To Robert Moody

Energy-momentum tensor for the toroidal Lie algebras.

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Abstract. We construct vertex operator representations for the full $(N+1)$-toroidal Lie algebra $\mathfrak{g}$. We associate with $\mathfrak{g}$ a toroidal vertex operator algebra, which is a tensor product of an affine VOA, a sub-VOA of a hyperbolic lattice VOA, affine sl_N VOA and a twisted Heisenberg-Virasoro VOA. The modules for the toroidal VOA are also modules for the toroidal Lie algebra $\mathfrak{g}$. We also construct irreducible modules for an important subalgebra $\mathfrak{g}_{\mathrm{div}}$ of the toroidal Lie algebra that corresponds to the divergence free vector fields. This subalgebra carries a non-degenerate invariant bilinear form. The VOA that controls the representation theory of $\mathfrak{g}_{\mathrm{div}}$ is a tensor product of an affine VOA $V_{\hat{\dot{\mathfrak{g}}}}(c)$ at level c, a sub-VOA of a hyperbolic lattice VOA, affine sl_N VOA and a Virasoro VOA at level $\overline{c}_L$ with the following condition on the central charges: $2(N+1) + \mathrm{rank}\, V_{\hat{\dot{\mathfrak{g}}}}(c) + \overline{c}_L = 26$.

0. Introduction.

Toroidal Lie algebras are very natural multi-variable generalizations of affine Kac-Moody algebras. The theory of affine Lie algebras is rich and beautiful, and has many important applications in physics. By large, applications of toroidal Lie algebras in physics are still to be discovered. We should mention however the papers [IKUX], [IKU], where the toroidal symmetry is discussed in the context of a 4-dimensional conformal field theory. We hope that the development of the representation theory of toroidal Lie algebras will help to find the proper place for these algebras in physical theories.

The construction of a toroidal Lie algebra is totally parallel to the well-known construction of an (untwisted) affine Kac-Moody algebra [K1]. One starts with a finite-dimensional simple Lie algebra $\dot{\mathfrak{g}}$ and considers maps from an $N+1$-dimensional torus into $\dot{\mathfrak{g}}$. We may identify the algebra of functions on a torus with the Laurent polynomial algebra $\mathcal{R} = \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_N^{\pm}]$, by taking the Fourier basis, setting $t_k = e^{ix_k}$. The Lie algebra of the $\dot{\mathfrak{g}}$-valued maps from a torus will then become $\mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_N^{\pm}] \otimes \dot{\mathfrak{g}}$. When $N = 0$, this yields the loop algebra.

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* Research supported by the Natural Sciences and Engineering Research Council of Canada.

Just as in affine case, one builds the universal central extension $(\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K}$ of $\mathcal{R} \otimes \dot{\mathfrak{g}}$. However when $N \geq 1$, the center $\mathcal{K}$ is infinite-dimensional. The infinite-dimensional center makes this Lie algebra highly degenerate. One can show, for example, that in an irreducible bounded weight module, most of the center should act trivially. To eliminate this degeneracy, we add the Lie algebra of the vector fields on a torus, $\mathcal{D} = \mathrm{Der}(\mathcal{R})$ to $(\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K}$. The resulting algebra,

$$\mathfrak{g} = (\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K} \oplus \mathcal{D}$$

is called the toroidal Lie algebra (see Section 1 for details). The action of $\mathcal{D}$ on $\mathcal{K}$ is non-trivial, making the center of the toroidal Lie algebra $\mathfrak{g}$ finite-dimensional. This enlarged algebra will have a much better representation theory.

A major obstruction to building the representation theory of toroidal Lie algebras is that these algebras, being $\mathbb{Z}^{N+1}$ graded, do not possess a triangular decomposition whenever $N \geq 1$. For this reason, the standard construction of the highest weight modules fails to work. The modules for the toroidal Lie algebras built by forcing the highest weight condition are not attractive [BC].

The true representation theory for toroidal Lie algebras was originated by Moody, Rao and Yokonuma in [MRY] and [EM], where they constructed a vertex operator representation in a homogeneous realization. The principal realization was later given in [B1]. Both realizations were unified and substantially generalized in [L] and [BB].

As the first application of this representation theory, one may use the vertex operator realizations to construct hierarchies of soliton equations as it was done in [B2], [ISW] and [IT].

The modules for toroidal Lie algebras introduced in these papers have weight decompositions with finite-dimensional weight spaces and are bounded, but do not possess a unique highest weight. To explain this, we consider a $\mathbb{Z}$ grading of $\mathfrak{g}$ by degree with respect to t_0, which is declared to be a special variable. The subalgebra of elements of degree 0 in this $\mathbb{Z}$ grading, is very close to an N-toroidal Lie algebra. For this subalgebra we may consider an irreducible module T, which is a "toroidal" module for $(\mathcal{R}_0 \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K}_0$ (a multi-variable analog of a loop module), and is a tensor module for $\mathcal{D}_0$. We let the elements of positive degree act on T trivially, and then induce T to the module over the whole toroidal Lie algebra. This induced module has a unique irreducible quotient that may be alternatively studied via the explicit vertex operator constructions. This approach to the representation theory of toroidal Lie algebras was laid out in [BB]. As we see, instead of a one-dimensional highest weight space, the whole of T will be the "top" of the resulting bounded module. The space T is infinite-dimensional, but has a $\mathbb{Z}^N$ grading with finite-dimensional subspaces.

One serious problem with this representation theory that has not been previously resolved is that the vertex operator realizations were constructed not for the full toroidal algebra, but only for its subalgebra $\mathfrak{g}^* = (\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K} \oplus \mathcal{D}^*$, where the derivations in t_0 are missing from the derivation part $\mathcal{D}$. Plausible candidates for representing the missing part yielded extremely messy cocycles with values in a certain complicated completion of $\mathcal{K}$ [MRY], [L].

This was leaving the whole picture incomplete and unsatisfactory from the physics perspective, because the missing derivations are responsible for the energy-momentum tensor for these modules. The main goal of the present paper is to resolve this problem and construct a class of representations for the full toroidal Lie algebras.

Often the representation theory of Lie algebras is used for the construction of the vertex operator algebras. In our case it is the opposite – the representation theory is developed using the machinery of the vertex operator algebras. This has been done in [BBS] for the subalgebra $\mathfrak{g}^*$ of the toroidal Lie algebra $\mathfrak{g}$. The VOA that controls the representation theory of $\mathfrak{g}^*$ is a tensor product of three fairly well-known VOAs – the affine $\dot{\mathfrak{g}}$ VOA $V_{\widehat{\dot{\mathfrak{g}}}}$, the affine gl_N VOA $V_{\widehat{gl}_N}$ and a sub-VOA V^+_{Hyp} of a hyperbolic lattice VOA.

After a careful analysis using the methods of [BB], it became clear that the irreducible modules for $\mathfrak{g}^*$ do not admit the action of the full toroidal Lie algebra $\mathfrak{g}$, and thus it is necessary to enlarge the representation space in order to get a module for $\mathfrak{g}$. A natural guess is that this enlarged space should be again a VOA or a VOA module. It turns out that the missing ingredient is a VOA that corresponds to the twisted Heisenberg-Virasoro algebra $\mathcal{HV}ir$, and the full toroidal VOA is a tensor product of four VOAs:

$$V_{tor} = V_{\widehat{\dot{\mathfrak{g}}}} \otimes V^+_{Hyp} \otimes V_{\widehat{sl}_N} \otimes V_{\mathcal{HV}ir}$$

with certain conditions on the central charges of these VOAs.

The twisted Heisenberg-Virasoro Lie algebra has a Virasoro subalgebra and a Heisenberg subalgebra, but the natural action of the Virasoro on the Heisenberg subalgebra is twisted with a cocycle (see Section 2.4 for the precise definition). The representation theory of $\mathcal{HV}ir$ has been studied by Arbarello et al. in [ACKP]. However one special case, namely when the central charge of the Heisenberg subalgebra is zero, was not fully investigated in that paper. It happens that this is precisely the case we need for the toroidal VOA. The structure of the irreducible modules for $\mathcal{HV}ir$ with the trivial action of the center of the Heisenberg subalgebra has been determined in [B3].

Using these ingredients we can easily write down the characters of the toroidal VOA and of its modules. This leads to the following open problem: while the explicit expressions for the characters of irreducible modules are known, there is no Weyl-type character formula for the toroidal Lie algebras. Obtaining such a formula may lead to interesting number-theoretic identities.

Toroidal Lie algebras are related to the class of extended affine Lie algebras. These Lie algebras have been extensively studied during the last decade (see [AABGP] and references therein). The main features of an extended affine Lie algebra is that it is graded by a finite root system and possesses a non-degenerate symmetric invariant bilinear form. The full toroidal Lie algebra $\mathfrak{g}$ does not possess a non-degenerate invariant form, but its subalgebra

$$\mathfrak{g}_{\text{div}} = (\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K} \oplus \mathcal{D}_{\text{div}}$$

does. Here $\mathcal{D}_{\text{div}}$ is the subalgebra of the divergence-free vector fields. Using the theory for the full toroidal algebra $\mathfrak{g}$, we are able to construct irreducible representations for its important subalgebra $\mathfrak{g}_{\text{div}}$ as well.

The vertex operator algebra that controls the representation theory of $\mathfrak{g}_{\text{div}}$ is a tensor product of an affine VOA $V_{\widehat{\dot{\mathfrak{g}}}}$ at level c, a sub-VOA of a hyperbolic lattice VOA V^+_{Hyp}, and a Virasoro VOA at level $\overline{c}_L$. The condition on the central charges that we get here is

$$2(N+1) + \frac{c \dim \dot{\mathfrak{g}}}{c + h^\vee} + \overline{c}_L = 26,$$

which has a striking resemblance to the formula for the critical dimension in the bosonic string theory.

Another interesting fact is that when $N = 12$, we get an exceptional module for the Lie algebra $\mathcal{D}_{\mathrm{div}} \oplus \mathcal{K}$. Only for this value of N we can represent $\mathcal{D}_{\mathrm{div}} \oplus \mathcal{K}$ just on a hyperbolic lattice sub-VOA V_{Hyp}^{+}, and the structure of the module becomes exceptionally simple. The character of this module is given by the -24-th power of the Dedekind η-function and has nice modular properties.

We should mention that the class of the modules for the full toroidal Lie algebra constructed in this paper is not exhaustive, but could be described as a toroidal counterpart of the level 1 representations for affine Lie algebras, so more research remains to be done.

The structure of the paper is the following. In Section 1 we give the definition of the toroidal Lie algebras. In Section 2 we recall the definition and the properties of VOAs and construct the vertex operator algebras corresponding to the twisted Heisenberg-Virasoro algebra using the technique of the vertex Lie algebras. In Section 3 we describe the tensor factors of the toroidal VOAs – the affine VOA $V_{\hat{\mathfrak{g}}}$, a sub-VOA of a hyperbolic lattice VOA V_{Hyp}^{+}, and the twisted $\widehat{gl}_N$-Virasoro VOA $V_{\widehat{gl}_N - \mathcal{V}ir}$. We conclude Section 3 with the definition of the toroidal VOA. In Section 4 we state and prove our main result – every module for the toroidal VOA is a module for the toroidal Lie algebra. In the last Section we describe the structure of the irreducible modules for the full toroidal Lie algebra, as well as for its subalgebra $\mathfrak{g}_{\mathrm{div}}$. We conclude the paper with the construction of an exceptional module for $\mathcal{D}_{\mathrm{div}} \oplus \mathcal{K}$ which is possible only when $N = 12$.

Acknowledgements: I am grateful to Stephen Berman for the stimulating discussions and encouragement.

1. Toroidal Lie algebras.

Toroidal Lie algebras are the natural multi-variable generalizations of affine Lie algebras. In this review of the toroidal Lie algebras we follow the work [BB]. Let $\dot{\mathfrak{g}}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$ with a non-degenerate invariant bilinear form $(\cdot|\cdot)$ and let $N \geq 1$ be an integer. We consider the Lie algebra $\mathcal{R} \otimes \dot{\mathfrak{g}}$ of maps of an $N+1$ dimensional torus into $\dot{\mathfrak{g}}$, where $\mathcal{R} = \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_N^{\pm}]$ is the algebra of functions on a torus (in the Fourier basis). The universal central extension of this Lie algebra may be described by means of the following construction which is due to Kassel [Kas]. Let $\Omega_{\mathcal{R}}$ be the space of 1-forms on a torus: $\Omega_{\mathcal{R}} = \mathop{\oplus}\limits_{p=0}^{N} \mathcal{R} dt_p$. We will choose the forms $\{k_p = t_p^{-1} dt_p | p = 0, \ldots, N\}$ as a basis of this free $\mathcal{R}$ module. There is a natural map d from the space of functions $\mathcal{R}$ into $\Omega_{\mathcal{R}}$: $d(f) = \sum\limits_{p=0}^{N} \frac{\partial f}{\partial t_p} dt_p = \sum\limits_{p=0}^{N} t_p \frac{\partial f}{\partial t_p} k_p$. The center $\mathcal{K}$ for the universal central extension $(\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K}$ of $\mathcal{R} \otimes \dot{\mathfrak{g}}$ is realized as

$$\mathcal{K} = \Omega_{\mathcal{R}} / d(\mathcal{R}),$$

and the Lie bracket is given by the formula

$$[f_1(t) g_1, f_2(t) g_2] = f_1(t) f_2(t) [g_1, g_2] + (g_1 | g_2) f_2 d(f_1).$$

Here and in the rest of the paper we will denote elements of $\mathcal{K}$ by the same symbols as elements of $\Omega_{\mathcal{R}}$, keeping in mind the canonical projection $\Omega_{\mathcal{R}} \to \Omega_{\mathcal{R}}/d(\mathcal{R})$.

Just as in affine case, we add to $(\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K}$ the algebra $\mathcal{D}$ of outer derivations

$$\mathcal{D} = \overset{N}{\underset{p=0}{\oplus}} \mathcal{R} d_p,$$

where $d_p = t_p \frac{\partial}{\partial t_p}$. We will denote the multi-indices by bold letters $\mathbf{r} = (r_0, r_1, \dots, r_N)$, etc., and by $\mathbf{t}^{\mathbf{r}}$ the corresponding monomials $t_0^{r_0} t_1^{r_1} \dots t_N^{r_N}$.

The natural action of $\mathcal{D}$ on $\mathcal{R} \otimes \dot{\mathfrak{g}}$

$$[\mathbf{t}^{\mathbf{r}} d_a, \mathbf{t}^{\mathbf{m}} g] = m_a \mathbf{t}^{\mathbf{r}+\mathbf{m}} g \tag{1.1}$$

uniquely extends to the action on the universal central extension $(\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K}$ by

$$[\mathbf{t}^{\mathbf{r}} d_a, \mathbf{t}^{\mathbf{m}} k_b] = m_a \mathbf{t}^{\mathbf{r}+\mathbf{m}} k_b + \delta_{ab} \sum_{p=0}^{N} r_p \mathbf{t}^{\mathbf{r}+\mathbf{m}} k_p. \tag{1.2}$$

This corresponds to the Lie derivative action of the vector fields on 1-forms.

It turns out that there is still an extra degree of freedom in defining the Lie algebra structure on $(\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K} \oplus \mathcal{D}$. The Lie bracket on $\mathcal{D}$ may be twisted with a $\mathcal{K}$-valued 2-cocycle:

$$[\mathbf{t}^{\mathbf{r}} d_a, \mathbf{t}^{\mathbf{m}} d_b] = m_a \mathbf{t}^{\mathbf{r}+\mathbf{m}} d_b - r_b \mathbf{t}^{\mathbf{r}+\mathbf{m}} d_a + \tau(\mathbf{t}^{\mathbf{r}} d_a, \mathbf{t}^{\mathbf{m}} d_b). \tag{1.3}$$

The space of these cocycles $H^2(\mathcal{D}, \mathcal{K})$ is two-dimensional and is spanned by the following cocycles τ_1 and τ_2:

$$\tau_1(\mathbf{t}^{\mathbf{r}} d_a, \mathbf{t}^{\mathbf{m}} d_b) = m_a r_b \sum_{p=0}^{N} m_p \mathbf{t}^{\mathbf{r}+\mathbf{m}} k_p,$$

$$\tau_2(\mathbf{t}^{\mathbf{r}} d_a, \mathbf{t}^{\mathbf{m}} d_b) = r_a m_b \sum_{p=0}^{N} m_p \mathbf{t}^{\mathbf{r}+\mathbf{m}} k_p.$$

We will write $\tau = \mu\tau_1 + \nu\tau_2$. The resulting algebra (or rather a family of algebras) is called the toroidal Lie algebra

$$\mathfrak{g} = \mathfrak{g}(\mu, \nu) = (\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K} \oplus \mathcal{D}.$$

Note that after adding the algebra of derivations $\mathcal{D}$, the center of the toroidal Lie $\mathfrak{g}$ becomes finite-dimensional with the basis $\{k_0, k_1, \dots, k_N\}$. This can be seen from the action (1.2) of $\mathcal{D}$ on $\mathcal{K}$, which is non-trivial.

In this paper we will consider only the multiples of the first cocycle τ_1, and we will be assuming $\nu = 0$ for most of our results here.

The toroidal Lie algebra $\mathfrak{g}(\mu, \nu) = (\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K} \oplus \mathcal{D}$ has an important subalgebra $\mathfrak{g}_{\mathrm{div}}(\mu)$ that has divergence free vector fields as the derivation part:

$$\mathfrak{g}_{\mathrm{div}}(\mu) = (\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K} \oplus \mathcal{D}_{\mathrm{div}},$$

where

$$\mathcal{D}_{\mathrm{div}} = \left\{ \sum_{p=0}^{N} f_p(\mathbf{t}) d_p \;\middle|\; \sum_{p=0}^{N} t_p \frac{\partial f_p}{\partial t_p} = 0 \right\}.$$

The expression $i \sum\limits_{p=0}^{N} t_p \frac{\partial f_p}{\partial t_p}$ becomes the divergence of a vector field in the angular coordinates $(x_0, \ldots, x_N)$ on a torus, where $t_j = e^{ix_j}$.

Note that the cocycle τ_2 trivializes on $\mathcal{D}_{\mathrm{div}}$, so we only get the restriction of $\mu\tau_1$.

The importance of this subalgebra is explained by the fact that unlike the full toroidal Lie algebra, $\mathfrak{g}_{\mathrm{div}}$ is an extended affine Lie algebra [BGK], i.e., $\mathfrak{g}_{\mathrm{div}}$ has a non-degenerate symmetric invariant bilinear form. The restrictions of this form to both $\mathcal{R} \otimes \dot{\mathfrak{g}}$ and to $\mathcal{D}_{\mathrm{div}} \oplus \mathcal{K}$ are non-degenerate:

$$(\mathbf{t}^{\mathbf{r}} g_1 | \mathbf{t}^{\mathbf{m}} g_2) = \delta_{\mathbf{r},-\mathbf{m}} (g_1 | g_2), \quad g_1, g_2 \in \dot{\mathfrak{g}},$$

while the vector fields pair with the 1-forms:

$$\big(\sum_{p=0}^{N} a_p \mathbf{t}^{\mathbf{r}} d_p | \mathbf{t}^{\mathbf{m}} k_q \big) = \delta_{\mathbf{r},-\mathbf{m}} a_q. \tag{1.4}$$

One can see that the above formula is ill-defined for the full $\mathcal{D}$, since $d(\mathbf{t}^{\mathbf{m}}) = \sum\limits_{q=0}^{N} m_q \mathbf{t}^{\mathbf{m}} k_q$, being zero in $\mathcal{K}$, must be in the kernel of the form. For the subalgebra $\mathcal{D}_{\mathrm{div}}$ this is precisely the case since

$$\big(\sum_{p=0}^{N} a_p \mathbf{t}^{\mathbf{r}} d_p | \sum_{q=0}^{N} r_q \mathbf{t}^{-\mathbf{r}} k_q \big) = \sum_{q=0}^{N} a_q r_q = 0.$$

All other values of the bilinear form are trivial:

$$(\mathcal{R} \otimes \dot{\mathfrak{g}} | \mathcal{D}_{\mathrm{div}} \oplus \mathcal{K}) = 0, \quad (\mathcal{D}_{\mathrm{div}} | \mathcal{D}_{\mathrm{div}}) = 0, \quad (\mathcal{K} | \mathcal{K}) = 0.$$

It is easy to verify that the resulting symmetric bilinear form is invariant and non-degenerate.

The study representation theory of toroidal Lie algebras has begun in [MRY] and [EM], with further developments in [B1], [L], [BB], [BBS]. In all of these papers there was one common difficulty that has not been resolved – the representations constructed there were not for the full toroidal algebra $\mathfrak{g}$, but only for a subalgebra

$$\mathfrak{g}^* = (\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K} \oplus \left(\bigoplus_{p=1}^{N} \mathcal{R} d_p \right),$$

where the piece $\mathcal{R} d_0$ that corresponds to the toroidal energy-momentum tensor was missing. This left the theory in a somewhat incomplete form, and the goal of the present paper is to construct a class of representations for the full toroidal Lie algebra.

2. Vertex operator algebra associated with the twisted Heisenberg- Virasoro Lie algebra.

2.1. Definitions and properties of a VOA.

Let us recall the basic notions of the theory of the vertex operator algebras. Here we are following [K2] and [Li].

Definition. *A vertex algebra is a vector space V with a distinguished vector* $\mathbf{1}$ *(vacuum vector) in V, an operator D (infinitesimal translation) on the space V, and a linear map Y (state-field correspondence)*

$$\begin{aligned} Y(\cdot, z): \quad & V \to (\mathrm{End}V)[[z, z^{-1}]], \\ & a \mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \quad (\textit{where} \ \ a_{(n)} \in \mathrm{End}V), \end{aligned}$$

such that the following axioms hold:
(V1) For any $a, b \in V$, $\quad a_{(n)}b = 0$ for n sufficiently large;
(V2) $[D, Y(a, z)] = Y(D(a), z) = \frac{d}{dz}Y(a, z)$ for any $a \in V$;
(V3) $Y(\mathbf{1}, z) = \mathrm{Id}_V$;
(V4) $Y(a, z)\mathbf{1} \in V[[z]]$ and $Y(a, z)\mathbf{1}|_{z=0} = a$ for any $a \in V$ (self-replication);
(V5) For any $a, b \in V$, the fields $Y(a, z)$ and $Y(b, z)$ are mutually local, that is,

$$(z-w)^n \left[Y(a, z), Y(b, w)\right] = 0, \quad \textit{for} \ \ n \ \textit{sufficiently large.}$$

A vertex algebra V is called a vertex operator algebra (VOA) if, in addition, V contains a vector ω (Virasoro element) such that
(V6) The components $L(n) = \omega_{(n+1)}$ of the field

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} \omega_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$$

satisfy the Virasoro algebra relations:

$$[L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m}\frac{n^3-n}{12}(\mathrm{rank}\ V)\mathrm{Id}, \quad \textit{where} \ \ \mathrm{rank}\ V \in \mathbb{C}; \qquad (2.1)$$

(V7) $D = L(-1)$;
(V8) V is graded by the eigenvalues of $L(0)$: $V = \underset{n\in\mathbb{Z}}{\oplus} V_n$ with $L(0)\big|_{V_n} = n\mathrm{Id}$.

This completes the definition of a VOA.

As a consequence of the axioms of the vertex algebra we have the following important commutator formula:

$$[Y(a, z_1), Y(b, z_2)] = \sum_{n\geq 0} \frac{1}{n!} Y(a_{(n)}b, z_2) \left[z_1^{-1} \left(\frac{\partial}{\partial z_2}\right)^n \delta\left(\frac{z_2}{z_1}\right)\right]. \qquad (2.2)$$

As usual, the delta function is

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n.$$

By (V1), the sum in the right hand side of the commutator formula is actually finite.

All the vertex operator algebras that appear in this paper have the gradings by non-negative integers: $V = \bigoplus_{n=0}^{\infty} V_n$. In this case the sum in the right hand side of the commutator formula (2.2) runs from $n = 0$ to $n = \deg\,(a) + \deg\,(b) - 1$, because

$$\deg\,(a_{(n)}b) = \deg\,(a) + \deg\,(b) - n - 1, \tag{2.3}$$

and the elements of negative degree vanish.

The commutator formula (2.2) may be written as the commutator relations between the components of the vertex operators:

$$[a_{(n)}, b_{(m)}] = \sum_{j\geq 0} \binom{n}{j} (a_{(j)}b)_{(n+m-j)}. \tag{2.4}$$

Equivalently,

$$a_{(n)}b_{(m)} = b_{(m)}a_{(n)} + \sum_{j\geq 0} \binom{n}{j} (a_{(j)}b)_{(n+m-j)}, \tag{2.5}$$

and also

$$a_{(n)}b_{(m)} = b_{(m)}a_{(n)} - \sum_{j\geq 0} \binom{m}{j} (b_{(j)}a)_{(n+m-j)}, \tag{2.6}$$

Another consequence of the axioms of a vertex algebra is the Borcherds' identity:

$$\sum_{j\geq 0} \binom{m}{j} (a_{(k+j)}b)_{(m+n-j)}c$$

$$= \sum_{j\geq 0} (-1)^{k+j+1} \binom{k}{j} b_{(n+k-j)}a_{(m+j)}c + \sum_{j\geq 0} (-1)^j \binom{k}{j} a_{(m+k-j)}b_{(n+j)}c, \qquad k, m, n \in \mathbb{Z}. \tag{2.7}$$

We will particularly need its special case when $m = 0$ and $k = -1$:

$$(a_{(-1)}b)_{(n)}c = \sum_{j\geq 0} b_{(n-j-1)}a_{(j)}c + \sum_{j\geq 0} a_{(-1-j)}b_{(n+j)}c, \qquad k \in \mathbb{Z}. \tag{2.8}$$

Let us list some other consequences of the axioms of a vertex algebra that we will be using in the paper. It follows from V7 and V8 that

$$\omega_{(0)}a = D(a) \tag{2.9}$$

and

$$\omega_{(1)}a = \deg\,(a)a \qquad \text{for } \; a \text{ homogeneous.} \tag{2.10}$$

The map D is a derivation of the n-th product:

$$D(a_{(n)}b) = (Da)_{(n)}b + a_{(n)}Db. \tag{2.11}$$

It could be easily derived from V2 that

$$(Da)_{(n)} = -na_{(n-1)} \tag{2.12}$$

and thus

$$a_{(-1-k)} = \frac{1}{k!}(D^k(a))_{(-1)}, \quad k \geq 0. \tag{2.13}$$

The last formula that we quote here is the skew-symmetry identity:

$$a_{(n)}b = \sum_{j\geq 0}(-1)^{n+j+1}\frac{1}{j!}D^j(b_{(n+j)}a). \tag{2.14}$$

2.2. Tensor products of VOAs.

The toroidal VOA that we introduce at the end of Section 3 is constructed by taking a tensor product of three VOAs. Let us review here the definition of the tensor product of two VOAs $(V', Y', \omega', \mathbf{1})$ and $(V'', Y'', \omega'', \mathbf{1})$ (the case of an arbitrary number of factors is a trivial generalization). The tensor product space $V = V' \otimes V''$ has the VOA structure under

$$Y(a\otimes b, z) = Y'(a, z) \otimes Y''(b, z), \qquad (2.15),$$

$$\omega = \omega' \otimes \mathbf{1} + \mathbf{1} \otimes \omega'', \tag{2.16}$$

and $\mathbf{1} = \mathbf{1}\otimes\mathbf{1}$ being the identity element.

It follows from (2.16) that the rank of V (see V6) is the sum of the ranks of the tensor factors.

We will be later using the following simple lemma:

Lemma 2.1. *Let $a, c \in V'$, $b, d \in V''$. Then*

(i) $(a\otimes\mathbf{1})_{(-1)}(\mathbf{1}\otimes b) = a\otimes b$.

(ii) $(a\otimes\mathbf{1})_{(n)}(\mathbf{1}\otimes b) = 0$ *for* $n \geq 0$.

(iii) Suppose $a_{(j)}c = 0$ *for* $j \geq 0$. *Then* $(a\otimes b)_{(n)}(c\otimes d) = \sum\limits_{j\geq 0}(a_{(-1-j)}c)\otimes(b_{(n+j)}d)$.

Proof. Part (i) follows from V3 and V4. Part (ii) is a consequence of the commutativity of $Y(a\otimes\mathbf{1}, z_1)$ and $Y(\mathbf{1}\otimes b, z_2)$. Part (iii) follows from (i), (ii) and (2.8).

2.3. Vertex Lie algebras.

An important source of the vertex algebras is provided by the vertex Lie algebras. In presenting this construction we will be following [DLM] (see also [P], [R], [K2], [FKRW]).

Let $\mathcal{L}$ be a Lie algebra with the basis $\{u(n), c(-1) \big| u \in \mathcal{U}, c \in \mathcal{C}, n \in \mathbb{Z}\}$ ($\mathcal{U}$, $\mathcal{C}$ are some index sets). Define the corresponding fields in $\mathcal{L}[[z, z^{-1}]]$:

$$u(z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1}, \quad c(z) = c(-1) z^0, \quad u \in \mathcal{U}, c \in \mathcal{C}.$$

Let $\mathcal{F}$ be a subspace in $\mathcal{L}[[z, z^{-1}]]$ spanned by all the fields $u(z), c(z)$ and their derivatives of all orders.

Definition. *A Lie algebra $\mathcal{L}$ with the basis as above is called a vertex Lie algebra if the following two conditions hold:*

(1) for all $u_1, u_2 \in \mathcal{U}$,

$$[u_1(z_1), u_2(z_2)] = \sum_{j=0}^{n} f_j(z_2) \left[z_1^{-1} \left(\frac{\partial}{\partial z_2} \right)^j \delta\left(\frac{z_2}{z_1} \right) \right], \tag{2.17}$$

where $f_j(z) \in \mathcal{F}$ and n depends on u_1, u_2,

(2) for all $c \in \mathcal{C}$, the elements $c(-1)$ are central in $\mathcal{L}$.

This definition is a simplified version of the one from [DLM] and is not quite as general as the original definition, but it is sufficient for our purposes.

Let $\mathcal{L}^+$ be a subspace in $\mathcal{L}$ with the basis $\{u(n) \big| u \in \mathcal{U}, n \geq 0\}$ and let $\mathcal{L}^-$ be a subspace with the basis $\{u(n), c(-1) \big| u \in \mathcal{U}, c \in \mathcal{C}, n < 0\}$. Then $\mathcal{L} = \mathcal{L}^+ \oplus \mathcal{L}^-$ and $\mathcal{L}^+, \mathcal{L}^-$ are in fact subalgebras in $\mathcal{L}$.

The universal enveloping vertex algebra $V_{\mathcal{L}}$ of a vertex Lie algebra $\mathcal{L}$ is defined as an induced module

$$V_{\mathcal{L}} = \mathrm{Ind}_{\mathcal{L}^+}^{\mathcal{L}}(\mathbb{C}\mathbf{1}) = U(\mathcal{L}^-) \otimes \mathbf{1},$$

where $\mathbb{C}\mathbf{1}$ is a trivial 1-dimensional $\mathcal{L}^+$ module.

Theorem 2.2. ([DLM], Theorem 4.8) *Let $\mathcal{L}$ be a vertex Lie algebra. Then*

(a) $V_{\mathcal{L}}$ has a structure of a vertex algebra with the vacuum vector $\mathbf{1}$, infinitesimal translation D being a natural extension of the derivation of $\mathcal{L}$ given by $D(u(n)) = -nu(n-1)$, $D(c(-1)) = 0$, $u \in \mathcal{U}$, $c \in \mathcal{C}$, and the state-field correspondence map Y defined by the formula:

$$Y\left(a_1(-1-n_1) \dots a_{k-1}(-1-n_{k-1}) a_k(-1-n_k)\mathbf{1}, z\right)$$
$$=: \left(\frac{1}{n_1!} \left(\frac{\partial}{\partial z} \right)^{n_1} a_1(z) \right) \dots : \left(\frac{1}{n_{k-1}!} \left(\frac{\partial}{\partial z} \right)^{n_{k-1}} a_{k-1}(z) \right) \left(\frac{1}{n_k!} \left(\frac{\partial}{\partial z} \right)^{n_k} a_k(z) \right) : \dots : \quad , \tag{2.18}$$

where $a_j \in \mathcal{U}, n_j \geq 0$ or $a_j \in \mathcal{C}, n_j = 0$.

(b) Any restricted $\mathcal{L}$ module is a vertex algebra module for $V_{\mathcal{L}}$.

(c) For an arbitrary character $\lambda : \mathcal{C} \to \mathbb{C}$, the factor module

$$V_{\mathcal{L}}(\lambda) = U(\mathcal{L}^-)\mathbf{1} / U(\mathcal{L}^-) \big\langle (c(-1) - \lambda(c))\mathbf{1} \big\rangle_{c \in \mathcal{C}}$$

is a quotient vertex algebra.

(d) Any restricted $\mathcal{L}$ module in which $c(-1)$ act as $\lambda(c)\mathrm{Id}$, for all $c \in \mathcal{C}$, is a vertex algebra module for $V_{\mathcal{L}}(\lambda)$.

In the formula (2.18) above, the normal ordering of two fields $: a(z)b(z) :$ is defined as

$$: a(z)b(z) := \sum_{n<0} a(n)z^{-n-1}b(z) + \sum_{n\geq 0} b(z)a(n)z^{-n-1}.$$

2.4. VOAs associated with the twisted Heisenberg-Virasoro algebra.

In the previous papers on the subject, no one has succeeded in constructing a vertex operator representation for the full toroidal Lie algebra. However, Theorem 1.12 from [BB] predicts that such representations should exist in this case as well. After a careful analysis using the methods from [BB], it became clear that irreducible $\mathfrak{g}^*$ modules do not admit the action of the full toroidal algebra $\mathfrak{g}$, and their spaces should be enlarged in order to make such an extension possible. The new ingredient turned out to be the VOA associated with the twisted Heisenberg-Virasoro algebra which we describe in this subsection.

We define the twisted Heisenberg-Virasoro algebra $\mathcal{HV}ir$ as a Lie algebra with the basis

$$\left\{ L(n), I(n), C_L, C_{LI}, C_I, \middle| n \in \mathbb{Z} \right\}$$

and Lie bracket given by

$$[L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m}\frac{n^3-n}{12}C_L, \tag{2.19}$$

$$[L(n), I(m)] = -mI(n+m) - \delta_{n,-m}(n^2+n)C_{LI}, \tag{2.20}$$

$$[I(n), I(m)] = n\delta_{n,-m}C_I, \tag{2.21}$$

$$[\mathcal{HV}ir, C_L] = [\mathcal{HV}ir, C_{LI}] = [\mathcal{HV}ir, C_I] = 0.$$

This Lie algebra has a Heisenberg subalgebra and a Virasoro subalgebra intertwined with the cocycle (2.20). The twisted Heisenberg-Virasoro algebra $\mathcal{HV}ir$ is the central extension of the Lie algebra $\{f(t)\frac{d}{dt} + g(t) | f, g \in \mathbb{C}[t, t^{-1}]\}$ of differential operators of order at most one. The corresponding projection is given by $L(n) \mapsto -t^{n+1}\frac{d}{dt}$, $I(n) \mapsto t^n$. The center of $\mathcal{HV}ir$ is four-dimensional and is spanned by $\{I(0), C_L, C_{LI}, C_I\}$.

Irreducible highest weight representations for $\mathcal{HV}ir$ have been studied by Arbarello et al. [ACKP], where the structure of these modules is determined in case when the action of C_I is non-zero. It turns out, however, that for our construction of the representations for the toroidal Lie algebras we need precisely the highest weight $\mathcal{HV}ir$ modules in which C_I acts as zero. The irreducible modules of this type were studied in [B3], and we quote the result from [B3] below.

We begin by recalling the standard construction of the Verma modules.

Introduce a $\mathbb{Z}$ grading on $\mathcal{HV}ir$ by $\deg L(n) = \deg I(n) = n$ and $\deg C_L = \deg C_{LI} = \deg C_I = 0$, and decompose $\mathcal{HV}ir$ with respect to this grading:

$$\mathcal{HV}ir = \mathcal{HV}ir_- \oplus \mathcal{HV}ir_0 \oplus \mathcal{HV}ir_+.$$

Fix arbitrary complex numbers h, h_I, c_L, c_{LI}, c_I. Let $\mathbb{C}\mathbf{1}$ be a 1-dimensional $\mathcal{HV}ir_0 \oplus \mathcal{HV}ir_+$ module defined by $L(0)\mathbf{1} = h\mathbf{1}$, $I(0)\mathbf{1} = h_I\mathbf{1}$, $C_L\mathbf{1} = c_L\mathbf{1}$, $C_{LI}\mathbf{1} = c_{LI}\mathbf{1}$, $C_I\mathbf{1} = c_I\mathbf{1}$, $\mathcal{HV}ir_+\mathbf{1} = 0$. As usual, the Verma module $M = M(h, h_I, c_L, c_{LI}, c_I)$ is the induced module

$$M(h, h_I, c_L, c_{LI}, c_I) = \mathrm{Ind}_{\mathcal{HV}ir_0 \oplus \mathcal{HV}ir_+}^{\mathcal{HV}ir}(\mathbb{C}\mathbf{1}) \cong U(\mathcal{HV}ir_-) \otimes \mathbf{1}.$$

The module M is $\mathbb{Z}$ graded by eigenvalues of the operator $L(0) - h\mathrm{Id}$: $M = \bigoplus_{n=0}^{\infty} M_n$ with $M_n = \{v \in M | L(0)v = (n+h)v\}$.

In order to understand the submodule structure of M, we need to study singular vectors in M. A non-zero homogeneous vector v in a highest weight $\mathcal{HV}ir$ module is called singular if $\mathcal{HV}ir_+ v = 0$. The Verma module $M(h, h_I, c_L, c_{LI}, c_I)$ has a unique irreducible factor which we denote $L(h, h_I, c_L, c_{LI}, c_I)$.

Theorem 2.3. ([B3], Theorem 1.) *Let $c_I = 0$ and $c_{LI} \neq 0$.*

(a) If $\frac{h_I}{c_{LI}} \notin \mathbb{Z}$ or $\frac{h_I}{c_{LI}} = 1$ then the $\mathcal{HV}ir$ module $M(h, h_I, c_L, c_{LI}, 0)$ is irreducible.

(b) If $\frac{h_I}{c_{LI}} \in \mathbb{Z}\backslash\{1\}$ then $M(h, h_I, c_L, c_{LI}, 0)$ possesses a singular vector $v \in M_n$, where $n = \left|\frac{h_I}{c_{LI}} - 1\right|$. The factor-module $L(h, h_I, c_L, c_{LI}, 0) = M(h, h_I, c_L, c_{LI}, 0)/U(\mathcal{HV}ir_-)v$ is irreducible and its character is

$$\mathrm{char}\ L(h, h_I, c_L, c_{LI}, 0) = (1 - q^n) \prod_{k \geq 1} (1 - q^k)^{-2}.$$

Using theorem 2.2 we can construct VOAs associated with the twisted Heisenberg-Virasoro algebra:

Theorem 2.4. *Let $c_{LI} \neq 0$.*

(a) The $\mathcal{HV}ir$ module $L(0, 0, c_L, c_{LI}, c_I)$ is a simple vertex operator algebra.

(b) The $\mathcal{HV}ir$ modules $M(h, h_I, c_L, c_{LI}, 0)$ and $L(h, h_I, c_L, c_{LI}, 0)$ are the VOA modules for $L(0, 0, c_L, c_{LI}, 0)$.

Proof. First let us show that the twisted Heisenberg-Virasoro algebra is a vertex Lie algebra with $\mathcal{U} = \{\omega, I\}$ and $\mathcal{C} = \{C_L, C_{LI}, C_I\}$, where we set $\omega(n) = L(n-1), C_L(-1) = C_L, C_{LI}(-1) = C_{LI}, C_I(-1) = C_I$. Then the set

$$\left\{\omega(n), I(n), C_L(-1), C_{LI}(-1), C_I(-1) \middle| n \in \mathbb{Z}\right\}$$

is the basis of $\mathcal{HV}ir$ compatible with the vertex structure.

Form the fields

$$\omega(z) = \sum_{n \in \mathbb{Z}} \omega(n) z^{n-1} = \sum_{n \in \mathbb{Z}} L(n) z^{n-2},$$

$$I(z) = \sum_{n \in \mathbb{Z}} I(n) z^{n-1},$$

as well as the constant fields $C_L(z) = C_L, C_{LI}(z) = C_{LI}, C_I(z) = C_I$. From the defining relations (2.19),(2.20),(2.21), we can derive the commutator relations between $\omega(z)$ and $I(z)$:

$$\begin{aligned}[\omega(z_1), \omega(z_2)] = \left(\frac{\partial}{\partial z_2}\omega(z_2)\right)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right] + 2\omega(z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right] \\ + \frac{C_L}{12}\left[z_1^{-1}\left(\frac{\partial}{\partial z_2}\right)^3\delta\left(\frac{z_2}{z_1}\right)\right],\end{aligned} \tag{2.22}$$

$$\begin{aligned}[\omega(z_1), I(z_2)] = \left(\frac{\partial}{\partial z_2}I(z_2)\right)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right] + I(z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right] \\ - C_{LI}\left[z_1^{-1}\left(\frac{\partial}{\partial z_2}\right)^2\delta\left(\frac{z_2}{z_1}\right)\right],\end{aligned} \tag{2.23}$$

$$[I(z_1), I(z_2)] = C_I\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right]. \tag{2.24}$$

Observe that the expressions in the right hand sides are of the form (2.17). Finally the elements C_L, C_{LI}, C_I are central and we see now that both properties of the vertex Lie algebra hold.

Let us prove that the irreducible $\mathcal{HV}ir$ module $L(0,0,c_L,c_{LI},c_I)$ is a VOA. Let λ be the character $\lambda(C_L) = c_L, \lambda(C_{LI}) = c_{LI}, \lambda(C_I) = c_I$. We are going to show that $L(0,0,c_L,c_{LI},c_I)$ is a homomorphic image of the vertex algebra $V_{\mathcal{HV}ir}(\lambda)$. As an $\mathcal{HV}ir$ module, $V_{\mathcal{HV}ir}(\lambda)$ is a factor of $U(\mathcal{HV}ir)$ modulo the left ideal generated by

$$\left\{L(k), I(n), C_L - c_L\mathbf{1}, C_{LI} - c_{LI}\mathbf{1}, C_I - c_I\mathbf{1} \middle| k \geq -1, n \geq 0\right\}.$$

The Verma module $M(0,0,c_L,c_{LI},c_I)$ on the other hand is a factor of $U(\mathcal{HV}ir)$ modulo the left ideal generated by

$$\left\langle L(n), I(n), C_L - c_L\mathbf{1}, C_{LI} - c_{LI}\mathbf{1}, C_I - c_I\mathbf{1} \middle| n \geq 0\right\rangle.$$

Thus

$$V_{\mathcal{HV}ir}(c_L, c_{LI}, c_I) \cong M(0,0,c_L,c_{LI},c_I)/ < L(-1)\mathbf{1} > \tag{2.25}$$

(note that $L(-1)\mathbf{1}$ is a singular vector in $M(0,0,c_L,c_{LI},c_I)$). Since $L(0,0,c_L,c_{LI},c_I)$ is a unique irreducible factor of $M(0,0,c_L,c_{LI},c_I)$, we conclude that it is also a factor-module of $V_{\mathcal{HV}ir}(\lambda)$. Since the vertex operator algebra $V_{\mathcal{HV}ir}(\lambda)$ is generated by the moments of the $\mathcal{HV}ir$ fields $\omega(z)$ and $I(z)$, then every $\mathcal{HV}ir$ submodule is a vertex algebra ideal in $V_{\mathcal{HV}ir}(\lambda)$. Thus every factor module of $V_{\mathcal{HV}ir}(\lambda)$ admits the structure of a (factor) vertex algebra, in particular the irreducible module $L(0,0,c_L,c_{LI},c_I)$ becomes a simple vertex algebra. Note that $V_{\mathcal{HV}ir}(\lambda)$ contains a Virasoro element $\omega(-1)\mathbf{1}$, so it is a VOA. The VOA structure on $L(0,0,c_L,c_{LI},c_I)$ is still given by (2.18).

When $c_I = 0$, the vertex operator algebras $V_{\mathcal{HV}ir}(c_L, c_{LI}, 0)$ and $L(0,0,c_L,c_{LI},0)$ are actually isomorphic by Theorem 2.3 and (2.25). The second part of the theorem is now an immediate consequence of Theorem 2.2(d). This completes the proof of the theorem.

Using the commutator formula (2.2) we derive from (2.23) the following relations:

Lemma 2.5.
(i) $\omega_{(0)}I(-1)\mathbf{1} = D(I(-1))\mathbf{1}$,
(ii) $\omega_{(1)}I(-1)\mathbf{1} = I(-1)\mathbf{1}$,
(iii) $\omega_{(2)}I(-1)\mathbf{1} = -2c_{LI}\mathbf{1}$,
(iv) $\omega_{(j)}I(-1)\mathbf{1} = 0$, *for* $j \geq 2$.

3. Toroidal vertex operator algebra.

In this section we discuss the tensor factors that will make up the vertex operator algebra associated with the toroidal Lie algebra $\mathfrak{g}$.

For a subalgebra $\mathfrak{g}^*$ of $\mathfrak{g}$, the structure of the associated vertex operator algebra and its modules was described in [BBS], however until now the attempts to construct vertex operator representations for the full toroidal algebra $\mathfrak{g}$ failed, which left the picture incomplete.

Let us now describe three tensor factors of a toroidal VOA – a sub-VOA of the hyperbolic lattice VOA, the affine VOA and the twisted $\widehat{gl}_N$-Virasoro VOA. The twisted $\widehat{gl}_N$-Virasoro VOA will be built using the twisted Heisenberg-Virasoro VOA introduced in the previous section.

3.1. Hyperbolic lattice VOA.

Here we present the construction of a hyperbolic lattice VOA. The general construction of a VOA corresponding to an arbitrary even lattice may be found in [FLM] or [K2].

Consider a hyperbolic lattice Hyp, which is a free abelian group on $2N$ generators $\{u_i, v_i\}_{1\leq i\leq N}$ with the symmetric bilinear form

$$(\cdot|\cdot): \quad Hyp \times Hyp \to \mathbb{Z},$$

defined by

$$(u_i|v_j) = \delta_{ij}, \quad (u_i|u_j) = (v_i|v_j) = 0.$$

Note that the form $(\cdot|\cdot)$ is non-degenerate and Hyp is an even lattice, i.e., $(x|x) \in 2\mathbb{Z}$.

The construction of the VOA associated to Hyp proceeds as follows.

First we complexify Hyp:

$$H = Hyp \otimes_{\mathbb{Z}} \mathbb{C},$$

and extend $(\cdot|\cdot)$ by linearity on H. Next, we "affinize" H by defining a Lie algebra $\widehat{H} = \mathbb{C}[t, t^{-1}] \otimes H \oplus \mathbb{C}K$ with the bracket

$$[x(n), y(m)] = n(x|y)\delta_{n,-m}K, \quad x, y \in H, \quad [\widehat{H}, K] = 0.$$

Here and in what follows, we are using the notation $x(n) = t^n \otimes x$. The algebra $\widehat{H}$ has a triangular decomposition $\widehat{H} = \widehat{H}_{-} \oplus \widehat{H}_0 \oplus \widehat{H}_{+}$, where $\widehat{H}_0 =< 1 \otimes H, K >$ and $\widehat{H}_{\pm} = t^{\pm 1}\mathbb{C}[t^{\pm}] \otimes H$.

We also need a twisted group algebra of Hyp, denoted by $\mathbb{C}[Hyp]$, which we now describe. The basis of $\mathbb{C}[Hyp]$ is $\{e^x | x \in Hyp\}$, and the multiplication is twisted with the 2-cocycle ϵ:

$$e^x e^y = \epsilon(x, y) e^{x+y}, \quad x, y \in Hyp, \tag{3.1}$$

where ϵ is a multiplicatively bilinear map

$$\epsilon : Hyp \times Hyp \to \{\pm 1\},$$

defined on the generators by $\epsilon(v_i, u_j) = (-1)^{\delta_{ij}}, \epsilon(u_i, v_j) = \epsilon(u_i, u_j) = \epsilon(v_i, v_j) = 1, \quad i, j = 1, \ldots, N$.

We define the structure of $\widehat{H}_0 \oplus \widehat{H}_+$ module on $\mathbb{C}[Hyp]$, letting $\widehat{H}_+$ act on $\mathbb{C}[Hyp]$ trivially and $\widehat{H}_0$ act by

$$x(0)e^y = (x|y)e^y, \quad Ke^y = e^y. \tag{3.2}$$

Finally let V_{Hyp} be the induced $\widehat{H}$ module:

$$V_{Hyp} = \mathrm{Ind}_{\widehat{H}_0 \oplus \widehat{H}_+}^{\widehat{H}} \left(\mathbb{C}[Hyp]\right).$$

This is the VOA attached to the lattice Hyp. As a space V_{Hyp} is isomorphic to the tensor product of the symmetric algebra $S(\widehat{H}_-)$ with the twisted group algebra $\mathbb{C}[Hyp]$:

$$V_{Hyp} = S(\widehat{H}_-) \otimes \mathbb{C}[Hyp].$$

The Y-map is defined on the basis elements of $\mathbb{C}[Hyp]$ by

$$Y(e^x, z) := \exp\left(\sum_{j\geq 1} \frac{x(-j)}{j} z^j\right) \exp\left(-\sum_{j\geq 1} \frac{x(j)}{j} z^{-j}\right) e^x z^x, \tag{3.3}$$

where e^x acts by twisted multiplication (3.1) and $z^x e^y = z^{(x|y)} e^y$. For a general basis element $a = x_1(-1-n_1) \ldots x_k(-1-n_k) \otimes e^y$, with $x_i, y \in Hyp, n_i \geq 0$, one defines (cf. (2.18))

$$Y(a, z) =: \left(\frac{1}{n_1!}\left(\frac{\partial}{\partial z}\right)^{n_1} x_1(z)\right) \ldots \left(\frac{1}{n_k!}\left(\frac{\partial}{\partial z}\right)^{n_k} x_k(z)\right) Y(e^y, z) :, \tag{3.4}$$

where $x(z) = \sum\limits_{j\in\mathbb{Z}} x(j) z^{-j-1}$. Note that $x_{(n)} = x(n)$ and sometimes the latter is more convenient typographically.

The Virasoro element in V_{Hyp} is $\omega = \sum\limits_{p=1}^{N} u_p(-1)v_p(-1) \otimes \mathbf{1}$, where $\mathbf{1} = e^0$ is the identity element of V_{Hyp}. The rank of V_{Hyp} is $2N$.

In the construction of the toroidal VOAs we would need not V_{Hyp} itself, but its sub-VOA V_{Hyp}^+:

$$V_{Hyp}^+ = S(\widehat{H}_-) \otimes \mathbb{C}[Hyp^+],$$

where Hyp^+ (resp. Hyp^-) is the isotropic sublattice of Hyp generated by $\{u_i, \quad 1 \leq i \leq N\}$ (resp. $\{v_i, \quad 1 \leq i \leq N\}$). One can verify immediately by inspecting (3.3) and (3.4) that V_{Hyp}^+ is indeed a sub-VOA of V_{Hyp}. Also note that the cocycle ϵ trivializes on $\mathbb{C}[Hyp^+]$, making

$\mathbb{C}[Hyp^+]$ the usual (untwisted) group algebra. The Virasoro element of V^+_{Hyp} is the same as in V_{Hyp}, and so the rank of V^+_{Hyp} is also $2N$.

Let us describe a class of modules for V^+_{Hyp}. Consider the group algebra $\mathbb{C}[H^+]$ of the vector space $H^+ = Hyp^+ \otimes_{\mathbb{Z}} \mathbb{C}$. The space $S(\widehat{H}_-) \otimes \mathbb{C}[H^+] \otimes \mathbb{C}[Hyp^-]$ has a structure of a VOA module for V^+_{Hyp}, where the action of V^+_{Hyp} is still given by (3.2),(3.3) and (3.4). Fix $\alpha \in \mathbb{C}^N, \beta \in \mathbb{Z}^N$. Then the subspace

$$M^+_{Hyp}(\alpha,\beta) = S(\widehat{H}_-) \otimes e^{\alpha\mathbf{u}+\beta\mathbf{v}}\mathbb{C}[Hyp^+] \tag{3.5}$$

in $S(\widehat{H}_-) \otimes \mathbb{C}[H^+] \otimes \mathbb{C}[Hyp^-]$ is an irreducible VOA module for V^+_{Hyp}. Here we are using the notations $\alpha\mathbf{u} = \alpha_1 u_1 + \ldots + \alpha_N u_N$, $\mathbf{mu} = m_1 u_1 + \ldots + m_N u_N$, etc.

The proof of the main result of this paper, Theorem 4.2 will be comprised of some fairly deep calculations in a toroidal VOA. The VOA V^+_{Hyp} which is a tensor factor in a toroidal VOA will play an essential role in these computations. We will present certain relations in V^+_{Hyp} in a sequence of two simple lemmas. These relations will be used in the proof of Theorem 4.2.

Lemma 3.1.
(i) $\omega_{(0)} e^{\mathbf{mu}} = De^{\mathbf{mu}} = \sum\limits_{p=1}^{N} m_p u_p(-1) e^{\mathbf{mu}}$,
(ii) $\omega_{(j)} e^{\mathbf{mu}} = 0 \quad$ *for* $j \geq 1$,
(iii) $\omega_{(0)} u_a(-1) e^{\mathbf{mu}} = D(u_a(-1)e^{\mathbf{mu}})$,
(iv) $\omega_{(1)} u_a(-1) e^{\mathbf{mu}} = u_a(-1)e^{\mathbf{mu}}$,
(v) $\omega_{(j)} u_a(-1) e^{\mathbf{mu}} = 0 \quad$ *for* $j \geq 2$,
(vi) $\omega_{(0)} v_a(-1) e^{\mathbf{mu}} = D(v_a(-1)e^{\mathbf{mu}})$,
(vii) $\omega_{(1)} v_a(-1) e^{\mathbf{mu}} = v_a(-1)e^{\mathbf{mu}}$,
(viii) $\omega_{(2)} v_a(-1) e^{\mathbf{mu}} = m_a e^{\mathbf{mu}}$,
(ix) $\omega_{(j)} v_a(-1) e^{\mathbf{mu}} = 0 \quad$ *for* $j \geq 3$.

Proof. Using (2.9) we get (iii), (vi) and the first part of (i). The second part of (i) follows from V2 and the fact that

$$\frac{d}{dz} Y(e^{\mathbf{mu}}, z) = \sum_{p=1}^{N} m_p u_p(z) Y(e^{\mathbf{mu}}, z) = Y\left(\sum_{p=1}^{N} m_p u_p(-1) e^{\mathbf{mu}}, z\right).$$

The claims (iv), (vii) and (ii) with $j = 1$ are the consequences of (2.10). The statements (ii) with $j > 1$, (v) with $j > 2$ and (ix) follow from (2.3). Finally, for (v) with $j = 2$ and (viii) we recall ([FLM], (8.7.13)), that in a lattice VOA

$$[\omega_{(n+1)}, x(m)] = [L(n), x(m)] = -m x(m+n).$$

Then for part (viii) we have

$$\omega_{(2)} v_a(-1) e^{\mathbf{mu}} = [\omega_{(2)}, v_a(-1)] e^{\mathbf{mu}} + v_a(-1)\omega_{(2)} e^{\mathbf{mu}} = v_a(0) e^{\mathbf{mu}} = m_a e^{\mathbf{mu}},$$

while for part (v) with $j = 2$ we get

$$\omega_{(2)} u_a(-1) e^{\mathbf{mu}} = [\omega_{(2)}, u_a(-1)] e^{\mathbf{mu}} + u_a(-1)\omega_{(2)} e^{\mathbf{mu}} = u_a(0) e^{\mathbf{mu}} = 0.$$

This completes the proof of the lemma.

Lemma 3.2.
(i) $e^{\mathbf{ru}}{}_{(-1)}e^{\mathbf{mu}} = e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}$,
(ii) $e^{\mathbf{ru}}{}_{(j)}e^{\mathbf{mu}} = 0 \quad$ *for* $j \geq 0$,
(iii) $e^{\mathbf{ru}}{}_{(0)}v_a(-1)e^{\mathbf{mu}} = -r_a e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}$,
(iv) $e^{\mathbf{ru}}{}_{(j)}v_a(-1)e^{\mathbf{mu}} = 0 \quad$ *for* $j \geq 1$,
(v) $e^{\mathbf{ru}}{}_{(-1)}v_a(-1)e^{\mathbf{mu}} = v_a(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} - r_a(De^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}}$,
(vi) $e^{\mathbf{ru}}{}_{(j)}\omega_{(n)} = \omega_{(n)}e^{\mathbf{ru}}{}_{(j)} - (De^{\mathbf{ru}})_{(n+j)}$,
(vii) $\left(u_s(-1)e^{\mathbf{ru}}\right)_{(0)} v_a(-1)e^{\mathbf{mu}} = \delta_{s,a}(De^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}} - r_a u_s(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}$,
(viii) $\left(u_s(-1)e^{\mathbf{ru}}\right)_{(1)} v_a(-1)e^{\mathbf{mu}} = \delta_{s,a}e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}$,
(ix) $\left(u_s(-1)e^{\mathbf{ru}}\right)_{(j)} v_a(-1)e^{\mathbf{mu}} = 0 \quad$ *for* $j \geq 2$,
(x) $e^{\mathbf{ru}}{}_{(0)}D(v_a(-1)e^{\mathbf{mu}}) = -r_a D(e^{(\mathbf{r}+\mathbf{m})\mathbf{u}})$,
(xi) $e^{\mathbf{ru}}{}_{(1)}D(v_a(-1)e^{\mathbf{mu}}) = -r_a e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}$.

Proof. Relation (i) follows from (3.3), while (ii) is a consequence of the commutativity of $Y(e^{\mathbf{ru}}, z_1)$ and $Y(e^{\mathbf{mu}}, z_2)$.

Using the commutator formula (2.6) we get

$$e^{\mathbf{ru}}{}_{(n)}v_a(-1) = v_a(-1)e^{\mathbf{ru}}{}_{(n)} - \sum_{j\geq 0}\binom{-1}{j}\left(v_a(j)e^{\mathbf{ru}}\right)_{(n-1-j)}$$

$$= v_a(-1)e^{\mathbf{ru}}{}_{(n)} - \left(v_a(0)e^{\mathbf{ru}}\right)_{(n-1)} = v_a(-1)e^{\mathbf{ru}}{}_{(n)} - r_a e^{\mathbf{ru}}{}_{(n-1)},$$

from which (iii),(iv) and (v) immediately follow.

The identity (vi) is obtained by applying (2.6) and Lemma 3.1(i),(ii).

To show equalities (vii)–(ix), we apply (2.8):

$$\left(u_s(-1)e^{\mathbf{ru}}\right)_{(n)} v_a(-1)e^{\mathbf{mu}}$$

$$= \sum_{j\geq 0} e^{\mathbf{ru}}{}_{(n-1-j)}u_s(j)v_a(-1)e^{\mathbf{mu}} + \sum_{j\geq 0} u_s(-1-j)e^{\mathbf{ru}}{}_{(n+j)}v_a(-1)e^{\mathbf{mu}}.$$

The expression $u_s(j)v_a(-1)e^{\mathbf{mu}}$ is non-zero only for $j = 1$, in which case $u_s(1)v_a(-1)e^{\mathbf{mu}} = \delta_{s,a}e^{\mathbf{mu}}$, while by (iii) and (iv) $e^{\mathbf{ru}}{}_{(n+j)}v_a(-1)e^{\mathbf{mu}}$ is non-zero only for $n = j = 0$. Thus we get

$$\left(u_s(-1)e^{\mathbf{ru}}\right)_{(n)} v_a(-1)e^{\mathbf{mu}} = \delta_{s,a}e^{\mathbf{ru}}{}_{(n-2)}e^{\mathbf{mu}} - r_a\delta_{n,0}u_s(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}.$$

The claims (vii), (viii) and (ix) follow from this equality.

For the last two statements of the lemma, we use (2.11):

$$e^{\mathbf{ru}}{}_{(0)}D(v_a(-1)e^{\mathbf{mu}}) = D\left(e^{\mathbf{ru}}{}_{(0)}v_a(-1)e^{\mathbf{mu}}\right) - (De^{\mathbf{ru}})_{(0)}v_a(-1)e^{\mathbf{mu}} = -r_a D(e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}).$$

On the last step we used part (iii) of the lemma and (2.12). Equality (xi) is obtained in a similar way using (iv) and (iii):

$$e^{\mathbf{ru}}{}_{(1)}D(v_a(-1)e^{\mathbf{mu}}) = D\left(e^{\mathbf{ru}}{}_{(1)}v_a(-1)e^{\mathbf{mu}}\right) - (De^{\mathbf{ru}})_{(1)}v_a(-1)e^{\mathbf{mu}}$$

$$= e^{\mathbf{ru}}{}_{(0)} v_a(-1) e^{\mathbf{mu}} = -r_a e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}.$$

This completes the proof of the lemma.

3.2. Affine VOAs.

The theory of affine VOA is by now standard, so we outline it very briefly. A detailed exposition may be found in [Li] or [K2].

Let $\dot{\mathfrak{g}}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ with the symmetric invariant bilinear form normalized by the condition $(\theta|\theta) = 2$ for the longest root θ of $\dot{\mathfrak{g}}$.

The untwisted affine algebra associated with $\dot{\mathfrak{g}}$ is

$$\widehat{\dot{\mathfrak{g}}} = \mathbb{C}[t_0, t_0^{-1}] \otimes \dot{\mathfrak{g}} \oplus \mathbb{C}k_0,$$

where k_0 is a central element and

$$[g_1(n), g_2(m)] = [g_1, g_2](n+m) + n\delta_{n,-m}(g_1|g_2)k_0.$$

As before, we write $g(n)$ for $t_0^n \otimes g$. The affine Lie algebra $\widehat{\dot{\mathfrak{g}}}$ may be identified with a subalgebra in the toroidal Lie algebra $\mathfrak{g}$.

It is easy to see that $\widehat{\dot{\mathfrak{g}}}$ is a vertex Lie algebra with the set $\mathcal{U}$ being a basis of $\dot{\mathfrak{g}}$ and $\mathcal{C} = \{k_0\}$.

Let $c \in \mathbb{C}$ be an arbitrary constant and let $\mathbb{C}\mathbf{1}$ be the one-dimensional module for $\mathbb{C}[t_0] \otimes \dot{\mathfrak{g}} \oplus \mathbb{C}k_0$ where $\mathbb{C}[t_0] \otimes \dot{\mathfrak{g}}$ acts trivially and k_0 acts as multiplication by c. By Theorem 2.2 the induced module

$$V_{\widehat{\dot{\mathfrak{g}}}}(c) := \mathrm{Ind}_{\mathbb{C}[t_0]\otimes\dot{\mathfrak{g}}\oplus\mathbb{C}k_0}^{\widehat{\dot{\mathfrak{g}}}}(\mathbb{C}\mathbf{1}) \cong U(t_0^{-1}\mathbb{C}[t_0^{-1}] \otimes \dot{\mathfrak{g}}) \otimes \mathbf{1}$$

has a structure of a vertex algebra.

If c is not equal to the negative of the dual Coxeter number $h^\vee$ of $\dot{\mathfrak{g}}$, one can use the Sugawara construction to show that $V_{\widehat{\dot{\mathfrak{g}}}}(c)$ contains a Virasoro element and thus turns into a VOA. The rank of this VOA is

$$\mathrm{rank}\, V_{\widehat{\dot{\mathfrak{g}}}}(c) = \frac{c \dim(\dot{\mathfrak{g}})}{c + h^\vee}.$$

Using (2.9), (2.10) and (2.3) we get the following

Lemma 3.3.
(i) $\omega_{(0)} g(-1)\mathbf{1} = D(g(-1))\mathbf{1}$,
(ii) $\omega_{(1)} g(-1)\mathbf{1} = g(-1)\mathbf{1}$,
(iii) $\omega_{(j)} g(-1)\mathbf{1} = 0$ *for* $j \geq 2$.

Finally, the irreducible quotient $L_{\widehat{\dot{\mathfrak{g}}}}(c)$ of $V_{\widehat{\dot{\mathfrak{g}}}}(c)$ is also a VOA of the same rank. By Theorem 2.2(d) any highest weight $\widehat{\dot{\mathfrak{g}}}$ module of level c is a VOA module for $V_{\widehat{\dot{\mathfrak{g}}}}(c)$.

3.3. $\widehat{gl}_N$ VOAs .

For the construction of the $(N+1)$-toroidal VOA we also need to consider the VOA associated with affine $\widehat{gl}_N$. The construction of this VOA follows the general scheme of the previous subsection, with the difference that gl_N is not simple, but only reductive. The Lie algebra gl_N decomposes into a direct sum: $gl_N = sl_N \oplus \mathbb{C}I$, where I is an identity matrix. Let ψ_1 denote the projection on the traceless matrices and ψ_2 be the projection on the scalar matrices in this decomposition:

$$\psi_2(A) = \frac{\mathrm{tr}(A)}{N} I, \quad \psi_1(A) = A - \psi_2(A), \quad A \in gl_N.$$

Accordingly, we define the affine algebra $\widehat{gl}_N$ to be the direct sum of the affine algebra $\widehat{sl}_N = \mathbb{C}[t_0, t_0^{-1}] \otimes sl_N \oplus \mathbb{C}C_1$ and a (degenerate) Heisenberg algebra $\mathcal{H}ei = \mathbb{C}[t_0, t_0^{-1}] \otimes I \oplus \mathbb{C}C_I$. The Lie bracket in $\widehat{gl}_N$ is given by

$$[g_1(n), g_2(m)] = [g_1, g_2](n+m) + n\delta_{n,-m} \left\{ tr(\psi_1(g_1)\psi_1(g_2))C_1 + (\psi_2(g_1)|\psi_2(g_2))C_I \right\},$$

where for the last term we will use normalization $(I|I) = 1$.

This can be rewritten as a commutator of the formal generating series:

$$\begin{aligned} [g_1(z_1), g_2(z_2)] =& [g_1, g_2](z_2) \left[z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) \right] \\ &+ \left\{ tr(\psi_1(g_1)\psi_1(g_2))C_1 + (\psi_2(g_1)|\psi_2(g_2))C_I \right\} \left[z_1^{-1} \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right) \right]. \end{aligned} \tag{3.6}$$

To construct a vertex algebra corresponding to $\widehat{gl}_N$, we will take a tensor product of a vertex algebra for affine $\widehat{sl}_N$ and a Heisenberg vertex algebra. We outline the construction of the latter.

The Heisenberg algebra $\mathbb{C}[t_0, t_0^{-1}] \otimes I \oplus \mathbb{C}C_I$ with the Lie bracket given by (2.21), is a vertex Lie algebra with $\mathcal{U} = \{I\}, \mathcal{C} = \{C_I\}$. Let c_I be an arbitrary constant in $\mathbb{C}$ and consider a one-dimensional module $\mathbb{C}\mathbf{1}$ for the subalgebra $\mathbb{C}[t_0] \otimes I \oplus \mathbb{C}C_I$ where $\mathbb{C}[t_0] \otimes I$ acts on $\mathbf{1}$ trivially and C_I acts as multiplication by c_I. By theorem 2.2, the induced module

$$V_{\mathcal{H}ei}(c_I) := \mathrm{Ind}_{\mathbb{C}[t_0] \otimes I \oplus \mathbb{C}C_I}^{\mathcal{H}ei} (\mathbb{C}\mathbf{1}) \cong U(t_0^{-1}\mathbb{C}[t_0^{-1}] \otimes I) \otimes \mathbf{1},$$

is a vertex algebra with

$$I(z) = \sum_{j \in \mathbb{Z}} I(j) z^{-j-1}$$

and the map Y defined by (2.18).

The $\widehat{gl}_N$ vertex algebra, defined as a tensor product

$$V_{\widehat{gl}_N}(c_1, c_I) := V_{\widehat{sl}_N}(c_1) \otimes V_{\mathcal{H}ei}(c_I),$$

is a vertex operator algebra when $c_1 \neq -N$ (the dual Coxeter number for sl_N is N) and $c_I \neq 0$. In this case the irreducible quotient $L_{\widehat{gl}_N}(c_1, c_I) = L_{\widehat{sl}_N}(c_1) \otimes V_{\mathcal{H}ei}(c_I)$ of $V_{\widehat{gl}_N}(c_1, c_I)$ is also a VOA. Any highest weight $\widehat{gl}_N$ module with C_1, C_I acting as multiplications by c_1, c_I, is a VOA module for $V_{\widehat{gl}_N}(c_1, c_I)$.

The following relations in $V_{\widehat{gl}_N}(c_1, c_I)$ are the consequences of (3.6) and the commutator formula (2.2):

Lemma 3.4.
(i) $E_{ab}(-1)_{(0)}E_{cd}(-1)\mathbf{1} = \delta_{bc}E_{ad}(-1)\mathbf{1} - \delta_{ad}E_{cb}(-1)\mathbf{1}$,
(ii) $E_{ab}(-1)_{(1)}E_{cd}(-1)\mathbf{1} = \delta_{ad}\delta_{bc}c_1\mathbf{1} + \delta_{ab}\delta_{cd}\left(\frac{c_I}{N^2} - \frac{c_1}{N}\right)\mathbf{1}$,
(iii) $E_{ab}(-1)_{(j)}E_{cd}(-1)\mathbf{1} = 0$, *for* $j \geq 2$.

3.4. Twisted $\widehat{gl}_N$ - Virasoro VOAs.

We define the twisted $\widehat{gl}_N$ - Virasoro VOA $V_{\widehat{gl}_N - \mathcal{V}ir}$ as the tensor product of affine $\widehat{sl}_N$ VOA with the twisted Heisenberg-Virasoro VOA introduced in Section 2.4:

$$V_{\widehat{gl}_N - \mathcal{V}ir}(c_1, c_L, c_{LI}, c_I) := V_{\widehat{sl}_N}(c_1) \otimes V_{\mathcal{HV}ir}(c_L, c_{LI}, c_I).$$

The rank of this VOA is

$$\mathrm{rank}\ V_{\widehat{gl}_N - \mathcal{V}ir}(c_1, c_L, c_{LI}, c_I) = \frac{c_1(N^2-1)}{c_1 + N} + c_L.$$

We turn $V_{\widehat{gl}_N - \mathcal{V}ir}$ into a $\widehat{gl}_N$ module by identifying the symbol I from the Heisenberg-Virasoro part with the identity matrix in gl_N:

$$\sum_{j \in \mathbb{Z}} A(j) z^{-j-1} \mapsto Y_{\widehat{sl}_N}(\psi_1(A)(-1)\mathbf{1}, z) + Y_{\mathcal{HV}ir}(\psi_2(A)(-1)\mathbf{1}, z) =$$

$$= \sum_{j \in \mathbb{Z}} \psi_1(A)(j) z^{-j-1} + \frac{\mathrm{tr}(A)}{N} \sum_{j \in \mathbb{Z}} I(j) z^{-j-1}, \quad \text{for } A \in gl_N.$$

We may view the vertex algebra $V_{\widehat{gl}_N}(c_1, c_I)$ as a sub-vertex algebra of the VOA $V_{\widehat{gl}_N - \mathcal{V}ir}(c_1, c_L, c_{LI}, c_I)$.

The VOA $V_{\widehat{gl}_N - \mathcal{V}ir}(c_1, c_L, c_{LI}, c_I)$ has a simple factor VOA $L_{\widehat{gl}_N - \mathcal{V}ir}(c_1, c_L, c_{LI}, c_I) \cong L_{\widehat{sl}_N}(c_1) \otimes L_{\mathcal{HV}ir}(0, 0, c_L, c_{LI}, c_I)$.

Let $L_{\widehat{sl}_N}(\lambda_1, c_1)$ be an irreducible $\widehat{sl}_N$ module of the highest weight (λ_1, c_1), where λ_1 is a linear functional on the Cartan subalgebra of sl_N. The tensor product

$$L_{\widehat{sl}_N}(\lambda_1, c_1) \otimes L_{\mathcal{HV}ir}(h, h_I, c_L, c_{LI}, c_I)$$

is an irreducible VOA module for $V_{\widehat{gl}_N - \mathcal{V}ir}(c_1, c_L, c_{LI}, c_I)$.

We will later need the following relations in the twisted $\widehat{gl}_N$-Virasoro VOA:

Lemma 3.5.
(i) $\omega_{(0)}E_{ab}(-1)\mathbf{1} = D(E_{ab}(-1))\mathbf{1}$,
(ii) $\omega_{(1)}E_{ab}(-1)\mathbf{1} = E_{ab}(-1)\mathbf{1}$,
(iii) $\omega_{(2)}E_{ab}(-1)\mathbf{1} = -\delta_{ab}\frac{2c_{LI}}{N}\mathbf{1}$,

(iv) $\omega_{(j)} E_{ab}(-1)\mathbf{1} = 0$ *for* $j \geq 3$.

Proof. Here (i) follows from (2.9), (ii) from (2.10), and (iv) from (2.3). Let us establish (iii). We project $E_{ab}(-1)\mathbf{1}$ into the affine $\widehat{sl}_N$ VOA and the twisted Heisenberg-Virasoro VOA: $E_{ab}(-1)\mathbf{1} = \psi_1(E_{ab})(-1)\mathbf{1} + \delta_{ab}\frac{1}{N} I(-1)\mathbf{1}$. Applying Lemma 3.3(iii) and Lemma 2.5(iii) we get

$$\omega_{(2)} \psi_1(E_{ab})(-1)\mathbf{1} = 0,$$

$$\omega_{(2)} I(-1)\mathbf{1} = -2c_{LI}\mathbf{1},$$

and the claim (iii) follows.

The main result of this paper is that for certain values of the parameters, the tensor product of VOAs

$$V_{tor} = V_{\hat{\dot{\mathfrak{g}}}}(c) \otimes V_{Hyp}^{+} \otimes V_{\widehat{gl}_N - \mathcal{V}ir}(0, c_L, c_{LI}, 0) \tag{3.7}$$

is a module for the toroidal Lie algebra $\mathfrak{g}(\mu, 0)$. The modules for this VOA naturally inherit the structure of $\mathfrak{g}(\mu, 0)$ modules.

4. Main theorem.

The goal of the present paper is to show that the toroidal VOA (3.7) has a structure of a module over the toroidal Lie algebra $\mathfrak{g}(\mu, 0)$. In order to establish the correspondence between V_{tor} and $\mathfrak{g}$, we consider the following fields in $\mathfrak{g}[[z, z^{-1}]]$:

$$k_0(\mathbf{r}, z) = \sum_{j \in \mathbb{Z}} t_0^j \mathbf{t}^{\mathbf{r}} k_0 z^{-j},$$

$$k_a(\mathbf{r}, z) = \sum_{j \in \mathbb{Z}} t_0^j \mathbf{t}^{\mathbf{r}} k_a z^{-j-1}, \quad a = 1, \ldots, N,$$

$$g(\mathbf{r}, z) = \sum_{j \in \mathbb{Z}} t_0^j \mathbf{t}^{\mathbf{r}} g z^{-j-1}, \quad g \in \dot{\mathfrak{g}},$$

$$d_a(\mathbf{r}, z) = \sum_{j \in \mathbb{Z}} t_0^j \mathbf{t}^{\mathbf{r}} d_a z^{-j-1}, \quad a = 1, \ldots, N,$$

$$\tilde{d}_0(\mathbf{r}, z) = \sum_{j \in \mathbb{Z}} \left(-t_0^j \mathbf{t}^{\mathbf{r}} d_0 + \mu(j + \frac{1}{2}) t_0^j \mathbf{t}^{\mathbf{r}} k_0 \right) z^{-j-2}.$$

Here $\mathbf{r} \in \mathbb{Z}^N$ and $\mathbf{t}^{\mathbf{r}} = t_1^{r_1} \ldots t_N^{r_N}$. In the last field we added the term $\mu(j + \frac{1}{2}) t_0^j \mathbf{t}^{\mathbf{r}} k_0$ in order to make the Lie bracket compatible with the VOA structure. This is analogous to choosing the "correct" basis in a vertex Lie algebra.

The moments of the above fields span the toroidal algebra and the Lie bracket structure of $\mathfrak{g}$ may be encoded in the commutator relations involving these fields. The commutators involving the first four fields only, $k_0(\mathbf{r}, z), k_a(\mathbf{r}, z), g(\mathbf{r}, z), d_a(\mathbf{r}, z)$, are given in [BBS] and will not be reproduced here. Let us write down only those commutators that involve the last field $\tilde{d}_0(\mathbf{r}, z)$. These relations are derived from (1.1),(1.2) and (1.3):

$$[\tilde{d}_0(\mathbf{r}, z_1), k_0(\mathbf{m}, z_2)] = \sum_{p=1}^{N} m_p k_p(\mathbf{r}+\mathbf{m}, z_2) \left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right], \tag{4.1}$$

$$[\tilde{d}_0(\mathbf{r}, z_1), k_a(\mathbf{m}, z_2)] = \frac{\partial}{\partial z_2}\left(k_a(\mathbf{r}+\mathbf{m}, z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right]\right), \tag{4.2}$$

$$[\tilde{d}_0(\mathbf{r}, z_1), g(\mathbf{m}, z_2)] = \frac{\partial}{\partial z_2}\left(g(\mathbf{r}+\mathbf{m}, z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right]\right), \tag{4.3}$$

$$\begin{aligned}
[\tilde{d}_0(\mathbf{r}, z_1), d_a(\mathbf{m}, z_2)] = {} & \frac{\partial}{\partial z_2}\left(d_a(\mathbf{r}+\mathbf{m}, z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right]\right) - r_a\tilde{d}_0(\mathbf{r}+\mathbf{m}, z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right] \\
& -\mu r_a\frac{\partial}{\partial z_2}\left(k_0(\mathbf{r}+\mathbf{m}, z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right]\right) \\
& -\mu r_a\frac{\partial}{\partial z_2}\left(\sum_{p=1}^{N} r_p k_p(\mathbf{r}+\mathbf{m}, z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right]\right),
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
[\tilde{d}_0(\mathbf{r}, z_1), \tilde{d}_0(\mathbf{m}, z_2)] = {} & \tilde{d}_0(\mathbf{r}+\mathbf{m}, z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right] \\
& +\frac{\partial}{\partial z_2}\left(\tilde{d}_0(\mathbf{r}+\mathbf{m}, z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right]\right) \\
& +\frac{\mu}{2}\left(\frac{\partial}{\partial z_2}\right)^2\left(k_0(\mathbf{r}+\mathbf{m}, z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right]\right) \\
& +\frac{\mu}{2}\frac{\partial}{\partial z_2}\left(k_0(\mathbf{r}+\mathbf{m}, z_2)\left[z_1^{-1}\left(\frac{\partial}{\partial z_2}\right)^2\delta\left(\frac{z_2}{z_1}\right)\right]\right) \\
& +\frac{\mu}{2}\sum_{p=1}^{N}(r_p-m_p)\frac{\partial}{\partial z_2}\left(k_p(\mathbf{r}+\mathbf{m}, z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right]\right).
\end{aligned} \tag{4.5}$$

Before we state the main theorem of the paper, we recall a result from [BBS], where the representation theory of a subalgebra $\mathfrak{g}^*$ of the toroidal Lie algebra $\mathfrak{g}$ is developed using the VOA approach.

Theorem 4.1. ([BBS], Theorem 5.1.) *Let $M_{\widehat{\mathfrak{g}}}$ be a module for affine vertex algebra $V_{\widehat{\mathfrak{g}}}(c)$ with $c \neq 0$. Let V_{Hyp}^{+} be a sub-VOA of a hyperbolic lattice VOA and let $M_{\widehat{gl}_N}$ be a module for the vertex algebra $V_{\widehat{gl}_N}(c_1, c_I)$. Then the tensor product*

$$M_{\widehat{\mathfrak{g}}} \otimes V_{Hyp}^{+} \otimes M_{\widehat{gl}_N}$$

is a module for the subalgebra $\mathfrak{g}^*(\mu,\nu)$ *of the toroidal Lie algebra* $\mathfrak{g}(\mu,\nu)$*, where the cocycle* $\tau = \mu\tau_1 + \nu\tau_2$ *has* $\mu = \frac{1-c_1}{c}$ *and* $\nu = \frac{c_1}{cN} - \frac{c_I}{cN^2}$*. The action of the Lie algebra* $\mathfrak{g}^*(\mu,\nu)$ *is given by the vertex operators:*

$$k_0(\mathbf{r},z) \mapsto cY(\mathbf{1}\otimes e^{\mathbf{ru}}\otimes \mathbf{1}, z), \tag{4.6}$$

$$k_a(\mathbf{r},z) \mapsto cY(\mathbf{1}\otimes u_a(-1)e^{\mathbf{ru}}\otimes \mathbf{1}, z), \tag{4.7}$$

$$g(\mathbf{r},z) \mapsto Y(g(-1)\otimes e^{\mathbf{ru}}\otimes \mathbf{1}, z), \tag{4.8}$$

$$d_a(\mathbf{r},z) \mapsto Y(\mathbf{1}\otimes v_a(-1)e^{\mathbf{ru}}\otimes \mathbf{1}, z) + \sum_{p=1}^{N} r_p Y(\mathbf{1}\otimes e^{\mathbf{ru}}\otimes E_{pa}(-1), z). \tag{4.9}$$

In this paper we are taking $c_1 = c_I = 0$, so the corresponding cocycle $\tau = \mu\tau_1 + \nu\tau_2$ has $\mu = \frac{1}{c}$ and $\nu = 0$.

By a slight abuse of notations we will drop, from now on, the symbols $\mathbf{1}$ from the tensor products, e.g., write $e^{\mathbf{ru}}\otimes E_{pa}(-1)$ instead of $\mathbf{1}\otimes e^{\mathbf{ru}}\otimes E_{pa}(-1)$.

We are now able to state the main result of the paper.

Theorem 4.2. *Let* c, c_L, c_{LI} *be complex numbers such that* $c\neq 0, c\neq -h^\vee, c_{LI} = \frac{N}{2}$ *and*

$$\frac{c\dim\dot{\mathfrak{g}}}{c+h^\vee} + 2N + c_L = 12.$$

Let also $c_1 = c_I = 0$*. Then the tensor product of the affine VOA, sub-VOA of a hyperbolic lattice VOA and the twisted* $\widehat{gl}_N$*-Virasoro VOA*

$$V_{tor} = V_{\widehat{\mathfrak{g}}}(c)\otimes V^+_{Hyp}\otimes V_{\widehat{gl}_N-\mathcal{V}ir}(0,c_L,c_{LI},0)$$

is a module for the toroidal Lie algebra $\mathfrak{g}(\frac{1}{c},0)$*. The action of the fields* $k_0(\mathbf{r},z)$*,* $k_a(\mathbf{r},z)$*,* $g(\mathbf{r},z)$ *and* $d_a(\mathbf{r},z)$ *is given by the formulas (4.6), (4.7), (4.8), (4.9) (however now in a different representation space), whereas the field* $\tilde{d}_0(\mathbf{r},z)$ *is represented by the vertex operator*

$$\tilde{d}_0(\mathbf{r},z) \mapsto Y(\omega_{(-1)}e^{\mathbf{ru}}, z) + \sum_{p,s=1}^{N} r_p Y(u_s(-1)e^{\mathbf{ru}}\otimes E_{ps}(-1), z), \tag{4.10}$$

where ω *is the Virasoro element of the tensor product (the sum of the Virasoro elements of* $V_{\widehat{\mathfrak{g}}}(c)$*,* V^+_{Hyp} *and* $V_{\widehat{gl}_N-\mathcal{V}ir}(0,c_L,c_{LI},0)$*.*

The following corollary can be immediately obtained from the theorem by applying the principle of the "preservation of identities" ([Li], Lemma 2.3.5).

Corollary 4.3. *Let* c, c_L, c_{LI} *satisfy the conditions of Theorem 4.2. Let* $M_{\widehat{\mathfrak{g}}}$ *be a VOA module for* $V_{\widehat{\mathfrak{g}}}(c)$*,* M^+_{Hyp} *be a VOA module for* V^+_{Hyp}*, and* $M_{\widehat{gl}_N-\mathcal{V}ir}$ *be a VOA module for* $V_{\widehat{gl}_N-\mathcal{V}ir}(0,c_L,c_{LI},0)$*. Then*

$$M_{\widehat{\mathfrak{g}}}\otimes M^+_{Hyp}\otimes M_{\widehat{gl}_N-\mathcal{V}ir}$$

is a module for the toroidal Lie algebra $\mathfrak{g}(\frac{1}{c}, 0)$.

Proof of Theorem 4.2. We need to prove that the vertex operators in (4.6)–(4.10) satisfy the same commutator identities as the corresponding fields in the toroidal Lie algebra. Relations involving only the operators corresponding to $k_0(\mathbf{r}, z), k_a(\mathbf{r}, z), g(\mathbf{r}, z)$ and $d_a(\mathbf{r}, z)$ follow from Theorem 4.1. Our main tool for proving the remaining relations (4.1)–(4.5) will be the commutator formula (2.2):

$$[Y(a, z_1), Y(b, z_2)] = \sum_{n=0}^{\deg\,(a)+\deg\,(b)-1} \frac{1}{n!} Y(a_n b, z_2) \left[z_1^{-1} \left(\frac{\partial}{\partial z_2} \right)^n \delta \left(\frac{z_1}{z_2} \right) \right]. \qquad (4.11)$$

Since our proof is somewhat lengthy, we organize it in four lemmas.

Lemma 4.4. *The vertex operators in (4.10), (4.8) representing* $\tilde{d}_0(\mathbf{r}, z)$ *and* $g(\mathbf{m}, z)$ *satisfy the relation (4.3).*

Proof. To verify that the vertex operators in (4.10), (4.8) satisfy (4.3), we need to compute the n-th products

$$\left(\omega_{(-1)} e^{\mathbf{ru}} + \sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1) \right)_{(n)} (g(-1) \otimes e^{\mathbf{mu}})$$

for $n = 0, 1, 2$. The vertex operators $Y(u_s(-1)e^{\mathbf{ru}} \otimes E_{ps}(-1), z_1)$ and $Y(g(-1) \otimes e^{\mathbf{mu}}, z_2)$ commute, and thus

$$\left(\sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1) \right)_{(n)} (g(-1) \otimes e^{\mathbf{mu}}) = 0, \quad \text{for} \quad n \geq 0.$$

Let us evaluate $\left(\omega_{(-1)} e^{\mathbf{ru}}\right)_{(n)} (g(-1) \otimes e^{\mathbf{mu}})$ with the help of the Borcherds' identity (2.8):

$$\left(\omega_{(-1)} e^{\mathbf{ru}}\right)_{(n)} (g(-1) \otimes e^{\mathbf{mu}})$$

$$= \sum_{j=0}^{\infty} e^{\mathbf{ru}}{}_{(n-j-1)} \omega_{(j)} (g(-1) \otimes e^{\mathbf{mu}}) + \sum_{j=0}^{\infty} \omega_{(-1-j)} e^{\mathbf{ru}}{}_{(j)} (g(-1) \otimes e^{\mathbf{mu}}). \qquad (4.12)$$

Again because $Y(e^{\mathbf{ru}}, z_1)$ and $Y(g(-1) \otimes e^{\mathbf{mu}}, z_2)$ commute, we get that $(e^{\mathbf{ru}})_{(j)} (g(-1) \otimes e^{\mathbf{mu}}) = 0$ for $j \geq 0$ and hence the last sum in (4.12) equals 0. For the first sum in (4.12) we write

$$\omega_{(j)} (g(-1) \otimes e^{\mathbf{mu}}) = (\omega_{(j)} g(-1)) \otimes e^{\mathbf{mu}} + g(-1) \otimes \omega_{(j)} e^{\mathbf{mu}}.$$

By Lemma 3.1(i), (ii) we have $\omega_{(0)} e^{\mathbf{mu}} = D e^{\mathbf{mu}}$, $\omega_{(j)} e^{\mathbf{mu}} = 0$ for $j \geq 1$, and by Lemma 3.3, $\omega_{(0)} g(-1) = D g(-1)$, $\omega_{(1)} g(-1) = g(-1)$ and $\omega_{(j)} g(-1) = 0$ for $j \geq 2$. Thus (4.12) becomes

$$\left(\omega_{(-1)} e^{\mathbf{ru}}\right)_{(n)} (g(-1) \otimes e^{\mathbf{mu}}) = e^{\mathbf{ru}}{}_{(n-1)} D (g(-1) \otimes e^{\mathbf{mu}}) + e^{\mathbf{ru}}{}_{(n-2)} (g(-1) \otimes e^{\mathbf{mu}}). \qquad (4.13)$$

Since $Y(e^{\mathbf{ru}}, z_1)$ commutes with $Y(g(-1)\otimes e^{\mathbf{mu}}, z_2)$, we get that the first term in the right hand side of (4.13) is non-zero for $n=0$, while the second term is non-zero for $n=0,1$. Using Lemma 3.2(i), we get that

$$\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(1)}\left(g(-1)\otimes e^{\mathbf{mu}}\right) = g(-1)\otimes {e^{\mathbf{ru}}}_{(-1)}e^{\mathbf{mu}} = g(-1)\otimes e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}, \tag{4.14}$$

$$\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(0)}\left(g(-1)\otimes e^{\mathbf{mu}}\right) = {e^{\mathbf{ru}}}_{(-1)}D(g(-1)\otimes e^{\mathbf{mu}}) + {e^{\mathbf{ru}}}_{(-2)}(g(-1)\otimes e^{\mathbf{mu}}) =$$

$$= {e^{\mathbf{ru}}}_{(-1)}D(g(-1)\otimes e^{\mathbf{mu}}) + (De^{\mathbf{ru}})_{(-1)}(g(-1)\otimes e^{\mathbf{mu}}) = D(g(-1)\otimes e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}). \tag{4.15}$$

Substituting (4.14) and (4.15) in the commutator formula (4.11) we get that

$$\left[Y(\omega_{(-1)}e^{\mathbf{ru}} + \sum_{s,p=1}^{N} r_s(u_p(-1)e^{\mathbf{ru}}\otimes E_{ps}(-1)), z_1), Y(g(-1)\otimes e^{\mathbf{mu}}, z_2)\right] =$$

$$\left(\frac{\partial}{\partial z_2}Y(g(-1)\otimes e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}, z_2)\right)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right] + Y(g(-1)\otimes e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}, z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right],$$

which corresponds to (4.3). This completes the proof of the Lemma.

Lemma 4.5. *The vertex operators in (4.10), (4.6), (4.7) representing $\tilde{d}_0(\mathbf{r}, z)$, $k_0(\mathbf{m}, z)$ and $k_a(\mathbf{m}, z)$ satisfy the relations (4.1) and (4.2).*

Proof. We could perform the direct calculations for (4.1) and (4.2), just as we did in the previous lemma, however this computation may be skipped altogether by referring to Lemma 2.1 of [BBS]. According to this lemma, the relations (4.1) and (4.2) follow from (4.3), which was established in Lemma 4.4.

Lemma 4.6. *The vertex operators in (4.10), (4.9), (4.6), (4.7) representing $\tilde{d}_0(\mathbf{r}, z)$, $d_a(\mathbf{m}, z)$, $k_0(\mathbf{m}, z)$ and $k_a(\mathbf{m}, z)$ satisfy the relation (4.4).*

Proof. Again, to make use of the commutator formula (4.11), we need to compute the following n-th products:

$$\left(\omega_{(-1)}e^{\mathbf{ru}} + \sum_{p,s=1}^{N} r_p u_s(-1)e^{\mathbf{ru}}\otimes E_{ps}(-1)\right)_{(n)}\left(v_a(-1)e^{\mathbf{mu}} + \sum_{q=1}^{N} m_q e^{\mathbf{mu}}\otimes E_{qa}(-1)\right) \tag{4.16}$$

for $n\geq 0$. We split (4.16) into four terms by expanding both sums. In the four steps below, we handle each of these terms.

Step 1. Let us evaluate $\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(n)}\left(v_a(-1)e^{\mathbf{mu}}\right)$. We apply the Borcherds' identity (2.8):

$$\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(n)}\left(v_a(-1)e^{\mathbf{mu}}\right)$$

$$= \sum_{j\geq 0}(e^{\mathbf{ru}})_{(n-1-j)}\omega_{(j)}(v_a(-1)e^{\mathbf{mu}}) + \sum_{j\geq 0}\omega_{(-1-j)}{e^{\mathbf{ru}}}_{(n+j)}v_a(-1)e^{\mathbf{mu}}.$$

Since the left factor in the n-th product (4.16) is of degree 2, and the right factor is of degree 1, then by (2.3), we need only to consider values $n = 0, 1, 2$. We apply Lemma 3.1(vi)–(ix) to the first sum in the right hand side, and Lemma 3.2(iii),(iv) to the second sum.

Let $n = 2$. In this case

$$\left(\omega_{(-1)} e^{\mathbf{ru}}\right)_{(2)} (v_a(-1)e^{\mathbf{mu}}) = e^{\mathbf{ru}}{}_{(1)} D(v_a(-1)e^{\mathbf{mu}}) + e^{\mathbf{ru}}{}_{(0)} v_a(-1)e^{\mathbf{mu}} + e^{\mathbf{ru}}{}_{(-1)}(m_a e^{\mathbf{mu}})$$

$$= -r_a e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} - r_a e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} + m_a e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} = (m_a - 2r_a) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}.$$

In the above calculation we also used Lemma 3.2(iii), (xi).

Next let $n = 1$.

$$\left(\omega_{(-1)}(e^{\mathbf{ru}})\right)_{(1)} (v_a(-1)e^{\mathbf{mu}}) = e^{\mathbf{ru}}{}_{(0)} D(v_a(-1)e^{\mathbf{mu}}) + e^{\mathbf{ru}}{}_{(-1)} v_a(-1)e^{\mathbf{mu}} + e^{\mathbf{ru}}{}_{(-2)}(m_a e^{\mathbf{mu}})$$

$$= -r_a D(e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}) + v_a(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} - r_a (De^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}} + m_a (De^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}}.$$

Here we used Lemma 3.2(x), (v) and (2.13).

Finally let $n = 0$.

$$\left(\omega_{(-1)}(e^{\mathbf{ru}})\right)_{(0)} (v_a(-1)e^{\mathbf{mu}})$$

$$= e^{\mathbf{ru}}{}_{(-1)} D(v_a(-1)e^{\mathbf{mu}}) + e^{\mathbf{ru}}{}_{(-2)} v_a(-1)e^{\mathbf{mu}} + e^{\mathbf{ru}}{}_{(-3)}(m_a e^{\mathbf{mu}}) + \omega_{(-1)} e^{\mathbf{ru}}{}_{(0)} v_a(-1)e^{\mathbf{mu}}$$

$$= D(e^{\mathbf{ru}}{}_{(-1)} v_a(-1)e^{\mathbf{mu}}) + \frac{m_a}{2}(D^2 e^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}} - r_a \omega_{(-1)} e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}$$

$$= D(v_a(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}) - r_a D((De^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}}) + \frac{m_a}{2}(D^2 e^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}} - r_a \omega_{(-1)} e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}.$$

To obtain the last two equalities we applied (2.11), (2.13) and Lemma 3.2(iii), (v).

The computations for Step 1 are now complete and we pass to

Step 2. We are going to evaluate

$$\left(\omega_{(-1)} e^{\mathbf{ru}}\right)_{(n)} \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1),$$

for $n = 0, 1, 2$. Just as in Step 1, we use the Borcherds' identity (2.8):

$$\left(\omega_{(-1)} e^{\mathbf{ru}}\right)_{(n)} \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1)$$

$$= \sum_{j \geq 0} e^{\mathbf{ru}}{}_{(n-1-j)} \sum_{q=1}^{N} m_q \left((\omega_{(j)} e^{\mathbf{mu}}) \otimes E_{qa}(-1) + e^{\mathbf{mu}} \otimes \omega_{(j)} E_{qa}(-1)\right)$$

$$+ \sum_{j \geq 0} \omega_{(-1-j)} e^{\mathbf{ru}}{}_{(n+j)} \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1).$$

The second sum is equal to zero, while the first one yields (we use Lemma 3.1(i), (ii), Lemma 3.5 and the condition $c_{LI} = \frac{N}{2}$):

$$\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(n)} \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1)$$

$$= e^{\mathbf{ru}}{}_{(n-1)} \sum_{q=1}^{N} m_q D(e^{\mathbf{mu}}) \otimes E_{qa}(-1) + e^{\mathbf{ru}}{}_{(n-1)} \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes DE_{qa}(-1)$$

$$+ e^{\mathbf{ru}}{}_{(n-2)} \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1) - m_a \frac{2c_{LI}}{N} e^{\mathbf{ru}}{}_{(n-3)} e^{\mathbf{mu}}$$

$$= e^{\mathbf{ru}}{}_{(n-1)} \sum_{q=1}^{N} m_q D(e^{\mathbf{mu}} \otimes E_{qa}(-1)) + e^{\mathbf{ru}}{}_{(n-2)} \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1) - m_a e^{\mathbf{ru}}{}_{(n-3)} e^{\mathbf{mu}}. \quad (4.17)$$

Let $n = 2$. Then the only non-zero term in the above expression is the last term:

$$\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(2)} \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1) = -m_a e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}.$$

When $n = 1$, the two last terms in (4.17) are non-zero:

$$\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(1)} \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1) = \sum_{q=1}^{N} m_q e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{qa}(-1) - m_a (De^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}}.$$

For $n = 0$ we get:

$$\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(0)} \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1)$$

$$= e^{\mathbf{ru}}{}_{(-1)} \sum_{q=1}^{N} m_q D(e^{\mathbf{mu}} \otimes E_{qa}(-1)) + (De^{\mathbf{ru}})_{(-1)} \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1) - \frac{m_a}{2} (D^2 e^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}}$$

$$= \sum_{q=1}^{N} m_q D(e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{qa}(-1)) - \frac{m_a}{2} (D^2 e^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}}.$$

Step 3. The next term from (4.16) that we need to handle is

$$\left(\sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1) \right)_{(n)} v_a(-1) e^{\mathbf{mu}}.$$

Using Lemma 2.1(iii) we get

$$\left(\sum_{p,s=1}^{N} r_p u_s(-1)e^{\mathbf{ru}} \otimes E_{ps}(-1)\right)_{(n)} v_a(-1)e^{\mathbf{mu}}$$

$$= \sum_{j\geq 0}\sum_{p,s=1}^{N} r_p((u_s(-1)e^{\mathbf{ru}})_{(n+j)} v_a(-1)e^{\mathbf{mu}}) \otimes E_{ps}(-1-j).$$

Next we apply Lemma 3.2(ix) and we see that the expression above turns into zero for $n \geq 2$. Let us consider now cases $n = 0, 1$.

Let $n = 1$. Then using Lemma 3.2(viii), (ix) we obtain

$$\left(\sum_{p,s=1}^{N} r_p u_s(-1)e^{\mathbf{ru}} \otimes E_{ps}(-1)\right)_{(1)} v_a(-1)e^{\mathbf{mu}} = \sum_{p=1}^{N} r_p e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{pa}(-1).$$

For $n = 0$ we have

$$\left(\sum_{p,s=1}^{N} r_p u_s(-1)e^{\mathbf{ru}} \otimes E_{ps}(-1)\right)_{(0)} v_a(-1)e^{\mathbf{mu}} = \sum_{p=1}^{N} r_p((De^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}}) \otimes E_{pa}(-1)$$

$$-r_a \sum_{p,s=1}^{N} r_p u_s(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{ps}(-1) + \sum_{p=1}^{N} r_p e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{pa}(-2).$$

Step 4. We evaluate

$$\left(\sum_{p,s=1}^{N} r_p u_s(-1)e^{\mathbf{ru}} \otimes E_{ps}(-1)\right)_{(n)} \left(\sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1)\right)$$

using Lemma 2.1(iii). Since we assume $c_1 = c_I = 0$ then by Lemma 3.4 we have $E_{ps}(-1)_{(j)}E_{qa}(-1) = 0$ for $j \geq 1$. Thus the above expression is non-zero only for $n = 0$, in which case we get:

$$\left(\sum_{p,s=1}^{N} r_p u_s(-1)e^{\mathbf{ru}} \otimes E_{ps}(-1)\right)_{(0)} \left(\sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1)\right)$$

$$= \sum_{p,s=1}^{N}\sum_{q=1}^{N} r_p m_q (u_s(-1)e^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}} \otimes E_{ps}(-1)_{(0)}E_{qa}(-1)$$

$$= \sum_{p,s=1}^{N} r_p m_s u_s(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{pa}(-1) - r_a \sum_{q,s=1}^{N} m_q u_s(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{qs}(-1)$$

$$= \sum_{p=1}^{N} r_p ({e^{\mathbf{ru}}}_{(-1)} De^{\mathbf{mu}}) \otimes E_{pa}(-1) - r_a \sum_{q,s=1}^{N} m_q u_s(-1) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{qs}(-1).$$

Now we combine the results of the four steps:

$$\left(\omega_{(-1)} e^{\mathbf{ru}} + \sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1) \right)_{(2)} \left(v_a(-1) e^{\mathbf{mu}} + \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1) \right)$$

$$= (m_a - 2r_a) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} - m_a e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} = -2r_a e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}.$$

This corresponds to the term in (4.4) with the second derivative of the delta function.

Now let $n = 1$:

$$\left(\omega_{(-1)} e^{\mathbf{ru}} + \sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1) \right)_{(1)} \left(v_a(-1) e^{\mathbf{mu}} + \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1) \right)$$

$$= v_a(-1) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} - r_a D(e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}) - r_a (De^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}} + m_a (De^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}}$$

$$+ \sum_{q=1}^{N} m_q e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{qa}(-1) - m_a (De^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}} + \sum_{p=1}^{N} r_p e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{pa}(-1)$$

$$= v_a(-1) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} + \sum_{p=1}^{N} (r_p + m_p) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{pa}(-1) - r_a D(e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}) - r_a (De^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}}.$$

This matches the terms in (4.4) with the first derivative of the delta function.

Finally let $n = 0$:

$$\left(\omega_{(-1)} e^{\mathbf{ru}} + \sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1) \right)_{(0)} \left(v_a(-1) e^{\mathbf{mu}} + \sum_{q=1}^{N} m_q e^{\mathbf{mu}} \otimes E_{qa}(-1) \right)$$

$$= D(v_a(-1) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}) - r_a D((De^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}}) + \frac{m_a}{2} (D^2 e^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}} - r_a \omega_{(-1)} e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}$$

$$+ \sum_{q=1}^{N} m_q D(e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{qa}(-1)) - \frac{m_a}{2} (D^2 e^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}} + \sum_{p=1}^{N} r_p ((De^{\mathbf{ru}})_{(-1)} e^{\mathbf{mu}}) \otimes E_{pa}(-1)$$

$$- r_a \sum_{p,s=1}^{N} r_p u_s(-1) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{ps}(-1) + \sum_{p=1}^{N} r_p e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes D(E_{pa}(-1))$$

$$+ \sum_{p=1}^{N} r_p ({e^{\mathbf{ru}}}_{(-1)} De^{\mathbf{mu}}) \otimes E_{pa}(-1) - r_a \sum_{q,s=1}^{N} m_q u_s(-1) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{qs}(-1)$$

$$= D\left(v_a(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} + \sum_{p=1}^{N}(r_p+m_p)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{pa}(-1)\right)$$

$$-r_a\left(\omega_{(-1)}e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} + \sum_{p,s=1}^{N}(r_p+m_p)u_s(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{ps}(-1)\right) - r_a D((De^{\mathbf{r}\mathbf{u}})_{(-1)}e^{\mathbf{m}\mathbf{u}}).$$

This expression matches the terms with the delta function in (4.4) and the Lemma is proved.

Lemma 4.7. *The vertex operators in (4.10), (4.6), (4.7) representing $\tilde{d}_0(\mathbf{r}, z), k_0(\mathbf{m}, z)$ and $k_a(\mathbf{m}, z)$ satisfy the relation (4.5).*

Proof. We will verify (4.5) using the commutator formula (4.11). For this we need to evaluate the following n-th products:

$$\left(\omega_{(-1)}e^{\mathbf{r}\mathbf{u}} + \sum_{p,s=1}^{N} r_p u_s(-1)e^{\mathbf{r}\mathbf{u}} \otimes E_{ps}(-1)\right)_{(n)} \left(\omega_{(-1)}e^{\mathbf{m}\mathbf{u}} + \sum_{q,k=1}^{N} m_q u_k(-1)e^{\mathbf{m}\mathbf{u}} \otimes E_{qk}(-1)\right). \tag{4.18}$$

We expand the sums in both factors and get four terms, which will be dealt with in the four steps below. Since both factors are of degree 2, we need to consider values $n = 0, 1, 2, 3$ (see (4.11)).

Step 1. The first term to be simplified is $\left(\omega_{(-1)}e^{\mathbf{r}\mathbf{u}}\right)_{(n)}\left(\omega_{(-1)}e^{\mathbf{m}\mathbf{u}}\right)$. We apply the Borcherds' identity (2.8):

$$\left(\omega_{(-1)}e^{\mathbf{r}\mathbf{u}}\right)_{(n)}\left(\omega_{(-1)}e^{\mathbf{m}\mathbf{u}}\right)$$

$$= \sum_{j\geq 0} e^{\mathbf{r}\mathbf{u}}{}_{(n-j-1)}\omega_{(j)}\omega_{(-1)}e^{\mathbf{m}\mathbf{u}} + \sum_{j\geq 0}\omega_{(-1-j)}e^{\mathbf{r}\mathbf{u}}{}_{(n+j)}\omega_{(-1)}e^{\mathbf{m}\mathbf{u}}. \tag{4.19}$$

For the first sum we use the relations in the Virasoro algebra:

$$\omega_{(j)}\omega_{(-1)} = \omega_{(-1)}\omega_{(j)} + (j+1)\omega_{(j-2)} + 6\delta_{j,3}\mathrm{Id}$$

(note that by the assumption of the theorem, the rank of the VOA V_{tor} is equal to 12). We transform the second sum in (4.19) using Lemma 3.2(vi):

$$\sum_{j\geq 0}\omega_{(-1-j)}e^{\mathbf{r}\mathbf{u}}{}_{(n+j)}\omega_{(-1)}e^{\mathbf{m}\mathbf{u}}$$

$$= \sum_{j\geq 0}\omega_{(-1-j)}\omega_{(-1)}e^{\mathbf{r}\mathbf{u}}{}_{(n+j)}e^{\mathbf{m}\mathbf{u}} - \sum_{j\geq 0}\omega_{(-1-j)}(De^{\mathbf{r}\mathbf{u}})_{(n+j-1)}e^{\mathbf{m}\mathbf{u}} = -\delta_{n,0}\omega_{(-1)}(De^{\mathbf{r}\mathbf{u}})_{(-1)}e^{\mathbf{m}\mathbf{u}}.$$

Thus (4.19) becomes

$$\left(\omega_{(-1)}e^{\mathbf{r}\mathbf{u}}\right)_{(n)}\left(\omega_{(-1)}e^{\mathbf{m}\mathbf{u}}\right)$$

$$= \sum_{j\geq 0} e^{\mathbf{r}\mathbf{u}}{}_{(n-j-1)}\omega_{(-1)}\omega_{(j)}e^{\mathbf{m}\mathbf{u}} + \sum_{j\geq 0}(j+1)e^{\mathbf{r}\mathbf{u}}{}_{(n-j-1)}\omega_{(j-2)}e^{\mathbf{m}\mathbf{u}}$$

$$+6e^{\mathbf{ru}}{}_{(n-4)}e^{\mathbf{mu}} - \delta_{n,0}\omega_{(-1)}(De^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}}.$$

This can be simplified further with the help of Lemma 3.1(i),(ii) and Lemma 3.2(vi):

$$\begin{aligned}
&\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(n)}\left(\omega_{(-1)}e^{\mathbf{mu}}\right)\\
&= e^{\mathbf{ru}}{}_{(n-1)}\omega_{(-1)}De^{\mathbf{mu}} + e^{\mathbf{ru}}{}_{(n-1)}\omega_{(-2)}e^{\mathbf{mu}} + 2e^{\mathbf{ru}}{}_{(n-2)}\omega_{(-1)}e^{\mathbf{mu}} + 3(e^{\mathbf{ru}})_{(n-3)}\omega_{(0)}e^{\mathbf{mu}}\\
&\quad +6e^{\mathbf{ru}}{}_{(n-4)}e^{\mathbf{mu}} - \delta_{n,0}\omega_{(-1)}((De^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}})\\
&= \omega_{(-1)}e^{\mathbf{ru}}{}_{(n-1)}De^{\mathbf{mu}} - (De^{\mathbf{ru}})_{(n-2)}De^{\mathbf{mu}} + \omega_{(-2)}e^{\mathbf{ru}}{}_{(n-1)}e^{\mathbf{mu}} - (De^{\mathbf{ru}})_{(n-3)}e^{\mathbf{mu}}\\
&\quad +2\omega_{(-1)}e^{\mathbf{ru}}{}_{(n-2)}e^{\mathbf{mu}} - 2(De^{\mathbf{ru}})_{(n-3)}e^{\mathbf{mu}} + 3e^{\mathbf{ru}}{}_{(n-3)}De^{\mathbf{mu}}\\
&\quad +6e^{\mathbf{ru}}{}_{(n-4)}e^{\mathbf{mu}} - \delta_{n,0}\omega_{(-1)}((De^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}})\\
&= \omega_{(-1)}e^{\mathbf{ru}}{}_{(n-1)}De^{\mathbf{mu}} + \omega_{(-2)}e^{\mathbf{ru}}{}_{(n-1)}e^{\mathbf{mu}} + 2\omega_{(-1)}e^{\mathbf{ru}}{}_{(n-2)}e^{\mathbf{mu}} - \delta_{n,0}\omega_{(-1)}((De^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}})\\
&\quad -(De^{\mathbf{ru}})_{(n-2)}De^{\mathbf{mu}} - 3(De^{\mathbf{ru}})_{(n-3)}e^{\mathbf{mu}} + 3e^{\mathbf{ru}}{}_{(n-3)}De^{\mathbf{mu}} + 6e^{\mathbf{ru}}{}_{(n-4)}e^{\mathbf{mu}}.
\end{aligned}$$

Now we consider particular values of n. Let $n = 3$. In this case the previous expression will simplify to just one term:

$$\left(\omega_{(-1)}(e^{\mathbf{ru}})\right)_{(3)}\left(\omega_{(-1)}(e^{\mathbf{mu}})\right) = 6e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}.$$

When $n = 2$ we obtain using (2.11) and (2.13):

$$\begin{aligned}
\left(\omega_{(-1)}(e^{\mathbf{ru}})\right)_{(2)}\left(\omega_{(-1)}(e^{\mathbf{mu}})\right) &= -3(De^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}} + 3e^{\mathbf{ru}}{}_{(-1)}De^{\mathbf{mu}} + 6e^{\mathbf{ru}}{}_{(-2)}e^{\mathbf{mu}}\\
&= 3De^{(\mathbf{r}+\mathbf{m})\mathbf{u}}.
\end{aligned}$$

For $n = 1$ we get:

$$\begin{aligned}
&\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(1)}\left(\omega_{(-1)}e^{\mathbf{mu}}\right)\\
&= 2\omega_{(-1)}e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} - (De^{\mathbf{ru}})_{(-1)}De^{\mathbf{mu}} - 3(De^{\mathbf{ru}})_{(-2)}e^{\mathbf{mu}} + 3e^{\mathbf{ru}}{}_{(-2)}De^{\mathbf{mu}} + 6e^{\mathbf{ru}}{}_{(-3)}e^{\mathbf{mu}}\\
&= 2\omega_{(-1)}e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} + 2(De^{\mathbf{ru}})_{(-1)}De^{\mathbf{mu}}.
\end{aligned}$$

Finally, for $n = 0$ we have

$$\begin{aligned}
&\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(0)}\left(\omega_{(-1)}e^{\mathbf{mu}}\right)\\
&= \omega_{(-1)}e^{\mathbf{ru}}{}_{(-1)}De^{\mathbf{mu}} + \omega_{(-2)}e^{\mathbf{ru}}{}_{(-1)}e^{\mathbf{mu}} + 2\omega_{(-1)}e^{\mathbf{ru}}{}_{(-2)}e^{\mathbf{mu}} - \omega_{(-1)}(De^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}}\\
&\quad -(De^{\mathbf{ru}})_{(-2)}De^{\mathbf{mu}} - 3(De^{\mathbf{ru}})_{(-3)}e^{\mathbf{mu}} + 3e^{\mathbf{ru}}{}_{(-3)}De^{\mathbf{mu}} + 6e^{\mathbf{ru}}{}_{(-4)}e^{\mathbf{mu}}\\
&= D(\omega_{(-1)}e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}) + \frac{1}{2}(D^2e^{\mathbf{ru}})_{(-1)}De^{\mathbf{mu}} - \frac{1}{2}(D^3e^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}}.
\end{aligned}$$

Step 2. As in the previous step, to compute the n-th product

$$\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(n)}\sum_{q,k=1}^{N} m_q u_k(-1)e^{\mathbf{mu}} \otimes E_{qk}(-1)$$

we use the Borcherds' identity (2.8), as well as (2.9), (2.10), Lemma 3.1(v) and Lemma 3.5 (with $c_{LI}=\frac{N}{2}$):

$$\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(n)}\sum_{q,k=1}^{N} m_q u_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1)$$

$$=\sum_{j\geq 0} e^{\mathbf{ru}}{}_{(n-j-1)}\sum_{q,k=1}^{N} m_q \omega_{(j)}(u_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1))$$

$$=e^{\mathbf{ru}}{}_{(n-1)}\sum_{q,k=1}^{N} m_q\omega_{(0)}(u_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1))+e^{\mathbf{ru}}{}_{(n-2)}\sum_{q,k=1}^{N} m_q\omega_{(1)}(u_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1))$$

$$+e^{\mathbf{ru}}{}_{(n-3)}\sum_{q=1}^{N} m_q u_k(-1)e^{\mathbf{mu}}\otimes\omega_{(2)}E_{qk}(-1))$$

$$=e^{\mathbf{ru}}{}_{(n-1)}\sum_{q,k=1}^{N} m_q D(u_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1))+2e^{\mathbf{ru}}{}_{(n-2)}\sum_{q,k=1}^{N} m_q u_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1)$$

$$-e^{\mathbf{ru}}{}_{(n-3)}De^{\mathbf{mu}}.$$

The above yields zero when $n\geq 3$. Let us evaluate this expression for $n=0,1,2$. First we let $n=2$:

$$\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(2)}\sum_{q,k=1}^{N} m_q u_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1)=-e^{\mathbf{ru}}{}_{(-1)}De^{\mathbf{mu}}.$$

For $n=1$ we get:

$$\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(1)}\sum_{q,k=1}^{N} m_q u_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1)$$

$$=2e^{\mathbf{ru}}{}_{(-1)}\sum_{q,k=1}^{N} m_q u_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1)-e^{\mathbf{ru}}{}_{(-2)}De^{\mathbf{mu}}$$

$$=2\sum_{q,k=1}^{N} m_q u_k(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}\otimes E_{qk}(-1)-(De^{\mathbf{ru}})_{(-1)}De^{\mathbf{mu}}.$$

Finally, for $n=0$ we obtain:

$$\left(\omega_{(-1)}e^{\mathbf{ru}}\right)_{(0)}\sum_{q,k=1}^{N} m_q u_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1)$$

$$=e^{\mathbf{ru}}{}_{(-1)}D\left(\sum_{q,k=1}^{N} m_q u_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1)\right)+2(De^{\mathbf{ru}})_{(-1)}\sum_{q,k=1}^{N} m_q u_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1)$$

$$-\frac{1}{2}(D^2 e^{\mathbf{ru}})_{(-1)} De^{\mathbf{mu}}.$$

Step 3. The n-th products

$$\left(\sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1)\right)_{(n)} \omega_{(-1)} e^{\mathbf{mu}}$$

can be computed using the skew symmetry (2.14) from the n-th products computed in the previous step. Indeed, letting $a = \sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1)$ and $b = \omega_{(-1)} e^{\mathbf{mu}}$ and noticing that $b_{(j)} a = 0$ for $j \geq 3$, we get from (2.14):

$$a_{(n)} b = 0 \quad \text{for} \quad n > 2,$$

$$\begin{aligned} a_{(2)} b &= -b_{(2)} a, \\ a_{(1)} b &= b_{(1)} a - D(b_{(2)} a), \\ a_{(0)} b &= -b_{(0)} a + D(b_{(1)} a) - \frac{1}{2} D^2 (b_{(2)} a). \end{aligned}$$

The expressions in the right hand sides are now available from Step 2 if we switch in those formulas $\mathbf{r}$ with $\mathbf{m}$. For $n = 2$ this will give us:

$$\left(\sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1)\right)_{(2)} \omega_{(-1)} e^{\mathbf{mu}} = e^{\mathbf{mu}}{}_{(-1)} De^{\mathbf{ru}}.$$

For $n = 1$:

$$\left(\sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1)\right)_{(1)} \omega_{(-1)} e^{\mathbf{mu}}$$

$$= 2 \sum_{p,s=1}^{N} r_p u_s(-1) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{ps}(-1) - (De^{\mathbf{mu}})_{(-1)} De^{\mathbf{ru}} + D(e^{\mathbf{mu}}{}_{(-1)} De^{\mathbf{ru}})$$

$$= 2 \sum_{p,s=1}^{N} r_p u_s(-1) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{ps}(-1) + e^{\mathbf{mu}}{}_{(-1)} D^2 e^{\mathbf{ru}}.$$

And finally for $n = 0$:

$$\left(\sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1)\right)_{(0)} \omega_{(-1)} e^{\mathbf{mu}}$$

$$= -e^{\mathbf{mu}}{}_{(-1)} D \left(\sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1)\right)$$

$$-2(De^{\mathbf{mu}})_{(-1)} \sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1) + \frac{1}{2}(D^2 e^{\mathbf{mu}})_{(-1)} De^{\mathbf{ru}}$$

$$+2D\left(\sum_{p,s=1}^{N} r_p u_s(-1) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{ps}(-1)\right) - D\left((De^{\mathbf{mu}})_{(-1)} De^{\mathbf{ru}}\right) + \frac{1}{2}D^2({e^{\mathbf{mu}}}_{(-1)} De^{\mathbf{ru}})$$

$$= {e^{\mathbf{mu}}}_{(-1)} D\left(\sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1)\right) + \frac{1}{2}{e^{\mathbf{mu}}}_{(-1)} D^3 e^{\mathbf{ru}}.$$

Step 4. The computation of the products

$$\left(\sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1)\right)_{(n)} \sum_{q,k=1}^{N} m_q u_k(-1) e^{\mathbf{mu}} \otimes E_{qk}(-1) \tag{4.20}$$

will be based on Lemma 2.1(iii). Since we have chosen the trivial cocycle on $\widehat{gl}_N$, $c_1 = c_I = 0$, then by Lemma 3.4, $E_{ps}(-1)_{(n)} E_{qk}(-1) = 0$ for $n \geq 1$. This implies that (4.20) is also zero for $n \geq 1$. The only case that we need to consider is $n = 0$. Applying Lemma 2.1(iii) and Lemma 3.4, we get

$$\left(\sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1)\right)_{(0)} \sum_{q,k=1}^{N} m_q u_k(-1) e^{\mathbf{mu}} \otimes E_{qk}(-1)$$

$$= \sum_{p,s,k=1}^{N} r_p m_s u_s(-1) u_k(-1) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{pk}(-1) - \sum_{q,p,s=1}^{N} r_p m_q u_s(-1) u_p(-1) e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} \otimes E_{qs}(-1)$$

$$= (De^{\mathbf{mu}})_{(-1)} \sum_{p,k=1}^{N} r_p u_k(-1) e^{\mathbf{ru}} \otimes E_{pk}(-1) - (De^{\mathbf{ru}})_{(-1)} \sum_{q,s=1}^{N} m_q u_s(-1) e^{\mathbf{mu}} \otimes E_{qs}(-1).$$

This completes Step 4.

Now we can combine the results of the four steps and obtain the desired n-th products (4.18). For $n = 3$ we get

$$\left(\omega_{(-1)} e^{\mathbf{ru}} + \sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1)\right)_{(3)} \left(\omega_{(-1)} e^{\mathbf{mu}} + \sum_{q,k=1}^{N} m_q u_k(-1) e^{\mathbf{mu}} \otimes E_{qk}(-1)\right)$$

$$= 6e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}.$$

This corresponds via the commutator formula (2.2) to the term in (4.5) with the third derivative of the delta function.

Collecting the terms with $n=2$ from all four steps we obtain

$$\left(\omega_{(-1)}e^{\mathbf{ru}}+\sum_{p,s=1}^{N}r_pu_s(-1)e^{\mathbf{ru}}\otimes E_{ps}(-1)\right)_{(2)}\left(\omega_{(-1)}e^{\mathbf{mu}}+\sum_{q,k=1}^{N}m_qu_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1)\right)$$

$$=3De^{(\mathbf{r}+\mathbf{m})\mathbf{u}}-e^{\mathbf{ru}}{}_{(-1)}De^{\mathbf{mu}}+(De^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}}=4(De^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}}+2e^{\mathbf{ru}}{}_{(-1)}De^{\mathbf{mu}}.$$

These match the terms with the second derivative of the delta function in (4.5).

The case $n=1$ yields:

$$\left(\omega_{(-1)}e^{\mathbf{ru}}+\sum_{p,s=1}^{N}r_pu_s(-1)e^{\mathbf{ru}}\otimes E_{ps}(-1)\right)_{(1)}\left(\omega_{(-1)}e^{\mathbf{mu}}+\sum_{q,k=1}^{N}m_qu_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1)\right)$$

$$=2\omega_{(-1)}e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}+2(De^{\mathbf{ru}})_{(-1)}De^{\mathbf{mu}}+2\sum_{q,k=1}^{N}m_qu_k(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}\otimes E_{qk}(-1)-(De^{\mathbf{ru}})_{(-1)}De^{\mathbf{mu}}$$

$$+2\sum_{p,s=1}^{N}r_pu_s(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}\otimes E_{ps}(-1)+e^{\mathbf{mu}}{}_{(-1)}D^2e^{\mathbf{ru}}=$$

$$=2\left(\omega_{(-1)}e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}+\sum_{p,s=1}^{N}(r_p+m_p)u_s(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}\otimes E_{ps}(-1)\right)+D\left((De^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}}\right).$$

This is in agreement with the terms in (4.5) with the first derivative of the delta function.

Finally, for $n=0$:

$$\left(\omega_{(-1)}e^{\mathbf{ru}}+\sum_{p,s=1}^{N}r_pu_s(-1)e^{\mathbf{ru}}\otimes E_{ps}(-1)\right)_{(0)}\left(\omega_{(-1)}e^{\mathbf{mu}}+\sum_{q,k=1}^{N}m_qu_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1)\right)$$

$$=D(\omega_{(-1)}e^{(\mathbf{r}+\mathbf{m})\mathbf{u}})+\frac{1}{2}(D^2e^{\mathbf{ru}})_{(-1)}De^{\mathbf{mu}}-\frac{1}{2}(D^3e^{\mathbf{ru}})_{(-1)}e^{\mathbf{mu}}$$

$$+e^{\mathbf{ru}}{}_{(-1)}D\left(\sum_{q,k=1}^{N}m_qu_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1)\right)+2(De^{\mathbf{ru}})_{(-1)}\sum_{q,k=1}^{N}m_qu_k(-1)e^{\mathbf{mu}}\otimes E_{qk}(-1)$$

$$-\frac{1}{2}(D^2e^{\mathbf{ru}})_{(-1)}De^{\mathbf{mu}}+e^{\mathbf{mu}}{}_{(-1)}D\left(\sum_{p,s=1}^{N}r_pu_s(-1)e^{\mathbf{ru}}\otimes E_{ps}(-1)\right)+\frac{1}{2}e^{\mathbf{mu}}{}_{(-1)}D^3e^{\mathbf{ru}}$$

$$+(De^{\mathbf{mu}})_{(-1)}\sum_{p,k=1}^{N}r_pu_k(-1)e^{\mathbf{ru}}\otimes E_{pk}(-1)-(De^{\mathbf{ru}})_{(-1)}\sum_{q,s=1}^{N}m_qu_s(-1)e^{\mathbf{mu}}\otimes E_{qs}(-1)$$

$$= D\left(\omega_{(-1)}e^{(\mathbf{r}+\mathbf{m})\mathbf{u}} + \sum_{p,s=1}^{N}(r_p+m_p)u_s(-1)e^{(\mathbf{r}+\mathbf{m})\mathbf{u}}\otimes E_{ps}(-1)\right).$$

This matches the term in (4.5) that comes with the delta function.

The proofs of both Lemma 4.7 and Theorem 4.2 are now complete.

5. Irreducible representations of the full toroidal Lie algebra and of its subalgebra with the divergence free vector fields.

5.1. Irreducible modules for the full toroidal Lie algebra.

We are now going to apply the results of the previous section for the construction of irreducible representations for the full toroidal Lie algebra $\mathfrak{g}$.

Theorem 5.1. *Let the constants c, c_L, c_{LI} satisfy the assumptions of Theorem 4.2. Let $L_{\widehat{\dot{\mathfrak{g}}}}(\lambda, c)$ be an irreducible highest weight module for $\widehat{\dot{\mathfrak{g}}}$ with the highest weight (λ, c), $\lambda \in \dot{\mathfrak{h}}^*$. Let for $\alpha \in \mathbb{C}^N, \beta \in \mathbb{Z}^N$, $M^+_{Hyp}(\alpha,\beta)$ be the irreducible VOA module for V^+_{Hyp}, defined in (3.5). Let $L_{\widehat{sl}_N}(\lambda_1, 0)$ be the irreducible highest weight $\widehat{sl}_N$ module of level 0, where λ_1 is a linear functional on the Cartan subalgebra of sl_N. Let $L_{\mathcal{HV}ir}(h, h_I, c_L, c_{LI}, 0)$ be the irreducible highest weight module for the twisted Heisenberg-Virasoro algebra. Then*

$$L_{tor} = L_{\widehat{\dot{\mathfrak{g}}}}(\lambda, c)\otimes M^+_{Hyp}(\alpha,\beta)\otimes L_{\widehat{sl}_N}(\lambda_1, 0)\otimes L_{\mathcal{HV}ir}(h, h_I, c_L, c_{LI}, 0)$$

has a structure of an irreducible module for the toroidal Lie algebra $\mathfrak{g}(\frac{1}{c}, 0)$.

Proof. First of all we note that L_{tor} is an irreducible VOA module for

$$V_{tor} = V_{\widehat{\dot{\mathfrak{g}}}}(c)\otimes V^+_{Hyp}\otimes V_{\widehat{sl}_N}(0)\otimes V_{\mathcal{HV}ir}(c_L, c_{LI}, 0),$$

and thus by Corollary 4.3, L_{tor} is a module for the toroidal Lie algebra $\mathfrak{g}(\frac{1}{c}, 0)$. We are going to show that the fields (4.6)-(4.10), corresponding to $\mathfrak{g}$, generate the VOA V_{tor}. After this is done, we use the Borcherds' identity (2.7) with $m = 0$ to see that each moment of every vertex operator from V_{tor} is a linear combination of associative products of the operators corresponding to $\mathfrak{g}$. Thus every $\mathfrak{g}$ submodule is also a submodule for V_{tor} and hence the simplicity of L_{tor} as a VOA module implies its irreducibility as a module for $\mathfrak{g}$.

So we need to show that the set

$$S = \left\{ e^{\mathbf{ru}}, g(-1)\otimes e^{\mathbf{ru}}, u_p(-1)e^{\mathbf{ru}}, v_a(-1)e^{\mathbf{ru}} + \sum_{p=1}^{N} r_p e^{\mathbf{ru}}\otimes E_{pa}(-1),\right.$$
$$\left.\omega_{(-1)}e^{\mathbf{ru}} + \sum_{p,s=1}^{N} r_p u_s(-1)e^{\mathbf{ru}}\otimes E_{ps}(-1)\right\}_{\mathbf{r}\in\mathbb{Z}^N}$$

generates V_{tor}. Denote by $\langle S\rangle$ the vertex subalgebra in V_{tor} generated by S. Since V_{tor} is a tensor product of four VOAs, it is sufficient to show that each of the tensor factors, $V_{\widehat{\dot{\mathfrak{g}}}}$, V^+_{Hyp}, $V_{\widehat{sl}_N}$ and $V_{\mathcal{HV}ir}$, are in $\langle S\rangle$.

By taking $\mathbf{r}=0$, we see that $g(-1), u_p(-1), v_p(-1)$ and ω are in S. Since $V_{\widehat{\mathfrak{g}}}$ is spanned by $g_1(-n_1)\dots g_k(-n_k)\mathbf{1} = g_1(-1)_{(-n_1)}\dots g_k(-1)_{(-n_k)}\mathbf{1}$, we conclude that $V_{\widehat{\mathfrak{g}}} \subset \langle S\rangle$. Since V^+_{Hyp} is generated by $u_p(-1), v_p(-1), e^{\mathbf{ru}}$, we get that $V^+_{Hyp} \subset \langle S\rangle$. To show that $V_{\widehat{sl}_N} \subset \langle S\rangle$, we choose $\mathbf{r}$ with $r_j=\delta_{pj}$, $j=1,\dots,N$. Then we get that $v_a(-1)e^{u_p} + e^{u_p}\otimes E_{pa}(-1) \in S$. Since $v_a(-1)e^{u_p} \in \langle S\rangle$, we obtain that $e^{u_p}\otimes E_{pa}(-1)\in\langle S\rangle$ and $e^{-u_p}_{(-1)}e^{u_p}\otimes E_{pa}(-1) = E_{pa}(-1)\in\langle S\rangle$. However $V_{\widehat{sl}_N}$ is generated by $\{E_{pa}(-1) | p\neq a\}$ and thus $V_{\widehat{sl}_N}\subset\langle S\rangle$.

For the last tensor factor we note that $\omega = \omega_{\widehat{\mathfrak{g}}} + \omega_{Hyp} + \omega_{\widehat{sl}_N} + \omega_{\mathcal{HV}ir} \in S$ and also $\omega_{\widehat{\mathfrak{g}}}, \omega_{Hyp}, \omega_{\widehat{sl}_N} \in \langle S\rangle$, hence $\omega_{\mathcal{HV}ir}\in\langle S\rangle$. Finally, we have $E_{aa}(-1) = \psi_1(E_{aa})(-1) + \psi_2(E_{aa})(-1) \in \langle S\rangle$, where $\psi_1(E_{aa})(-1)\in V_{\widehat{sl}_N}\subset\langle S\rangle$ and $\psi_2(E_{aa})(-1) = \frac{1}{N}I(-1)$. But $I(-1)$ and $\omega_{\mathcal{HV}ir}$ generate $V_{\mathcal{HV}ir}$, thus $V_{\mathcal{HV}ir}\subset\langle S\rangle$.

Since all four tensor factors are in $\langle S\rangle$, we conclude that V_{tor} is generated by the set S. The theorem is now proved.

5.2. Vertex operator algebra and irreducible representations for $\mathfrak{g}_{\mathrm{div}}$.

Now we will study the restriction of the modules for the toroidal Lie algebra to the subalgebra $\mathfrak{g}_{\mathrm{div}} = (\dot{\mathfrak{g}}\otimes R)\oplus\mathcal{K}\oplus\mathcal{D}_{\mathrm{div}}$. This subalgebra is spanned by the elements $t_0^j\mathbf{t}^{\mathbf{r}}k_0$, $t_0^j\mathbf{t}^{\mathbf{r}}k_p$, $t_0^j\mathbf{t}^{\mathbf{r}}g$, d_0, $t_0^j\mathbf{t}^{\mathbf{r}}d_a$ with $r_a=0$ and

$$t_0^j\mathbf{t}^{\mathbf{r}}\widehat{d}_a = -r_a t_0^j\mathbf{t}^{\mathbf{r}}d_0 + \frac{1}{c}r_a(j+1)t_0^j\mathbf{t}^{\mathbf{r}}k_0 + jt_0^j\mathbf{t}^{\mathbf{r}}d_a. \tag{5.1}$$

The elements $t_0^j\mathbf{t}^{\mathbf{r}}k_0, t_0^j\mathbf{t}^{\mathbf{r}}k_p, t_0^j\mathbf{t}^{\mathbf{r}}g$ and $t_0^j\mathbf{t}^{\mathbf{r}}d_a$ with $r_a=0$ correspond to the fields (4.6), (4.7), (4.8) and (4.9) with $r_a=0$. Just as in the previous theorem we get that the moments of these fields generate $V_{\widehat{\mathfrak{g}}}, V^+_{Hyp}$ and $V_{\widehat{sl}_N}$. Collect the elements of the form (5.1) into the fields:

$$\widehat{d}_a(\mathbf{r},z) = \sum_{j\in\mathbb{Z}} t_0^j\mathbf{t}^{\mathbf{r}}\widehat{d}_a z^{-j-2}.$$

Using (4.10) and (4.9) we obtain that $\widehat{d}_a(\mathbf{r},z)$ is represented in the following way:

$$\widehat{d}_a(\mathbf{r},z)\mapsto r_a Y\left(\omega_{(-1)}e^{\mathbf{ru}} + \sum_{p,s=1}^N r_p u_s(-1)e^{\mathbf{ru}}\otimes E_{ps}(-1), z\right)$$

$$-\left(z^{-1}+\frac{\partial}{\partial z}\right)Y\left(v_a(-1)e^{\mathbf{ru}} + \sum_{p=1}^N r_p e^{\mathbf{ru}}\otimes E_{pa}(-1), z\right) + r_a\frac{z^{-2}}{2}Y\left(e^{\mathbf{ru}}, z\right). \tag{5.2}$$

Since $V_{\widehat{\mathfrak{g}}}\otimes V^+_{Hyp}\otimes V_{\widehat{sl}_N}$ is generated by $\mathfrak{g}_{\mathrm{div}}$, we shall consider in (5.2) only the terms that involve $V_{\mathcal{HV}ir}$:

$$r_a Y\left(e^{\mathbf{ru}}\otimes\omega_{\mathcal{HV}ir}, z\right) + \frac{r_a}{N}Y\left(\sum_{p=1}^N r_p u_p(-1)e^{\mathbf{ru}}\otimes I(-1), z\right)$$

$$-\frac{r_a}{N}Y\left(D(e^{\mathbf{ru}}\otimes I(-1)),z\right)-z^{-1}\frac{r_a}{N}Y\left(e^{\mathbf{ru}}\otimes I(-1),z\right)+r_a\frac{z^{-2}}{2}Y\left(e^{\mathbf{ru}},z\right)$$

$$=r_a\Bigg(Y\left(e^{\mathbf{ru}}\otimes\omega_{\mathcal{HV}ir},z\right)-\frac{1}{N}Y\left(e^{\mathbf{ru}}\otimes D(I(-1)),z\right)$$

$$-z^{-1}\frac{1}{N}Y\left(e^{\mathbf{ru}}\otimes I(-1),z\right)+\frac{z^{-2}}{2}Y\left(e^{\mathbf{ru}},z\right)\Bigg)$$

$$=r_aY\left(e^{\mathbf{ru}},z\right)\otimes\left(Y\left(\omega_{\mathcal{HV}ir},z\right)-\frac{1}{N}Y\left(D(I(-1)),z\right)-z^{-1}\frac{1}{N}Y\left(I(-1),z\right)+\frac{z^{-2}}{2}\mathrm{Id}\right). \tag{5.3}$$

To understand the structure of the expression

$$Y\left(\omega_{\mathcal{HV}ir},z\right)-\frac{1}{N}\left(z^{-1}+\frac{\partial}{\partial z}\right)Y\left(I(-1),z\right)+\frac{z^{-2}}{2}\mathrm{Id}, \tag{5.4}$$

we consider the following

Lemma 5.2. *Let $\overline{\mathcal{V}ir}$ be the Virasoro algebra with the basis $\left\{\overline{L}(n),\overline{C}_L\right\}$. For any $\gamma\in\mathbb{C}$ the map*

$$\rho_\gamma:\quad \overline{\mathcal{V}ir}\to\mathcal{HV}ir/<C_I>,$$

given by

$$\rho_\gamma(\overline{L}(n))=L(n)+\gamma nI(n)+\delta_{n,0}\gamma C_{LI},$$
$$\rho_\gamma(\overline{C}_L)=C_L+24\gamma C_{LI},$$

is an embedding of Lie algebras.

Corollary 5.3. *(a) Let $h,h_I,c_L,c_{LI}\in\mathbb{C}$. The homomorphism ρ_γ extends to the embedding of the Verma module $M_{\overline{\mathcal{V}ir}}(h+\gamma c_{LI},c_L+24\gamma c_{LI})$ for the Virasoro algebra $\overline{\mathcal{V}ir}$, into the Verma module $M_{\mathcal{HV}ir}(h,h_I,c_L,c_{LI},0)$ for the twisted Heisenberg-Virasoro algebra $\mathcal{HV}ir$.*
(b) Let $c_{LI}=\frac{N}{2},\gamma=\frac{1}{N}$. Under the map ρ_γ we have the correspondence of the fields

$$\sum_{n\in\mathbb{Z}}\overline{L}(n)z^{-n-2}\mapsto\sum_{n\in\mathbb{Z}}L(n)z^{-n-2}-\frac{1}{N}\left(z^{-1}+\frac{\partial}{\partial z}\right)\sum_{n\in\mathbb{Z}}I(n)z^{-n-1}+\frac{z^{-2}}{2}\mathrm{Id}. \tag{5.5}$$

The proof of the Lemma is a straightforward computation, and the Corollary is an immediate consequence. Note that the field in the right hand side in part (b) of the Corollary coincides precisely with (5.4). Thus when $c_{LI}=\frac{N}{2}$ (as we have in Theorem 4.2), the components of this field satisfy the relations of the Virasoro algebra with the value of the central charge $\overline{c}_L=c_L+12$.

The field (5.4) involves vertex operators that are shifted by powers of z. To deal with such expressions we need the following generalization of the "preservation of identities" principle (cf. [Li], Lemma 2.3.5):

Lemma 5.4. *Let V be a VOA and let M be a VOA module for V. Let $a^s,b^k,c^{nj}\in V$, where s,k,n,j run over finite subsets of $\mathbb{Z}$ and $n\geq 0$.*

(i) If M is a faithful VOA module and $\sum_s z^{-s} Y_M(a^s, z) = 0$ (finite sum), then $a^s = 0$ for all s.

(ii) If

$$\left[\sum_s z^{-s} Y_V(a^s, z), \sum_k w^{-k} Y_V(b^k, w)\right] = \sum_{n\geq 0} \sum_j w^{-j} Y_V(c^{nj}, w) \left[z^{-1} \left(\frac{\partial}{\partial w}\right)^n \delta\left(\frac{w}{z}\right)\right]$$

$$\textit{(all sums finite)} \tag{5.6}$$

then

$$\left[\sum_s z^{-s} Y_M(a^s, z), \sum_k w^{-k} Y_M(b^k, w)\right] = \sum_{n\geq 0} \sum_j w^{-j} Y_M(c^{nj}, w) \left[z^{-1} \left(\frac{\partial}{\partial w}\right)^n \delta\left(\frac{w}{z}\right)\right]. \tag{5.7}$$

(iii) If M is a faithful VOA module then (5.7) implies (5.6).

Proof. Let us prove (i). Let D_M be the infinitesimal translation operator on M. If $\sum_s Y_M(a^s, z) z^{-s} = 0$ then

$$0 = z \sum_s \left[D_M, Y_M(a^s, z)\right] z^{-s} = z \sum_s \left(\frac{\partial}{\partial z} Y_M(a^s, z)\right) z^{-s}$$

$$= z \sum_s \left(\frac{\partial}{\partial z} Y_M(a^s, z)\right) z^{-s} - z \frac{\partial}{\partial z} \left(\sum_s Y_M(a^s, z) z^{-s}\right)$$

$$= - \sum_s Y_M(a^s, z) z \frac{\partial}{\partial z} (z^{-s}) = \sum_s s Y_M(a^s, z) z^{-s}.$$

Repeating this argument, we get that for any $m = 0, 1, 2, \ldots$

$$\sum_s s^m Y_M(a^s, z) z^{-s} = 0.$$

Since the sum in s is finite, we can apply the Vandermonde determinant argument and derive that $Y_M(a^s, z) = 0$ for all s. By the definition of the faithful module, this implies that all $a^s = 0$.

To prove (ii), we use the commutator formula (2.2) and the basic properties of the delta-function:

$$\left[\sum_s z^{-s} Y_V(a^s, z), \sum_k w^{-k} Y_V(b^k, w)\right]$$

$$= \sum_{n,i\geq 0} \sum_{s,k} \frac{1}{n!} \binom{-s}{i} w^{-k-s-i} Y_V(a^s_{(n+i)} b^k, w) \left[z^{-1} \left(\frac{\partial}{\partial w}\right)^n \delta\left(\frac{w}{z}\right)\right] \quad \text{(all sums finite)}. \tag{5.8}$$

By Corollary 2.2 from [K2], we obtain that for all $n \geq 0$,

$$\sum_j w^{-j} Y_V(c^{nj}, w) = \sum_{i\geq 0}\sum_{s,k} \frac{1}{n!}\begin{pmatrix} -s \\ i \end{pmatrix} w^{-k-s-i} Y_V(a^s_{(n+i)} b^k, w).$$

Since V is a faithful VOA module over itself, we get using part (i) of the Lemma that

$$c^{nj} = \sum_{s,k} \sum_{\substack{i\geq 0 \\ s+k+i=j}} \frac{1}{n!}\begin{pmatrix} -s \\ i \end{pmatrix} a^s_{(n+i)} b^k. \tag{5.9}$$

However the relation (5.8) holds in every VOA module M. Taking (5.9) into account, we see that (5.7) also holds.

The proof for part (iii) is similar. We first see that

$$\sum_{n\geq 0}\sum_j w^{-j} Y_M(c^{nj}, w) \left[z^{-1} \left(\frac{\partial}{\partial w}\right)^n \delta\left(\frac{w}{z}\right)\right] =$$

$$\sum_{n,i\geq 0}\sum_{s,k} \frac{1}{n!}\begin{pmatrix} -s \\ i \end{pmatrix} w^{-k-s-i} Y_M(a^s_{(n+i)} b^k, w) \left[z^{-1} \left(\frac{\partial}{\partial w}\right)^n \delta\left(\frac{w}{z}\right)\right].$$

Again using Corollary 2.2 from [K2] and part (i) of the Lemma, we obtain that the relation (5.9) holds in V. Thus (5.6) also holds. This completes the proof of the Lemma.

Now we have done all the preparatory work and now ready to describe the representations for $\mathfrak{g}_{\text{div}}$.

Theorem 5.5. *Let $c, \overline{c}_L$ be complex numbers such that $c \neq 0, c \neq -h^\vee$ and*

$$\frac{c \dim \dot{\mathfrak{g}}}{c + h^\vee} + 2N + \overline{c}_L = 24.$$

Let $M_{\hat{\dot{\mathfrak{g}}}}$ be a VOA module for affine VOA $V_{\hat{\dot{\mathfrak{g}}}}(c)$, M^+_{Hyp} be a VOA module for the sub-VOA V^+_{Hyp} of the hyperbolic lattice VOA, $M_{\widehat{sl}_N}$ be a module for affine $\widehat{sl}_N$ VOA $V_{\widehat{sl}_N}(0)$, and $M_{\overline{\mathcal{V}ir}}$ be a VOA module for the Virasoro VOA $V_{\overline{\mathcal{V}ir}}(\overline{c}_L)$. Then

$$M_{\mathfrak{g}_{\text{div}}} = M_{\hat{\dot{\mathfrak{g}}}} \otimes M^+_{Hyp} \otimes M_{\widehat{sl}_N} \otimes M_{\overline{\mathcal{V}ir}}$$

is a module for the divergence free subalgebra $\mathfrak{g}_{\text{div}}(\frac{1}{c})$ of the toroidal Lie algebra. The action of the fields $k_0(\mathbf{r}, z), k_a(\mathbf{r}, z), g(\mathbf{r}, z)$ is given by the formulas (4.6),(4.7),(4.8). The action of $d_a(\mathbf{r}, z)$ with $r_a = 0$ is given by (4.9). The action of d_0 is given by

$$d_0 \mapsto \text{Id} - \omega_{(1)}, \tag{5.10}$$

where ω is the Virasoro element of the tensor product VOA

$$V_{\mathfrak{g}_{\text{div}}} = V_{\hat{\dot{\mathfrak{g}}}}(c) \otimes V^+_{Hyp} \otimes V_{\widehat{sl}_N}(0) \otimes V_{\overline{\mathcal{V}ir}}(\overline{c}_L).$$

Finally, the field $\widehat{d}_a(\mathbf{r}, z)$ is represented by

$$r_a Y_M \left(\omega_{(-1)} e^{\mathbf{ru}} + \sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes \psi_1(E_{ps})(-1), z \right)$$

$$- \left(z^{-1} + \frac{\partial}{\partial z} \right) Y_M \left(v_a(-1) e^{\mathbf{ru}} + \sum_{p=1}^{N} r_p e^{\mathbf{ru}} \otimes \psi_1(E_{pa})(-1), z \right). \tag{5.11}$$

Proof. The Lie bracket in $\mathfrak{g}_{\text{div}}$ may be encoded in the commutator relations between the fields $k_0(\mathbf{r}, z), k_a(\mathbf{r}, z), g(\mathbf{r}, z)$, $\widehat{d}_a(\mathbf{r}, z)$ $d_a(\mathbf{r}, z)$ with $r_a = 0$, and the element d_0, analogous to (4.1)-(4.5). We need to show that the same commutator relations hold for their images (4.6), (4.7), (4.8), (4.9) with $r_a = 0$, (5.10) and (5.11). It is easy to see that the relations involving d_0 in the left hand sides, hold due to (2.10). Also, d_0 does not belong to the commutant of $\mathfrak{g}_{\text{div}}$ and will not appear in the right hand sides of the commutator relations. The commutator relations that should be verified for the remaining fields (4.6), (4.7), (4.8), (4.9) with $r_a = 0$, and (5.11) are of the form (5.7). Our strategy is to embed one of the modules for $V_{\mathfrak{g}_{\text{div}}}$ into a module for the full toroidal Lie algebra $\mathfrak{g}$. This embedding will have the property that the restriction of the action of $\mathfrak{g}$ to subalgebra $\mathfrak{g}_{\text{div}}$ will coincide with (4.6), (4.7), (4.8), (4.9) with $r_a = 0$, (5.10) and (5.11). This will imply that the necessary commutator relations hold in the chosen module for $V_{\mathfrak{g}_{\text{div}}}$. Since the module that we will consider will be faithful, then by preservation of identities (Lemma 5.4), the same required relations will hold in all VOA modules for $V_{\mathfrak{g}_{\text{div}}}$.

Let us carry out this plan. Consider the embedding given by Corollary 5.3 with $h = -\frac{1}{2}$, $\gamma = \frac{1}{N}, c_{LI} = \frac{N}{2}$, $h_I = c_I = 0$, of the Verma module $M_{\overline{\mathcal{V}ir}}(0, \overline{c}_L)$ for the Lie algebra $\overline{\mathcal{V}ir}$, into the Verma module $M_{\mathcal{HV}ir}(-\frac{1}{2}, 0, \overline{c}_L - 12, \frac{N}{2}, 0)$ for the twisted Heisenberg-Virasoro algebra. Under this homomorphism

$$\overline{L}(z) = \sum_{n \in \mathbb{Z}} \overline{L}(n) z^{-n-2} \mapsto L(z) - \frac{1}{N} \left(z^{-1} + \frac{\partial}{\partial z} \right) I(z) + \frac{z^{-2}}{2} \text{Id}.$$

This map extends to the embedding

$$V_{\widehat{\mathfrak{g}}}(c) \otimes V_{Hyp}^{+} \otimes V_{\widehat{sl}_N}(0) \otimes M_{\overline{\mathcal{V}ir}}(0, \overline{c}_L) \subset V_{\widehat{\mathfrak{g}}}(c) \otimes V_{Hyp}^{+} \otimes V_{\widehat{sl}_N}(0) \otimes M_{\mathcal{HV}ir}(-\frac{1}{2}, 0, \overline{c}_L - 12, \frac{N}{2}, 0).$$

By Corollary 4.3, the latter is a module for the full toroidal Lie algebra $\mathfrak{g}(\frac{1}{c}, 0)$. We consider the restriction of this representation to the subalgebra $\mathfrak{g}_{\text{div}}(\frac{1}{c})$ and claim that $V_{\widehat{\mathfrak{g}}}(c) \otimes V_{Hyp}^{+} \otimes V_{\widehat{sl}_N}(0) \otimes M_{\overline{\mathcal{V}ir}}(0, \overline{c}_L)$ is invariant under the action of $\mathfrak{g}_{\text{div}}(\frac{1}{c})$. The action of $k_0(\mathbf{r}, z)$, $k_a(\mathbf{r}, z)$, $g(\mathbf{r}, z)$ and $d_a(\mathbf{r}, z)$ with $r_a = 0$ is given by (4.6),(4.7),(4.8) and (4.9). Let us show that the action of $\widehat{d}_a(\mathbf{r}, z)$ on $V_{\widehat{\mathfrak{g}}}(c) \otimes V_{Hyp}^{+} \otimes V_{\widehat{sl}_N}(0) \otimes M_{\mathcal{HV}ir}(-\frac{1}{2}, 0, \overline{c}_L - 12, \frac{N}{2}, 0)$ coincides with (5.11). Indeed, following the computations (5.2)–(5.5), we get:

$$\widehat{d}_a(\mathbf{r}, z) \mapsto r_a Y \left((\omega_{\widehat{\mathfrak{g}}} + \omega_{Hyp} + \omega_{\widehat{sl}_N} + \omega_{\mathcal{HV}ir})_{(-1)} e^{\mathbf{ru}} + \sum_{p,s=1}^{N} r_p u_s(-1) e^{\mathbf{ru}} \otimes E_{ps}(-1), z \right)$$

$$-\left(z^{-1}+\frac{\partial}{\partial z}\right)Y\left(v_a(-1)e^{\mathbf{ru}}+\sum_{p=1}^{N}r_pe^{\mathbf{ru}}\otimes E_{pa}(-1),z\right)+r_a\frac{z^{-2}}{2}Y(e^{\mathbf{ru}},z)$$

$$=r_aY\left((\omega_{\widehat{\dot{\mathfrak{g}}}}+\omega_{Hyp}+\omega_{\widehat{sl}_N})_{(-1)}e^{\mathbf{ru}}+\sum_{p,s=1}^{N}r_pu_s(-1)e^{\mathbf{ru}}\otimes\psi_1(E_{ps})(-1),z\right)$$

$$-\left(z^{-1}+\frac{\partial}{\partial z}\right)Y\left(v_a(-1)e^{\mathbf{ru}}+\sum_{p=1}^{N}r_pe^{\mathbf{ru}}\otimes\psi_1(E_{pa})(-1),z\right)$$

$$+r_aY(e^{\mathbf{ru}}\otimes\omega_{\mathcal{HV}ir},z)+r_a\frac{1}{N}Y((De^{\mathbf{ru}})\otimes I(-1),z)$$

$$-r_a\left(z^{-1}+\frac{\partial}{\partial z}\right)\frac{1}{N}Y(e^{\mathbf{ru}}\otimes I(-1),z)+r_a\frac{z^{-2}}{2}Y(e^{\mathbf{ru}},z)$$

$$=r_aY\left((\omega_{\widehat{\dot{\mathfrak{g}}}}+\omega_{Hyp}+\omega_{\widehat{sl}_N}+\omega_{\overline{\mathcal{V}ir}})_{(-1)}e^{\mathbf{ru}}+\sum_{p,s=1}^{N}r_pu_s(-1)e^{\mathbf{ru}}\otimes\psi_1(E_{ps})(-1),z\right)$$

$$-\left(z^{-1}+\frac{\partial}{\partial z}\right)Y\left(v_a(-1)e^{\mathbf{ru}}+\sum_{p=1}^{N}r_pe^{\mathbf{ru}}\otimes\psi_1(E_{pa})(-1),z\right),$$

which is the same as (5.11). Thus the specified action defines a representation of $\mathfrak{g}_{\rm div}$ on $V_{\widehat{\dot{\mathfrak{g}}}}(c)\otimes V^+_{Hyp}\otimes V_{\widehat{sl}_N}(0)\otimes M_{\overline{\mathcal{V}ir}}(0,\overline{c}_L)$, and the fields (4.6), (4.7), (4.8), (4.9) with $r_a=0$, and (5.11) satisfy the relations that reflect the Lie bracket in $\mathfrak{g}_{\rm div}$. This module is a faithful VOA module for $V_{\mathfrak{g}_{\rm div}}$, since $V_{\mathfrak{g}_{\rm div}}$ itself is its factor module. Thus by the preservation of identities, Lemma 5.4, the required commutator relations hold in $V_{\mathfrak{g}_{\rm div}}$ and in all VOA modules for $V_{\mathfrak{g}_{\rm div}}$. This establishes the claim of the theorem.

In the next theorem we give the description of the irreducible modules for $\mathfrak{g}_{\rm div}$.

Theorem 5.6. *Let constants $c,\overline{c}_L$ satisfy the assumptions of Theorem 5.5. Let $L_{\widehat{\dot{\mathfrak{g}}}}(\lambda,c)$ be an irreducible highest weight module for $\widehat{\dot{\mathfrak{g}}}$. Let for $\alpha\in\mathbb{C}^N,\beta\in\mathbb{Z}^N$, $M^+_{Hyp}(\alpha,\beta)$ be the irreducible VOA module for V^+_{Hyp}, defined in (3.5). Let $L_{\widehat{sl}_N}(\lambda_1,0)$ be the irreducible highest weight $\widehat{sl}_N$ module of level 0, where λ_1 is a linear functional on the Cartan subalgebra of sl_N. Let $L_{\overline{\mathcal{V}ir}}(h,\overline{c}_L)$ be the irreducible highest weight module for the Virasoro Lie algebra, $h\in\mathbb{C}$. Then*

$$L_{\mathfrak{g}_{\rm div}}=L_{\widehat{\dot{\mathfrak{g}}}}(\lambda,c)\otimes M^+_{Hyp}(\alpha,\beta)\otimes L_{\widehat{sl}_N}(\lambda_1,0)\otimes L_{\overline{\mathcal{V}ir}}(h,\overline{c}_L)$$

has a structure of an irreducible module for the Lie algebra $\mathfrak{g}_{\rm div}(\frac{1}{c})$.

The proof of this theorem is completely analogous to the proof of Theorem 5.1 and will be omitted.

We conclude the paper with two observations. The relation for the central charges for the modules for the $N+1$-toroidal Lie algebra with the divergence-free vector fields $\mathfrak{g}_{\text{div}}$ constructed in Theorem 5.5, may be rewritten as

$$\frac{c \dim \dot{\mathfrak{g}}}{c + h^\vee} + 2(N+1) + \overline{c}_L = 26. \tag{5.12}$$

This has a striking resemblance to the "no ghost" theorem in string theory. If we choose the modules for the affine algebras $\widehat{\dot{\mathfrak{g}}}$, $\widehat{sl}_N$ and for the Virasoro algebra to be unitary, this would require $c > 0$ and $\overline{c}_L \geq 0$ and thus we get that such modules exist only when

$$N < 12.$$

For example, if we choose the basic module for $\widehat{\dot{\mathfrak{g}}}$ and trivial modules for $\widehat{sl}_N$ and for the Virasoro algebra then the condition (5.12) will become

$$\text{rank}\,(\dot{\mathfrak{g}}) + 2(N+1) = 26.$$

Unfortunately, even when the affine and the Virasoro parts are unitary, the module for $\mathfrak{g}_{\text{div}}$ is not unitarizable because the hyperbolic lattice is not positive-definite and the lattice VOA used for the construction of such modules does not possess a unitary structure.

The second curious fact is that at the critical value $N = 12$ we get an exceptional module for the Lie algebra

$$\mathcal{D}_{\text{div}} \oplus \mathcal{K}.$$

The Lie bracket in this algebra is given by (1.2) and (1.3) with cocycle τ_1. Note that this Lie algebra has a non-degenerate symmetric invariant bilinear form given by (1.4). If we take trivial modules for the Virasoro and for the affine algebras $\widehat{\dot{\mathfrak{g}}}$, $\widehat{sl}_N$, then only when $N = 12$ we arrive at the representation of $\mathcal{D}_{\text{div}} \oplus \mathcal{K}$ just on the lattice part V^+_{Hyp}. We get the following remarkable result:

Theorem 5.7. *Let $N = 12$. Then V^+_{Hyp} has a structure of a module for $\mathcal{D}_{\text{div}} \oplus \mathcal{K}$ with cocycle τ_1. The action of the Lie algebra is given by*

$$k_0(\mathbf{r}, z) \mapsto Y(e^{\mathbf{ru}}, z), \quad k_p(\mathbf{r}, z) \mapsto Y(u_p(-1)e^{\mathbf{ru}}, z),$$

$$d_0 \mapsto \text{Id} - \omega_{(1)}, \quad d_a \mapsto v_a(0),$$

$$\widehat{d_a}(\mathbf{r}, z) \mapsto r_a Y(\omega_{(-1)} e^{\mathbf{ru}}, z) - \left(z^{-1} + \frac{\partial}{\partial z}\right) Y(v_a(-1)e^{\mathbf{ru}}, z).$$

The character of this module with respect to the diagonalizable operators $d_0, d_1, \ldots, d_N$ has nice modular properties – it is a product of 12 delta-functions with the -24-th power of the Dedekind η-function:

$$\text{char}\, V^+_{Hyp} = q_0 \prod_{k=1}^{\infty} \left(1 - q_0^{-k}\right)^{-24} \times \prod_{p=1}^{12} \sum_{j \in \mathbb{Z}} q_p^j.$$

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