Modules for a sheaf of Lie algebras on loop manifolds.

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Abstract. We consider a central extension of the sheaf of Lie algebras of maps from a manifold $\mathbb{C}^* \times X$ into a finite-dimensional simple Lie algebra, together with the sheaf of vector fields on $\mathbb{C}^* \times X$. Using vertex algebra methods we construct sheaves of modules for this sheaf of Lie algebras. Our results extend the work of Malikov-Schechtman-Vaintrob on the chiral de Rham complex.

0. Introduction.

Two most interesting examples of infinite-dimensional Lie algebras, affine Kac-Moody algebras and the Virasoro algebra, are associated with a circle as an underlying geometric object. In this paper we are going to make a transition from the circle to more general manifolds. To construct an analogue of (untwisted) affine Kac-Moody algebra in this case we start with the algebra of functions on a manifold with values in a finite-dimensional simple Lie algebra $\mathfrak{g}$, then take its central extension and add the Lie algebra of vector fields on the manifold, acting as derivations.

Our goal is to develop a representation theory for this class of Lie algebras. Since we would like to retain the features of the theory of the highest weight modules, we still need the concept of positive/negative Fourier modes, and for this reason as the underlying manifold we take $\widehat{X} = \mathbb{C}^* \times X$, where X is a smooth irreducible complex algebraic variety of dimension N. The punctured complex line $\mathbb{C}^*$ here is a complex analogue of a circle. We choose to work with complex manifolds only as a matter of technical convenience, and one could just as well consider $\widehat{X} = S^1 \times X$, where X is a real manifold.

When we look at functions on an algebraic manifold, taking the algebra of globally defined functions may be inadequate (for example, in case of complex projective manifolds this algebra contains only constant functions). Instead, it is natural to use the language of sheaves.

We begin by taking the sheaf $\mathrm{Map}(\widehat{X}, \mathfrak{g})$ of functions on $\widehat{X}$ with values in $\mathfrak{g}$, or, equivalently, functions on X with values in the loop algebra $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$. As the space of the central extension we take the sheaf $\overline{\Omega}^1_{\widehat{X}}$ associated with the quotient of 1-forms by differentials of functions $\Omega^1_{\widehat{X}} / d\mathcal{O}_{\widehat{X}}$. There is a 2-cocycle on $\mathrm{Map}(\widehat{X}, \mathfrak{g})$ with values in $\overline{\Omega}^1_{\widehat{X}}$ that naturally generalizes the central cocycle on loop Lie algebras. Finally we take a semidirect product $\mathcal{G}$ of this central extension with the sheaf $\mathrm{Vect}(\widehat{X})$ of vector fields on $\widehat{X}$.

To get a representation theory for this sheaf of Lie algebras we need to construct the sheaves of modules. This is done using vertex algebras. We introduce a sheaf of vertex

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algebras $\mathcal{V}$ on X and show that the sheaf of Lie algebras $\mathcal{G}$ acts on $\mathcal{V}$. We then construct sheaves of modules for $\mathcal{V}$, which automatically become modules for $\mathcal{G}$.

If we take $\mathfrak{g}$ to be a trivial Lie algebra, $\mathfrak{g} = (0)$, we get representations for the sheaf of vector fields on $\widehat{X}$. We would like to point out here a connection with the important construction of the chiral de Rham complex. Chiral de Rham complex was introduced by Malikov-Schechtman-Vaintrob in [9]. It is a sheaf of vertex superalgebras on X with a $\mathbb{Z}$-grading and a differential. Malikov and Schechtman [8] show that the chiral de Rham complex admits the action of the sheaf of Lie algebras $\mathbb{C}[t, t^{-1}] \otimes \mathrm{Vect}(X)$. Our result yield a stronger statement that in fact a larger sheaf $\mathrm{Vect}(\widehat{X})$ acts on the chiral de Rham complex. In addition to this we note that the chiral differential is a homomorphism of $\mathrm{Vect}(\widehat{X})$-modules.

In classical differential geometry the Lie algebra of vector fields acts on modules of tensor fields of a fixed type, and modules of differential forms appearing in the classical de Rham complex are a special case of this. Likewise the modules that we construct here for $\mathrm{Vect}(\widehat{X})$ could be thought of as chiralizations of tensor modules and we get a wider class of representations than those appearing in the chiral de Rham complex.

In case when X is a torus, the representation theory of toroidal Lie algebras and Lie algebra of vector fields was developed in [10, 4, 6, 1, 2, 3]. Since $(\mathbb{C}^*)^N$ can be covered with a single chart, there was no need to work with sheaves of Lie algebras. In the toroidal case one gets strong results on irreducibility of the modules [2, 3].

In the present paper we work with the algebraic varieties, however, this theory also works if X is taken to be an analytic or a C^∞ manifold. The loop component $\mathbb{C}^*$ should be still viewed in the algebraic setting with the ring of Laurent polynomials as the algebra of functions.

The structure of this paper is as follows: in Section 1 we introduce the sheaf $\mathcal{G}$ of Lie algebras, generalizing the construction of affine Kac-Moody algebras, in Section 2 we construct a sheaf $\mathcal{V}$ of vertex algebras on X and in Section 3 we define the sheaves of the generalized Verma modules $\mathcal{M}$ and their quotients $\mathcal{L}$. In Section 4 we prove that the sheaves $\mathcal{V}$, $\mathcal{M}$ and $\mathcal{L}$ are modules for the sheaf $\mathcal{G}$ of Lie algebras. In the final section we consider a version of our construction in the setting of rational functions on a manifold.

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1. A sheaf of Lie algebras.

Let X be a smooth irreducible algebraic variety over $\mathbb{C}$ of dimension N. Let $\widehat{X} = \mathbb{C}^* \times X$. Fix a finite-dimensional simple Lie algebra $\mathfrak{g}$ with a symmetric invariant bilinear form $(\cdot|\cdot)$. We consider the sheaf $\mathrm{Map}(\widehat{X}, \mathfrak{g})$ of locally regular functions on $\widehat{X}$ with values in $\mathfrak{g}$ (or, equivalently, functions on X with values in the loop algebra $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$). This becomes a sheaf of Lie algebras over X with pointwise multiplication.

Even though our base manifold is $\widehat{X}$, all sheaves that we consider throughout the paper are over X, and for each open set $U \subset X$, we will consider functions defined over $\widehat{U} = \mathbb{C}^* \times U$.

Next we are going to model in this setting the construction of affine Kac-Moody algebras from the loop Lie algebras. Let $\overline{\Omega}^1_{\widehat{X}}$ be the sheaf associated with the presheaf $\Omega^1_{\widehat{X}}/d\mathcal{O}_{\widehat{X}}$, where $\mathcal{O}_{\widehat{X}}$ is the sheaf of rational functions on $\widehat{X}$ and $\Omega^1_{\widehat{X}}$ is the sheaf of 1-forms on $\widehat{X}$.

We define a central extension sheaf of Lie algebras

$$\mathrm{Map}(\widehat{X}, \mathfrak{g}) \oplus \overline{\Omega}^1_{\widehat{X}},$$

where $\overline{\Omega}^1_{\widehat{X}}$ is central, while the new Lie bracket on $\mathrm{Map}(\widehat{X}, \mathfrak{g})$ is defined by

$$[f_1 \otimes g_1, f_2 \otimes g_2] = f_1 f_2 \otimes [g_1, g_2] + (g_1|g_2)\overline{f_2 df_1},$$

where f_1, f_2 are functions on an open set $\widehat{U}$, $g_1, g_2 \in \mathfrak{g}$, and $\overline{}$ is the canonical projection $\Omega^1_{\widehat{X}} \to \Omega^1_{\widehat{X}}/d\mathcal{O}_{\widehat{X}}$.

The sheaf of vector fields $\mathrm{Vect}(\widehat{X})$ acts on $\mathrm{Map}(\widehat{X}, \mathfrak{g}) \oplus \overline{\Omega}^1_{\widehat{X}}$ and we can form a semidirect product sheaf

$$\left(\mathrm{Map}(\widehat{X}, \mathfrak{g}) \oplus \overline{\Omega}^1_{\widehat{X}}\right) \rtimes \mathrm{Vect}(\widehat{X}).$$

Here the action of $\mathrm{Vect}_{\widehat{X}}$ on $\mathrm{Map}(\widehat{X}, \mathfrak{g})$ is the natural action of vector fields on functions, while the action on $\overline{\Omega}^1_{\widehat{X}}$ is via Lie derivative

$$\eta(\overline{f_1 df_2}) = \overline{\eta(f_1) df_2} + \overline{f_1 d\eta(f_2)},$$

for $\eta \in \mathrm{Vect}_{\widehat{X}}(\widehat{U})$, $f_1, f_2 \in \mathcal{O}_{\widehat{X}}(\widehat{U})$.

The variety X admits a finite covering by open affine sets $\{U_i\}$ where each U_i has local (uniformizing) parameters $x_1, \dots, x_N \in \mathcal{O}_X(U_i)$ such that $\Omega^1_X(U_i)$ is a free $\mathcal{O}_X(U_i)$-module of rank N with generators $dx_1, \dots, dx_N$ [11].

An open covering $\{U_i\}$ of X yields an open covering $\{\widehat{U}_i\}$ of $\widehat{X}$. We fix a local parameter t on $\mathbb{C}^*$, and we identify functions on $\mathbb{C}^*$ with $\mathbb{C}[t, t^{-1}]$, so that

$$\mathcal{O}_{\widehat{X}}(\widehat{U}_i) = \mathbb{C}[t, t^{-1}] \otimes \mathcal{O}_X(U_i).$$

Let $\mathcal{G}$ be a sheaf of Lie algebras on X and let $\mathcal{M}$ be a sheaf of vector spaces on X.

Definition. A representation $(\rho, \mathcal{M})$ of $\mathcal{G}$ is a sheaf morphism

$$\rho : \mathcal{G} \times \mathcal{M} \to \mathcal{M},$$

such that for every open set $U \subset X$, the map

$$\rho_U : \ \mathcal{G}(U) \times \mathcal{M}(U) \to \mathcal{M}(U)$$

is a representation of the Lie algebra $\mathcal{G}(U)$.

The main goal of this paper is to construct representations of the sheaf of Lie algebras

$$\mathcal{G} = \left(\mathrm{Map}(\widehat{X}, \mathfrak{g}) \oplus \overline{\Omega}^1_{\widehat{X}}\right) \rtimes \mathrm{Vect}(\widehat{X}).$$

2. A sheaf of vertex algebras.

We will construct representations using the vertex algebra techniques. Our main object will be a sheaf $\mathcal{V}$ of vertex algebras. Locally this sheaf will be defined as the space of functions with values in a certain vertex algebra V.

Let us recall the basic notions of the theory of the vertex operator algebras. Here we are following [5] and [7].

Definition. A vertex algebra is a vector space V with a distinguished vector $\mathbf{1}$ (vacuum vector) in V, an operator D (infinitesimal translation) on the space V, and a linear map Y (state-field correspondence)

$$\begin{aligned} Y(\cdot, z): \quad & V \to (\mathrm{End}V)[[z, z^{-1}]], \\ & a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \quad (\text{where } \ a_{(n)} \in \mathrm{End}V), \end{aligned}$$

such that the following axioms hold:
(V1) For any $a, b \in V$, $\quad a_{(n)}b = 0$ for n sufficiently large;
(V2) $[D, Y(a,z)] = Y(D(a), z) = \frac{d}{dz} Y(a,z)$ for any $a \in V$;
(V3) $Y(\mathbf{1}, z) = \mathrm{Id}_V$;
(V4) $Y(a,z)\mathbf{1} \in V[[z]]$ and $Y(a,z)\mathbf{1}|_{z=0} = a$ for any $a \in V$ (self-replication);
(V5) For any $a, b \in V$, the fields $Y(a,z)$ and $Y(b,z)$ are mutually local, that is,

$$(z-w)^n \left[Y(a,z), Y(b,w)\right] = 0, \qquad \text{for } \ n \text{ sufficiently large.}$$

A vertex algebra V is called a vertex operator algebra (VOA) if, in addition, V contains a vector ω (Virasoro element) such that
(V6) The components $L(n) = \omega_{(n+1)}$ of the field

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} \omega_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$$

satisfy the Virasoro algebra relations:

$$[L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m} \frac{n^3-n}{12} (\mathrm{rank}\, V)\mathrm{Id}, \quad \text{where } \ \mathrm{rank}\, V \in \mathbb{C}; \quad (2.1)$$

(V7) $D = L(-1)$;
(V8) V is graded by the eigenvalues of $L(0)$: $V = \underset{n\in\mathbb{Z}}{\oplus} V_n$ with $L(0)\big|_{V_n} = n\mathrm{Id}$.

This completes the definition of a VOA.

As a consequence of the axioms of the vertex algebra we have the following important commutator formula:

$$[Y(a,z_1),Y(b,z_2)] = \sum_{n\geq 0} \frac{1}{n!} Y(a_{(n)}b,z_2) \left[z_1^{-1} \left(\frac{\partial}{\partial z_2} \right)^n \delta\left(\frac{z_2}{z_1} \right) \right]. \tag{2.2}$$

As usual, the delta function is

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n.$$

By (V1), the sum in the right hand side of the commutator formula is actually finite.

All the vertex operator algebras that appear in this paper have the gradings by non-negative integers: $V = \mathop{\oplus}\limits_{n=0}^{\infty} V_n$. In this case the sum in the right hand side of the commutator formula (2.2) runs from $n = 0$ to $n = \deg(a) + \deg(b) - 1$, because

$$\deg(a_{(n)}b) = \deg(a) + \deg(b) - n - 1, \tag{2.3}$$

and the elements of negative degree vanish.

Another consequence of the axioms of a vertex algebra is the Borcherds' identity:

$$(a_{(k)}b)_{(n)}c = \sum_{j\geq 0} (-1)^{k+j+1} \binom{k}{j} b_{(n+k-j)}a_{(j)}c + \sum_{j\geq 0} (-1)^j \binom{k}{j} a_{(k-j)}b_{(n+j)}c, \quad k,n\in\mathbb{Z}. \tag{2.4}$$

Let us list some other consequences of the axioms of a vertex algebra that we will be using in the paper. It follows from (V7) and (V8) that

$$\omega_{(0)}a = D(a) \tag{2.5}$$

and

$$\omega_{(1)}a = \deg(a)a, \qquad \text{for } \ a \text{ homogeneous.} \tag{2.6}$$

The map D is a derivation of the n-th product:

$$D(a_{(n)}b) = (Da)_{(n)}b + a_{(n)}Db. \tag{2.7}$$

It could be easily derived from (V2) that

$$(Da)_{(n)} = -na_{(n-1)}. \tag{2.8}$$

The vertex algebra V that we need for the construction of the sheaf $\mathcal{V}$, is the tensor product of four well-known vertex algebras:

$$V = V_{\mathfrak{Hei}} \otimes V_{\mathfrak{gl}_N} \otimes V_{\mathfrak{g}} \otimes V_{\mathfrak{Vir}}. \tag{2.9}$$

Let us describe each tensor factor.

Consider a Heisenberg Lie algebra $\mathfrak{Hei}$ with the basis $\{u_p(j), v_p(j), C_{\mathfrak{Hei}}|_{j\in\mathbb{Z}}^{p=1,\ldots,N}\}$ and the Lie bracket

$$[u_p(m), v_q(s)] = m\delta_{pq}\delta_{m,-s}C_{\mathfrak{Hei}},$$

$$[u_p(m), u_q(s)] = [v_p(m), v_q(s)] = 0, \;\; p, q = 1, \ldots, N, \; m, s \in \mathbb{Z},$$

and the element $C_{\mathfrak{Hei}}$ being central. The vertex algebra

$$V_{\mathfrak{Hei}} = \mathbb{C}[u_p(-j), v_p(-j)|_{j=1,2,\ldots}^{p=1,\ldots,N}]$$

is a Fock space module for this Heisenberg algebra, in which $C_{\mathfrak{Hei}}$ acts as an identity operator and the raising operators $u_p(j), v_p(j)$ with $j \geq 1$ annihilate the highest weight vector $\mathbf{1}$. The generating fields for this vertex algebra are

$$u_p(z) = Y(u_p(-1)\mathbf{1}, z) = \sum_{j\in\mathbb{Z}} u_p(j)z^{-j-1},$$

$$v_p(z) = Y(v_p(-1)\mathbf{1}, z) = \sum_{j\in\mathbb{Z}} v_p(j)z^{-j-1}, \;\; p = 1, \ldots, N,$$

with $u_p(0)$ and $v_p(0)$ acting on $V_{\mathfrak{Hei}}$ trivially.

The Virasoro element in $V_{\mathfrak{Hei}}$ is

$$\omega^{\mathfrak{Hei}} = \sum_{p=1}^{N} v_p(-1)u_p(-1)\mathbf{1}$$

and rank $(V_{\mathfrak{Hei}}) = 2N$.

Consider next the affine Lie algebra

$$\widehat{\mathfrak{gl}}_N = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{gl}_N \oplus \mathbb{C}C_{\mathfrak{gl}_N}$$

with the Lie bracket

$$[t^m \otimes A, t^s \otimes B] = t^{m+s} \otimes [A, B] + m\delta_{m,-s}\mathrm{Tr}(AB)C_{\mathfrak{gl}_N},$$

where $A, B \in \mathfrak{gl}_N(\mathbb{C})$ and $C_{\mathfrak{gl}_N}$ is central.

The second tensor factor $V_{\mathfrak{gl}_N}$ in (2.9) is the universal enveloping vertex algebra for $\widehat{\mathfrak{gl}}_N$ at level 1. It is a highest weight module for $\widehat{\mathfrak{gl}}_N$, with the highest weight vector being annihilated by the subalgebra $\mathbb{C}[t] \otimes \mathfrak{gl}_N$ and $C_{\mathfrak{gl}_N}$ acting as the identity operator. As a vector space, it is realized as

$$V_{\mathfrak{gl}_N} = U(t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{gl}_N) \otimes \mathbf{1}.$$

The generating fields of this vertex algebra are

$$E_{ab}(z) = Y(E_{ab}(-1)\mathbf{1}, z) = \sum_{j\in\mathbb{Z}} E_{ab}(j)z^{-j-1}, \;\; a, b = 1, \ldots, N,$$

where E_{ab} is a matrix with entry 1 in position (a, b) and zeros elsewhere, and $E_{ab}(j) = t^j \otimes E_{ab}$. It follows from this formula that $(E_{ab}(-1)\mathbf{1})_{(n)} = E_{ab}(n)$.

The commutator relations between the generating fields are encoded in n-th products:

$$E_{ab}(0)E_{cd}(-1)\mathbf{1} = \delta_{bc}E_{ad}(-1)\mathbf{1} - \delta_{ad}E_{cb}(-1)\mathbf{1},$$

$$E_{ab}(1)E_{cd}(-1)\mathbf{1} = \delta_{ad}\delta_{bc}\mathbf{1},$$
$$E_{ab}(n)E_{cd}(-1)\mathbf{1} = 0 \quad \text{for} \quad n \geq 2. \tag{2.10}$$

We consider the following (non-standard) Virasoro element in $V_{\mathfrak{gl}_N}$:

$$\omega^{\mathfrak{gl}_N} = \frac{1}{2(N+1)} \left(I(-1)I(-1)\mathbf{1} + \sum_{a,b=1}^{N} E_{ab}(-1)E_{ba}(-1)\mathbf{1} \right) + \frac{1}{2}I(-2)\mathbf{1},$$

where I is the identity matrix. The rank of $V_{\mathfrak{gl}_N}$ is $-2N$ (see [2] for details).

The third tensor factor is the universal enveloping vertex algebra for the affine Kac-Moody algebra

$$\widehat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}C_{\mathfrak{g}}$$

at level c. As a vector space $V_{\mathfrak{g}} = U(t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{g}) \otimes \mathbf{1}$ with $\mathbb{C}[t] \otimes \mathfrak{g}$ annihilating the vacuum vector $\mathbf{1}$.

For $g_1, g_2 \in \mathfrak{g}$, the n-th products in this case are:

$$g_1(0)g_2(-1)\mathbf{1} = [g_1, g_2](-1)\mathbf{1},\ g_1(1)g_2(-1)\mathbf{1} = c(g_1|g_2)\mathbf{1},\ g_1(n)g_2(-1)\mathbf{1} = 0 \quad \text{for} \quad n \geq 2.$$

When the level c is non-critical, $c \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, the vertex algebra $V_{\mathfrak{g}}$ has a Virasoro element $\omega^{\mathfrak{g}}$ and its rank is

$$\text{rank } (V_{\mathfrak{g}}) = \frac{c \dim \mathfrak{g}}{c + h^\vee}.$$

To define the last tensor factor, consider the Virasoro Lie algebra $\mathfrak{Vir}$ with the basis $\{L(j), C_{\mathfrak{Vir}} | j \in \mathbb{Z}\}$ and Lie bracket

$$[L(m), L(s)] = (m-s)L(m+s) + \frac{m^3 - m}{12}\delta_{m,-s}C_{\mathfrak{Vir}}, \ \ m, s \in \mathbb{Z},$$

and $C_{\mathfrak{Vir}}$ being a central element.

The vertex algebra $V_{\mathfrak{Vir}}$ is the universal enveloping vertex algebra for the Virasoro Lie algebra where the central element $C_{\mathfrak{Vir}}$ acts as a scalar $-\frac{c \dim \mathfrak{g}}{c+h^\vee}$. It is a highest weight module for the Virasoro algebra in which the operators $L(j)$ with $j \geq -1$ annihilate the highest weight vector $\mathbf{1}$. As a space, it is realized as

$$V_{\mathfrak{Vir}} = U(\mathfrak{Vir}^{(-)}) \otimes \mathbf{1},$$

where $\mathfrak{Vir}^{(-)}$ is the subalgebra spanned by $L(j)$ with $j \leq -2$.

The generator of this vertex algebra is $\omega^{\mathfrak{Vir}} = L(-2)\mathbf{1}$ and the generating field of this vertex algebra is

$$Y(\omega^{\mathfrak{Vir}}, z) = \sum_{j \in \mathbb{Z}} L(j) z^{-j-2}.$$

The Virasoro element of V is the sum of the Virasoro elements of its tensor factors:

$$\omega = \omega^{\mathfrak{Hei}} + \omega^{\mathfrak{gl}_N} + \omega^{\mathfrak{g}} + \omega^{\mathfrak{Vir}}.$$

The rank of the Virasoro tensor factor was chosen in a way to make the total rank of V to be 0.

The n-th products for the rank 0 Virasoro element are:

$$\omega_{(0)}\omega = D\omega, \quad \omega_{(1)}\omega = 2\omega, \quad \omega_{(n)}\omega = 0 \quad \text{for} \quad n \geq 2.$$

We begin the construction of the sheaf $\mathcal{V}$ with its local description. We have fixed a covering of X with open affine sets admitting local parameters. Let U_i be one of these open sets with local parameters $x_1, \ldots, x_N$.

We set

$$\mathcal{V}(U_i) = V \otimes \mathcal{O}_X(U_i). \tag{2.11}$$

The fields $v_p(z)$, $u_q(z)$, $E_{ab}(z)$ are the "chiralizations" of the vector fields, 1-forms and $(1,1)$-tensors on X respectively, and transform under the changes of coordinates (see (2.17) below).

Let us define the vertex algebra structure on the space $V \otimes \mathcal{O}_X(U_i)$. The vertex algebra V is embedded as subalgebra $V \otimes 1$ in $V \otimes \mathcal{O}_X(U_i)$. The state-field correspondence map Y on the elements of V is defined as above, with the only difference that the action of $v_p(0)$ is now defined as

$$v_p(0) = \frac{\partial}{\partial x_p},$$

while $u_p(0)$ still acts as zero.

We define the state-field correspondence map on $\mathbf{1} \otimes \mathcal{O}_X(U_i)$ as

$$Y(\mathbf{1} \otimes f, z) = \sum_{s \in \mathbb{Z}_+^N} \frac{1}{s!} \overline{u}(z)^s \otimes \frac{\partial^s f}{\partial x^s},$$

where $\overline{u}_p(z)$ is an antiderivative of $u_p(z)$:

$$\overline{u}_p(z) = \sum_{j \in \mathbb{Z} \backslash \{0\}} \frac{1}{j} u_p(-j) z^j,$$

and in general,

$$Y(\nu \otimes f, z) =: Y(\nu \otimes 1, z) Y(\mathbf{1} \otimes f, z) :,$$

for $f \in \mathcal{O}_X(U_i)$, $\nu \in V$. Here and throughout the paper we use the multi-index notation, for $s = (s_1, \ldots, s_N) \in \mathbb{Z}_+^N$ we set $s! = s_1! \ldots s_N!$, $\frac{\partial^s}{\partial x^s} = \left(\frac{\partial}{\partial x_1}\right)^{s_1} \ldots \left(\frac{\partial}{\partial x_N}\right)^{s_N}$, $\overline{u}(z)^s = \overline{u}_1(z)^{s_1} \ldots \overline{u}_N(z)^{s_N}$, etc.

Note that for $f, h \in \mathcal{O}_X(U_i)$,

$$Y(\mathbf{1} \otimes fh, z) = Y(\mathbf{1} \otimes f, z) Y(\mathbf{1} \otimes h, z).$$

Proposition 2.1. $V \otimes \mathcal{O}_X(U_i)$ is a vertex algebra.

Proof. We are going to apply the Existence theorem ([5], Theorem 4.5). The infinitesimal translation map D on $V \otimes \mathcal{O}_X(U_i)$ is defined in the following way:

$$D(\nu \otimes f) = D(\nu) \otimes f + \sum_{p=1}^{N} u_p(-1)\nu \otimes \frac{\partial f}{\partial x_p},$$

where $\nu \in V,\ f \in \mathcal{O}_X(U_i)$.

One has to verify that the generating fields for $V \otimes \mathcal{O}_X(U_i)$ are mutually local. The only non-trivial relation is between $v_a(z)$ and $Y(\mathbf{1} \otimes f, z)$. It is easy to check that

$$[v_a(z_1), \overline{u}_b(z_2)] = \delta_{ab} z_1^{-1} \sum_{j \in \mathbb{Z} \setminus \{0\}} \left(\frac{z_2}{z_1}\right)^j,$$

which implies

$$[v_a(z_1), \overline{u}(z_2)^s] = s_a \overline{u}(z_2)^{s-\epsilon_a} z_1^{-1} \sum_{j \in \mathbb{Z} \setminus \{0\}} \left(\frac{z_2}{z_1}\right)^j,$$

while

$$[v_a(z_1), \mathrm{Id} \otimes f] = z_1^{-1} \mathrm{Id} \otimes \frac{\partial f}{\partial x_a}.$$

In the above, ϵ_a is an element of $\mathbb{Z}^N$ with 1 in position a and zeros elsewhere. Combining these, we get

$$\begin{aligned} &[v_a(z_1), Y(\mathbf{1} \otimes f, z_2)] \\ &= \sum_{s \in \mathbb{Z}_+^N} \frac{s_a}{s!} \overline{u}(z)^{s-\epsilon_a} \otimes \frac{\partial^s f}{\partial x^s} z_1^{-1} \sum_{j \in \mathbb{Z} \setminus \{0\}} \left(\frac{z_2}{z_1}\right)^j + \sum_{s \in \mathbb{Z}_+^N} \frac{1}{s!} \overline{u}(z)^s \otimes z_1^{-1} \frac{\partial^s}{\partial x^s} \frac{\partial f}{\partial x_a} \\ &= \sum_{s \in \mathbb{Z}_+^N} \frac{1}{s!} \overline{u}(z)^s \otimes \frac{\partial^s}{\partial x^s} \frac{\partial f}{\partial x_a} z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) = Y(\mathbf{1} \otimes \frac{\partial f}{\partial x_a}, z_2) z_1^{-1} \delta\left(\frac{z_2}{z_1}\right), \end{aligned} \tag{2.12}$$

which implies locality. Verification of other conditions of the Existence theorem is straightforward.

As a corollary of (2.12) we obtain

$$[v_a(k), f_{(m)}] = \left(\frac{\partial f}{\partial x_a}\right)_{(m+k)}. \tag{2.13}$$

Also note that

$$u_p(-1)\mathbf{1} = Dx_p. \tag{2.14}$$

Clearly, for any open subset $U \subset U_i$ we have a natural restriction homomorphism of vertex algebras

$$\mathrm{res}_{U_i,U}: \quad V \otimes \mathcal{O}_X(U_i) \to V \otimes \mathcal{O}_X(U).$$

Over the intersection $U_i \cap U_j$ we have defined two vertex algebra structures on $V \otimes \mathcal{O}_X(U_i \cap U_j)$. We are now going to construct the gluing isomorphism between these structures. Let $x_1, \dots, x_N$ be the local parameters on U_i and $\tilde{x}_1, \dots, \tilde{x}_N$ be the local parameters on U_j. In order to emphasize the fact that the fields $v_p(z)$, $u_q(z)$, etc., transform under the change of the local parameters, we will denote them as $\tilde{v}_p(z)$, $\tilde{u}_q(z)$, etc., when working in coordinates $\tilde{x}_1, \dots, \tilde{x}_N$.

Under the coordinate changes, the partial derivative operators $\partial_k = \frac{\partial}{\partial x_k}$, $\tilde{\partial}_s = \frac{\partial}{\partial \tilde{x}_s}$ transform in the standard way:

$$\tilde{\partial}_a = (\tilde{\partial}_a x_p)\partial_p, \quad \partial_b = (\partial_b \tilde{x}_s)\tilde{\partial}_s.$$

Throughout this paper we use Einstein notations on summation over repeated indices.

The product of the jacobians of the coordinate changes is the identity:

$$\delta_{ab} = \partial_b x_a = (\partial_b \tilde{x}_s)(\tilde{\partial}_s x_a), \quad \delta_{ab} = \tilde{\partial}_a \tilde{x}_b = (\tilde{\partial}_a x_p)(\partial_p \tilde{x}_b).$$

Further differentiating the last equality, we get

$$0 = \partial_q \delta_{ab} = (\partial_q \tilde{\partial}_a x_p)(\partial_p \tilde{x}_b) + (\tilde{\partial}_a x_p)(\partial_q \partial_p \tilde{x}_b) = (\partial_q \tilde{\partial}_a x_p)(\partial_p \tilde{x}_b) + \tilde{\partial}_a \partial_q \tilde{x}_b. \tag{2.15}$$

The operators ∂_r and $\tilde{\partial}_p$ do not commute and their commutator may be expressed as follows:

$$[\partial_r, \tilde{\partial}_p] = (\partial_r \tilde{\partial}_p x_q)\partial_q = -(\tilde{\partial}_p \partial_r \tilde{x}_s)\tilde{\partial}_s. \tag{2.16}$$

We now define the gluing isomorphism $\Phi_{ij} : V \otimes \mathcal{O}_X(U_j \cap U_i) \to V \otimes \mathcal{O}_X(U_i \cap U_j)$, where in the first copy we use local parameters $\tilde{x}_1, \dots, \tilde{x}_N$, while in the second we use $x_1, \dots, x_N$. This map will be first defined on the generators of this vertex algebra and then extended as a vertex algebra homomorphism. In order to simplify notations we will drop from now on the tensor product symbol, as well as the symbol $\mathbf{1}$, and write $v_a(-1)f$ instead of $v_a(-1)\mathbf{1} \otimes f$, etc.

$$\begin{aligned}
&\Phi_{ij}(f) = f, \quad f \in \mathcal{O}_X(U_i \cap U_j),\\
&\Phi_{ij}(\tilde{u}_a(-1)\mathbf{1}) = u_p(-1)\partial_p \tilde{x}_a,\\
&\Phi_{ij}(\tilde{v}_a(-1)\mathbf{1}) = v_p(-1)\tilde{\partial}_a x_p + E_{sp}(-1)\partial_s \tilde{\partial}_a x_p,\\
&\Phi_{ij}(\tilde{E}_{ab}(-1)\mathbf{1}) = E_{sp}(-1)(\partial_s \tilde{x}_a)(\tilde{\partial}_b x_p) + u_s(-1)\tilde{\partial}_b \partial_s \tilde{x}_a\\
&\qquad\qquad\qquad = E_{sp}(-1)(\partial_s \tilde{x}_a)(\tilde{\partial}_b x_p) - (\tilde{\partial}_b x_p)_{(-2)}(\partial_p \tilde{x}_a),\\
&\Phi_{ij}(g(-1)\mathbf{1}) = g(-1)\mathbf{1}, \quad g \in \mathfrak{g},\\
&\Phi_{ij}(\tilde{\omega}^{\mathfrak{Vir}}) = \omega^{\mathfrak{Vir}}.
\end{aligned} \tag{2.17}$$

Lemma 2.2. The map $\Phi = \Phi_{ij}$ extends to a homomorphism of vertex algebras.

Proof. We need to show that the fields corresponding to the images of the generators satisfy the same relations as the original generating fields. Taking into account the commutator formula (2.2), we need to show that

$$\Phi(a)_{(n)}\Phi(b) = \Phi(a_{(n)}b)$$

for any pair a, b of the generators and all $n \geq 0$.

The relations between $\Phi(v_p(-1)\mathbf{1})$ and $\Phi(u_q(-1)\mathbf{1})$ have been established in ([9], Theorem 3.7). Let us verify other relations.

Let us check that

$$\Phi(\tilde{v}_m(-1)\mathbf{1})_{(0)}\Phi(f) = \Phi(\tilde{v}_m(0)f).$$

In the proof of this lemma we will use extensively the Borcherds' identity (2.4):

$$\Phi(\tilde{v}_m(-1)\mathbf{1})_{(0)}\Phi(f) = \left(v_p(-1)\tilde{\partial}_m x_p + E_{sp}(-1)\partial_s\tilde{\partial}_m x_p\right)_{(0)} f$$

$$= \left(\tilde{\partial}_m x_p\right)_{(-1)} v_p(0)f = (\tilde{\partial}_m x_p)\partial_p f = \tilde{\partial}_m f = \Phi(\tilde{v}_m(0)f).$$

The relations

$$\Phi(\tilde{v}_m(-1)\mathbf{1})_{(n)}\Phi(f) = 0$$

for $n \geq 1$ follow from the degree considerations since the degree of the left hand side is $-n$.

Let us show now that

$$\Phi(\tilde{v}_a(-1)\mathbf{1})_{(n)}\Phi(\tilde{E}_{bc}(-1)\mathbf{1}) = 0 \quad \text{for} \quad n \geq 0.$$

We have

$$\Phi(\tilde{v}_a(-1)\mathbf{1})_{(0)}\Phi(\tilde{E}_{bc}(-1)\mathbf{1})$$

$$= \left(v_p(-1)\tilde{\partial}_a x_p + E_{sp}(-1)\partial_s\tilde{\partial}_a x_p\right)_{(0)} \left(E_{qr}(-1)(\partial_q\tilde{x}_b)(\tilde{\partial}_c x_r) + u_q(-1)\tilde{\partial}_c\partial_q\tilde{x}_b\right)$$

$$= (\tilde{\partial}_a x_p)_{(-1)} v_p(0) E_{qr}(-1)(\partial_q\tilde{x}_b)(\tilde{\partial}_c x_r) + (\partial_s\tilde{\partial}_a x_p)_{(-1)} E_{sp}(0) E_{qr}(-1)(\partial_q\tilde{x}_b)(\tilde{\partial}_c x_r)$$

$$+(\tilde{\partial}_a x_p)_{(-1)} v_p(0) u_q(-1)\tilde{\partial}_c\partial_q\tilde{x}_b + (\tilde{\partial}_a x_p)_{(-2)} v_p(1) u_q(-1)\tilde{\partial}_c\partial_q\tilde{x}_b$$

$$+(\partial_s\tilde{\partial}_a x_p)_{(-2)} E_{sp}(1) E_{qr}(-1)(\partial_q\tilde{x}_b)(\tilde{\partial}_c x_r)$$

$$= E_{qr}(-1)(\tilde{\partial}_a x_p)\partial_p((\partial_q\tilde{x}_b)(\tilde{\partial}_c x_r)) + E_{sr}(-1)(\partial_s\tilde{\partial}_a x_p)(\partial_p\tilde{x}_b)(\tilde{\partial}_c x_r)$$

$$-E_{qp}(-1)(\partial_s\tilde{\partial}_a x_p)(\partial_q\tilde{x}_b)(\tilde{\partial}_c x_s)$$

$$+u_q(-1)(\tilde{\partial}_a x_p)(\partial_p\tilde{\partial}_c\partial_q\tilde{x}_b) + u_q(-1)(\partial_q\tilde{\partial}_a x_p)(\tilde{\partial}_c\partial_p\tilde{x}_b) + u_q(-1)(\partial_q\partial_s\tilde{\partial}_a x_p)(\partial_p\tilde{x}_b)(\tilde{\partial}_c x_s)$$

$$= E_{qr}(-1)\tilde{\partial}_a((\partial_q\tilde{x}_b)(\tilde{\partial}_c x_r)) + E_{qr}(-1)(\partial_q\tilde{\partial}_a x_p)(\partial_p\tilde{x}_b)(\tilde{\partial}_c x_r) - E_{qr}(-1)(\tilde{\partial}_c\tilde{\partial}_a x_r)(\partial_q\tilde{x}_b)$$

$$+u_q(-1)\tilde{\partial}_a\tilde{\partial}_c\partial_q\tilde{x}_b + u_q(-1)(\partial_q\tilde{\partial}_a x_p)(\tilde{\partial}_c\partial_p\tilde{x}_b) + u_q(-1)(\tilde{\partial}_c\partial_q\tilde{\partial}_a x_p)(\partial_p\tilde{x}_b) = 0.$$

In the above we used the relation (2.15).

$$\Phi(\tilde{v}_a(-1)\mathbf{1})_{(1)}\Phi(\tilde{E}_{bc}(-1)\mathbf{1})$$
$$= \Big(v_p(-1)\tilde{\partial}_a x_p + E_{sp}(-1)\partial_s\tilde{\partial}_a x_p\Big)_{(1)} \Big(E_{qr}(-1)(\partial_q\tilde{x}_b)(\tilde{\partial}_c x_r) + u_q(-1)\tilde{\partial}_c\partial_q\tilde{x}_b\Big)$$
$$= (\tilde{\partial}_a x_p)_{(-1)} v_p(1) u_q(-1)\tilde{\partial}_c\partial_q\tilde{x}_b + (\partial_s\tilde{\partial}_a x_p)_{(-1)} E_{sp}(1)E_{qr}(-1)(\partial_q\tilde{x}_b)(\tilde{\partial}_c x_r)$$
$$= (\tilde{\partial}_a x_p)(\tilde{\partial}_c\partial_p\tilde{x}_b) + (\partial_s\tilde{\partial}_a x_p)(\partial_p\tilde{x}_b)(\tilde{\partial}_c x_s) = \tilde{\partial}_c\Big((\tilde{\partial}_a x_p)(\partial_p\tilde{x}_b)\Big) = 0.$$

The relations
$$\Phi(\tilde{v}_a(-1)\mathbf{1})_{(n)}\Phi(\tilde{E}_{bc}(-1)\mathbf{1}) = 0 \quad \text{for} \quad n \geq 2$$
follow trivially from the degree considerations (2.3).

Let us now consider the analogues of (2.10):

$$\Phi(\tilde{E}_{ab}(-1)\mathbf{1})_{(0)}\Phi(\tilde{E}_{cd}(-1)\mathbf{1})$$
$$= \Big(E_{sp}(-1)(\partial_s\tilde{x}_a)(\tilde{\partial}_b x_p) + u_s(-1)\tilde{\partial}_b\partial_s\tilde{x}_a\Big)_{(0)} \Big(E_{qr}(-1)(\partial_q\tilde{x}_c)(\tilde{\partial}_d x_r) + u_q(-1)\tilde{\partial}_d\partial_q\tilde{x}_c\Big)$$
$$= \big((\partial_s\tilde{x}_a)(\tilde{\partial}_b x_p)\big)_{(-1)} E_{sp}(0)E_{qr}(-1)(\partial_q\tilde{x}_c)(\tilde{\partial}_d x_r)$$
$$+\big((\partial_s\tilde{x}_a)(\tilde{\partial}_b x_p)\big)_{(-2)} E_{sp}(1)E_{qr}(-1)(\partial_q\tilde{x}_c)(\tilde{\partial}_d x_r)$$
$$= E_{sr}(-1)(\partial_s\tilde{x}_a)(\tilde{\partial}_b x_p)(\partial_p\tilde{x}_c)(\tilde{\partial}_d x_r) - E_{qp}(-1)(\partial_s\tilde{x}_a)(\tilde{\partial}_b x_p)(\partial_q\tilde{x}_c)(\tilde{\partial}_d x_s)$$
$$+u_j(-1)\partial_j((\partial_s\tilde{x}_a)(\tilde{\partial}_b x_p))(\partial_p\tilde{x}_c)(\tilde{\partial}_d x_s)$$
$$= \delta_{bc}E_{sr}(-1)(\partial_s\tilde{x}_a)(\tilde{\partial}_d x_r) - \delta_{ad}E_{qp}(-1)(\partial_q\tilde{x}_c)(\tilde{\partial}_b x_p) - \delta_{ad}u_j(-1)\tilde{\partial}_b\partial_j\tilde{x}_c + \delta_{bc}u_j(-1)\tilde{\partial}_d\partial_j\tilde{x}_a$$
$$= \delta_{bc}\Phi(\tilde{E}_{ad}(-1)\mathbf{1}) - \delta_{ad}\Phi(\tilde{E}_{cb}(-1)\mathbf{1}),$$

and

$$\Phi(\tilde{E}_{ab}(-1)\mathbf{1})_{(1)}\Phi(\tilde{E}_{cd}(-1)\mathbf{1}) = \big((\partial_s\tilde{x}_a)(\tilde{\partial}_b x_p)\big)_{(-1)} E_{sp}(1)E_{qr}(-1)(\partial_q\tilde{x}_c)(\tilde{\partial}_d x_r)$$
$$= (\partial_s\tilde{x}_a)(\tilde{\partial}_b x_p)(\partial_p\tilde{x}_c)(\tilde{\partial}_d x_s) = \delta_{ad}\delta_{bc}.$$

Verification of the remaining relations is trivial.

Lemma 2.3. Over the triple intersection $U_i \cap U_j \cap U_k$ we have $\Phi_{ij} \circ \Phi_{jk} = \Phi_{ik}$.

Proof. Let us denote $\hat{x}_1, \ldots, \hat{x}_N$ the local parameters on U_k. We need to verify the equality $\Phi_{ij} \circ \Phi_{jk} = \Phi_{ik}$ on the generators of the vertex algebra $V \otimes \mathcal{O}_X(U_k \cap U_j \cap U_i)$. For the functions $f \in \mathcal{O}_X(U_k \cap U_j \cap U_i)$ this equality holds since the functions do not transform under the coordinate changes. Let us carry out the calculations for the generators of V.

$$\Phi_{ij}\left(\Phi_{jk}(\hat{u}_a(-1)\mathbf{1})\right) = \Phi_{ij}(\tilde{u}_p(-1)\tilde{\partial}_p\hat{x}_a)$$

$$= (u_s(-1)\partial_s\tilde{x}_p)_{(-1)}\tilde{\partial}_p\hat{x}_a = u_s(-1)(\partial_s\tilde{x}_p)(\tilde{\partial}_p\hat{x}_a)$$
$$= u_s(-1)\partial_s\hat{x}_a = \Phi_{ik}(\hat{u}_a(-1)\mathbf{1}).$$

$$\Phi_{ij}\left(\Phi_{jk}(\hat{v}_a(-1)\mathbf{1})\right) = \Phi_{ij}\left(\tilde{v}_p(-1)\hat{\partial}_a\tilde{x}_p + \tilde{E}_{sp}(-1)\tilde{\partial}_s\hat{\partial}_a\tilde{x}_p\right)$$
$$= \left(v_q(-1)\tilde{\partial}_p x_q\right)_{(-1)}\hat{\partial}_a\tilde{x}_p + \left(E_{rq}(-1)\partial_r\tilde{\partial}_p x_q\right)_{(-1)}\hat{\partial}_a\tilde{x}_p$$
$$+\left(E_{rq}(-1)(\partial_r\tilde{x}_s)(\tilde{\partial}_p x_q)\right)_{(-1)}\tilde{\partial}_s\hat{\partial}_a\tilde{x}_p + \left(u_r(-1)\tilde{\partial}_p\partial_r\tilde{x}_s\right)_{(-1)}\tilde{\partial}_s\hat{\partial}_a\tilde{x}_p$$
$$= v_q(-1)(\tilde{\partial}_p x_q)(\hat{\partial}_a\tilde{x}_p) + (\tilde{\partial}_p x_q)_{(-2)}v_q(0)\hat{\partial}_a\tilde{x}_p + E_{rq}(-1)(\partial_r\tilde{\partial}_p x_q)(\hat{\partial}_a\tilde{x}_p)$$
$$+E_{rq}(-1)(\partial_r\tilde{x}_s)(\tilde{\partial}_p x_q)(\tilde{\partial}_s\hat{\partial}_a\tilde{x}_p) + u_r(-1)(\tilde{\partial}_p\partial_r\tilde{x}_s)(\tilde{\partial}_s\hat{\partial}_a\tilde{x}_p)$$
$$= v_q(-1)\hat{\partial}_a x_q + u_r(-1)(\partial_r\tilde{\partial}_p x_q)(\partial_q\hat{\partial}_a\tilde{x}_p) + u_r(-1)(\tilde{\partial}_p\partial_r\tilde{x}_s)(\tilde{\partial}_s\hat{\partial}_a\tilde{x}_p)$$
$$+E_{rq}(-1)(\partial_r\tilde{\partial}_p x_q)(\hat{\partial}_a\tilde{x}_p) + E_{rq}(-1)(\tilde{\partial}_p x_q)(\partial_r\hat{\partial}_a\tilde{x}_p)$$
$$= v_q(-1)\hat{\partial}_a x_q + E_{rq}(-1)\partial_r\hat{\partial}_a x_q = \Phi_{ik}(\hat{v}_a(-1)\mathbf{1}).$$

In the above one can use (2.16) to see that the terms with $u_r(-1)$ cancel.

$$\Phi_{ij}\left(\Phi_{jk}(\hat{E}_{ab}(-1)\mathbf{1})\right) = \Phi_{ij}\left(\tilde{E}_{sp}(-1)(\tilde{\partial}_s\hat{x}_a)(\hat{\partial}_b\tilde{x}_p) + \tilde{u}_s(-1)\hat{\partial}_b\tilde{\partial}_s\hat{x}_a\right)$$
$$= \left(E_{rq}(-1)(\partial_r\tilde{x}_s)(\tilde{\partial}_p x_q) + u_r(-1)\tilde{\partial}_p\partial_r\tilde{x}_s\right)_{(-1)}(\tilde{\partial}_s\hat{x}_a)(\hat{\partial}_b\tilde{x}_p) + (u_r(-1)\partial_r\tilde{x}_s)_{(-1)}\hat{\partial}_b\tilde{\partial}_s\hat{x}_a$$
$$= E_{rq}(-1)(\partial_r\tilde{x}_s)(\tilde{\partial}_p x_q)(\tilde{\partial}_s\hat{x}_a)(\hat{\partial}_b\tilde{x}_p) + u_r(-1)(\hat{\partial}_b\partial_r\tilde{x}_s)(\tilde{\partial}_s\hat{x}_a) + u_r(-1)(\partial_r\tilde{x}_s)(\hat{\partial}_b\tilde{\partial}_s\hat{x}_a)$$
$$= E_{rq}(-1)(\partial_r\hat{x}_a)(\hat{\partial}_b x_q) + u_r(-1)\hat{\partial}_b\partial_r\hat{x}_a = \Phi_{ik}(\hat{E}_{ab}(-1)\mathbf{1}).$$

Corollary 2.4. The map Φ_{ij} is an isomorphism.
Proof. Setting $k = i$ in the previous Lemma, we get $\Phi_{ij}^{-1} = \Phi_{ji}$.

As a result of our construction, we get the following
Theorem 2.5. The local data (2.11) together with the gluing maps Φ_{ij} define a sheaf $\mathcal{V}$ of vertex algebras over X.

3. A sheaf of modules of chiral tensor fields.

In this section we will construct sheaves of modules for the sheaf $\mathcal{V}$ of vertex algebras.

First let us discuss the local situation. Let U be an open set contained in U_i, and let $M = \bigoplus_{n=0}^{\infty} M_n$ be a module for the vertex Lie algebra $\mathcal{V}(U)$. The module M is a module for the Lie algebra

$$\mathfrak{s} = \mathfrak{Hei} \oplus \widehat{\mathfrak{gl}}_N \oplus \widehat{\mathfrak{g}} \oplus \mathfrak{Vir}.$$

This Lie algebra has a natural $\mathbb{Z}$-grading $\mathfrak{s} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{s}_n$ and a triangular decomposition $\mathfrak{s} = \mathfrak{s}_+ \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_-$ associated with this grading.

Using the standard methods of vertex algebras, we get the following

Proposition 3.1. (i) M_0 is a module for the commutative algebra $\mathcal{O}_X(U)$ with the action

$$fm = f_{(-1)}m, \quad f \in \mathcal{O}_X(U), m \in M_0.$$

(ii) M_0 is a module for the Lie algebra $\mathfrak{s}_0$. The actions of $\mathcal{O}_X(U)$ and $\mathfrak{s}_0$ on M_0 are compatible in the following way:

$$v_p(0)fm - fv_p(0)m = (\partial_p f)m,$$

while the remaining basis elements of $\mathfrak{s}_0$ commute with $\mathcal{O}_X(U)$.

(iii) For $f \in \mathcal{O}_X(U)$ introduce the operator $T(f,z)$ on the space $U(\mathfrak{s}_-)M_0$:

$$T(f,z)ym = \sum_{k\in\mathbb{Z}_+^N} \frac{1}{k!}\overline{u}(z)^k y(\partial^k f)m,$$

where $y \in U(\mathfrak{s}_-)$, $m \in M_0$. If M is generated by M_0 as a $\mathcal{V}(U)$-module then

$$Y(f,z)ym = T(f,z)ym.$$

(iv) If M is generated by M_0 as a $\mathcal{V}(U)$-module then $M = U(\mathfrak{s}_-)M_0$.

Proof. Part (i) follows from the relation

$$Y(f,z)Y(h,z) = Y(fh,z)$$

and the fact that $f_{(n)}m = 0$ for $n \geq 0$, $m \in M_0$. Part (ii) is a consequence of the statement that M is a graded $\mathfrak{s}$-module and (2.13).

We shall prove part (iii) by induction on the degree of y. For the basis of induction, $\deg(y) = 0$, so that $y = 1$, we need to show that

$$Y(f,z)m = \sum_{k\in\mathbb{Z}_+^N} \frac{1}{k!}\overline{u}(z)^k (\partial^k f)m, \tag{3.1}$$

for $m \in M_0$. It is clear that both sides involve only non-negative powers of z. Let us reason by induction on n the equality of terms up to z^n in (3.1). The coefficients at z^0 in (3.1) coincide by the definition of the action of $\mathcal{O}_X(U)$ on M_0. The equality of the coefficients at z^{n+1} in (3.1) will follow from the equality of z^n terms in

$$\frac{\partial}{\partial z}Y(f,z)m = \frac{\partial}{\partial z}T(f,z)m. \tag{3.2}$$

Note that

$$\frac{\partial}{\partial z}Y(f,z)m = \sum_{p=1}^{N} u_p(z)Y(\partial_p f, z)m,$$

and also

$$\frac{\partial}{\partial z} T(f,z)m = \sum_{p=1}^{N} u_p(z) T(\partial_p f, z)m.$$

Since in the above terms in $u_p(z)$ with the negative powers of z act trivially (note that $u_p(0)$ acts trivially on M since $u_p(z) = \frac{d}{dz} Y(x_p, z)$), the equality of z^n terms in (3.2) follows from the induction assumption.

Let us now complete the induction on the degree of $y \in U(\mathfrak{s}_-)$. Suppose $y = y'y''$, where $y' \in \mathfrak{s}_-$, $y'' \in U(\mathfrak{s}_-)$. If y' is one of $u_a(-n)$, $E_{ab}(-n)$ or $L(-n)$, then both operators $Y(f,z)$ and $T(f,z)$ commute with y' and we get

$$Y(f,z)y'y''m = y'Y(f,z)y''m = y'T(f,z)y''m = T(f,z)y'y''m.$$

The only non-trivial case is $y' = v_a(-n)$. However it follows from (2.12) that

$$[v_a(-n), Y(f,z)] = z^{-n} Y(\partial_a f, z),$$

and also

$$[v_a(-n), T(f,z)] = z^{-n} T(\partial_a f, z),$$

thus

$$Y(f,z)v_a(-n)y''m = v_a(-n)Y(f,z)y''m - z^{-n}Y(\partial_a f, z)y''m$$

$$= v_a(-n)T(f,z)y''m - z^{-n}T(\partial_a f, z)y''m = T(f,z)v_a(-n)y''m.$$

Part (iv) follows immediately from (iii).

Corollary 3.2. Let M', M'' be two $\mathcal{V}(U)$-modules, $M' = \mathop{\oplus}\limits_{n=0}^{\infty} M'_n$, $M'' = \mathop{\oplus}\limits_{n=0}^{\infty} M''_n$, that are generated by M'_0 and M''_0 respectively. Let $\psi : M' \to M''$ be a homomorphism of $\mathfrak{s}$-modules preserving the grading, such that $\psi : M'_0 \to M''_0$ is a homomorphism of $\mathcal{O}_X(U)$-modules. Then ψ is a homomorphism of $\mathcal{V}(U)$-modules.

The previous Proposition essentially tells us how the vertex algebra $\mathcal{V}(U)$ may act on its modules. Let us now give an explicit construction.

Let W be a rational finite-dimensional simple $GL_N(\mathbb{C})$-module and let $M_{\mathfrak{gl}_N}(W)$ be the generalized Verma module at level 1 for $\widehat{\mathfrak{gl}}_N$, induced from the $\mathfrak{gl}_N$-module W. Let $M_{\mathfrak{g}}(S)$ be the generalized Verma module for $\widehat{\mathfrak{g}}$ at level c, induced from an irreducible $\mathfrak{g}$-module S, and $M_{\mathfrak{Vir}}(h)$ be the Verma module for the Virasoro Lie algebra of rank $-\frac{c \dim \mathfrak{g}}{c+h^\vee}$ with the highest weight vector v_h such that $L(0)v_h = hv_h$, $h \in \mathbb{C}$.

For an open set $U \subset U_i$, the space

$$\mathcal{M}(U) = V_{\mathfrak{Hei}} \otimes M_{\mathfrak{gl}_N}(W) \otimes M_{\mathfrak{g}}(S) \otimes M_{\mathfrak{Vir}}(h) \otimes \mathcal{O}_X(U)$$

has a natural structure of a module for the vertex algebra $\mathcal{V}(U)$.

In order to construct the sheaf of modules $\mathcal{M}$, we need to define the gluing isomorphisms

$$\Psi_{ij} : \mathcal{M}(U_j \cap U_i) \to \mathcal{M}(U_i \cap U_j).$$

Both $\mathcal{M}(U_j \cap U_i)$ and $\mathcal{M}(U_i \cap U_j)$ are the modules for the vertex algebra $\mathcal{V}(U_j \cap U_i)$, where the action on the second module is defined via the isomorphism Φ_{ij} of vertex algebras. The isomorphism Ψ_{ij} of modules that we need to construct must be an isomorphism of $\mathcal{V}(U_j \cap U_i)$-modules. The map Ψ_{ij} will be constructed using Corollary 3.2. Note that both $\mathcal{M}' = \mathcal{M}(U_j \cap U_i)$ and $\mathcal{M}'' = \mathcal{M}(U_i \cap U_j)$ are $\mathbb{Z}$-graded and their top components are:

$$\mathcal{M}'_0 = W \otimes S \otimes v_h \otimes \mathcal{O}_X(U_j \cap U_i),$$

$$\mathcal{M}''_0 = W \otimes S \otimes v_h \otimes \mathcal{O}_X(U_i \cap U_j).$$

We first construct the map $\Psi_{ij} : \mathcal{M}'_0 \to \mathcal{M}''_0$, which is an isomorphism of $\mathcal{O}_X(U_j \cap U_i)$-modules. Note that the jacobian matrix $J^{(ij)} = J = (\partial_r \tilde{x}_s) E_{rs}$ is an element of $GL_N(\mathcal{O}_X(U_j \cap U_i))$. We set

$$\Psi_{ij}(\tilde{w} \otimes s \otimes v_h \otimes f) = (J^{(ij)} \tilde{w}) \otimes s \otimes v_h \otimes f. \tag{3.3}$$

We claim that this is a homomorphism of $\mathfrak{s}_0$-modules. Indeed, the action of $\mathfrak{g}$ on both $\mathcal{M}'_0$ and $\mathcal{M}''_0$ is the natural action on S, while $L(0)$ acts as multiplication by h on both spaces. To see that $\Psi_{ij} : \mathcal{M}'_0 \to \mathcal{M}''_0$ is a homomorphism of $\mathfrak{gl}_N$-modules, we need to check that

$$\Phi_{ij}(\tilde{E}_{ab}(0)) \Psi_{ij}(\tilde{w} \otimes s \otimes v_h \otimes f) = \Psi_{ij}(\tilde{E}_{ab}(0) \tilde{w} \otimes s \otimes v_h \otimes f).$$

Applying (2.17), we see that the left hand side equals

$$E_{sp}(0) J \tilde{w} \otimes s \otimes v_h \otimes (\partial_s \tilde{x}_a)(\tilde{\partial}_b x_p) f,$$

and to compute the right hand side we use the connection between the action of the group GL_N and its Lie algebra $\mathfrak{gl}_N$ on W:

$$J E_{ab}(0) \tilde{w} \otimes s \otimes v_h \otimes f = (J E_{ab}(0) J^{-1}) J \tilde{w} \otimes s \otimes v_h \otimes f$$

$$= E_{sp}(0) J \tilde{w} \otimes s \otimes v_h \otimes (\partial_s \tilde{x}_a)(\tilde{\partial}_b x_p) f.$$

Since $u_a(0)$ acts on both $\mathcal{M}'_0$ and $\mathcal{M}''_0$ trivially, the last thing to check is the equality

$$\Phi_{ij}(\tilde{v}_a(0)) \Psi_{ij}(\tilde{w} \otimes s \otimes v_h \otimes f) = \Psi_{ij}(\tilde{v}_a(0) \tilde{w} \otimes s \otimes v_h \otimes f). \tag{3.4}$$

In the right hand side $\tilde{v}_a(0)$ acts as $\tilde{\partial}_a$, which gives

$$J \tilde{w} \otimes s \otimes v_h \otimes \tilde{\partial}_a f,$$

while in the left hand side $\Phi_{ij}(\tilde{v}_a(0))$ acts as $\tilde{\partial}_a + E_{sp}(0) \partial_s \tilde{\partial}_a x_p$. The left hand side then becomes

$$J \tilde{w} \otimes s \otimes v_h \otimes \tilde{\partial}_a f + (\tilde{\partial}_a J) \tilde{w} \otimes s \otimes v_h \otimes f + E_{sp}(0) J \tilde{w} \otimes s \otimes v_h \otimes (\partial_s \tilde{\partial}_a x_p) f.$$

In order to evaluate the action of $\tilde{\partial}_a J$ we note that $(\tilde{\partial}_a J)J^{-1}$ belongs to the Lie algebra $\mathfrak{gl}_N$, and we get

$$\tilde{\partial}_a J = ((\tilde{\partial}_a J)J^{-1})J = (\tilde{\partial}_a \partial_s \tilde{x}_k)(\tilde{\partial}_k x_p) E_{sp}(0) J.$$

Since $(\tilde{\partial}_a \partial_s \tilde{x}_k)(\tilde{\partial}_k x_p) + \partial_s \tilde{\partial}_a x_p = 0$, we establish (3.4).

Having established the homomorphism $\Psi_{ij} : \mathcal{M}'_0 \to \mathcal{M}''_0$ as both $\mathfrak{s}_0$- and $\mathcal{O}_X(U_j \cap U_i)$-modules, we note that $\mathcal{M}'$ is a generalized Verma module for the Lie algebra $\mathfrak{s}$, generated by $\mathcal{M}'_0$,

$$\mathcal{M}' = U(\mathfrak{s}_-) \otimes \mathcal{M}'_0,$$

thus, Ψ_{ij} extends uniquely to a homomorphism

$$\Psi_{ij} : \mathcal{M}' \to \mathcal{M}''$$

of $\mathfrak{s}$-modules. By Corollary 3.2, this is a homomorphism of modules for the vertex algebra $\mathcal{V}(U_j \cap U_i)$. One can immediately see that $\Psi_{ji} \circ \Psi_{ij}$ is the identity map on $\mathcal{M}(U_j \cap U_i)$, so Ψ_{ij} is in fact an isomorphism of modules.

The $\mathcal{V}(U_i)$-module $\mathcal{M}(U_i)$ has a unique maximal submodule that trivially intersects with $\mathcal{M}_0(U_i)$. The quotient module $\mathcal{L}(U_i)$ can be written as a tensor product

$$\mathcal{L}(U_i) = V_{\mathfrak{Hei}} \otimes L_{\mathfrak{gl}_N}(W) \otimes L_{\mathfrak{g}}(S) \otimes L_{\mathfrak{Vir}}(h) \otimes \mathcal{O}_X(U_i),$$

where $L_{\mathfrak{gl}_N}(W)$, $L_{\mathfrak{g}}(S)$ and $L_{\mathfrak{Vir}}(h)$ are the simple quotients of the corresponding $\widehat{\mathfrak{gl}}_N$-, $\widehat{\mathfrak{g}}$- and Virasoro modules. It is clear that taking this quotient is compatible with the coordinate change map Ψ_{ij}, and we obtain a sheaf $\mathcal{L}$ of modules for $\mathcal{V}$.

We established the following

Theorem 3.3. Let $L_{\mathfrak{gl}_N}$ be an irreducible highest weight module for the Lie algebra $\widehat{\mathfrak{gl}}_N$ at level 1, such that its $\mathfrak{gl}_N$-submodule W generated the the highest weight vector of $L_{\mathfrak{gl}_N}$ is a finite-dimensional rational GL_N-module. Let $L_{\mathfrak{g}}$ be an irreducible highest weight module for $\widehat{\mathfrak{g}}$ at level $c \neq 0, -h^\vee$, and let $L_{\mathfrak{Vir}}$ be an irreducible highest weight module for the Virasoro algebra with central charge $-\frac{c \dim \mathfrak{g}}{c+h^\vee}$. There is a sheaf $\mathcal{L}$ of modules for the sheaf $\mathcal{V}$ of vertex algebras, where for an open set U_i with a system of local parameters, the module $\mathcal{L}(U_i)$ is defined as

$$\mathcal{L}(U_i) = V_{\mathfrak{Hei}} \otimes L_{\mathfrak{gl}_N} \otimes L_{\mathfrak{g}} \otimes L_{\mathfrak{Vir}} \otimes \mathcal{O}_X(U_i),$$

and the coordinate change map Ψ_{ij} is defined on the top graded component by (3.3) and extended to $\mathcal{L}(U_j \cap U_i)$ as a homomorphism of $\mathfrak{s}$-modules.

4. Representations of the sheaf $\mathcal{G}$ of Lie algebras.

In this section we are going to show that the sheaf $\mathcal{V}$ of vertex algebras admits an action of the sheaf $\mathcal{G}$ of Lie algebras. As an immediate consequence we get representations of $\mathcal{G}$ on the sheaves of modules $\mathcal{M}$ and $\mathcal{L}$.

Let $U \subset U_i$. For $f \in \mathcal{O}_X(U)$ we set the following formal generating series which coefficients span $\mathcal{G}(U)$:

$$
\begin{aligned}
g(f,z) &= \sum_{j\in\mathbb{Z}} t^j f \otimes g z^{-j-1}, \quad g \in \mathfrak{g}, \\
k_0(f,z) &= \sum_{j\in\mathbb{Z}} t^j f dt z^{-j-1}, \\
k_a(f,z) &= \sum_{j\in\mathbb{Z}} t^j f dx_a z^{-j-1}, \\
d_a(f,z) &= \sum_{j\in\mathbb{Z}} t^j f \partial_a z^{-j-1}, \quad a = 1,\ldots,N, \\
d_0(f,z) &= -\sum_{j\in\mathbb{Z}} t^j f \frac{\partial}{\partial t} z^{-j-1}.
\end{aligned}
$$

The negative sign in the last formula is chosen to conform with the Virasoro algebra conventions.

Theorem 4.1. Let $V = V_{\mathfrak{Hei}} \otimes V_{\mathfrak{gl}_N} \otimes V_{\mathfrak{g}} \otimes V_{\mathfrak{Vir}}$ be a tensor product of vertex operator algebras, where $V_{\mathfrak{gl}_N}$ is the universal enveloping vertex operator algebra for $\widehat{\mathfrak{gl}}_N$ at level 1 and rank $-2N$, $V_{\mathfrak{g}}$ be the universal enveloping algebra for $\widehat{\mathfrak{g}}$ at a non-zero, non-critical level c, and $V_{\mathfrak{Vir}}$ be the universal enveloping Virasoro vertex algebra of rank $-\frac{c \dim \mathfrak{g}}{c+h^\vee}$, so that the total rank of V is zero. Let $\mathcal{V}$ be the corresponding sheaf of vertex algebras on X. There is a representation ρ of the sheaf of Lie algebras

$$
\mathcal{G} = \left(\mathrm{Map}(\widehat{X}, \mathfrak{g}) \oplus \overline{\Omega}^1_{\widehat{X}} \right) \rtimes \mathrm{Vect}(\widehat{X})
$$

on the sheaf of vertex algebras $\mathcal{V}$, given locally by the correspondence:

$$
\rho(g(f,z)) = Y(g(-1)f, z), \tag{4.1}
$$

$$
\rho(k_0(f,z)) = cY(f,z), \tag{4.2}
$$

$$
\rho(k_a(f,z)) = cY(u_a(-1)f, z), \tag{4.3}
$$

$$
\rho(d_a(f,z)) = Y(v_a(-1)f, z) + \sum_{p=1}^{N} Y(E_{pa}(-1)\partial_p f, z), \tag{4.4}
$$

$$
\rho(d_0(f,z)) = Y(\omega_{(-1)} f, z) + \sum_{s,k=1}^{N} Y(u_k(-1)E_{sk}(-1)\partial_s f, z) - \sum_{p=1}^{N} Y(u_p(-2)\partial_p f, z). \tag{4.5}
$$

Proof. We need to prove that everything is well-defined and that the Lie brackets of the vertex operators in the right hand sides of (4.1)-(4.5) match the Lie brackets of the left hand sides.

Note that the relation

$$\frac{d}{dz}Y(f,z) = \sum_{p=1}^{N} Y(u_p(-1)\partial_p f, z)$$

ensures that the elements of $d(\mathbb{C}[t,t^{-1}]\mathcal{O}_X(U_i))$ act trivially.

We need to show that both sides of (4.1)-(4.5) transform in a compatible way under the coordinate changes in $U_i \cap U_j$, i.e.,

$$\Phi \circ \rho = \rho \circ \Theta,$$

where Θ is the coordinate transformation $\Theta:\ \mathcal{G}(U_j \cap U_i) \to \mathcal{G}(U_i \cap U_j)$. For (4.1) and (4.2) this holds trivially. Let us verify this for (4.3)-(4.5).

$$\Phi(\rho(\sum_{j\in\mathbb{Z}} t^j f d\tilde{x}_a z^{-j-1})) = c\Phi(Y(\tilde{u}_a(-1)f,z)) = cY((u_p(-1)\partial_p\tilde{x}_a)_{(-1)}f,z)$$

$$= \rho(\sum_{j\in\mathbb{Z}} t^j f(\partial_p\tilde{x}_a)dx_p z^{-j-1}) = \rho(\Theta(\sum_{j\in\mathbb{Z}} t^j f d\tilde{x}_a z^{-j-1})).$$

Since $\Theta(\sum_{j\in\mathbb{Z}} t^j f\tilde{\partial}_a z^{-j-1}) = \sum_{j\in\mathbb{Z}} t^j(\tilde{\partial}_a x_s)f\partial_s z^{-j-1}$, the verification of compatibility for (4.4) amounts to checking the equality

$$\Phi\big(\tilde{v}_a(-1)f + \tilde{E}_{qa}(-1)\tilde{\partial}_q f\big) = v_s(-1)(\tilde{\partial}_a x_s)f + E_{ks}(-1)\partial_k((\tilde{\partial}_a x_s)f). \tag{4.6}$$

Let us prove this equality:

$$\Phi\big(\tilde{v}_a(-1)f + \tilde{E}_{qa}(-1)\tilde{\partial}_q f\big)$$

$$= \big(v_p(-1)(\tilde{\partial}_a x_p) + E_{ks}(-1)\partial_k\tilde{\partial}_a x_s\big)_{(-1)}f + \big(E_{ks}(-1)(\partial_k\tilde{x}_q)(\tilde{\partial}_a x_s) + u_s(-1)\tilde{\partial}_a\partial_s\tilde{x}_q\big)_{(-1)}\tilde{\partial}_q f$$

$$= (\tilde{\partial}_a x_p)_{(-2)}v_p(0)f + v_p(-1)(\tilde{\partial}_a x_p)f + E_{ks}(-1)(\partial_k\tilde{\partial}_a x_s)f + E_{ks}(-1)(\partial_k\tilde{x}_q)(\tilde{\partial}_a x_s)(\tilde{\partial}_q f)$$

$$+u_s(-1)(\tilde{\partial}_a\partial_s\tilde{x}_q)(\tilde{\partial}_q f)$$

$$= u_s(-1)(\partial_s\tilde{\partial}_a x_p)(\partial_p f) + u_s(-1)(\tilde{\partial}_a\partial_s\tilde{x}_q)(\tilde{\partial}_q f) + v_p(-1)(\tilde{\partial}_a x_p)f$$

$$+E_{ks}(-1)(\partial_k\tilde{\partial}_a x_s)f + E_{ks}(-1)(\tilde{\partial}_a x_s)(\partial_k f)$$

$$= v_p(-1)(\tilde{\partial}_a x_p)f + E_{ks}(-1)\partial_k((\tilde{\partial}_a x_s)f).$$

Here we used (2.16) in the last step.

Finally, for (4.5) we need to show that

$$\Phi\big(\tilde{\omega}_{(-1)}f + \tilde{u}_k(-1)\tilde{E}_{sk}(-1)\tilde{\partial}_s f - \tilde{u}_s(-2)\tilde{\partial}_s f\big)$$

$$= \omega_{(-1)}f + u_b(-1)E_{ab}(-1)\partial_a f - u_a(-2)\partial_a f. \tag{4.7}$$

Lemma 4.2. $\Phi_{ij}(\tilde{\omega}) = \omega$.

Proof. The Virasoro element in $\mathcal{V}(U)$ is the sum of the Virasoro elements for the tensor factors:

$$\omega = \omega^{\mathfrak{Hei}} + \omega^{\mathfrak{g}} + \omega^{\mathfrak{gl}_N} + \omega^{\mathfrak{Vir}}.$$

The coordinate change map Φ does not affect the components $V_{\mathfrak{g}}$ and $V_{\mathfrak{Vir}}$, so

$$\Phi(\tilde{\omega}^{\mathfrak{g}}) = \omega^{\mathfrak{g}}, \;\; \Phi(\tilde{\omega}^{\mathfrak{Vir}}) = \omega^{\mathfrak{Vir}}. \tag{4.8}$$

One can verify that

$$\Phi(\tilde{\omega}^{\mathfrak{Hei}}) = \omega^{\mathfrak{Hei}} - (\partial_k \tilde{x}_p)_{(-3)}(\tilde{\partial}_p x_k) - E_{sm}(-1)(\partial_s \tilde{x}_a)_{(-2)}(\tilde{\partial}_a x_m) \tag{4.9}$$

and

$$\Phi(\tilde{\omega}^{\mathfrak{gl}_N}) = \omega^{\mathfrak{gl}_N} + (\partial_k \tilde{x}_p)_{(-3)}(\tilde{\partial}_p x_k) + E_{sm}(-1)(\partial_s \tilde{x}_a)_{(-2)}(\tilde{\partial}_a x_m). \tag{4.10}$$

Adding (4.8), (4.10) and (4.9) together we get the claim of the Lemma.

Let us check (4.9). Recall that

$$\omega^{\mathfrak{Hei}} = v_p(-1)u_p(-1)\mathbf{1} = v_p(-1)(x_p)_{(-2)}\mathbf{1}.$$

Then

$$\begin{aligned}
\Phi(\tilde{\omega}^{\mathfrak{Hei}}) &= \Phi_{ij}(\tilde{v}_a(-1)(\tilde{x}_a)_{(-2)}\mathbf{1}) \\
&= (v_m(-1)(\tilde{\partial}_a x_m))_{(-1)}(\tilde{x}_a)_{(-2)}\mathbf{1} + (E_{sm}(-1)(\partial_s \tilde{\partial}_a x_m))_{(-1)}(\tilde{x}_a)_{(-2)}\mathbf{1} \\
&= v_m(-1)u_r(-1)(\tilde{\partial}_a x_m)(\partial_r \tilde{x}_a) + (\tilde{\partial}_a x_m)_{(-2)} v_m(0)(\tilde{x}_a)_{(-2)}\mathbf{1} \\
&\quad + (\tilde{\partial}_a x_m)_{(-3)} v_m(1)(\tilde{x}_a)_{(-2)}\mathbf{1} + E_{sm}(-1)u_r(-1)(\partial_s \tilde{\partial}_a x_m)(\partial_r \tilde{x}_a) \\
&= \omega^{\mathfrak{Hei}} + (\tilde{\partial}_a x_m)_{(-2)}(\partial_m \tilde{x}_a)_{(-2)}\mathbf{1} + (\tilde{\partial}_a x_m)_{(-3)}(\partial_m \tilde{x}_a)_{(-1)}\mathbf{1} \\
&\quad - E_{sm}(-1)u_r(-1)(\tilde{\partial}_a x_m)(\partial_s \partial_r \tilde{x}_a) \\
&= \omega^{\mathfrak{Hei}} - (\tilde{\partial}_a x_m)_{(-1)}(\partial_m \tilde{x}_a)_{(-3)}\mathbf{1} - E_{sm}(-1)(\partial_s \tilde{x}_a)_{(-2)}(\tilde{\partial}_a x_m).
\end{aligned}$$

Next let us derive (4.10). The Virasoro element in $V_{\mathfrak{gl}_N}$ is

$$\omega^{\mathfrak{gl}_N} = \frac{1}{2(N+1)}\left(I(-1)I(-1)\mathbf{1} + \sum_{a,b=1}^{N} E_{ab}(-1)E_{ba}(-1)\mathbf{1} \right) + \frac{1}{2} I(-2)\mathbf{1}.$$

We have

$$\begin{aligned}
\Phi(\tilde{I}(-1)\mathbf{1}) &= E_{sp}(-1)(\partial_s \tilde{x}_a)(\tilde{\partial}_a x_p) - (\tilde{\partial}_a x_p)_{(-2)}(\partial_p \tilde{x}_a) \\
&= I(-1) - (\tilde{\partial}_a x_p)_{(-2)}(\partial_p \tilde{x}_a), \\
\Phi(\tilde{I}(-2)\mathbf{1}) &= I(-2)\mathbf{1} - (D\tilde{\partial}_a x_p)_{(-2)}(\partial_p \tilde{x}_a) - (\tilde{\partial}_a x_p)_{(-2)}(D\partial_p \tilde{x}_a) \\
&= I(-2)\mathbf{1} - 2(\tilde{\partial}_a x_p)_{(-3)}(\partial_p \tilde{x}_a) - (\tilde{\partial}_a x_p)_{(-2)}(\partial_p \tilde{x}_a)_{(-2)}\mathbf{1},
\end{aligned}$$

and

$$\Phi(\tilde{\omega}^{\mathfrak{gl}_N}) = \frac{1}{2(N+1)}\Big(\Phi(\tilde{I}(-1))_{(-1)}\Phi(\tilde{I}(-1)\mathbf{1}) + \Phi(\tilde{E}_{ab}(-1))_{(-1)}\Phi(\tilde{E}_{ba}(-1))\mathbf{1}\Big)$$

$$+\frac{1}{2}\Phi(\tilde{I}(-2)\mathbf{1})$$

$$= \frac{1}{2(N+1)}\bigg(\Big(I(-1)\mathbf{1} - (\tilde{\partial}_a x_p)_{(-2)}(\partial_p\tilde{x}_a)\Big)_{(-1)}\Big(I(-1)\mathbf{1} - (\tilde{\partial}_b x_s)_{(-2)}(\partial_s\tilde{x}_b)\Big)$$

$$+\big(E_{sp}(-1)(\partial_s\tilde{x}_a)(\tilde{\partial}_b x_p) - (\tilde{\partial}_b x_p)_{(-2)}(\partial_p\tilde{x}_a)\big)_{(-1)}\big(E_{km}(-1)(\partial_k\tilde{x}_b)(\tilde{\partial}_a x_m) - (\tilde{\partial}_a x_q)_{(-2)}(\partial_q\tilde{x}_b)\big)\bigg)$$

$$+\frac{1}{2}I(-2) - (\tilde{\partial}_a x_p)_{(-3)}(\partial_p\tilde{x}_a) - \frac{1}{2}(\tilde{\partial}_a x_p)_{(-2)}(\partial_p\tilde{x}_a)_{(-2)}\mathbf{1}$$

$$= \frac{1}{2(N+1)}\bigg(I(-1)I(-1)\mathbf{1} - 2I(-1)(\tilde{\partial}_a x_p)_{(-2)}(\partial_p\tilde{x}_a)$$

$$+(\tilde{\partial}_a x_p)_{(-2)}(\tilde{\partial}_b x_s)_{(-2)}(\partial_p\tilde{x}_a)(\partial_s\tilde{x}_b) + E_{sp}(-1)E_{km}(-1)(\partial_s\tilde{x}_a)(\tilde{\partial}_b x_p)(\partial_k\tilde{x}_b)(\tilde{\partial}_a x_m)$$

$$+((\partial_s\tilde{x}_a)(\tilde{\partial}_b x_p))_{(-2)}E_{sp}(0)E_{km}(-1)(\partial_k\tilde{x}_b)(\tilde{\partial}_a x_m)$$

$$+((\partial_s\tilde{x}_a)(\tilde{\partial}_b x_p))_{(-3)}E_{sp}(1)E_{km}(-1)(\partial_k\tilde{x}_b)(\tilde{\partial}_a x_m)$$

$$-E_{km}(-1)(\tilde{\partial}_b x_p)_{(-2)}(\partial_p\tilde{x}_a)(\partial_k\tilde{x}_b)(\tilde{\partial}_a x_m) - E_{sp}(-1)(\tilde{\partial}_a x_q)_{(-2)}(\tilde{\partial}_b x_p)(\partial_q\tilde{x}_b)(\partial_s\tilde{x}_a)$$

$$+(\tilde{\partial}_b x_p)_{(-2)}(\tilde{\partial}_a x_q)_{(-2)}(\partial_p\tilde{x}_a)(\partial_q\tilde{x}_b)\bigg)$$

$$+\frac{1}{2}I(-2)\mathbf{1} - (\tilde{\partial}_a x_p)_{(-3)}(\partial_p\tilde{x}_a) - \frac{1}{2}(\tilde{\partial}_a x_p)_{(-2)}(\partial_p\tilde{x}_a)_{(-2)}\mathbf{1}$$

$$= \frac{1}{2(N+1)}\bigg(I(-1)I(-1)\mathbf{1} - 2I(-1)(\tilde{\partial}_a x_p)_{(-2)}(\partial_p\tilde{x}_a)$$

$$+(\tilde{\partial}_a x_p)_{(-2)}(\tilde{\partial}_b x_s)_{(-2)}(\partial_p\tilde{x}_a)(\partial_s\tilde{x}_b) + E_{sp}(-1)E_{ps}(-1)\mathbf{1}$$

$$+E_{sm}(-1)(\partial_s\tilde{x}_a)_{(-2)}(\tilde{\partial}_b x_p)(\tilde{\partial}_a x_m)(\partial_p\tilde{x}_b) + E_{sm}(-1)(\tilde{\partial}_b x_p)_{(-2)}(\partial_s\tilde{x}_a)(\tilde{\partial}_a x_m)(\partial_p\tilde{x}_b)$$

$$-E_{kp}(-1)(\partial_s\tilde{x}_a)_{(-2)}(\tilde{\partial}_b x_p)(\tilde{\partial}_a x_s)(\partial_k\tilde{x}_b) - E_{kp}(-1)(\tilde{\partial}_b x_p)_{(-2)}(\partial_s\tilde{x}_a)(\tilde{\partial}_a x_s)(\partial_k\tilde{x}_b)$$

$$-2E_{kp}(-1)(\tilde{\partial}_b x_p)_{(-2)}(\partial_k\tilde{x}_b) + (\partial_s\tilde{x}_a)_{(-3)}(\tilde{\partial}_b x_p)(\partial_p\tilde{x}_b)(\tilde{\partial}_a x_s)$$

$$+(\tilde{\partial}_b x_p)_{(-2)}(\partial_s\tilde{x}_a)_{(-2)}(\partial_p\tilde{x}_b)(\tilde{\partial}_a x_s) + (\tilde{\partial}_b x_p)_{(-3)}(\partial_s\tilde{x}_a)(\partial_p\tilde{x}_b)(\tilde{\partial}_a x_s)$$

$$+(\tilde{\partial}_b x_p)_{(-2)}(\tilde{\partial}_a x_q)_{(-2)}(\partial_p\tilde{x}_a)(\partial_q\tilde{x}_b)\bigg)$$

$$+\frac{1}{2}I(-2)\mathbf{1} - (\tilde{\partial}_a x_p)_{(-3)}(\partial_p\tilde{x}_a) - \frac{1}{2}(\tilde{\partial}_a x_p)_{(-2)}(\partial_p\tilde{x}_a)_{(-2)}\mathbf{1}$$

$$= \omega^{\mathfrak{gl}_N} + \frac{1}{2(N+1)}\Big(-2I(-1)(\tilde\partial_a x_p)_{(-2)}(\partial_p \tilde x_a) + NE_{sm}(-1)(\partial_s \tilde x_a)_{(-2)}(\tilde\partial_a x_m)$$

$$+I(-1)(\tilde\partial_b x_p)_{(-2)}(\partial_p \tilde x_b) - I(-1)(\partial_s \tilde x_a)_{(-2)}(\tilde\partial_a x_s)$$

$$-NE_{kp}(-1)(\tilde\partial_b x_p)_{(-2)}(\partial_k \tilde x_b) - 2E_{kp}(-1)(\tilde\partial_b x_p)_{(-2)}(\partial_k \tilde x_b) + N(\partial_s \tilde x_a)_{(-3)}(\tilde\partial_a x_s)$$

$$+(\tilde\partial_b x_p)_{(-2)}((\partial_s \tilde x_a)(\tilde\partial_a x_s))_{(-2)}(\partial_p \tilde x_b) + N(\tilde\partial_b x_p)_{(-3)}(\partial_p \tilde x_b)$$

$$-(\tilde\partial_b x_p)_{(-2)}(\partial_p \tilde x_a)_{(-2)}(\tilde\partial_a x_q)(\partial_q \tilde x_b)\Big) - (\tilde\partial_a x_p)_{(-3)}(\partial_p \tilde x_a) - \frac{1}{2}(\tilde\partial_a x_p)_{(-2)}(\partial_p \tilde x_a)_{(-2)}\mathbf{1}$$

$$= \omega^{\mathfrak{gl}_N} + E_{sm}(-1)(\partial_s \tilde x_a)_{(-2)}(\tilde\partial_a x_m) - \frac{1}{2}(\tilde\partial_a x_p)_{(-2)}(\partial_p \tilde x_a)_{(-2)}\mathbf{1}$$

$$-(\tilde\partial_a x_p)_{(-3)}(\partial_p \tilde x_a) - \frac{1}{2}(\tilde\partial_a x_p)_{(-2)}(\partial_p \tilde x_a)_{(-2)}\mathbf{1}$$

$$= \omega^{\mathfrak{gl}_N} + E_{sm}(-1)(\partial_s \tilde x_a)_{(-2)}(\tilde\partial_a x_m) + (\partial_p \tilde x_a)_{(-3)}(\tilde\partial_a x_p).$$

Now let us establish (4.7):

$$\Phi\left(\tilde\omega_{(-1)} f + \tilde u_k(-1)\tilde E_{sk}(-1)\tilde\partial_s f - \tilde u_s(-2)\tilde\partial_s f\right)$$

$$= \omega_{(-1)} f + (u_b(-1)(\partial_b \tilde x_k))_{(-1)}\left(E_{ac}(-1)(\partial_a \tilde x_s)(\tilde\partial_k x_c) + u_a(-1)\tilde\partial_k \partial_a \tilde x_s\right)_{(-1)}(\tilde\partial_s f)$$

$$-(u_a(-1)(\partial_a \tilde x_s))_{(-2)}(\tilde\partial_s f)$$

$$= \omega_{(-1)} f + u_b(-1)E_{ac}(-1)(\partial_b \tilde x_k)(\partial_a \tilde x_s)(\tilde\partial_k x_c)(\tilde\partial_s f) + u_b(-1)u_a(-1)(\partial_b \tilde x_k)(\tilde\partial_k \partial_a \tilde x_s)(\tilde\partial_s f)$$

$$-u_a(-2)(\partial_a \tilde x_s)(\tilde\partial_s f) - u_a(-1)u_b(-1)(\partial_b \partial_a \tilde x_s)(\tilde\partial_s f)$$

$$= \omega_{(-1)} f + u_b(-1)E_{ab}(-1)(\partial_a f) + u_b(-1)u_a(-1)(\partial_b \partial_a \tilde x_s)(\tilde\partial_s f)$$

$$-u_a(-1)u_b(-1)(\partial_b \partial_a \tilde x_s)(\tilde\partial_s f) - u_a(-2)(\partial_a f)$$

$$= \omega_{(-1)} f + u_b(-1)E_{ab}(-1)(\partial_a f) - u_a(-2)(\partial_a f).$$

To complete the proof of Theorem 4.1, we need to show that locally (4.1)-(4.5) define a representation of the Lie algebra $\mathcal{G}(U_i)$.

The Lie bracket in $\mathcal{G}(U_i)$ may be encoded using the commutators of the generating series:

$$[g_1(f,z_1), g_2(h,z_2)] = [g_1,g_2](fh,z_2)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)$$

$$+(g_1|g_2)k_0(fh,z_2)z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right) + (g_1|g_2)\sum_{p=1}^{N} k_p(h\partial_p f, z_2)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right), \qquad (4.11)$$

$$[d_a(f,z_1), g(h,z_2)] = g(f\partial_a h, z_2)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right), \qquad (4.12)$$

$$[d_a(f,z_1),k_0(h,z_2)]=k_0(f\partial_a h,z_2)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right), \tag{4.13}$$

$$[d_a(f,z_1),k_b(h,z_2)]=k_b(f\partial_a h,z_2)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)$$

$$+\delta_{ab}k_0(fh,z_2)z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)+\delta_{ab}\sum_{p=1}^{N}k_p(h\partial_p f,z_2)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right), \tag{4.14}$$

$$[d_a(f,z_1),d_b(h,z_2)]=\big(d_b(f\partial_a h,z_2)-d_a(h\partial_b f,z_2)\big)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right), \tag{4.15}$$

$$[k_i(f,z_1),k_j(h,z_2)]=0, \tag{4.16}$$

$$[g(f,z_1),k_i(h,z_2)]=0, \tag{4.17}$$

$$[d_0(f,z_1),g(h,z_2)]=\left(\frac{\partial}{\partial z_2}g(fh,z_2)\right)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)+g(fh,z_2)z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right), \tag{4.18}$$

$$[d_0(f,z_1),k_0(h,z_2)]=\sum_{p=1}^{N}k_p(f\partial_p h,z_2)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right), \tag{4.19}$$

$$[d_0(f,z_1),k_a(h,z_2)]=\left(\frac{\partial}{\partial z_2}k_a(fh,z_2)\right)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)+k_a(fh,z_2)z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right), \tag{4.20}$$

$$[d_0(f,z_1),d_a(h,z_2)]=\left(\frac{\partial}{\partial z_2}d_a(fh,z_2)\right)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)$$

$$+d_a(fh,z_2)z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)-d_0(h\partial_a f,z_2)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right), \tag{4.21}$$

$$[d_0(f,z_1),d_0(h,z_2)]=\left(\frac{\partial}{\partial z_2}d_0(fh,z_2)\right)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)+2d_0(fh,z_2)z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right), \tag{4.22}$$

where $g,g_1,g_2\in\mathfrak{g},\ f,h\in\mathcal{O}_X(U_i),\ a,b=1,\dots,N,\ i,j=0,\dots,N.$

We will use the commutator formula (2.2) in order to prove that ρ preserves these relations. For (4.11) we need to verify in $\mathcal{V}(U_i)$ the following relations for n-th products:

$$(g_1(-1)f)_{(0)}(g_2(-1)h)=[g_1,g_2](-1)fh+(g_1|g_2)cu_p(-1)h\partial_p f,$$

$$(g_1(-1)f)_{(1)}(g_2(-1)h)=c(g_1|g_2)fh,$$

$$(g_1(-1)f)_{(n)}(g_2(-1)h)=0\ \text{ for }\ n>1.$$

Let us check these equalities:

$$(g_1(-1)f)_{(0)}(g_2(-1)h)=f_{(-1)}g_1(0)g_2(-1)h+f_{(-2)}g_1(1)g_2(-1)h$$

$$=[g_1,g_2](-1)fh+c(g_1|g_2)u_p(-1)h\partial_p f,$$

$$(g_1(-1)f)_{(1)}(g_2(-1)h) = f_{(-1)}g_1(1)g_2(-1)h = c(g_1|g_2)fh.$$

The last relation holds trivially since the degree of the left hand side becomes negative.

We verify (4.12) in an analogous way:

$$\big(v_a(-1)f + E_{pa}(-1)\partial_p f\big)_{(0)} g(-1)h = f_{(-1)}v_a(0)g(-1)h = g(-1)f\partial_a h.$$

It is easy to see that

$$\big(v_a(-1)f + E_{pa}(-1)\partial_p f\big)_{(n)} g(-1)h = 0 \text{ for } n > 0.$$

For the remaining n-th products we will verify only those that have a non-negative degree. For (4.13) we have

$$\big(v_a(-1)f + E_{pa}(-1)\partial_p f\big)_{(0)} h = f_{(-1)}v_a(0)h = f\partial_a h.$$

To prove (4.14), we compute

$$\big(v_a(-1)f + E_{pa}(-1)\partial_p f\big)_{(0)} u_b(-1)h = f_{(-1)}v_a(0)u_b(-1)h + f_{(-2)}v_a(1)u_b(-1)h$$
$$= u_b(-1)f\partial_a h + \delta_{ab}u_p(-1)h\partial_p f,$$

and

$$\big(v_a(-1)f + E_{pa}(-1)\partial_p f\big)_{(1)} u_b(-1)h = f_{(-1)}v_a(1)u_b(-1)h = \delta_{ab}fh.$$

Let us now verify (4.15):

$$\big(v_a(-1)f + E_{pa}(-1)\partial_p f\big)_{(0)} \big(v_b(-1)h + E_{kb}(-1)\partial_k h\big)$$

$$= f_{(-1)}v_a(0)\big(v_b(-1)h + E_{kb}(-1)\partial_k h\big) + v_a(-1)f_{(0)}v_b(-1)h$$
$$+(\partial_p f)_{(-1)}E_{pa}(0)E_{kb}(-1)\partial_k h + (\partial_p f)_{(-2)}E_{pa}(1)E_{kb}(-1)\partial_k h + E_{pa}(-1)(\partial_p f)_{(0)}v_b(-1)h$$
$$= f_{(-1)}v_b(-1)\partial_a h + E_{kb}(-1)f\partial_a\partial_k h - v_a(-1)(\partial_b f)_{(-1)}h$$
$$+(\partial_p f)_{(-1)}E_{pb}(-1)\partial_a h - (\partial_b f)_{(-1)}E_{ka}(-1)\partial_k h + (\partial_b f)_{(-2)}\partial_a h - E_{pa}(-1)(\partial_b\partial_p f)h$$
$$= v_b(-1)f\partial_a h - (\partial_b f)_{(-2)}\partial_a h + E_{kb}(-1)f\partial_a\partial_k h + E_{kb}(-1)(\partial_k f)(\partial_a h)$$
$$-v_a(-1)(\partial_b f)h - E_{ka}(-1)(\partial_b f)(\partial_k h) - E_{ka}(-1)(\partial_b\partial_k f)h + (\partial_b f)_{(-2)}(\partial_a h)$$
$$= v_b(-1)f\partial_a h + E_{kb}(-1)\partial_k(f\partial_a h) - v_a(-1)(\partial_b f)h - E_{ka}(-1)\partial_k((\partial_b f)h),$$

and for $n = 1$:

$$\big(v_a(-1)f + E_{pa}(-1)\partial_p f\big)_{(1)} \big(v_b(-1)h + E_{kb}(-1)\partial_k h\big)$$

$$= f_{(0)}v_a(0)\big(v_b(-1)h + E_{kb}(-1)\partial_k h\big) + (\partial_p f)_{(-1)}E_{pa}(1)E_{kb}(-1)\partial_k h$$
$$= f_{(0)}v_b(-1)\partial_a h + (\partial_b f)_{(-1)}\partial_a h = -(\partial_b f)_{(-1)}\partial_a h + (\partial_b f)_{(-1)}\partial_a h = 0.$$

Relations (4.16) and (4.17) hold trivially.

For the commutators involving $d_0(f,z)$, we will be using the properties (2.5) and (2.6) of the Virasoro element. We will also need the following commutator relations (see [2]):

$$[\omega_{(n)}, f_{(m)}] = -(n+m)f_{(n+m-1)},$$

$$[\omega_{(n)}, g(m)] = -mg(n+m-1),$$
$$[\omega_{(n)}, u_a(m)] = -mu_a(n+m-1),$$
$$[\omega_{(n)}, v_a(m)] = -mv_a(n+m-1),$$
$$[\omega_{(n)}, E_{ab}(m)] = -mE_{ab}(n+m-1) - \delta_{ab}\delta_{n+m-1,0}\frac{n(n-1)}{2}\mathrm{Id},$$
$$[\omega_{(n)}, \omega_{(m)}] = (n-m)\omega_{(n+m-1)}.$$

For (4.18) we get

$$\big(\omega_{(-1)}f + u_k(-1)E_{sk}(-1)\partial_s f - u_k(-2)\partial_k f\big)_{(0)} g(-1)h$$

$$= f_{(-1)}\omega_{(0)}g(-1)h + f_{(-2)}\omega_{(1)}g(-1)h$$
$$= fD(g(-1)h) + g(-1)(Df)_{(-1)}h = D(g(-1)fh)$$

and

$$\big(\omega_{(-1)}f + u_k(-1)E_{sk}(-1)\partial_s f - u_k(-2)\partial_k f\big)_{(1)} g(-1)h$$
$$= f_{(-1)}\omega_{(1)}g(-1)h = g(-1)fh.$$

For (4.19) we have

$$\big(\omega_{(-1)}f + u_k(-1)E_{sk}(-1)\partial_s f - u_k(-2)\partial_k f\big)_{(0)} h$$

$$= f_{(-1)}\omega_{(0)}h + f_{(-2)}\omega_{(1)}h = f_{(-1)}Dh = u_p(-1)f\partial_p h,$$

and

$$\big(\omega_{(-1)}f + u_k(-1)E_{sk}(-1)\partial_s f - u_k(-2)\partial_k f\big)_{(1)} h$$
$$= f_{(0)}\omega_{(0)}h + f_{(-1)}\omega_{(1)}h = 0.$$

To verify that (4.20) holds for ρ, we calculate:

$$\big(\omega_{(-1)}f + u_k(-1)E_{sk}(-1)\partial_s f - u_k(-2)\partial_k f\big)_{(0)} u_a(-1)h$$

$$= f_{(-1)}\omega_{(0)}u_a(-1)h + f_{(-2)}\omega_{(1)}u_a(-1)h + f_{(-3)}\omega_{(2)}u_a(-1)h$$
$$= f_{(-1)}D(u_a(-1)h) + (Df)_{(-1)}u_a(-1)h + f_{(-3)}u_a(0)h = D(u_a(-1)fh),$$

for $n = 1$:

$$\big(\omega_{(-1)}f + u_k(-1)E_{sk}(-1)\partial_s f - u_k(-2)\partial_k f\big)_{(1)} u_a(-1)h$$

$$= f_{(0)}\omega_{(0)}u_a(-1)h + f_{(-1)}\omega_{(1)}u_a(-1)h + f_{(-2)}\omega_{(2)}u_a(-1)h$$
$$= f_{(0)}D(u_a(-1)h) + f_{(-1)}u_a(-1)h + f_{(-2)}u_a(0)h = u_a(-1)fh,$$

and for $n = 2$:

$$\big(\omega_{(-1)}f + u_k(-1)E_{sk}(-1)\partial_s f - u_k(-2)\partial_k f\big)_{(2)}u_a(-1)h$$
$$= f_{(-1)}\omega_{(2)}u_a(-1)h = f_{(-1)}u_a(0)h = 0.$$

To establish (4.21) we need to compute the corresponding $n = 0, 1$ and 2 products:

$$\big(\omega_{(-1)}f + u_k(-1)E_{sk}(-1)\partial_s f - u_k(-2)\partial_k f\big)_{(0)}\big(v_a(-1)h + E_{pa}(-1)\partial_p h\big)$$
$$= f_{(-1)}\omega_{(0)}\big(v_a(-1)h + E_{pa}(-1)\partial_p h\big) + f_{(-2)}\omega_{(1)}\big(v_a(-1)h + E_{pa}(-1)\partial_p h\big)$$
$$+f_{(-3)}\omega_{(2)}\big(v_a(-1)h + E_{pa}(-1)\partial_p h\big) + \omega_{(-1)}f_{(0)}v_a(-1)h$$
$$+(E_{sk}(-1)\partial_s f)_{(-2)}u_k(1)v_a(-1)h + u_k(-1)(E_{sk}(-1)\partial_s f)_{(0)}\big(v_a(-1)h + E_{pa}(-1)\partial_p h\big)$$
$$+u_k(-2)(E_{sk}(-1)\partial_s f)_{(1)}E_{pa}(-1)\partial_p h + 2(\partial_k f)_{(-3)}u_k(1)v_a(-1)h - u_k(-2)(\partial_k f)_{(0)}v_a(-1)h$$
$$= f_{(-1)}D\big(v_a(-1)h + E_{pa}(-1)\partial_p h\big) + (Df)_{(-1)}(v_a(-1)h + E_{pa}(-1)\partial_p h)$$
$$+f_{(-3)}v_a(0)h - f_{(-3)}\partial_a h - \omega_{(-1)}(\partial_a f)h + \big(D(E_{pa}(-1)\partial_p f)\big)_{(-1)}h$$
$$+u_k(-1)E_{sk}(-1)(\partial_s f)_{(0)}v_a(-1)h + u_k(-1)(\partial_s f)_{(-1)}E_{sk}(0)E_{pa}(-1)\partial_p h$$
$$+u_k(-1)(\partial_s f)_{(-2)}E_{sk}(1)E_{pa}(-1)\partial_p h + u_k(-2)(\partial_s f)_{(-1)}E_{sk}(1)E_{pa}(-1)\partial_p h$$
$$+2(\partial_a f)_{(-3)}h + u_k(-2)(\partial_a\partial_k f)h$$
$$= D(f_{(-1)}v_a(-1)h) + D(f_{(-1)}E_{pa}(-1)\partial_p h) + f_{(-3)}\partial_a h - f_{(-3)}\partial_a h - \omega_{(-1)}(\partial_a f)h$$
$$+\big(D(E_{pa}(-1)\partial_p f)\big)_{(-1)}h - u_k(-1)E_{sk}(-1)(\partial_a\partial_s f)h + u_k(-1)(\partial_s f)_{(-1)}E_{sa}(-1)\partial_k h$$
$$-u_k(-1)(\partial_a f)_{(-1)}E_{pk}(-1)\partial_p h + u_k(-1)(\partial_a f)_{(-2)}\partial_k h$$
$$+u_k(-2)(\partial_a f)(\partial_k h) + 2(\partial_a f)_{(-3)}h + u_k(-2)(\partial_k\partial_a f)h$$
$$= D(v_a(-1)fh) - D((\partial_a f)_{(-2)}h) + D(E_{pa}(-1)f\partial_p h) - \omega_{(-1)}(\partial_a f)h$$
$$+\big(D(E_{pa}(-1)\partial_p f)\big)_{(-1)}h + E_{pa}(-1)(\partial_p f)D(h) - u_k(-1)E_{sk}(-1)(\partial_s\partial_a f)h$$
$$-u_k(-1)E_{sk}(-1)(\partial_a f)(\partial_s h) + (\partial_a f)_{(-2)}Dh + u_k(-2)\partial_k((\partial_a f)h) + 2(\partial_a f)_{(-3)}h$$
$$= D\big(v_a(-1)fh + E_{pa}(-1)\partial_p(fh)\big)$$
$$-\big(\omega_{(-1)}(\partial_a f)h + u_k(-1)E_{sk}(-1)\partial_s((\partial_a f)h) - u_k(-2)\partial_k((\partial_a f)h)\big).$$

For $n = 1$:

$$\big(\omega_{(-1)}f + u_k(-1)E_{sk}(-1)\partial_s f - u_k(-2)\partial_k f\big)_{(1)}\big(v_a(-1)h + E_{pa}(-1)\partial_p h\big)$$

$$= f_{(0)}\omega_{(0)}\big(v_a(-1)h + E_{pa}(-1)\partial_p h\big) + f_{(-1)}\omega_{(1)}\big(v_a(-1)h + E_{pa}(-1)\partial_p h\big)$$
$$+f_{(-2)}\omega_{(2)}\big(v_a(-1)h + E_{pa}(-1)\partial_p h\big) + (E_{sk}(-1)\partial_s f)_{(-1)}u_k(1)v_a(-1)h$$
$$+u_k(-1)(E_{sk}(-1)\partial_s f)_{(1)}\big(v_a(-1)h + E_{pa}(-1)\partial_p h\big) + 2(\partial_k f)_{(-2)}u_k(1)v_a(-1)h$$
$$= f_{(0)}v_a(-2)h + f_{(0)}v_a(-1)Dh + f_{(-1)}v_a(-1)h + f_{(-1)}E_{pa}(-1)\partial_p h$$
$$+f_{(-2)}v_a(-0)h - f_{(-2)}\partial_a h + E_{sa}(-1)(\partial_s f)h$$
$$+u_k(-1)(\partial_s f)_{(-1)}E_{sk}(1)E_{pa}(-1)\partial_p h + 2(\partial_a f)_{(-2)}h$$
$$= -(\partial_a f)_{(-2)}h - (\partial_a f)_{(-1)}Dh + v_a(-1)fh - (\partial_a f)_{(-2)}h + E_{pa}(-1)f\partial_p h$$
$$+f_{(-2)}\partial_a h - f_{(-2)}\partial_a h + E_{pa}(-1)(\partial_p f)h + u_k(-1)(\partial_a f)(\partial_k h) + 2(\partial_a f)_{(-2)}h$$
$$= v_a(-1)fh + E_{pa}(-1)\partial_p(fh),$$

and for $n = 2$:

$$\big(\omega_{(-1)}f + u_k(-1)E_{sk}(-1)\partial_s f - u_k(-2)\partial_k f\big)_{(2)}\big(v_a(-1)h + E_{pa}(-1)\partial_p h\big)$$

$$= f_{(1)}\omega_{(0)}(v_a(-1)h + E_{pa}(-1)\partial_p h) + f_{(0)}\omega_{(1)}(v_a(-1)h + E_{pa}(-1)\partial_p h)$$
$$+f_{(-1)}\omega_{(2)}\big(v_a(-1)h + E_{pa}(-1)\partial_p h\big) + (E_{sk}(-1)\partial_s f)_{(0)}u_k(1)v_a(-1)h$$
$$+2(\partial_k f)_{(-1)}u_k(1)v_a(-1)h$$
$$= f_{(1)}v_a(-2)h + f_{(1)}v_a(-1)Dh + f_{(0)}v_a(-1)h + f_{(-1)}v_a(0)h - f_{(-1)}\partial_a h + 2(\partial_a f)h$$
$$= -(\partial_a f)h - (\partial_a f)_{(0)}Dh - (\partial_a f)h + f\partial_a h - f\partial_a h + 2(\partial_a f)h = 0,$$

which proves (4.21).

In the following computation, which establishes (4.22), we will be using (2.8) and the Borcherds identity (2.4).

We begin with calculating $n = 0$ product of the elements of the vertex algebra corresponding to $d_0(f, z)$ and $d_0(h, z)$:

$$\big(\omega_{(-1)}f + u_k(-1)E_{sk}(-1)\partial_s f - u_k(-2)\partial_k f\big)_{(0)}\big(\omega_{(-1)}h + u_p(-1)E_{mp}(-1)\partial_m h - u_p(-2)\partial_p h\big)$$

$$= f_{(-1)}\omega_{(0)}\big(\omega_{(-1)}h + u_p(-1)E_{mp}(-1)\partial_m h - u_p(-2)\partial_p h\big)$$
$$+f_{(-2)}\omega_{(1)}\big(\omega_{(-1)}h + u_p(-1)E_{mp}(-1)\partial_m h - u_p(-2)\partial_p h\big)$$
$$+f_{(-3)}\omega_{(2)}\big(\omega_{(-1)}h + u_p(-1)E_{mp}(-1)\partial_m h - u_p(-2)\partial_p h\big)$$
$$+f_{(-4)}\omega_{(3)}\big(\omega_{(-1)}h + u_p(-1)E_{mp}(-1)\partial_m h - u_p(-2)\partial_p h\big)$$
$$+\omega_{(-1)}f_{(0)}\omega_{(-1)}h + \omega_{(-2)}f_{(1)}\omega_{(-1)}h$$
$$+(E_{sk}(-1)\partial_s f)_{(-2)}u_k(1)\omega_{(-1)}h + (E_{sk}(-1)\partial_s f)_{(-3)}u_k(2)\omega_{(-1)}h$$
$$+u_k(-1)(E_{sk}(-1)\partial_s f)_{(0)}\big(\omega_{(-1)}h + u_p(-1)E_{mp}(-1)\partial_m h\big)$$

$$+u_k(-2)(E_{sk}(-1)\partial_s f)_{(1)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h\big)$$
$$+u_k(-3)(E_{sk}(-1)\partial_s f)_{(2)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h\big)$$
$$+3(\partial_k f)_{(-4)}u_k(2)\omega_{(-1)}h+2(\partial_k f)_{(-3)}u_k(1)\omega_{(-1)}h$$
$$-u_k(-2)(\partial_k f)_{(0)}\omega_{(-1)}h-2u_k(-3)(\partial_k f)_{(1)}\omega_{(-1)}h$$
$$=f_{(-1)}D\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+2(Df)_{(-1)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+3f_{(-3)}\omega_{(0)}h+f_{(-3)}u_p(0)E_{mp}(-1)\partial_m h+f_{(-3)}u_p(-1)\omega_{(2)}E_{mp}(-1)\partial_m h$$
$$-2f_{(-3)}u_p(-1)\partial_p h+4f_{(-4)}\omega_{(1)}h+f_{(-4)}u_p(1)E_{mp}(-1)\partial_m h-2f_{(-4)}u_p(0)\partial_p h$$
$$-\omega_{(-1)}f_{(-2)}h+(E_{sk}(-1)\partial_s f)_{(-2)}u_k(-1)h+2(E_{sk}(-1)\partial_s f)_{(-3)}u_k(0)h$$
$$+u_k(-1)(\partial_s f)_{(-1)}E_{sk}(0)\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h\big)$$
$$+u_k(-1)(\partial_s f)_{(-2)}E_{sk}(1)\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h\big)+u_k(-1)(\partial_s f)_{(-3)}E_{sk}(2)\omega_{(-1)}h$$
$$+u_k(-1)E_{sk}(-1)(\partial_s f)_{(0)}\omega_{(-1)}h+u_k(-1)E_{sk}(-2)(\partial_s f)_{(1)}\omega_{(-1)}h$$
$$+u_k(-2)(\partial_s f)_{(0)}E_{sk}(0)\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h\big)$$
$$+u_k(-2)(\partial_s f)_{(-1)}E_{sk}(1)\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h\big)+u_k(-2)(\partial_s f)_{(-2)}E_{sk}(2)\omega_{(-1)}h$$
$$+u_k(-2)E_{sk}(-1)(\partial_s f)_{(1)}\omega_{(-1)}h$$
$$+u_k(-3)(\partial_s f)_{(1)}E_{sk}(0)\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h\big)$$
$$+u_k(-3)(\partial_s f)_{(0)}E_{sk}(1)\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h\big)+u_k(-3)(\partial_s f)_{(-1)}E_{sk}(2)\omega_{(-1)}h$$
$$+6(\partial_k f)_{(-4)}u_k(0)h+2(\partial_k f)_{(-3)}u_k(-1)h+u_k(-2)(\partial_k f)_{(-2)}h$$
$$=D(f_{(-1)}\omega_{(-1)}h+u_p(-1)E_{mp}(-1)f\partial_m h-u_p(-2)f\partial_p h)$$
$$+f_{(-2)}\omega_{(-1)}h+u_p(-1)E_{mp}(-1)f_{(-2)}\partial_m h-u_p(-2)f_{(-2)}\partial_p h$$
$$+3f_{(-3)}Dh-f_{(-3)}u_p(-1)\partial_p h-2f_{(-3)}Dh$$
$$-\omega_{(-1)}f_{(-2)}h+(D(E_{sk}(-1)\partial_s f))_{(-1)}u_k(-1)h$$
$$+u_k(-1)u_p(-1)E_{sp}(-1)(\partial_s f)(\partial_k h)-u_k(-1)u_p(-1)E_{mk}(-1)(\partial_p f)(\partial_m h)$$
$$+u_k(-1)(\partial_s f)_{(-2)}E_{sk}(-1)h+u_k(-1)(\partial_s f)_{(-2)}u_s(-1)\partial_k h+u_k(-1)(\partial_k f)_{(-3)}h$$
$$-u_k(-1)E_{sk}(-1)(\partial_s f)_{(-2)}h+u_k(-2)(\partial_s f)_{(-1)}E_{sk}(-1)h+u_k(-2)u_s(-1)(\partial_s f)(\partial_k h)$$
$$+u_k(-2)(\partial_k f)_{(-2)}h+u_k(-3)(\partial_k f)h+2u_k(-1)(\partial_k f)_{(-3)}h+u_k(-2)(\partial_k f)_{(-2)}h$$
$$=D\big(\omega_{(-1)}fh+u_p(-1)E_{mp}(-1)f\partial_m h-u_p(-2)f\partial_p h\big)$$
$$-2D(f_{(-3)}h)-3f_{(-4)}h$$
$$+(D(E_{sk}(-1)\partial_s f))_{(-1)}u_k(-1)h+u_p(-1)E_{sp}(-1)(\partial_s f)Dh+(Du_k(-1))_{(-1)}E_{sk}(-1)(\partial_s f)h$$

$$+u_s(-1)(\partial_s f)_{(-2)}Dh+3u_k(-1)(\partial_k f)_{(-3)}h+2u_k(-2)(\partial_k f)_{(-2)}h+u_k(-3)(\partial_k f)h$$
$$=D\big(\omega_{(-1)}fh+u_p(-1)E_{mp}(-1)f\partial_m h-u_p(-2)f\partial_p h\big)$$
$$+D(u_p(-1)E_{sp}(-1)(\partial_s f)h)-3(Df)_{(-3)}h-(Df)_{(-2)}Dh+u_p(-1)(\partial_p f)_{(-2)}Dh$$
$$+3u_p(-1)(\partial_p f)_{(-3)}h+2u_p(-2)(\partial_p f)_{(-2)}h+u_p(-3)(\partial_p f)h$$
$$=D\big(\omega_{(-1)}fh+u_p(-1)E_{mp}(-1)\partial_m(fh)-u_p(-2)f\partial_p h\big)$$
$$-3(u_p(-1)\partial_p f)_{(-3)}h-(u_p(-1)\partial_p f)_{(-2)}Dh+u_p(-1)(\partial_p f)_{(-2)}Dh$$
$$+3u_p(-1)(\partial_p f)_{(-3)}h+2u_p(-2)(\partial_p f)_{(-2)}h+u_p(-3)(\partial_p f)h$$
$$=D\big(\omega_{(-1)}fh+u_p(-1)E_{mp}(-1)\partial_m(fh)-u_p(-2)f\partial_p h\big)$$
$$-3u_p(-3)(\partial_p f)h-3u_p(-2)(\partial_p f)_{(-2)}h-3u_p(-1)(\partial_p f)_{(-3)}h$$
$$-u_p(-2)(\partial_p f)Dh-u_p(-1)(\partial_p f)_{(-2)}Dh+u_p(-1)(\partial_p f)_{(-2)}Dh$$
$$+3u_p(-1)(\partial_p f)_{(-3)}h+2u_p(-2)(\partial_p f)_{(-2)}h+u_p(-3)(\partial_p f)h$$
$$=D\big(\omega_{(-1)}fh+u_p(-1)E_{mp}(-1)\partial_m(fh)-u_p(-2)f\partial_p h\big)-D(u_p(-2)(\partial_p f)h)$$
$$=D\big(\omega_{(-1)}fh+u_p(-1)E_{mp}(-1)\partial_m(fh)-u_p(-2)\partial_p(fh)\big).$$

Let us do the computations for $n=1$ product in (4.22):

$$\big(\omega_{(-1)}f+u_k(-1)E_{sk}(-1)\partial_s f-u_k(-2)\partial_k f\big)_{(1)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$=f_{(0)}\omega_{(0)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+f_{(-1)}\omega_{(1)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+f_{(-2)}\omega_{(2)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+f_{(-3)}\omega_{(3)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+\omega_{(-1)}f_{(1)}\omega_{(-1)}h+(E_{sk}(-1)\partial_s f)_{(-1)}u_k(1)\omega_{(-1)}h+(E_{sk}(-1)\partial_s f)_{(-2)}u_k(2)\omega_{(-1)}h$$
$$+u_k(-1)(E_{sk}(-1)\partial_s f)_{(1)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h\big)$$
$$+u_k(-2)(E_{sk}(-1)\partial_s f)_{(2)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h\big)$$
$$+3(\partial_k f)_{(-3)}u_k(2)\omega_{(-1)}h+2(\partial_k f)_{(-2)}u_k(1)\omega_{(-1)}h-u_k(-2)(\partial_k f)_{(1)}\omega_{(-1)}h$$
$$=f_{(0)}\omega_{(-2)}h+f_{(0)}\omega_{(-1)}Dh+2f_{(-1)}\omega_{(-1)}h$$
$$+2f_{(-1)}u_p(-1)E_{mp}(-1)\partial_m h-2f_{(-1)}u_p(-2)\partial_p h+3f_{(-2)}\omega_{(0)}h+f_{(-2)}u_p(0)E_{mp}(-1)\partial_m h$$
$$+f_{(-2)}u_p(-1)\omega_{(2)}E_{mp}(-1)\partial_m h-2f_{(-2)}u_p(-1)\partial_p h+4f_{(-3)}\omega_{(1)}h+f_{(-3)}u_p(1)E_{mp}(-1)\partial_m h$$
$$-2f_{(-3)}u_p(0)\partial_p h+(E_{sk}(-1)\partial_s f)_{(-1)}u_k(-1)h+2(E_{sk}(-1)\partial_s f)_{(-2)}u_k(0)h$$
$$+u_k(-1)(\partial_s f)_{(0)}E_{sk}(0)(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h)$$

$$+u_k(-1)(\partial_s f)_{(-1)}E_{sk}(1)(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h)$$
$$+u_k(-1)(\partial_s f)_{(-2)}E_{sk}(2)\omega_{(-1)}h+u_k(-1)E_{sk}(-1)(\partial_s f)_{(1)}\omega_{(-1)}h$$
$$+u_k(-2)(\partial_s f)_{(1)}E_{sk}(0)\omega_{(-1)}h+u_k(-2)(\partial_s f)_{(0)}E_{sk}(1)\omega_{(-1)}h$$
$$+u_k(-2)(\partial_s f)_{(-1)}E_{sk}(2)\omega_{(-1)}h+6(\partial_k f)_{(-3)}u_k(0)h+2(\partial_k f)_{(-2)}u_k(-1)h$$
$$=-2f_{(-3)}h-f_{(-2)}Dh+2\omega_{(-1)}fh-4f_{(-3)}h$$
$$+2u_p(-1)E_{mp}(-1)f\partial_m h-2u_p(-2)f\partial_p h+3f_{(-2)}Dh$$
$$-u_p(-1)f_{(-2)}\partial_p h-2u_p(-1)f_{(-2)}\partial_p h+E_{sk}(-1)u_k(-1)(\partial_s f)h$$
$$+u_k(-1)(\partial_s f)_{(-1)}E_{sk}(-1)h+u_k(-1)(\partial_s f)_{(-1)}u_s(-1)\partial_k h+u_k(-1)(\partial_k f)_{(-2)}h$$
$$+u_k(-2)(\partial_k f)h+2u_k(-1)(\partial_k f)_{(-2)}h$$
$$=2\omega_{(-1)}fh+2u_k(-1)E_{sk}(-1)\partial_s(fh)-3(Df)_{(-2)}h$$
$$-2u_k(-2)f\partial_k h+u_k(-2)(\partial_k f)h+3u_k(-1)(\partial_k f)_{(-2)}h$$
$$=2\omega_{(-1)}fh+2u_k(-1)E_{sk}(-1)\partial_s(fh)-3u_k(-2)(\partial_k f)h$$
$$-3u_k(-1)(\partial_k f)_{(-2)}h-2u_k(-2)f\partial_k h+u_k(-2)(\partial_k f)h+3u_k(-1)(\partial_k f)_{(-2)}h$$
$$=2\big(\omega_{(-1)}fh+u_k(-1)E_{sk}(-1)\partial_s(fh)-u_k(-2)\partial_k(fh)\big).$$

For $n=2$:

$$\big(\omega_{(-1)}f+u_k(-1)E_{sk}(-1)\partial_s f-u_k(-2)\partial_k f\big)_{(2)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$=f_{(1)}\omega_{(0)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+f_{(0)}\omega_{(1)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+f_{(-1)}\omega_{(2)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+f_{(-2)}\omega_{(3)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+(E_{sk}(-1)\partial_s f)_{(0)}u_k(1)\omega_{(-1)}h+(E_{sk}(-1)\partial_s f)_{(-1)}u_k(2)\omega_{(-1)}h$$
$$+u_k(-1)(E_{sk}(-1)\partial_s f)_{(2)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h\big)$$
$$+2(\partial_k f)_{(-1)}u_k(1)\omega_{(-1)}h+3(\partial_k f)_{(-2)}u_k(2)\omega_{(-1)}h$$
$$=f_{(1)}\omega_{(-2)}h+f_{(1)}\omega_{(-1)}Dh+2f_{(0)}\omega_{(-1)}h+3f_{(-1)}\omega_{(0)}h$$
$$+f_{(-1)}u_p(0)E_{mp}(-1)\partial_m h+f_{(-1)}u_p(-1)\omega_{(2)}E_{mp}(-1)\partial_m h-2f_{(-1)}u_p(-1)\partial_p h$$
$$+4f_{(-2)}\omega_{(1)}h+f_{(-2)}u_p(1)E_{mp}(-1)\partial_m h-2f_{(-2)}u_p(0)\partial_p h$$
$$+(E_{sk}(-1)\partial_s f)_{(0)}u_k(-1)h+2(E_{sk}(-1)\partial_s f)_{(-1)}u_k(0)h$$
$$+u_k(-1)(\partial_s f)_{(1)}E_{sk}(0)\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h\big)$$

$$+u_k(-1)(\partial_s f)_{(0)}E_{sk}(1)\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h\big)$$
$$+u_k(-1)(\partial_s f)_{(-1)}E_{sk}(2)\omega_{(-1)}h+2(\partial_k f)_{(-1)}u_k(-1)h+6(\partial_k f)_{(-2)}u_k(0)h$$
$$=-f_{(-2)}h-2f_{(-2)}h+3f_{(-1)}Dh-f_{(-1)}u_p(-1)\partial_p h$$
$$-2f_{(-1)}Dh+u_k(-1)(\partial_k f)_{(-1)}h+2f_{(-2)}h=0.$$

And finally for $n=3$:

$$\big(\omega_{(-1)}f+u_k(-1)E_{sk}(-1)\partial_s f-u_k(-2)\partial_k f\big)_{(3)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$=f_{(2)}\omega_{(0)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+f_{(1)}\omega_{(1)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+f_{(0)}\omega_{(2)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+f_{(-1)}\omega_{(3)}\big(\omega_{(-1)}h+u_p(-1)E_{mp}(-1)\partial_m h-u_p(-2)\partial_p h\big)$$
$$+(E_{sk}(-1)\partial_s f)_{(1)}u_k(1)\omega_{(-1)}h+(E_{sk}(-1)\partial_s f)_{(0)}u_k(2)\omega_{(-1)}h$$
$$+2(\partial_k f)_{(0)}u_k(1)\omega_{(-1)}h+3(\partial_k f)_{(-1)}u_k(2)\omega_{(-1)}h$$
$$=f_{(2)}\omega_{(-2)}h+f_{(2)}\omega_{(-1)}Dh+2f_{(1)}\omega_{(-1)}h+3f_{(0)}\omega_{(0)}h+4f_{(-1)}\omega_{(1)}h$$
$$+(E_{sk}(-1)\partial_s f)_{(1)}u_k(-1)h+2(E_{sk}(-1)\partial_s f)_{(0)}u_k(0)h$$
$$+2(\partial_k f)_{(0)}u_k(-1)h+6(\partial_k f)_{(-1)}u_k(0)h$$
$$=f_{(0)}Dh+3f_{(0)}Dh+(\partial_s f)_{(0)}E_{sk}(0)u_k(-1)h+(\partial_s f)_{(-1)}E_{sk}(1)u_k(-1)h=0.$$

This completes the proof of Theorem 4.1.

Let us discuss some applications of our results to the chiral de Rham complex, constructed by Malikov-Schechtman-Vaintrob [9, 8]. Here we specialize to the case $\mathfrak{g}=(0)$. Under this assumption, the sheaf $\mathcal{V}$ has a local description

$$\mathcal{V}(U_i)=V_{\mathfrak{Hei}}\otimes V_{\mathfrak{gl}_N}\otimes V_{\mathfrak{Vir}}\otimes\mathcal{O}_X(U_i),$$

where $V_{\mathfrak{Vir}}$ is the universal enveloping vertex algebra for $\mathfrak{Vir}$ of rank 0. The sheaf $\mathcal{V}$ is a module for the sheaf $\mathrm{Vect}(\widehat{X})$ of Lie algebras. The chiral de Rham complex is a sheaf Ω_{ch} of vertex superalgebras on X with a local description

$$\Omega_{ch}(U_i)=V_{\mathfrak{Hei}}\otimes V_{\mathbb{Z}^N}\otimes\mathcal{O}_X(U_i),$$

where $V_{\mathbb{Z}^N}$ is a vertex superalgebra associated with the standard Euclidean lattice $\mathbb{Z}^N$. Malikov-Schechtman-Vaintrob use a fermionic realization of $V_{\mathbb{Z}^N}$ with $V_{\mathbb{Z}^N}$ being $\mathbb{Z}$-graded by fermionic degree,

$$V_{\mathbb{Z}^N}=\bigoplus_{k=-\infty}^{\infty}V_{\mathbb{Z}^N}^{(k)}.$$

Each component $V_{\mathbb{Z}^N}^{(k)}$ is an irreducible highest weight module for $\widehat{\mathfrak{gl}}_N$ at level 1. This induces a $\mathbb{Z}$-grading on the chiral de Rham complex, $\Omega_{ch} = \bigoplus_{k=-\infty}^{\infty} \Omega_{ch}^{(k)}$. The chiral differential is a map

$$d: \ \Omega_{ch}^{(k)} \to \Omega_{ch}^{(k+1)}$$

(see [9] for details).

Theorem 4.3. (i) Each component $\Omega_{ch}^{(k)}$ of the chiral de Rham complex is a module for the sheaf $\mathrm{Vect}(\widehat{X})$ of Lie algebras of vector fields.

(ii) The differential $d: \ \Omega_{ch}^{(k)} \to \Omega_{ch}^{(k+1)}$ is a homomorphism of modules for the sheaf $\mathrm{Vect}(\widehat{X})$.

Proof. Applying Theorem 4.1 with $\mathfrak{g} = (0)$, $L_{\mathfrak{gl}_N} = V_{\mathbb{Z}^N}^{(k)}$ and $L_{\mathfrak{Vir}}$ being trivial 1-dimensional module for the Virasoro algebra, we obtain that $\Omega_{ch}^{(k)}$ is a module for the sheaf of Lie algebras $\overline{\Omega}^1_{\widehat{X}} \rtimes \mathrm{Vect}(\widehat{X})$, from which the first claim follows.

The proof of the second claim is completely analogous to Theorem 10.3 in [3], where the case of a torus is considered.

Remark. Note that although $d: \ \Omega_{ch}^{(k)} \to \Omega_{ch}^{(k+1)}$ is a homomorphism of $\mathrm{Vect}(\widehat{X})$-modules, it does not commute with the action of $\overline{\Omega}^1_{\widehat{X}}$.

5. Lie algebras associated with the field of rational functions on X.

In conclusion of the paper we outline a version of our construction in the setting of rational functions. Let $\mathbb{C}(X)$ be the field of rational functions on X and let $R = \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}(X)$. Consider the corresponding R-modules Ω^1_R of 1-forms on $\widehat{X}$, and Vect_R of vector fields on $\widehat{X}$. Formally these objects may be defined as direct limits over non-empty open subsets $U \subset X$:

$$\Omega^1_R = \varinjlim_U \Omega^1_{\widehat{X}}(U), \qquad \mathrm{Vect}_R = \varinjlim_U \mathrm{Vect}_{\widehat{X}}(U).$$

We consider the Lie algebra

$$\mathcal{G}_R = R \otimes \mathfrak{g} \oplus \left(\Omega^1_R / dR\right) \oplus \mathrm{Vect}_R,$$

where the definition of the Lie bracket is completely analogous to one given in section 1. Alternatively, $\mathcal{G}_R$ may be defined as a direct limit

$$\mathcal{G}_R = \varinjlim_U \mathcal{G}(U).$$

The field $\mathbb{C}(X)$ is a finite extension of a purely transcendental extension of $\mathbb{C}$, $\mathbb{C}(X) = \mathbb{C}(x_1, \ldots, x_N; y_1, \ldots, y_s)$, where $\{x_1, \ldots, x_N\}$ are algebraically independent and y_i's are algebraic over $\mathbb{C}(x_1, \ldots, x_N)$. Note that Ω^1_R is a free R-module with the free generators $dt, dx_1, \ldots, dx_N$, and Vect_R is a free R-module with the generators $\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}$.

We can use $\{x_1, \ldots, x_N\}$ in place of the local coordinates to define the vertex algebra

$$\mathcal{V}_R = V_{\mathfrak{Hei}} \otimes V_{\mathfrak{gl}_N} \otimes V_{\mathfrak{g}} \otimes V_{\mathfrak{Vir}} \otimes \mathbb{C}(X).$$

Our construction yields the following result:

Theorem 5.1. Let $\mathcal{V}_R = V_{\mathfrak{Hei}} \otimes V_{\mathfrak{gl}_N} \otimes V_{\mathfrak{g}} \otimes V_{\mathfrak{Vir}} \otimes \mathbb{C}(X)$ be the vertex algebra with the central charges of the tensor factors the same as in Theorem 4.1.

(i) $\mathcal{V}_R$ is a module for the Lie algebra $\mathcal{G}_R$, where the action is given by (4.1)-(4.5).

(ii) Let $L_{\mathfrak{gl}_N}$, $L_{\mathfrak{g}}$, $L_{\mathfrak{Vir}}$ be irreducible modules for the vertex algebras $V_{\mathfrak{gl}_N}$, $V_{\mathfrak{g}}$, $V_{\mathfrak{Vir}}$ respectively. Then

$$\mathcal{L}_R = V_{\mathfrak{Hei}} \otimes L_{\mathfrak{gl}_N} \otimes L_{\mathfrak{g}} \otimes L_{\mathfrak{Vir}} \otimes \mathbb{C}(X)$$

is an irreducible module for the Lie algebra $\mathcal{G}_R$.

Proof. The only claim that requires a proof here is the irreducibility of $\mathcal{L}_R$. Everything else follows immediately from Theorems 4.1 and 4.3 by passing to the direct limit.

To show that $\mathcal{L}_R$ is irreducible as a $\mathcal{G}_R$-module, we note that the fields in (4.1)-(4.5) that define the action of $\mathcal{G}_R$ on $\mathcal{V}_R$ generate the vertex algebra $\mathcal{V}_R$. Since $L_{\mathfrak{gl}_N}$, $L_{\mathfrak{g}}$, $L_{\mathfrak{Vir}}$ are irreducible modules for the vertex algebras $V_{\mathfrak{gl}_N}$, $V_{\mathfrak{g}}$, $V_{\mathfrak{Vir}}$ respectively, and the vertex algebra $V_{\mathfrak{Hei}} \otimes \mathbb{C}(X)$ is simple, we conclude that $\mathcal{L}_R$ is an irreducible $\mathcal{G}_R$-module.

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