a})(\eta^{\mathbf{b}}) = 1$ whenever $\eta^{\mathbf{b}} = 1$ in H. However,

$$\mathbf{T}^{\mathbf{a}}\mathbf{T}^{\mathbf{b}} = \gamma(\mathbf{a})(\eta^{\mathbf{b}})\mathbf{T}^{\mathbf{b}}\mathbf{T}^{\mathbf{a}}.$$

If $\eta^{\mathbf{b}} = 1$, then $\mathbf{T}^{\mathbf{b}}$ is a multiple of the identity and thus commutes with $\mathbf{T}^{\mathbf{a}}$. This implies that $\gamma(\mathbf{a})(\eta^{\mathbf{b}}) = 1$.

Note that the kernel of γ corresponds to the centre of the quantum torus $\mathcal{R}_q$:

$$\mathcal{Z}(\mathcal{R}_q) = \operatorname{Span}\{\mathbf{t}^{\mathbf{a}} \mid \mathbf{a} \in \ker \gamma\}.$$

By Schur's Lemma, the centre $\mathcal{Z}(\mathcal{R}_q)$ acts on a finite-dimensional simple $\mathcal{R}_q$-module N by scalar operators.

The following proposition is essentially equivalent to the classification of gradings on the matrix algebra ([4], Thm. 6).

Proposition 3.3.16 *Let $M = N \otimes \left(\bigoplus_{p \in H} V^p \right)$ be the isotypic decomposition of M as an $\mathcal{R}_q$-module. Then the space $\operatorname{End}_{\mathbb{C}}(M)$ decomposes into a direct sum of eigenspaces with respect to conjugation by the operators T_j, $j = 1, \dots, r$:*

$$\operatorname{End}_{\mathbb{C}}(M) = \bigoplus_{h \in H} \rho(\mathcal{A}_h),$$

where

$$\rho(\mathcal{A}_h) = \sum_{\mathbf{a} \in \mathbb{Z}^r} \mathbf{T}^{\mathbf{a}} \otimes \left(\bigoplus_{p \in H} \operatorname{Hom}_{\mathbb{C}} \left(V^p, V^{p+h+\gamma(\mathbf{a})} \right) \right),$$

is the eigenspace with eigenvalue $h(\eta_j)$ with respect to conjugation by T_j.

Proof It is easy to see that the subspaces

$$\mathbf{T}^{\mathbf{a}} \otimes \left(\bigoplus_{p \in H} \mathrm{Hom}_{\mathbb{C}} \left(V^p, V^{p+h+\gamma(\mathbf{a})} \right) \right)$$

span $\mathrm{End}_{\mathbb{C}}(M) = \mathrm{End}_{\mathbb{C}}(N) \otimes \mathrm{End}_{\mathbb{C}} \left(\bigoplus_{h \in H} V^h \right)$. It remains only to verify that these subspaces are the eigenspaces for the conjugation by T_j's.

Let $S \in \mathrm{Hom}_{\mathbb{C}} \left(V^p, V^{p+h+\gamma(\mathbf{a})} \right)$, $n \in N$, $v \in V^p$. Then

$$\begin{aligned} T_j(\mathbf{T}^{\mathbf{a}} \otimes S) T_j^{-1}(n \otimes v) &= T_j(\mathbf{T}^{\mathbf{a}} \otimes S)(T_j^{-1}(n) \otimes p^{-1}(\eta_j) v) \\ &= T_j(\mathbf{T}^{\mathbf{a}} T_j^{-1}(n) \otimes p^{-1}(\eta_j) S v) \\ &= p(\eta_j) h(\eta_j) \gamma(\mathbf{a})(\eta_j) p^{-1}(\eta_j) T_j \mathbf{T}^{\mathbf{a}} T_j^{-1}(n) \otimes S v \\ &= h(\eta_j)(\mathbf{T}^{\mathbf{a}} \otimes S)(n \otimes v). \end{aligned}$$

□

Using this result, we can now describe the thin H-coverings of the module M.

Theorem 3.3.17 *Let M be a finite-dimensional simple $\mathcal{A}$-module, and let $\widehat{H}$ be a finite abelian subgroup of $\mathrm{Aut}\,\mathcal{A}$, such that the twisted modules M^η are isomorphic to M for all $\eta \in \widehat{H}$. Let*

$$M = \bigoplus_{h \in H} \left(N \otimes V^h \right)$$

be the isotypic decomposition of M with respect to the action of the cyclotomic quantum torus $\mathcal{R}_q$ given by (3.3.13). Then up to equivalence, thin H-coverings of M are parametrized by one-dimensional subspaces in N. Namely, let $n \in N$, $n \neq 0$. Define

$$M_h = \sum_{\mathbf{a} \in \mathbb{Z}^r} \mathbf{T}^{\mathbf{a}}(n) \otimes V^{h+\gamma(\mathbf{a})}. \tag{3.3.18}$$

Then $\{M_h \mid h \in H\}$ is a thin H-covering of M, and every thin H-covering of M is equivalent to a covering of this type.

Remark 3.3.19 The sum in (3.3.18) may be replaced with a finite sum taken over $\mathbf{a} \in \mathbb{Z}^r / \ker \gamma$.

Proof By Proposition 3.3.16,

$$\rho(\mathcal{A}_m) = \sum_{\mathbf{b}\in\mathbb{Z}^r} \mathbf{T}^{\mathbf{b}} \otimes \left(\bigoplus_{p\in H} \mathrm{Hom}_{\mathbb{C}}\big(V^p, V^{p+m+\gamma(\mathbf{b})}\big) \right)$$

for any $m \in H$. Thus

$$\begin{aligned}
\rho(\mathcal{A}_m)M_h &= \sum_{\mathbf{a},\mathbf{b}\in\mathbb{Z}^r} \mathbf{T}^{\mathbf{b}}\mathbf{T}^{\mathbf{a}}(n) \otimes V^{h+m+\gamma(\mathbf{a})+\gamma(\mathbf{b})} && (3.3.20)\\
&= \sum_{\mathbf{c}\in\mathbb{Z}^r} \mathbf{T}^{\mathbf{c}}(n) \otimes V^{h+m+\gamma(\mathbf{c})} && (3.3.21)\\
&= M_{h+m}. && (3.3.22)
\end{aligned}$$

The subspaces M_h clearly span M:

$$\begin{aligned}
\sum_{h\in H} M_h &= \sum_{h\in H}\sum_{\mathbf{a}\in\mathbb{Z}^r} \mathbf{T}^{\mathbf{a}}(n) \otimes V^{h+\gamma(\mathbf{a})}\\
&= \left(\sum_{\mathbf{a}\in\mathbb{Z}^r} \mathbf{T}^{\mathbf{a}}(n)\right) \otimes \left(\bigoplus_{h\in H} V^h\right)\\
&= N \otimes \left(\bigoplus_{h\in H} V^h\right)\\
&= M.
\end{aligned}$$

Thus $\{M_h \mid h \in H\}$ is an H-covering of M.

We will use Lemma 1.1.2 to show that this covering is thin. All that remains is to determine the simple $\rho(\mathcal{A}_0)$-submodules of M. Let $L \subseteq M$ be a simple $\rho(\mathcal{A}_0)$-submodule, and let u be a nonzero vector in L. Fix a basis $\{v_{hj} \mid 1 \le j \le \dim V^h\}$ for each of the spaces V^h. Expand u according to this basis:

$$u = \sum_{h,j} n_{hj} \otimes v_{hj} \text{ with } n_{hj} \in N.$$

Since the algebra $\rho(\mathcal{A}_0)$ contains a subalgebra

$$\mathrm{id}_N \otimes \left(\bigoplus_{h\in H} \mathrm{End}_{\mathbb{C}}\left(V^h\right)\right),$$

we see that L contains $n_{hj} \otimes v_{hj}$ for all h, j. Thus without loss of generality, we may assume that $u = n \otimes v$, where $n \in N$, $v \in V^h$. Since

$$\rho(\mathcal{A}_0) = \sum_{\mathbf{a}\in\mathbb{Z}^r} \mathbf{T}^{\mathbf{a}} \otimes \left(\bigoplus_{p\in H} \mathrm{Hom}_{\mathbb{C}}\left(V^p, V^{p+\gamma(\mathbf{a})}\right)\right),$$

L contains the subspace

$$U^h(n) = \sum_{\mathbf{a} \in \mathbb{Z}^r} \mathbf{T}^{\mathbf{a}}(n) \otimes V^{h+\gamma(\mathbf{a})}. \tag{3.3.23}$$

The subspace $U^h(n)$ is $\rho(\mathcal{A}_0)$-invariant, and hence every simple $\rho(\mathcal{A}_0)$-submodule in M coincides with $U^h(n)$ for some $n \in N$, $h \in H$, $n \neq 0$, $V^h \neq 0$.

Let us show that $U^h(n)$ is, in fact, a simple $\rho(\mathcal{A}_0)$-module for all $n \in N$, $h \in H$ such that $n \neq 0$ and $V^h \neq 0$. Fix $\mathbf{b} \in \mathbb{Z}^r$. The intersection of $U^h(n)$ with $N \otimes V^{h+\gamma(\mathbf{b})}$ is

$$\sum_{\mathbf{a} \in \ker \gamma} \mathbf{T}^{\mathbf{b}} \mathbf{T}^{\mathbf{a}}(n) \otimes V^{h+\gamma(\mathbf{b})}.$$

The centre $\mathcal{Z}(\mathcal{R}_q)$ acts on the simple $\mathcal{R}_q$-module N by scalars. Thus for all $\mathbf{a} \in \ker \gamma$, $\mathbf{T}^{\mathbf{a}}(n)$ is a multiple of n, so the intersection of $U^h(n)$ with $N \otimes V^{h+\gamma(\mathbf{b})}$ is $\mathbf{T}^{\mathbf{b}}(n) \otimes V^{h+\gamma(\mathbf{b})}$. By the argument given above, every nonzero vector in $U^h(n)$ generates $U^h(n)$. Therefore the spaces $U^h(n)$ with $V^h \neq 0$, $n \neq 0$, exhaust all the simple $\rho(\mathcal{A}_0)$-submodules in M.

This proves that the components M_h of the H-covering (3.3.18) are simple $\rho(\mathcal{A}_0)$-modules whenever $M_h \neq 0$. Thus for any nonzero $m \in M_h$, we have

$$\rho(\mathcal{A}_{g-h})m \supseteq \rho(\mathcal{A}_{g-h})\rho(\mathcal{A}_0)m = \rho(\mathcal{A}_{g-h})M_h = M_g,$$

using (3.3.20). Hence the covering $\{M_h \mid h \in H\}$ is thin by Lemma 1.1.2.

Since the described components of the thin coverings exhaust all the simple $\rho(\mathcal{A}_0)$-submodules in M, we obtain that up to equivalence, these are all the thin H-coverings of M. □

Corollary 3.3.24 *Let M be a finite-dimensional simple $\mathcal{A}$-module, and let $\widehat{H}$ be a finite abelian subgroup of* $\operatorname{Aut} \mathcal{A}$*, such that the twisted modules M^η are isomorphic to M for all $\eta \in \widehat{H}$. Let $\{M_h \mid h \in H\}$ and $\{M'_h \mid h \in H\}$ be two thin coverings of M. Then for some $g \in H$, the graded-simple modules*

$$\widetilde{M} = \bigoplus_{h \in H} M_h \text{ and } \widetilde{M'} = \bigoplus_{h \in H} M'_{h+g}$$

are isomorphic as graded $\mathcal{A}$-modules.

Proof Clearly, we are allowed to replace the thin coverings $\{M_h\}$ and $\{M'_h\}$ with any equivalent thin coverings. By Theorem 3.3.17, we may assume that for all $h \in H$,

$$\widetilde{M}_h = \sum_{\mathbf{a} \in \mathbb{Z}^r} \mathbf{T}^{\mathbf{a}}(n) \otimes V^{h+\gamma(\mathbf{a})} \text{ and } \widetilde{M}'_h = \sum_{\mathbf{a} \in \mathbb{Z}^r} \mathbf{T}^{\mathbf{a}}(n') \otimes V^{h+\gamma(\mathbf{a})}$$

for some nonzero $n, n' \in N$.

We construct a grading-preserving isomorphism $\psi : \widetilde{M} \to \widetilde{M}'$ defined on $\widetilde{M}_h$ by

$$\psi\left(\mathbf{T}^{\mathbf{a}}(n) \otimes v\right) = \mathbf{T}^{\mathbf{a}}(n') \otimes v$$

for all $h \in H, \mathbf{a} \in \mathbb{Z}^r, v \in V^{h+\gamma(\mathbf{a})}$. Using Proposition 3.3.16, it is easy to see that ψ commutes with the action of $\mathcal{A}$. $\square$

We now consider some basic examples of thin H-coverings.

Example 3.3.25 Suppose $\mathcal{A}$ is graded by a finite cyclic group $\widehat{H} = \langle \eta \rangle \subseteq \mathrm{Aut}\mathcal{A}$, and suppose that there is a (normalized) isomorphism $T : M \to M^{\eta}$. Then, up to equivalence, there is a unique thin $H = \mathrm{Hom}(\widehat{H}, \mathbb{C}^*)$-covering $\{M_h\}$ of M, where M_h is the eigenspace

$$M_h = \{u \in M \mid Tu = h(\eta)u\}.$$

In this case, the module M admits an H-grading: $M = \bigoplus_{h \in H} M_h$.

Proof Since the sum $M = \sum_{h \in H} M_h$ is direct, the thinness of the covering $\{M_h\}$ in Example 3.3.25 is obvious. However, even in this simplest of examples, the *uniqueness* of the thin covering requires Theorem 3.3.17. Note that, in this case, the quantum torus $\mathcal{R}_q$ is just the algebra of Laurent polynomials in a single variable, so $\dim N = 1$, and M may be identified with $\bigoplus_{h \in H} V_h$. $\square$

From the statement of Theorem 3.3.17, it is clear that uniqueness of thin coverings is the exception, rather than the rule, as is illustrated by the next pair of examples.

Example 3.3.26 Let $\mathcal{A} = M_n(\mathbb{C})$ be the associative algebra of $n \times n$ matrices. If $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ is any grading satisfying conditions (i) and (ii) of Theorem 2.2.2, then by Theorem 2.2.3, every graded-simple module for $\mathcal{A}$ comes from a thin covering of a simple module for $\mathcal{A}$. By the Artin-Wedderburn Theorem, there is only one simple module for $\mathcal{A}$; and by Theorem 2.2.2, there is only one graded-simple module (up to graded-isomorphism) that can come from the thin coverings of this module. Therefore (up to graded isomorphism), $\mathcal{A}$ has only one graded-simple module.

In spite of the uniqueness of the graded-simple module for $\mathcal{A}$, the thin coverings for $\mathcal{A}$ are far from unique. For example, $\mathcal{A}$ has a $\mathbb{Z}_n \times \mathbb{Z}_n$-grading with

$$\mathcal{A}_{(\overline{a}, \overline{b})} := \mathrm{Span}\{E^a F^b\}$$

where

$$E := \begin{pmatrix} 1 & 0 & \cdot & \cdot & 0 \\ 0 & \zeta & \cdot & \cdot & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdot & \cdot & \zeta^{d-1} \end{pmatrix}, \quad F := \begin{pmatrix} 0 & 0 & \cdot & 0 & 1 \\ 1 & 0 & \cdot & 0 & 0 \\ 0 & 1 & \cdot & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdot & 1 & 0 \end{pmatrix},$$

ζ is a primitive nth root of unity, and $\overline{a}$ (resp., $\overline{b}$) is the image of $a \in \mathbb{Z}$ (resp., $b \in \mathbb{Z}$) under the canonical homomorphism $\mathbb{Z} \to \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

Then the graded ring $\mathcal{A}$ satisfies conditions (i) and (ii) of Theorem 2.2.2, so any thin covering of the natural module $M = \mathbb{C}^n$ induces the same (unique) graded-simple module of $\mathcal{A}$ (relative to the grading group $G = \mathbb{Z}_n \times \mathbb{Z}_n$). Let v be a nonzero vector in M. Then since M is irreducible,

$$M_g := \mathcal{A}_g v$$

defines a covering of M. Let $\{N_g \mid g \in G\}$ be another covering of M with $N_g \subseteq M_g$ for all $g \in G$. Then $N_h \neq 0$ for some $h \in G$, so $N_g = \mathcal{A}_{g-h}.N_h \neq 0$ for all $g \in G$, since the one-dimensional space $\mathcal{A}_{g-h}$ is spanned by the invertible matrix $E^a F^b$ (where $g - h = (\overline{a}, \overline{b})$). Moreover, N_g is a nonzero subspace of the one-dimensional space M_g, so $N_g = M_g$ for all $g \in G$. Thus the covering $\{M_g\}$ is thin.

Any choice of v results in a thin covering consisting of n^2 one-dimensional subspaces M_g of M. The space M_0 is the span of the vector v. Since there are infinitely many distinct one-dimensional subspaces we could choose for M_0, there are infinitely many nonequivalent thin coverings of M.

More generally, using Theorem 3.3.17, we have the following example.

Example 3.3.27 Suppose $\widehat{H}$ is generated by two automorphisms η_1, η_2 of order d. Let T_1, T_2 be isomorphisms from M to M^{η_1}, M^{η_2}, respectively. Assume that these operators satisfy the relation

$$T_1 T_2 = \zeta T_2 T_1,$$

where ζ is a primitive root of unity of order d. Consider the isotypic decomposition

$$M = N \otimes \left(\bigoplus_{h \in H} V^h \right)$$

of M as an $\mathcal{R}_\zeta$-module. We obtain from Theorem 3.3.17 that the thin H-coverings are parametrized by one-dimensional subspaces in a d-dimensional space N. For any nonzero vector $n \in N$ we get a thin H-covering with

$$M_h = \sum_{\mathbf{a} \in \mathbb{Z}^2} \mathbf{T}^{\mathbf{a}}(n) \otimes V^{h+\gamma(\mathbf{a})}.$$

Up to equivalence, these are all thin H-coverings of M.

We are finally ready to describe the thin G-coverings of M in the general case–that is, when the twists of M by automorphisms in $\widehat{G}$ are not necessarily isomorphic to M. The following theorem shows that G-coverings of M can be described via H-coverings, where $\widehat{H}$ is the subgroup of $\widehat{G}$ defined in Lemma 3.3.1. In order to distinguish between G-coverings and H-coverings (resp. gradings), we will add an appropriate superscript to our notation.

The inclusion $\widehat{H} \subseteq \widehat{G}$ yields a natural epimorphism

$$\psi : \operatorname{Hom}(\widehat{G}, \mathbb{C}^*) \to \operatorname{Hom}(\widehat{H}, \mathbb{C}^*)$$

by restriction of the maps to $\widehat{H}$.

Theorem 3.3.28 *Let M be a finite-dimensional simple $\mathcal{A}$-module, and let $\widehat{G}$ be a finite abelian subgroup of* $\operatorname{Aut}\mathcal{A}$. *Let $\widehat{H}$ be its subgroup defined in Lemma 3.3.1. Then*

(i) $\rho(\mathcal{A}_g^G) = \rho\big(\mathcal{A}_{\psi(g)}^H\big)$ *for all $g \in G$.*

(ii) There is a bijective correspondence between thin G-coverings of M and thin H-coverings of M. Given a thin H-covering $\{M_h^H\}$ we get a thin G-covering $\{M_g^G\}$ with $M_g^G = M_{\psi(g)}^H$. Every thin G-covering is of this form.

(iii) All graded-simple $\mathcal{A}$-modules associated with various thin coverings of M (as in Theorem 1.1.3) are isomorphic (up to shifts in the gradings on the modules).

Proof Let $\tau_1, \ldots, \tau_\ell$ be representatives of the cosets of $\widehat{H}$ in $\widehat{G}$, where $\ell = |\widehat{G}|/|\widehat{H}|$. By Lemma 3.3.1, the modules $M^{\tau_1}, \ldots, M^{\tau_\ell}$ are pairwise non-isomorphic. Consider the representation $\widehat{\rho}$ of $\mathcal{A}$ on

$$\widehat{M} = M^{\tau_1} \oplus \cdots \oplus M^{\tau_\ell}.$$

An element $a \in \mathcal{A}_g^G$ acts on $\widehat{M}$ by

$$\widehat{\rho}(a) = \left(g(\tau_1)\rho(a), \ldots, g(\tau_\ell)\rho(a)\right).$$

Thus $\dim\widehat{\rho}(\mathcal{A}_g^G) = \dim\rho(\mathcal{A}_g^G)$. Moreover, by the definition of the map ψ, $\mathcal{A}_g^G \subseteq \mathcal{A}_{\psi(g)}^H$, so $\rho(\mathcal{A}_g^G) \subseteq \rho\big(\mathcal{A}_{\psi(g)}^H\big)$. Hence

$$
\begin{aligned}
\dim\widehat{\rho}(\mathcal{A}) &\leq \sum_{g\in G} \dim\widehat{\rho}(\mathcal{A}_g^G) && (3.3.29)\\
&= \sum_{g\in G} \dim\rho(A_g^G) && (3.3.30)\\
&\leq \sum_{g\in G} \dim\rho(\mathcal{A}_{\psi(g)}^H) && (3.3.31)\\
&= \big(|G|/|H|\big)\sum_{h\in H} \dim\rho(\mathcal{A}_h^H) && (3.3.32)\\
&\leq \ell\dim\operatorname{End}_{\mathbb{C}}(M). && (3.3.33)
\end{aligned}
$$

However, by the density theorem,

$$\widehat{\rho}(\mathcal{A}) = \operatorname{End}_{\mathbb{C}}(M^{\tau_1}) \oplus \cdots \oplus \operatorname{End}_{\mathbb{C}}(M^{\tau_\ell}).$$

This implies that all inequalities in (3.3.29)–(3.3.33) are in fact equalities, and in particular,

$$\dim\rho(\mathcal{A}_g^G) = \dim\rho(\mathcal{A}_{\psi(g)}^H).$$

This completes the proof of Part (i) of the theorem.

To prove the second part, we note that $\rho(\mathcal{A}_0^G) = \rho(\mathcal{A}_0^H)$. This means that the simple $\rho(\mathcal{A}_0^G)$-submodules in M are the simple $\rho(\mathcal{A}_0^H)$-submodules. Let $\{M_g^G\}$ be a thin G-covering of M. Without loss of generality, we may assume that $M_0^G \neq 0$. Then by Lemma 1.1.2, M_0^G is a simple $\rho(\mathcal{A}_0^H)$-submodule. We have seen in the proof of Theorem 3.3.17 that every simple $\rho(\mathcal{A}_0^H)$-submodule in M occurs as a component of a thin H-covering. Since we can always replace such a covering by an equivalent thin covering, we see that every simple $\rho(\mathcal{A}_0^H)$-submodule in M occurs as the 0-component of a thin H-covering. Thus there exists a thin H-covering $\{M_h^H\}$, such that $M_0^H = M_0^G$. Moreover, $M_h^H = \rho\left(\mathcal{A}_h^H\right) M_0^H$ for all $h \in H$. By Lemma 1.1.2 and Part (i), we get that

$$M_g^G = \rho\left(\mathcal{A}_g^G\right) M_0^G = \rho\left(\mathcal{A}_{\psi(g)}^H\right) M_0^H = M_{\psi(g)}^H.$$

Thus every thin G-covering comes from a thin H-covering. Conversely, if we construct a G-covering from a thin H-covering by taking $M_g^G = M_{\psi(g)}^H$, then Lemma 1.1.2 implies that this G-covering is thin.

Part (iii) follows from (i), (ii), and Corollary 3.3.24. $\square$

When $\widehat{G}$ is finite cyclic, Theorem 3.3.28 reduces to the following eigenspace decomposition.

Example 3.3.34 Suppose $\widehat{G}$ is a finite cyclic group generated by $\eta \in \mathrm{Aut}\mathcal{A}$, $\widehat{H} \subseteq \widehat{G}$ is generated by η^s, and $T : M \to M^{\eta^s}$ is a (normalized) module isomorphism. Then (up to equivalence) M has a unique thin G-covering given by

$$M_g = \{u \in M \mid Tu = g(\eta)^s u\}.$$

4 Thin coverings of quasifinite modules

In this section we generalize our results to an infinite-dimensional setting. We will assume that the algebra $\mathcal{A}$ has an additional grading by an abelian group Z,

$$\mathcal{A} = \bigoplus_{k \in Z} \mathcal{A}_k,$$

and that M is a Z-graded $\mathcal{A}$-module,

$$M = \bigoplus_{n \in Z} M_n,$$

with finite-dimensional graded components M_n. Such modules are called *quasifinite.*

Note that M may be infinite-dimensional, since we allow Z to be infinite. We are primarily interested in the case when $Z = \mathbb{Z}$, but our results hold for arbitrary abelian grading groups.

We will work in the categories of Z-graded algebras and modules and consider only those automorphisms σ which preserve the Z-grading. That is,

$$\sigma(\mathcal{A}_k) = \mathcal{A}_k.$$

Let M be a quasifinite graded-simple module (with no proper Z-graded submodules). In this section, we show that Theorems 3.3.17 and 3.3.28 still hold in this more general setting. We start with some obvious adjustments to the definition of a thin G-covering.

Let $\widehat{G}$ be a finite abelian group of (Z-grading-preserving) automorphisms of $\mathcal{A}$. Then $\mathcal{A}$ is graded by the group $G = \mathrm{Hom}(\widehat{G}, \mathbb{C}^*)$, and the two gradings are compatible:

$$\mathcal{A} = \bigoplus_{(k,g) \in Z \times G} \mathcal{A}_{k,g}.$$

A *covering* of M is a set of Z-graded subspaces $\{M_g \mid g \in G\}$ of M that span M and satisfy

$$\rho(\mathcal{A}_g)M_h \subseteq M_{g+h}, \quad \text{for all } g, h \in G. \tag{4.4.1}$$

As before, the set of coverings is partially ordered, and the minimal elements of the poset of coverings are called *thin coverings.*

For all $n, k \in Z$, let ρ_{nk} be the restriction of the action of $\mathcal{A}_{k-n}$ to M_n:

$$\rho_{nk} : \mathcal{A}_{k-n} \to \mathrm{Hom}_{\mathbb{C}}(M_n, M_k).$$

It is easy to see that G-coverings can be defined in terms of ρ_{nk}. Indeed, condition (4.4.1) can be replaced by

$$\rho_{nk}(\mathcal{A}_{k-n,h})M_{n,g} \subseteq M_{k,g+h},$$

and "local" information about $\rho_{nk}(\mathcal{A}_{k-n}) \subseteq \mathrm{Hom}_{\mathbb{C}}(M_n, M_k)$ can substitute for "global" information about $\rho(\mathcal{A}) \subseteq \mathrm{End}_{\mathbb{C}}(M)$. The proofs of the previous section are essentially based on Schur's Lemma and the Jacobson Density Theorem. We will use the quasifinite version of the density theorem presented in the appendix.

Remark 4.4.2 The assumption on the existence of a maximal submodule made in Theorem 1.1.4 holds for the quasifinite modules described above.

Proof Let M be a Z-graded quasifinite module, which also has a compatible G-grading and is graded-simple (has no proper $Z \times G$-homogeneous submodules). Fix $n \in Z$ such that $M_n \neq 0$. Note that by Zorn's Lemma, M has a maximal Z-graded submodule. Indeed, if U is a proper Z-graded submodule in M then $U_n \neq M_n$. Otherwise, if $U_n = M_n$ then the space U_n is homogeneous both with respect to Z and G and will generate the whole module M since it is graded-simple.

Now if we consider an increasing chain of Z-graded submodules in M,

$$\cdots \subseteq U^{(i)} \subseteq U^{(i+1)} \subseteq \cdots,$$

the corresponding chain of components

$$\cdots \subseteq U_n^{(i)} \subseteq U_n^{(i+1)} \subseteq \cdots \subseteq M_n$$

will stabilize, since $\dim M_n < \infty$. Thus the union $\bigcup_i U^{(i)}$ is a Z-graded submodule in M, with nth component properly contained in M_n. Applying

Zorn's Lemma, we conclude that the partially ordered set of Z-graded proper submodules of M contains a maximal element. $\Box$

Our main results about quasifinite modules, Theorems 4.4.3 and 4.4.4, can be proven using the same arguments as for the finite case. Only the proof of Theorem 4.4.4(i) requires nontrivial modification, since the corresponding finite result, Theorem 3.3.28(i), used a dimension argument. We write down the details of the modified proof below, leaving the other proofs as straightforward exercises to the reader.

Theorem 4.4.3 *Let M be a Z-graded-simple quasifinite module for a Z-graded algebra $\mathcal{A}$. Let $\widehat{H}$ be a finite abelian group of automorphisms of $\mathcal{A}$ preserving the Z-grading, such that there exist Z-grading-preserving isomorphisms T_η between M and M^η for all $\eta \in \widehat{H}$. Let*

$$M = \bigoplus_{h \in H} N \otimes V^h$$

be the isotypic decomposition of M with respect to the action of the cyclotomic quantum torus $\mathcal{R}_q$ given by the operators T_η. In this decomposition, each subspace V^h is Z-graded. Then up to equivalence, thin H-coverings of M are parametrized by one-dimensional subspaces in N. Namely, for $n \in N$, $n \neq 0$, define

$$M_h = \sum_{a \in \mathbb{Z}^r} \mathbf{T}^{\mathbf{a}}(n) \otimes V^{h+\gamma(a)}.$$

Then $\{M_h \mid h \in H\}$ is a thin H-covering of M, and every thin H-covering of M is equivalent to a covering of this type. $\Box$

Theorem 4.4.4 *Let M be a Z-graded-simple quasifinite module for a Z-graded algebra $\mathcal{A}$. Let $\widehat{G}$ be a finite abelian group of automorphisms of $\mathcal{A}$ preserving the Z-grading. Let $\widehat{H}$ be its subgroup defined in Lemma 3.3.1. Then*

(i) $\rho_{nk}(\mathcal{A}^G_{k-n,g}) = \rho_{nk}(\mathcal{A}^H_{k-n,\psi(g)})$ *for all $g \in G$.*

(ii) There is a bijective correspondence between thin G-coverings of M and thin H-coverings of M. Given a thin H-covering $\{M^H_h\}$ we get a thin G-covering $\{M^G_g\}$ with $M^G_g := M^H_{\psi(g)}$. Every thin G-covering of M is of this form.

(iii) All graded-simple $\mathcal{A}$-modules associated with various thin coverings of M (as in Theorem 1.1.3) are isomorphic (up to shifts in the G-gradings on the modules).

Proof We will give a proof for Part (i) of the theorem. Parts (ii) and (iii) may be proven analogously to Theorem 3.3.28.

Let $\tau_1, \dots, \tau_\ell$ be representatives of the cosets of $\widehat{H}$ in $\widehat{G}$, where $\ell = |\widehat{G}|/|\widehat{H}|$. The modules $M^{\tau_1}, \dots, M^{\tau_\ell}$ do not admit (Z-grading-preserving) isomorphisms between any distinct pair of them.

Consider the representation $\widehat{\rho}$ of $\mathcal{A}$ on

$$\widehat{M} = M^{\tau_1} \oplus \cdots \oplus M^{\tau_\ell}.$$

For $n, k \in Z$, let $\widehat{\rho}_{nk}$ be the restriction of $\widehat{\rho}(\mathcal{A}_{k-n})$ to $\widehat{M}_n = M_n^{\tau_1} \oplus \cdots \oplus M_n^{\tau_\ell}$:

$$\widehat{\rho}_{nk} : \mathcal{A}_{k-n} \to \mathrm{Hom}_{\mathbb{C}}(\widehat{M}_n, \widehat{M}_k).$$

An element $a \in \mathcal{A}^G_{k-n,g}$ acts on $\widehat{M}_n$ by

$$\widehat{\rho}_{nk}(a) = \big(g(\tau_1)\rho_{nk}(a), \dots, g(\tau_k)\rho_{nk}(a)\big).$$

Thus $\dim \widehat{\rho}_{nk}(\mathcal{A}^G_{k-n,g}) = \dim \rho_{nk}(\mathcal{A}^G_{k-n,g})$. Hence

$$\dim \widehat{\rho}_{nk}(\mathcal{A}_{k-n}) \le \sum_{g \in G} \dim \widehat{\rho}_{nk}(\mathcal{A}^G_{k-n,g}) = \sum_{g \in G} \dim \rho_{nk}(\mathcal{A}^G_{k-n,g})$$

$$\le \sum_{g \in G} \dim \rho_{nk}\big(\mathcal{A}^H_{k-n,\psi(g)}\big) = \big(|G|/|H|\big) \sum_{h \in H} \dim \rho_{nk}\big(\mathcal{A}^H_{k-n,h}\big), \qquad (4.4.5)$$

using the fact that $\mathcal{A}^G_{k-n,g} \subseteq \mathcal{A}^H_{k-n,\psi(g)}$ by the definition of ψ. The space $\rho_{nk}(\mathcal{A}_{k-n})$ decomposes into a direct sum

$$\rho_{nk}(\mathcal{A}_{k-n}) = \bigoplus_{h \in H} \rho_{nk}(\mathcal{A}^H_{k-n,h})$$

(cf. Proposition 3.3.16), so

$$\big(|G|/|H|\big) \sum_{h \in H} \dim \rho_{nk}\big(\mathcal{A}^H_{n-k,h}\big) = \ell \dim \rho_{nk}\big(\mathcal{A}_{k-n}\big).$$

But by the Quasifinite Density Theorem (A.A.2),

$$\rho_{nk}(\mathcal{A}_{k-n}) = \mathrm{Hom}_{\mathbb{C}}(M_n, M_k)$$

and

$$\widehat{\rho}_{nk}(\mathcal{A}_{k-n}) = \bigoplus_{j=1}^{\ell} \mathrm{Hom}_{\mathbb{C}}(M_n^{\tau_j}, M_k^{\tau_j}),$$

so $\dim \widehat{\rho}_{nk}(\mathcal{A}_{n-k}) = \ell \dim \rho_{nk}(\mathcal{A}_{n-k})$.

Therefore all the inequalities in (4.4.5) are in fact equalities, and in particular,

$$\dim \rho_{nk}(\mathcal{A}^G_{k-n,g}) = \dim \rho_{nk}(\mathcal{A}^H_{k-n,\psi(g)}).$$

This completes the proof of Part (i). $\square$

5 Application to multiloop Lie algebras

Let $\widehat{G}$ be a finite abelian group of grading-preserving automorphisms of a Z-graded complex Lie algebra $\mathfrak{g}$. Consider the dual group $G = \mathrm{Hom}(\widehat{G}, \mathbb{C}^*)$. The Lie algebra $\mathfrak{g}$ is then graded by the group $Z \times G$. By choosing a set of N generators of G, we get an epimorphism $\mathbb{Z}^N \to G$. We will denote by $\overline{\mathbf{a}}$ the image of an element $\mathbf{a} \in \mathbb{Z}^N$ under this map.

Consider the multiloop Lie algebra $L(\mathfrak{g}, \widehat{G})$:

$$L(\mathfrak{g}, \widehat{G}) = \bigoplus_{\mathbf{a}\in\mathbb{Z}^N} \mathfrak{g}_{\overline{\mathbf{a}}} \otimes \mathbf{t}^{\mathbf{a}},$$

which is a Lie subalgebra of $\mathfrak{g} \otimes \mathbb{C}[t_1^{\pm}, \dots, t_N^{\pm}]$.

Let M be a quasifinite Z-graded-simple module for $\mathfrak{g}$. Using Theorem 4.4.4, we can construct a G-covering $\{M_g\}$ of M. Then the space

$$\bigoplus_{\mathbf{a}\in\mathbb{Z}^N} M_{\overline{\mathbf{a}}} \otimes \mathbf{t}^{\mathbf{a}}$$

becomes a $(Z \times \mathbb{Z}^N)$-graded-simple module for the multiloop algebra $L(\mathfrak{g}, \widehat{G})$.

A Appendix

In this appendix, we prove the quasifinite density theorem used in the proof of Theorem 4.4.4.

Let M be a Z-graded module over a Z-graded associative ring R. We call such a module *graded-semisimple* if it is a direct sum of a finite number of graded-simple modules. Consider the set of endomorphisms of M that shift the grading by an element $p \in Z$:

$$\mathrm{End}\,(M)_p = \{X \in \mathrm{End}\,(M) \mid XM_n \subseteq M_{n+p} \text{ for all } \ n \in Z\}.$$

In this notation, $\mathrm{End}\,(M)_0$ is the ring of grading-preserving endomorphisms of M.

The following is a graded version of the usual density theorem of Jacobson and Chevalley (cf. [11], Thm. 4.11.16, for instance):

Theorem A.A.1 (Graded Density Theorem) *Let M be a Z-graded-semi-simple module over a Z-graded associative ring R. Let $K = End_R(M)_0$ and let $E = \bigoplus_{p\in Z} End_K(M)_p$. Suppose that $\{x_1, \dots, x_m\} \subseteq M_s$ for some $s \in Z$. Then for every $f \in E$ there exists an element $r \in R$ such that $rx_j = f(x_j)$ for $j = 1, 2, \dots, m$.*

Proof The argument given in [11] is valid in the graded set-up as well. □

Theorem A.A.2 (Quasifinite Density Theorem) *Let $\mathcal{A}$ be an associative algebra graded by an abelian group Z. Let $M^{(1)}, ..., M^{(n)}$ be Z-graded-simple quasifinite $\mathcal{A}$-modules, such that there is no grading-preserving isomorphism between any distinct pair of them. Fix $s, p \in Z$, and for each $i = 1, \dots, n$, let $\{x^i_j\}_{j=1,\dots,m_i}$ be a $\mathbb{C}$-linearly independent subset of $M^{(i)}_s$, and let $\{y^i_j\}_{j=1,\dots,m_i}$ be a set of vectors in $M^{(i)}_{s+p}$. Then there exists an element $a \in \mathcal{A}_p$, such that $ax^i_j = y^i_j$ for all i, j.*

Proof By Schur's Lemma, every nonzero grading-preserving homomorphism of graded-simple modules is an isomorphism. Thus

$$\mathrm{Hom}_{\mathcal{A}}\left(M^{(i)}, M^{(\ell)}\right)_0 = 0 \text{ for } i \neq \ell.$$

Applying Schur's Lemma again, we see that every grading-preserving endomorphism of a quasifinite graded-simple module is a multiple of the identity. Take $M = M^{(1)} \oplus \dots \oplus M^{(n)}$. Then

$$K = \mathrm{End}_{\mathcal{A}}(M)_0 = \mathbb{C}\pi_1 \oplus \dots \oplus \mathbb{C}\pi_n,$$

where π_j is the projection $\pi_j : M \to M^{(j)}$. It follows that

$$\mathrm{End}_K(M)_p = \bigoplus_{j=1}^{n} \mathrm{End}_{\mathbb{C}}\big(M^{(j)}\big)_p,$$

and now Theorem A.A.2 follows immediately from Theorem A.A.1. □

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