CLASSIFICATION OF SIMPLE CUSPIDAL MODULES FOR SOLENOIDAL LIE ALGEBRAS

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Abstract. We give a new conceptual proof of the classification of cuspidal modules for the solenoidal Lie algebra. This classification was originally published by Y. Su [7]. Our proof is based on the theory of modules for the solenoidal Lie algebras that admit a compatible action of the commutative algebra of functions on a torus.

1. Introduction.

In [2] we classify simple W_n-modules with finite-dimensional weight spaces. This generalizes the classical result of Mathieu [6] on simple modules with finite-dimensional weight spaces for the Virasoro algebra. Mathieu proved that simple weight modules fall into two classes: (1) highest/lowest weight modules and (2) modules of tensor fields on a circle and their quotients.

In [2] we use solenoidal Lie algebras as a bridge between the Lie algebra W_1 of vector fields on a circle, and the Lie algebra W_n of vector fields on an n-dimensional torus. A solenoidal Lie algebra W_μ is a subalgebra in W_n which consists of vector fields collinear to a generic vector μ at each point of the torus. Integral curves of such vector fields are dense windings of a torus, which motivates the term "solenoidal". Solenoidal Lie algebras are also known in the literature as the centerless higher rank Virasoro algebras.

In many ways solenoidal Lie algebras behave like W_1, yet they are graded by $\mathbb{Z}^n$, like W_n. In fact, W_n can be decomposed (as a vector space) into a direct sum of n solenoidal subalgebras.

The main result of [2] relies on the classification of simple weight modules for the solenoidal Lie algebras with a property that the dimensions of all weight spaces are uniformly bounded. We call such modules cuspidal.

Classification of cuspidal modules for the solenoidal Lie algebras was published by Su [7]. However the methods of [7] are extremely computational and we were unable to verify all the details. In the present paper we offer an alternative conceptual proof of this classification. Our proof is based on the technique developed in [1] and [2].

Using the classification of simple cuspidal modules, Lu and Zhao [5] classified all simple modules for the solenoidal Lie algebras with finite-dimensional weight spaces.

It follows from the result of Mathieu [6] that the classification of simple cuspidal W_1-modules is given by the modules of tensor fields on a circle $T(\alpha,\beta)$, $\alpha,\beta\in\mathbb{C}$. The modules $T(\alpha,\beta)$ have bases $\{v_s \mid s\in\beta+\mathbb{Z}\}$ and the action of W_1 is given by

$$e_k v_s = (s+\alpha k)v_{s+k}, \quad k\in\mathbb{Z},\ s\in\beta+\mathbb{Z}. \tag{1.1}$$

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Here $\{e_k \mid k \in \mathbb{Z}\}$ is a basis of W_1 and its Lie bracket is

$$[e_k, e_m] = (m-k)e_{m+k}, \quad k, m \in \mathbb{Z}. \tag{1.2}$$

A simple cuspidal W_1-module is isomorphic to one of the following:

• a module of tensor fields $T(\alpha, \beta)$ with $\alpha \neq 0, 1$ or $\alpha = 0, 1$ and $\beta \notin \mathbb{Z}$,

• the quotient $\overline{T}(0,0)$ of the module $T(0,0)$ of functions by a one-dimensional submodule of constant functions,

or

• the trivial 1-dimensional module.

For a generic vector $\mu \in \mathbb{C}^n$ we consider a lattice $\Gamma_\mu \subset \mathbb{C}$, where Γ_μ is the image of $\mathbb{Z}^n$ under the map $r \in \mathbb{Z}^n \mapsto \mu \cdot r \in \mathbb{C}$. Then the solenoidal Lie algebra W_μ may be presented as the Lie algebra with a basis $\{e_k \mid k \in \Gamma_\mu\}$ and the Lie bracket (1.2) with $k, m \in \Gamma_\mu$ instead of $\mathbb{Z}$. Likewise, the W_μ-module $T(\alpha, \beta)$, $\alpha, \beta \in \mathbb{C}$ has a basis $\{v_s \mid s \in \beta + \Gamma_\mu\}$ and the action of W_μ is still given by the formula (1.1) but with a change $k \in \Gamma_\mu$, $s \in \beta + \Gamma_\mu$.

The classification of simple cuspidal W_μ-modules is the same as in the case of W_1, with a replacement of the condition $\beta \notin \mathbb{Z}$ for the modules $T(\alpha, \beta)$, $\alpha = 0, 1$, with the condition $\beta \notin \Gamma_\mu$.

A special class of modules plays a crucial role in our approach – these are W_μ-modules that admit a compatible action of the commutative algebra A of functions on a torus. One of our key results is that every simple cuspidal W_μ-module is a homomorphic image of a simple AW_μ-module. Using the methods developed in [1] we show that simple cuspidal AW_μ-modules are precisely the modules of tensor fields $T(\alpha, \beta)$. These two results combined yield the classification of simple cuspidal W_μ-modules.

The present paper is intertwined with our recent work [2]. We use in [2] the classification of cuspidal W_μ-modules. In the present work we use some of the methods and techniques developed in [2], which are independent of the classification of W_μ-modules.

The paper is organized as follows. In Section 2 we review solenoidal Lie algebras, introduce the family $T(\alpha, \beta)$ of W_μ-modules and exhibit their elementary properties. In Section 3 we study the structure of cuspidal AW_μ-modules and prove that every simple cuspidal AW_μ-module is isomorphic to $T(\alpha, \beta)$ for some $\alpha, \beta \in \mathbb{C}$. In Section 4 for each cuspidal W_μ-module we construct its A-cover and use this construction to establish our main result.

2. Solenoidal Lie algebras and their cuspidal modules

Consider the Lie algebra W_n of vector fields on an n-dimensional torus. The algebra of (complex-valued) Fourier polynomials on $\mathbb{T}^n$ is isomorphic to the algebra of Laurent polynomials

$$A = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}],$$

and W_n is the Lie algebra of derivations of A. Thus W_n has a natural structure of an A-module, which is free of rank n. We choose $d_1 = t_1 \frac{\partial}{\partial t_1}, \ldots, d_n = t_n \frac{\partial}{\partial t_n}$ as a basis of this A-module:

$$W_n = \mathop{\oplus}_{p=1}^{n} A d_p.$$

In case when $n = 1$ we get the Witt algebra W_1 with basis $e_k = t_1^k d_1$, $k \in \mathbb{Z}$, and the bracket

$$[e_k, e_m] = (m-k)e_{m+k}. \tag{2.1}$$

This paper is devoted to one particular family of subalgebras in W_n.

Definition 2.1. We call a vector $\mu \in \mathbb{C}^n$ *generic* if $\mu \cdot r \neq 0$ for all $r \in \mathbb{Z}^n \backslash \{0\}$.

Definition 2.2. Let μ be a generic vector in $\mathbb{C}^n$ and let $d_\mu = \mu_1 d_1 + \ldots + \mu_n d_n$. A *solenoidal* Lie algebra W_μ is the subalgebra in W_n which consists of vector fields collinear to μ at each point of $\mathbb{T}^n$, $W_\mu = A d_\mu$.

The Lie bracket in W_μ is

$$[t^r d_\mu, t^s d_\mu] = \mu \cdot (s - r) t^{r+s} d_\mu, \tag{2.2}$$

with $r, s \in \mathbb{Z}^n$. Here we are using the notation $t^r = t_1^{r_1} \ldots t_n^{r_n}$ for $r = (r_1, \ldots, r_n) \in \mathbb{Z}^n$.

Denote by Γ_μ the image of $\mathbb{Z}^n$ under the embedding $\mathbb{Z}^n \to \mathbb{C}$ given by $r \mapsto \mu \cdot r$. Then we can view the solenoidal Lie algebra W_μ as a version of the Witt algebra W_1 where the indices of the basis elements e_r run not over $\mathbb{Z}$, but over the lattice $\Gamma_\mu \subset \mathbb{C}$. Here we make the identification $t^r d_\mu = e_{\mu \cdot r}$ and the formula (2.1) remains valid.

Let us now discuss modules for the solenoidal Lie algebras. A W_μ-module M is called a *weight* module if $M = \bigoplus\limits_{\lambda \in \mathbb{C}} M_\lambda$, where the weight space M_λ is defined as

$$M_\lambda = \{ v \in M \mid d_\mu v = \lambda v \}.$$

In particular, A is a weight W_μ-module and W_μ is a weight module over itself.

Any weight W_μ-module can be decomposed into a direct sum of submodules corresponding to distinct cosets of Γ_μ in $\mathbb{C}$. It is then sufficient to study modules supported on a single coset $\beta + \Gamma_\mu$, $\beta \in \mathbb{C}$. We are going to impose such a condition on M and fix β for the rest of this paper.

Definition 2.3. A weight module is called *cuspidal* if the dimensions of its weight spaces are uniformly bounded by some constant.

Let us construct a family of cuspidal W_μ-modules. Fix $\alpha, \beta \in \mathbb{C}$.

Definition 2.4. A module of *tensor fields* $T(\alpha, \beta)$ is a vector space with a basis $\{ v_s \mid s \in \beta + \Gamma_\mu \}$ and the following action of W_μ:

$$e_k v_s = (s + \alpha k) v_{s+k}, \quad k \in \Gamma_\mu, \ s \in \beta + \Gamma_\mu. \tag{2.3}$$

It is easy to check that $T(\alpha, \beta)$ is indeed a W_μ-module. These modules are cuspidal since every weight space in $T(\alpha, \beta)$ is 1-dimensional.

Proposition 2.5. *(1) The W_μ-module $T(\alpha, \beta)$ is simple unless $\alpha = 0, 1$ and $\beta + \Gamma_\mu = \Gamma_\mu$.*

(2) The W_μ-module $T(0,0)$ has a trivial 1-dimensional submodule $\mathbb{C}v_0$, and the quotient $\overline{T}(0,0) = T(0,0)/\mathbb{C}v_0$ is a simple W_μ-module.

(3) Let $\{v_s | s \in \beta + \Gamma_\mu\}$ be a basis of $T(0, \beta)$ and $\{v'_s | s \in \beta + \Gamma_\mu\}$ be a basis of $T(1, \beta)$. The map $\theta : T(0, \beta) \to T(1, \beta)$,

$$\theta(v_s) = s v'_s$$

is a homomorphism of W_μ-modules.

(4) $\theta(T(0,0)) \cong \overline{T}(0,0)$ is a submodule of codimension 1 in $T(1,0)$.

Proof. Clearly, every submodule in $T(\alpha, \beta)$ is homogeneous with respect to the weight decomposition. Thus $T(\alpha, \beta)$ is simple if and only if for every $s \in \beta + \Gamma_\mu$ the vector v_s generates $T(\alpha, \beta)$. Let us assume $\alpha \neq 0$. Fix $s \in \beta + \Gamma_\mu$. Since

$$e_k v_s = (s + \alpha k) v_{s+k},$$

and $s + \alpha k$ may vanish only for $k = -s/\alpha$, we see that a W_μ-submodule generated by v_s is either $T(\alpha, \beta)$ or has codimension 1 in $T(\alpha, \beta)$. The latter happens precisely when for some $m \in \beta + \Gamma_\mu$

$$e_k v_{m-k} = (m - k + \alpha k) v_m = 0$$

for all non-zero $k \in \Gamma_\mu$. It follows from this that $T(\alpha, \beta)$ with $\alpha \neq 0$ has a submodule of codimension 1 only when $\alpha = 1$ and $m = 0 \in \beta + \Gamma_\mu$.

Consider now the case $\alpha = 0$. Then $e_k v_s = s v_{s+k}$, and every v_s with $s \neq 0$ generates $T(0, \beta)$, thus $T(0, \beta)$ could only have a 1-dimensional submodule $\mathbb{C} v_0$. This shows that $T(0, \beta)$ is reducible if and only if $0 \in \beta + \Gamma_\mu$. This establishes parts (1) and (2) of the proposition. Part (3) can be verified by a straightforward direct calculation. Claim (4) is then an immediate consequence of (3). □

3. AW_μ-modules

In this section we are going to describe the structure of cuspidal W_μ-modules that admit a compatible action of the commutative algebra A of functions on a torus.

Definition 3.1. We call M an AW_μ-module if it is a module for both the solenoidal Lie algebra W_μ and the commutative unital algebra $A = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with these two structures being compatible:

$$x(fv) = (xf)v + f(xv), \quad f \in A,\ x \in W_\mu,\ v \in M. \tag{3.1}$$

If an AW_μ-module M is a weight module then (3.1) implies that the action of A is compatible with the weight grading of M: $A_\gamma M_\lambda \subset M_{\lambda+\gamma}$, $\gamma \in \Gamma_\mu$, $\lambda \in \beta + \Gamma_\mu$. Suppose an AW_μ-module M has a weight decomposition with one of the weight spaces being finite-dimensional. Since all non-zero homogeneous elements of A are invertible, we conclude that all weight spaces of M have the same dimension and that M is a free A-module of a finite rank.

Let us now assume that M is a cuspidal AW_μ-module. Let $U = M_\beta$, $\dim U < \infty$. Since M is a free A-module, it can be presented as $M \cong A \otimes U$.

For each $s \in \mathbb{Z}^n$ consider an operator

$$D(s) : U \to U,$$

given by $D(s) = t^{-s} \circ (t^s d_\mu)$. Note that $D(0) = \beta \mathrm{Id}$.

The finite-dimensional operator $D(s)$ completely determines the action of $t^s d_\mu$ on M since by (3.1)

$$(t^s d_\mu)(t^m u) = (t^s d_\mu t^m) u + t^m (t^s d_\mu u) = \mu \cdot m t^{m+s} u + t^{m+s} D(s) u. \tag{3.2}$$

From (3.2) and the commutator relations (2.2) we can easily derive the Lie bracket (cf., Lemma 1 in [1]):

$$[D(s), D(m)] = \mu \cdot m \left(D(m+s) - D(m)\right) - \mu \cdot s \left(D(m+s) - D(s)\right). \tag{3.3}$$

We are going to show that $D(s)$ can be expressed as a polynomial in $s = (s_1, \dots, s_n)$.

Let us state two standard results on polynomial and rational functions on $\mathbb{Z}^n$. The proofs are elementary and we omit them.

Lemma 3.2. *Let P_1, P_2 be two polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ such that the degree of P_1, P_2 in each variable does not exceed $K \in \mathbb{N}$. Let $B = B_1 \times \ldots \times B_n \subset \mathbb{Z}^n$ with $|B_i| > K$. If $P_1(x) = P_2(x)$ for all $x \in B$ then $P_1 = P_2$ in $\mathbb{C}[x_1, \ldots, x_n]$.*

Corollary 3.3. *Let $R_1 = P_1/Q_1$, $R_2 = P_2/Q_2$ with $P_1, P_2, Q_1, Q_2 \in \mathbb{C}[x_1, \ldots, x_n]$ such that the degree of P_1, P_2, Q_1, Q_2 in each variable does not exceed K. Let $B = B_1 \times \ldots \times B_n \subset \mathbb{Z}^n$ with $|B_i| > 2K$ and suppose $Q_1(x), Q_2(x) \neq 0$ for all $x \in B$. If $R_1(x) = R_2(x)$ for all $x \in B$ then $R_1 = R_2$ in $\mathbb{C}(x_1, \ldots, x_n)$.*

Theorem 3.4. *Let M be a cuspidal AW_μ-module, $M = A \otimes U$, where $U = M_\beta$. Then the action of W_μ on M is given by*

$$(t^s d_\mu)(t^m u) = t^{m+s}(\mu \cdot m + D(s))u, \quad u \in U,$$

where the family of operators $D(s) : U \to U$ can be expressed as an $\mathrm{End}(U)$*-valued polynomial in $s = (s_1, \ldots, s_n)$ with the constant term $D(0) = \beta\mathrm{Id}$.*

Proof. We are going to prove this theorem by induction on n. In case when $n = 1$, the solenoidal Lie algebra W_μ is isomorphic to the Witt algebra W_1, and the claim follows from Theorem 1 in [1]. Applying this result to a subalgebra spanned by $\left\{t^{jm} d_\mu \,|\, j \in \mathbb{Z}\right\}$ with a fixed $m \in \mathbb{Z}^n \backslash \{0\}$, which is isomorphic to W_1, we conclude that the family $\{D(xm) \,|\, x \in \mathbb{Z}\}$ may be expressed as a polynomial in x.

Now let us establish the step of induction. By induction assumption the operators $D(x_1, \ldots, x_{n-1}, 0)$ have polynomial dependence on $x_1, \ldots, x_{n-1}$, while $D(0, \ldots, 0, x_n)$ is a polynomial in x_n. Thus

$$\begin{aligned}
&[D(x_1, \ldots, x_{n-1}, 0), D(0, \ldots, 0, x_n)] + \mu_n x_n D(0, \ldots, 0, x_n) \\
&\quad - (\mu_1 x_1 + \ldots \mu_{n-1} x_{n-1}) D(x_1, \ldots, x_{n-1}, 0) \\
&= (\mu_n x_n - \mu_{n-1} x_{n-1} - \ldots - \mu_1 x_1) D(x_1, \ldots, x_{n-1}, x_n)
\end{aligned}$$

is a polynomial in $x_1, \ldots, x_n$.

Since μ is generic then linear function on $\mathbb{Z}^n$

$$L_1(x) = \mu_n x_n - \mu_{n-1} x_{n-1} - \ldots - \mu_1 x_1$$

vanishes only at $x = 0$. Thus for all $x \in \mathbb{Z}^n \backslash \{0\}$, $D(x_1, \ldots, x_n)$ can be expressed as a rational function $\frac{P_1(x)}{L_1(x)}$ with $P_1 \in \mathrm{End}(U) \otimes \mathbb{C}[x_1, \ldots, x_n]$.

Likewise,

$$\begin{aligned}
&[D(x_1, \ldots, x_{n-1} - x_n, 0), D(0, \ldots, x_n, x_n)] + (\mu_{n-1} + \mu_n) x_n D(0, \ldots, x_n, x_n) \\
&\quad - (\mu_1 x_1 + \ldots + \mu_{n-1} x_{n-1} - \mu_{n-1} x_{n-1}) D(x_1, \ldots, x_{n-1} - x_n, 0) \\
&= (\mu_n x_n + 2\mu_{n-1} x_n - \mu_{n-1} x_{n-1} - \ldots - \mu_1 x_1) D(x_1, \ldots, x_{n-1}, x_n)
\end{aligned}$$

is a polynomial in $x_1, \ldots, x_n$ and on the set $\mathbb{Z}^n \backslash \{0\}$, $D(x_1, \ldots, x_n)$ may be expressed as a rational function $\frac{P_2(x)}{L_2(x)}$, where

$$L_2(x) = \mu_n x_n + 2\mu_{n-1} x_n - \mu_{n-1} x_{n-1} - \ldots - \mu_1 x_1.$$

By Corollary 3.3 we have $\frac{P_1(x)}{L_1(x)} = \frac{P_2(x)}{L_2(x)}$ in $\mathrm{End}(U) \otimes \mathbb{C}(x_1, \ldots, x_n)$, and hence $P_1(x) L_2(x) = P_2(x) L_1(x)$. Since $\mathbb{C}[x_1, \ldots, x_n]$ is a unique factorization domain and the greatest common divisor of $L_1(x)$ and $L_2(x)$ is 1, we conclude that $P_1(x)$ is divisible by $L_1(x)$. Hence $D(x_1, \ldots, x_n)$ coincides with some polynomial on the set $\mathbb{Z}^n \backslash \{0\}$. It only remains to show that the value of this polynomial at $x = 0$ coincides with $D(0)$. However the constant term of this polynomial is the same as the constant term of the polynomial $D(x_1, 0, \ldots, 0)$, and the constant term of the latter polynomial is $D(0)$ by the result for W_1. This completes the proof of the theorem. $\square$

Expand the polynomial $D(s)$ in s:

$$D(s) = \sum_{k \in \mathbb{Z}_+^n} \frac{s^k}{k!} \partial^k D,$$

where $\partial^k D \in \operatorname{End}(U)$ with only a finite number of these operators being non-zero. Here $k! = k_1! \times \ldots \times k_n!$. Expanding the commutator $[D(s), D(m)]$ in s, m,

$$\sum_{k,r \in \mathbb{Z}_+^n} \frac{s^k}{k!} \frac{m^r}{r!} [\partial^k D, \partial^r D]$$
$$= \left(\sum_{i=1}^n \mu_i m_i \right) \sum_{p \in \mathbb{Z}_+^n} \frac{(m+s)^p - m^p}{p!} \partial^p D - \left(\sum_{i=1}^n \mu_i s_i \right) \sum_{p \in \mathbb{Z}_+^n} \frac{(m+s)^p - s^p}{p!} \partial^p D,$$

we get the Lie brackets between $\partial^k D$ by equating the coefficients at $\frac{s^k}{k!} \frac{m^r}{r!}$:

$$[\partial^k D, \partial^r D] = \begin{cases} \sum\limits_{i=1}^n \mu_i (r_i - k_i) \partial^{k+r-\epsilon_i} D & \text{if } \ k, r \neq 0, \\ 0 & \text{if } \ k = 0 \text{ or } \ r = 0. \end{cases}$$

Here ϵ_i is the i-th standard basis vector of $\mathbb{Z}^n$. We can identify $\operatorname{Span}\left\{ \partial^k D \,|\, k \in \mathbb{Z}_+^n \backslash \{0\} \right\}$ with a subalgebra in $\operatorname{Der} \mathbb{C}[x_1, \ldots, x_n]$, where $\partial^k D \mapsto x^k \partial_\mu$, $\partial_\mu = \mu_1 \frac{\partial}{\partial x_1} + \ldots + \mu_n \frac{\partial}{\partial x_n}$. Set $\mathcal{L} = A \partial_\mu \subset \operatorname{Der} \mathbb{C}[x_1, \ldots, x_n]$. The Lie algebra $\mathcal{L}$ has a $\mathbb{Z}$-grading defined by assigning $\deg(x_i) = 1$, $i = 1, \ldots, n$, $\deg \partial_\mu = -1$:

$$\mathcal{L} = \bigoplus_{j=-1}^{\infty} \mathcal{L}_j.$$

Then $\operatorname{Span}\left\{ \partial^k D \,|\, k \in \mathbb{Z}_+^n \backslash \{0\} \right\}$ is identified with the subalgebra

$$\mathcal{L}_+ = \bigoplus_{j=0}^{\infty} \mathcal{L}_j.$$

We obtained the following result

Theorem 3.5. *(cf., [1], Theorem 3) There exists an isomorphism between the category of cuspidal AW_μ-modules with support $\beta + \Gamma_\mu$ and the category of finite-dimensional $\mathcal{L}_+$-modules (U, ρ) satisfying*

$$\rho(x^k \partial_\mu) = 0 \qquad \text{for} \ \ k_1 + \ldots + k_n \gg 0. \tag{3.4}$$

Given such an $\mathcal{L}_+$-module U, we associate to it an AW_μ-module

$$M = A \otimes U$$

with the following action of W_μ:

$$(t^s d_\mu)(t^m \otimes u) = t^{m+s} \otimes \left((\mu \cdot m + \beta) \mathrm{Id} + \sum_{k \in \mathbb{Z}_+^n \backslash \{0\}} \frac{s^k}{k!} \rho(x^k \partial_\mu) \right) u. \tag{3.5}$$

Remark 3.6. We shall see below that condition (3.4) automatically holds in every finite-dimensional $\mathcal{L}_+$-module U and hence can be dropped from the statement of the theorem.

Remark 3.7. The Lie algebra $\mathcal{L}$ is the jet Lie algebra for the solenoidal Lie algebra W_μ (see [3] for the definition of the jet Lie algebra).

Now let us focus on the description of simple cuspidal AW_μ-modules. According to Theorem 3.5, such modules correspond to simple finite-dimensional $\mathcal{L}_+$-modules satisfying (3.4).

Theorem 3.8. *(1) The commutant $\mathcal{L}_+^{(1)}$ has codimension 1 in $\mathcal{L}_+$:*

$$\mathcal{L}_+^{(1)} = \mathcal{L}_0^{(1)} \oplus \sum_{j=1}^{\infty} \mathcal{L}_j,$$

where the commutant $\mathcal{L}_0^{(1)}$ of $\mathcal{L}_0$ is an abelian subalgebra

$$\mathcal{L}_0^{(1)} = \left\{ \sum_{i=1}^{n} c_i x_i \partial_\mu \;\middle|\; \sum_{i=1}^{n} \mu_i c_i = 0 \right\}.$$

(2) Every finite-dimensional module U for $\mathcal{L}_+$ satisfies the condition (3.4).

(3) Every finite-dimensional simple $\mathcal{L}_+$-module has dimension 1.

(4) All 1-dimensional representations of $\mathcal{L}_+$ are parametrized by $\alpha \in \mathbb{C}$ with the action of $\mathcal{L}_+$ given by $x_i\partial_\mu \mapsto \alpha\mu_i$, $i = 1, \dots, n$, $\mathcal{L}_j \mapsto 0$ for $j \geq 1$.

Corollary 3.9. *Every simple cuspidal AW_μ-module is isomorphic to a tensor module $T(\alpha, \beta)$ for some $\alpha, \beta \in \mathbb{C}$.*

Proof. Combining Theorems 3.5 and 3.8, we see that a simple cuspidal AW_μ-module M is isomorphic to $A \otimes U$, where $U = \mathbb{C}u$ with the action

$$(t^s d_\mu)(t^m \otimes u) = (\mu \cdot m + \beta + \alpha\mu \cdot s) t^{m+s} \otimes u.$$

This module is isomorphic to $T(\alpha, \beta)$. □

Proof of Theorem 3.8. First of all, let us compute the commutant of $\mathcal{L}_0$. It is easy to see that the commutators

$$[x_i\partial_\mu, x_j\partial_\mu] = \mu_j x_i \partial_\mu - \mu_i x_j \partial_\mu$$

span the subspace

$$\left\{ \sum_{i=1}^{n} c_i x_i \partial_\mu \;\middle|\; \sum_{i=1}^{n} \mu_i c_i = 0 \right\}.$$

Next, let us show that for every $p \geq 1$, $[\mathcal{L}_0, \mathcal{L}_p] = \mathcal{L}_p$. Note that

$$[x_j\partial_\mu, x^k\partial_\mu] = \mu_j(k_j - 1)x^k\partial_\mu + f(x)\partial_\mu, \tag{3.6}$$

where all monomials in f have degree $k_j + 1$ in x_j. Using a descending induction on k_j we conclude that $x^k\partial_\mu \in \mathcal{L}_+^{(1)}$ if $k_j \geq 2$. It remains to show that $x_{i_1} x_{i_2} \dots x_{i_{p+1}} \partial_\mu \in \mathcal{L}_+^{(1)}$, $i_1 < \ldots < i_{p+1}$.

We have

$$[x_{i_1}\partial_\mu, x_{i_2}^2 x_{i_3} \dots x_{i_{p+1}} \partial_\mu] = 2\mu_{i_2} x_{i_1} x_{i_2} \dots x_{i_{p+1}} \partial_\mu + g(x)\partial_m u,$$

where $g(x)$ is divisible by $x_{i_2}^2$. Since $g(x)\partial_\mu \in \mathcal{L}_+^{(1)}$, we get that $x_{i_1} \dots x_{i_{p+1}} \partial_\mu \in \mathcal{L}_+^{(1)}$ and part (1) is proved.

Let us prove part (2). Consider the action of $\mathrm{ad}(x_j\partial_\mu)$ on $\mathcal{L}_p$. It follows from (3.6) that $x^k\partial_\mu$ belongs to the subspace spanned by the eigenvectors for $\mathrm{ad}(x_j\partial_\mu)$ with eigenvalues $\mu_j s$, where $s \geq k_j - 1$. By Lemma 2 from [1], given a finite-dimensional $\mathcal{L}_+$-module (U, ρ), there exists $m \in \mathbb{N}$ such that for all $j = 1, \dots, n$, every eigenvector of $\mathrm{ad}(x_j\partial_\mu)$ with an eigenvalue $\mu_j s$, with $s \geq m$, acts trivially on U. Thus $\rho(x^k\partial_\mu) = 0$ if $k_j - 1 \geq m$ for some j. This implies $\rho(\mathcal{L}_p) = 0$ if $p \geq nm$, which establishes claim (2).

It follows from (2) that every finite-dimensional representation of $\mathcal{L}_+$ factors through the quotient $\mathcal{L}_+ / \sum\limits_{j\geq p} \mathcal{L}_j$ for some p. This quotient is a finite-dimensional solvable Lie algebra and by the Lie's Theorem ([4], Corollary I.5.3.2), its finite-dimensional irreducible representations have dimension one.

Finally, to prove part (4), we note that 1-dimensional representations of $\mathcal{L}_+$ are determined by 1-dimensional representations of $\mathcal{L}^+/\mathcal{L}_+^{(1)}$. Since $\dim\left(\mathcal{L}^+/\mathcal{L}_+^{(1)}\right) = 1$, such representations are described by a single parameter $\alpha \in \mathbb{C}$. The map

$$\varphi : \quad \mathcal{L}_+/\mathcal{L}_+^{(1)} \to \mathbb{C}$$

is given by $\varphi(x_j\partial_\mu) = \mu_j$, $1 \leq j \leq n$, $\varphi(\mathcal{L}_p) = 0$ for $p \geq 1$. Claim (4) now follows. $\square$

4. A-cover of a cuspidal W_μ-module

In this section we shall establish a relation between cuspidal W_μ-modules and cuspidal AW_μ-modules. We are going to prove the following theorem:

Theorem 4.1. *Let M be a cuspidal W_μ-module satisfying $W_\mu M = M$. Then there exists a cuspidal AW_μ-module $\widehat{M}$ and a surjective homomorphism of W_μ-modules $\widehat{M} \to M$.*

Following [2], we define certain elements in the universal enveloping algebra of W_μ.

Definition 4.2. For $k, s, h \in \Gamma_\mu$ we define a *differentiator* of order m as

$$\Omega_{k,s}^{(m,h)} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} e_{k-ih} e_{s+ih} \in U(W_\mu).$$

The following theorem shows that a subspace in $U(W_\mu)$ spanned by products of differentiators of a given order, contains differentiators of higher orders. Here we denote by $\{X, Y\}$ the anticommutator $XY + YX$.

Theorem 4.3. *([2]) Let $r \geq 2$, $k, s, p, q, h \in \Gamma_\mu$. Then*

$$\sum_{i=0}^{r}\sum_{j=0}^{r} (-1)^{i+j} \binom{r}{i}\binom{r}{j} \left(\left\{ \Omega_{k-ih,s-jh}^{(r,h)}, \Omega_{q+ih,p+jh}^{(r,h)} \right\} - \left\{ \Omega_{k-ih,q-jh}^{(r,h)}, \Omega_{s+ih,p+jh}^{(r,h)} \right\} \right)$$
$$= (q-s)(p-k+2rh)\Omega_{k+p+2rh,s+q-2rh}^{(4r,h)}. \tag{4.1}$$

Using this theorem we are going to prove that every cuspidal W_μ-module is annihilated by the differentiators of high enough orders. For the Witt algebra W_1 we can quote a result from [2]:

Theorem 4.4. *([2]) For every $\ell \in \mathbb{N}$ there exists $r \in \mathbb{N}$ such that for all $k, s \in \mathbb{Z}$ the differentiators*

$$\Omega^{(r)}_{k,s} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} e_{k-i} e_{s+i}$$

annihilate all cuspidal W_1-modules with a composition series of length ℓ.

Note that this result may be stated in terms of the bound on the dimensions of weight spaces instead of the length of the composition series, since for a cuspidal W_1-module M the length of its composition series does not exceed $\dim M_0 + \dim M_\lambda$, where $\lambda \in \beta + \Gamma_\mu$, $\lambda \neq 0$.

For any non-zero $h \in \Gamma_\mu$, we can view a W_μ-module as a module for the subalgebra with basis $\{e_{ih} \,|\, i \in \mathbb{Z}\}$, which is isomorphic to W_1. Then a cuspidal W_μ-module becomes a direct sum of cuspidal W_1-submodules, corresponding to the cosets of $\mathbb{Z}h$ in Γ_μ, and all summands have a common bound on the dimensions of weight spaces. As a consequence we get

Corollary 4.5. *Let M be a cuspidal W_μ-module. Then there exists $r \in \mathbb{N}$, which depends only on the bound for the dimensions of weight spaces of M, such that $\Omega^{(r,h)}_{q,p}$ annihilates M for all $h \in \Gamma_\mu$, $q, p \in \mathbb{Z}h \subset \Gamma_\mu$.*

Now let us establish a generalization of the above corollary:

Proposition 4.6. *Let M be a cuspidal W_μ-module. Then there exists $m \in \mathbb{N}$, which depends only on the bound on the dimensions of weight spaces of M, such that $\Omega^{(m,h)}_{k,s}$ annihilates M for all $h, k \in \Gamma_\mu$ and $s \in \mathbb{Z}h$.*

Proof. Fix $h \in \Gamma_\mu$. By Corollary 4.5 there exists $r \in \mathbb{N}$ such that for all $q, p \in \mathbb{Z}h$, $\Omega^{(r,h)}_{q,p} \in \mathrm{Ann}(M)$. Then for $k \in \Gamma_\mu$, $s, q, p \in \mathbb{Z}h$, the left hand side of (4.1) annihilates M. Thus $\Omega^{(4r,h)}_{k+p+2rh, s+q-2rh} \in \mathrm{Ann}(M)$. Setting $m = 4r$, we get the claim of the proposition. $\square$

Now let us proceed with the proof of Theorem 4.1.

Given a cuspidal W_μ-module M, we construct the coinduced module $\mathrm{Hom}(A, M)$, which is an AW_μ-module with the following action of A and W_μ ([2], Proposition 4.3):

$$\begin{aligned} (x\phi)(f) &= x(\phi(f)) - \phi(xf), \\ (g\phi)(f) &= \phi(gf), \quad \phi \in \mathrm{Hom}(A, M), x \in W_\mu, f, g \in A. \end{aligned}$$

The map $\pi : \mathrm{Hom}(A, M) \to M$, $\pi(\phi) = \phi(1)$, is a surjective homomorphism of W_μ-modules.

Definition 4.7. ([2]) An A-cover of a cuspidal W_μ-module M is an AW_μ-submodule $\widehat{M}$ in the coinduced module $\mathrm{Hom}(A, M)$, spanned by

$$\{\psi(x, u) \,|\, x \in W_\mu, u \in M\},$$

where $\psi(x, u) : A \to M$ is given by

$$\psi(x, u)(f) = (fx)u.$$

The AW_μ-action on $\widehat{M}$ is

$$\begin{aligned} y\psi(x, u) &= \psi([y, x], u) + \psi(x, yu), \\ g\psi(x, u) &= \psi(gx, u), \quad x, y \in W_\mu, u \in M, g \in A. \end{aligned}$$

The map $\pi : \widehat{M} \to M$, $\pi(\psi(x,u)) = \psi(x,u)(1) = xu$, is a homomorphism of W_μ-modules with $\pi(\widehat{M}) = W_\mu M$.

It turns out that $\widehat{M}$ is a cuspidal AW_μ-module. The proof of this fact is the same as given in [2], and its key ingredient is Proposition 4.6. This establishes Theorem 4.1. □

Now we can prove the main result of the paper.

Theorem 4.8. *Let $\mu \in \mathbb{C}$ be generic. Every simple cuspidal W_μ-module is isomorphic to either*

- *a module of tensor fields $T(\alpha,\beta)$ with $\alpha \neq 0,1$ or $\alpha = 0,1$ and $\beta \notin \Gamma_\mu$,*
- *the quotient $\overline{T}(0,0)$ of the module of functions by the 1-dimensional submodule $\mathbb{C}v_0$ of constant functions,*

or

- *the trivial 1-dimensional module.*

Proof. Let M be a simple cuspidal W_μ-module. Let us assume that it is different from the trivial 1-dimensional module. Then $W_\mu M = M$. Construct the A-cover $\widehat{M}$ of M with a surjective homomorphism of W_μ-modules $\pi : \widehat{M} \to M$. Consider the composition series

$$0 = \widehat{M}_0 \subset \widehat{M}_1 \subset \ldots \subset \widehat{M}_\ell = \widehat{M}$$

with the quotients $\widehat{M}_{i+1}/\widehat{M}_i$ being simple AW_μ-modules. Let k be the smallest integer such that $\pi(\widehat{M}_k) \neq 0$. Since M is a simple W_μ-module, we have $\pi(\widehat{M}_k) = M$ and $\pi(\widehat{M}_{k-1}) = 0$. The map π factors through to

$$\overline{\pi} : \widehat{M}_k/\widehat{M}_{k-1} \to M.$$

The module $\widehat{M}_k/\widehat{M}_{k-1}$ is a simple cuspidal AW_μ-module, and by Corollary 3.9 is isomorphic to a module of tensor fields $T(\alpha,\beta)$ for some $\alpha,\beta \in \mathbb{C}$. Thus M is a homomorphic image of $T(\alpha,\beta)$.

By Proposition 2.5, $T(\alpha,\beta)$ is simple as a W_μ-module if $\alpha \neq 0,1$ or $\alpha = 0,1$ and $\beta \notin \Gamma_\mu$, thus $M \cong T(\alpha,\beta)$ in this case. If $\alpha = 0$ and $\beta \in \Gamma_\mu$ then the unique simple quotient of $T(0,0)$ is $\overline{T}(0,0)$. Finally, in case when $\alpha = 1$ and $\beta \in \Gamma_\mu$, the unique simple quotient of $T(1,0)$ is the trivial 1-dimensional module. This shows that the list given in the statement of the theorem is complete. □

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