

Sine-Gordon equation and representations of affine Kac-Moody algebra \widehat{sl}_2 .

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We give a representation-theoretic interpretation of the sine-Gordon equation using the action of affine Kac-Moody algebra \widehat{sl}_2 on the Weyl algebra. In this setup τ -functions become functions in non-commuting variables. The skew Casimir operators that we introduce here, give rise to a hierarchy of partial differential equations that includes the sine-Gordon equation and two copies of the Korteweg-de Vries equation.

0. Introduction

One of the most fascinating applications of the Kac-Moody theory is the use of affine Lie algebras and their groups to exhibit hidden symmetries of soliton equations. In 1981 M. Sato [20] and Drinfeld-Sokolov [6] (see also [7]) discovered fundamental links between soliton equations and infinite-dimensional Lie groups. In an important sequence of papers [2], [3], [4], Date, Jimbo, Kashiwara and Miwa gave a construction of the Kadomtsev-Petviashvili and Korteweg-de Vries hierarchies based on the representation theory of affine Kac-Moody algebras. The affine algebra a_∞ produces the KP hierarchy, while the KdV hierarchy is linked to \widehat{sl}_2 .

Date, Jimbo, Kashiwara and Miwa used the vertex operator realization of the basic highest weight module, which was discovered earlier by Lepowsky and Wilson [18]. In this realization for \widehat{sl}_2 , the space of the basic module is identified with the polynomial algebra in infinitely many variables $\mathbf{x} = (x_1, x_3, x_5, \dots)$. A completion of this space is interpreted as the space of τ -functions. In this framework, the Bäcklund transformation that raises the soliton number of an N-soliton τ -function, is given by the exponential of a vertex operator.

Kac and Wakimoto [16] discovered that the Casimir operator equation

$$\Omega(\tau \otimes \tau) = 0 \tag{0.1}$$

decomposes in a hierarchy of PDEs in Hirota form.

The sine-Gordon equation

$$u_{xt} = \sin(u) \quad (0.2)$$

is of great significance to both mathematics and physics. This equation appeared first in geometry and describes the surfaces of constant negative curvature in \mathbb{R}^3 . The physical importance of the sine-Gordon equation is related to the fact that it is a soliton equation which is manifestly Lorentz-invariant.

The sine-Gordon equation played a special role in the development of the soliton theory. The Bäcklund transformation method, which is a precursor for the Lie theory approach, was first developed in the geometric setup for this equation. Before the connection between soliton equations and infinite-dimensional Lie algebras was discovered in 1980's, Mandelstam wrote a paper [19] on quantized sine-Gordon equation, in which he used operators somewhat similar to the vertex operators.

From the AKNS method [1] it is known that the sine-Gordon equation is related to \widehat{sl}_2 , but the representation-theoretic interpretation of this equation was missing.

The main goal of the present paper is to fill this gap. Our starting point is the following observation. From the original Hirota's paper [13] we know that the sine-Gordon equation (0.2) can be written as a system of equations in Hirota form on a pair of functions (τ_0, τ_1)

$$\begin{cases} D_x D_t (\tau_0 \circ \tau_0 - \tau_1 \circ \tau_1) = 0 \\ D_x D_t (\tau_0 \circ \tau_1) = \tau_0 \tau_1, \end{cases} \quad (0.3)$$

where u is related to (τ_0, τ_1) by $u = 4 \arctan\left(\frac{\tau_1}{\tau_0}\right)$. To describe the soliton solutions in terms of τ -functions, we put τ_0 and τ_1 together into $\tau = \tau_0 + i\tau_1$. Then the N -soliton solution for the sine-Gordon is given by (cf. [13]):

$$\begin{aligned} \tau &= \tau_0 + i\tau_1 = \\ &= 1 + \sum_{k=1}^N \sum_{1 \leq j_1 < \dots < j_k \leq N} \alpha_{j_1} \dots \alpha_{j_k} i^k \prod_{1 \leq r < s \leq k} \left(\frac{z_{j_r} - z_{j_s}}{z_{j_r} + z_{j_s}} \right)^2 \exp \left(\left(\sum_{s=1}^k z_{j_s}^{-1} \right) t + \left(\sum_{s=1}^k z_{j_s} \right) x \right) \end{aligned}$$

with real-valued α_j 's and z_j 's.

If we compare this with the expression for the N -soliton solution for the KdV hierarchy [11], [14]

$$\tau = 1 + \sum_{k=1}^N \sum_{1 \leq j_1 < \dots < j_k \leq N} \alpha_{j_1} \dots \alpha_{j_k} \prod_{1 \leq r < s \leq k} \left(\frac{z_{j_r} - z_{j_s}}{z_{j_r} + z_{j_s}} \right)^2 \exp \left(\sum_{m \in \mathbb{N}_{\text{odd}}} \left(\sum_{s=1}^k z_{j_s}^m \right) x_m \right),$$

then it becomes clear that time t in the sine-Gordon equation is nothing but variable x_{-1} missing in the KdV picture! The “correct” set of variables should include both positive and negative odd integers: $\mathbf{x} = (\dots, x_{-3}, x_{-1}, x_1, x_3, \dots)$. We shall use the symbols $\mathbb{Z}_{\text{odd}}, \mathbb{Z}_{\text{ev}}, \mathbb{N}_{\text{odd}}, \mathbb{N}_{\text{ev}}$ to denote odd/even integers/natural numbers.

Our extended hierarchy will be completely symmetric in positive and negative directions. This means that the \widehat{sl}_2 -module that we need to construct should not be of the highest/lowest weight. However, the Casimir operator is a crucial ingredient for the construction of the hierarchy of differential equations (see (0.1) above). Formally, this operator is well-defined only for the highest/lowest weight modules. Nevertheless, we are still able to use the Casimir operator in our situation.

In the vertex operator realization of the basic highest weight module, \widehat{sl}_2 acts by differential operators on the algebra of polynomials $\mathbb{C}[x_1, x_3, x_5, \dots]$. In this paper we represent \widehat{sl}_2 by the same differential operators, but now acting on the algebra of differential operators in variables x_1, x_3, x_5, \dots , by left multiplication. The algebra of differential operators is isomorphic to the Weyl algebra \mathcal{W} generated by p_i, q_j , $i, j \in \mathbb{N}_{\text{odd}}$ with relations

$$p_i p_j = p_j p_i, \quad q_i q_j = q_j q_i, \quad p_i q_j - q_j p_i = c_i \delta_{ij} \cdot 1, \quad c_i \in \mathbb{C} \setminus \{0\}.$$

Though the Weyl algebra is non-commutative, still the partial differentiations $\frac{\partial}{\partial p_i}$, $\frac{\partial}{\partial q_j}$ are well-defined. This allows us to view the elements of the Weyl algebra as functions on a quantized space.

The Weyl algebra has a Poincaré-Birkhoff-Witt decomposition

$$\mathcal{W} = \mathbb{C}[q_1, q_3, \dots] \otimes \mathbb{C}[p_1, p_3, \dots].$$

Since both subalgebras $\mathbb{C}[q_1, q_3, \dots]$ and $\mathbb{C}[p_1, p_3, \dots]$ are commutative, there is a linear bijection between \mathcal{W} and the polynomial algebra $\mathbb{C}[\dots, x_{-3}, x_{-1}, x_1, x_3, \dots]$:

$$\pi : \mathcal{W} \rightarrow \mathbb{C}[x_1, x_3, \dots] \otimes \mathbb{C}[x_{-1}, x_{-3}, \dots].$$

The image of a differential operator under map π is called the symbol of a differential operator (see e.g., [17]).

The idea of our approach is to write a Kac-Moody group invariant equation on a Weyl algebra element $\hat{\tau}$ and then convert it into a hierarchy of partial differential equations in Hirota form on the symbol $\tau = \pi(\hat{\tau})$. Then we can use the Kac-Moody group action to generate the soliton solutions from the trivial solution $\hat{\tau} = 1$.

The function $\hat{\tau} = 1$ is not anymore a solution of (0.1), since we have changed the action to be the left multiplication and 1 now represents the identity operator. However, $\hat{\tau} = 1$ is trivially a solution of

$$\Omega(\hat{\tau} \otimes \hat{\tau}) - (\hat{\tau} \otimes \hat{\tau})\Omega = 0. \quad (0.4)$$

This algebraic equation transforms into a hierarchy of the non-linear PDEs in the variables $\mathbf{x} = (\dots, x_{-3}, x_{-1}, x_1, x_3, \dots)$ which contains two KdV subhierarchies going in positive and negative directions.

The simplest new equation in this double KdV hierarchy is the following (cf. (4.34)):

$$u_{yt} = \frac{\partial}{\partial x} (u_{xxy} + u_x u_y + u_x) \quad (0.5)$$

with $x = x_1, y = x_{-1}, t = x_3$. This is a generalization of the KdV with two spatial variables, but different from the Kadomtsev-Petviashvili equation.

The sine-Gordon equation can not possibly occur in the double KdV hierarchy (0.4) since the system of Hirota equations (0.3) can not be expressed in terms of $\tau = \tau_0 + i\tau_1$ alone.

Using the boson-fermion correspondence we construct “skew” Casimir operators that split τ into even and odd parts (Proposition 2.2). It remains a mystery whether the skew Casimir operators are related to the Dirac operator. We construct a hierarchy of equations by considering skew analogs of (0.4) (see (2.7), (2.8)).

As expected, the sine-Gordon equation with respect to the variables x_1 and x_{-1} appears in it, “linking” the two KdV hierarchies.

The generalization of the present work to the case of \widehat{sl}_n should yield a hierarchy containing the finite Toda chain. The infinite Toda chain was derived by Kac and van de Leur [15] from the two component vertex operator representation of a_∞ . It would be interesting to understand the connection between these two approaches.

Here is a brief description of the structure of the paper. In Section 1 we construct a representation of affine Lie algebra a_∞ on a completion of the Weyl algebra \mathcal{W} . We also review the boson-fermion correspondence and the construction of the Casimir operator. In Section 2 we consider the reduction of this representation to \widehat{sl}_2 , introduce the skew Casimir operators and study their properties. In Section 3 we show how to convert the skew Casimir operator equations into a generating series of PDEs in Hirota form and construct N -soliton solutions for these equations. In the final section we derive non-linear partial differential equations from the Hirota equations. We also obtain the Leibnitz formula for the Hirota formalism and use it for the treatment of systems of Hirota equations.

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1. Boson-fermion correspondence and representations of a_∞

In the previous works on applications of infinite-dimensional Lie algebras to soliton equations the main idea is to represent Lie algebras by differential operators. In this setting the Casimir operator equation becomes an infinite hierarchy of non-linear PDEs in Hirota form. The fact that the Casimir operator commutes with the Lie algebra allows one to construct soliton solutions of this hierarchy.

The approach that we take here is to use the action not on the space of functions, but rather on the Weyl algebra, or, equivalently, on the space of differential operators themselves.

We consider the Weyl algebra \mathcal{W} , an associative algebra with 1 generated by the elements $p_i, q_i, i \in \mathbb{N}$ with the defining relations

$$p_i q_j - q_j p_i = -\frac{4}{j} \delta_{ij} 1, \quad p_i p_j = p_j p_i, \quad q_i q_j = q_j q_i, \quad i, j \in \mathbb{N}. \quad (1.1)$$

The scaling factor $-\frac{4}{j}$ in the first of these relations is chosen to put the vertex operators considered below, in a more symmetric form. Essentially, \mathcal{W} is a factor of the universal enveloping algebra of the Heisenberg algebra in which the central element is identified with 1.

The Weyl algebra has a natural representation on the space of polynomials in infinitely many variables (Fock space) $F = \mathbb{C}[x_1, x_2, \dots]$ by the operators of differentiation and multiplication:

$$p_j \mapsto -\frac{2}{j} \frac{\partial}{\partial x_j}, \quad q_j \mapsto 2x_j, \quad j \in \mathbb{N}.$$

Thus the Weyl algebra can be also interpreted as the algebra of differential operators on F .

In spite of the non-commutativity of the Weyl algebra, the partial derivatives $\frac{\partial}{\partial p_j}, \frac{\partial}{\partial q_j}$ are well defined (which follows from the fact that the relations (1.1) survive the partial differentiation). These partial derivatives are inner derivations of \mathcal{W} ([5], section 4.6):

$$\frac{\partial}{\partial p_j} (f(\mathbf{q}, \mathbf{p})) = \frac{j}{4} [q_j, f(\mathbf{q}, \mathbf{p})], \quad \frac{\partial}{\partial q_j} (f(\mathbf{q}, \mathbf{p})) = -\frac{j}{4} [p_j, f(\mathbf{q}, \mathbf{p})], \quad f(\mathbf{q}, \mathbf{p}) \in \mathcal{W}. \quad (1.2)$$

Here $\mathbf{q} = (q_1, q_2, \dots)$, $\mathbf{p} = (p_1, p_2, \dots)$.

The exponentials $\exp(\alpha \frac{\partial}{\partial p_j})$, $\alpha \in \mathbb{C}$, (resp. $\exp(\alpha \frac{\partial}{\partial q_j})$) are the shift operators $p_i \mapsto p_i + \delta_{ij}\alpha$, $q_i \mapsto q_i$, (resp. $p_i \mapsto p_i$, $q_i \mapsto q_i + \delta_{ij}\alpha$), which are well-defined commuting automorphisms of \mathcal{W} .

Clearly, an element $f(\mathbf{q}, \mathbf{p}) \in \mathcal{W}$ can be written in many ways. We call the normal form of $f(\mathbf{q}, \mathbf{p})$ its presentation in which the generators q_i 's are grouped to the left of p_j 's in all monomials. The normal form can be found using the standard Poincaré-Birkhoff-Witt procedure. It also corresponds to the decomposition $\mathcal{W} = \mathcal{W}^- \otimes \mathcal{W}^+$, where \mathcal{W}^- (resp. \mathcal{W}^+) is an abelian algebra generated by q_1, q_2, \dots (resp. p_1, p_2, \dots).

The subalgebra \mathcal{W}^- (resp. \mathcal{W}^+) is isomorphic to the algebra of polynomials in infinitely many variables $\mathbb{C}[x_1, x_2, \dots]$ (resp. $\mathbb{C}[x_{-1}, x_{-2}, \dots]$). Consider the isomorphism of vector spaces

$$\begin{aligned} \pi : \quad \mathcal{W}^- \otimes \mathcal{W}^+ &\rightarrow \mathbb{C}[x_1, x_2, \dots, x_{-1}, x_{-2}, \dots], \\ \pi(f(q_1, q_2, \dots)g(p_1, p_2, \dots)) &= f(x_1, x_2, \dots)g(x_{-1}, x_{-2}, \dots). \end{aligned}$$

The map π coincides with the notion of a symbol of a differential operator expressed in the Weyl algebra language. This map allows us to convert the elements of the Weyl algebra into ordinary functions in commuting variables.

We define the normally ordered product $:fg:$ of $f(\mathbf{q}, \mathbf{p})$ and $g(\mathbf{q}, \mathbf{p})$ to be

$$:f(\mathbf{q}, \mathbf{p})g(\mathbf{q}, \mathbf{p}): := \pi^{-1}(\pi(f(\mathbf{q}, \mathbf{p}))\pi(g(\mathbf{q}, \mathbf{p}))).$$

We will consider a \mathbb{Z} -grading of $\mathcal{W} = \bigoplus_{n \in \mathbb{Z}} \mathcal{W}_n$ by assigning degrees to the generators as follows:

$$\deg(p_i) = i, \quad \deg(q_i) = -i, \quad i \in \mathbb{N}.$$

Next we introduce a completion $\overline{\mathcal{W}}$ of the Weyl algebra, in which the vertex operators will be later defined. We set the completion of \mathcal{W}_n to be

$$\overline{\mathcal{W}}_n = \prod_{\substack{i \leq 0, j \geq 0 \\ i+j=n}} \mathcal{W}_i^- \otimes \mathcal{W}_j^+ \tag{1.3}$$

and define $\overline{\mathcal{W}}$ as

$$\overline{\mathcal{W}} = \bigoplus_{n \in \mathbb{Z}} \overline{\mathcal{W}}_n. \tag{1.4}$$

It follows from the Poincaré-Birkhoff-Witt argument that $\overline{\mathcal{W}}$ has a well-defined structure of an associative algebra.

In physics literature the generators of the Weyl algebra p_i, q_i are called free bosons. The boson-fermion correspondence is a way of constructing free fermions out of free bosons and vice versa ([9],[14]). The free fermions are the generators ψ_i, ψ_i^* , $i \in \mathbb{Z}$, of the Clifford algebra \mathcal{C}_∞ satisfying the relations

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij}, \quad \psi_i \psi_j + \psi_j \psi_i = 0, \quad \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0, \quad i, j \in \mathbb{Z}.$$

Form the formal generating series (fermion fields)

$$\psi(z) = \sum_{k \in \mathbb{Z}} \psi_k z^{-k} \quad \text{and} \quad \psi^*(z) = \sum_{k \in \mathbb{Z}} \psi_k^* z^k.$$

Proposition 1.1. (cf. [14], Theorem 14.10) The Clifford algebra \mathcal{C}_∞ can be represented on the space $B = \mathbb{C}[u, u^{-1}] \otimes \overline{\mathcal{W}}$ by vertex operators

$$\psi(z) \mapsto z^{u \frac{\partial}{\partial u}} u \exp \left(\frac{1}{2} \sum_{j \in \mathbb{N}} q_j z^j \right) \exp \left(\frac{1}{2} \sum_{j \in \mathbb{N}} p_j z^{-j} \right), \tag{1.5}$$

$$\psi^*(z) \mapsto u^{-1} z^{-u \frac{\partial}{\partial u}} \exp \left(-\frac{1}{2} \sum_{j \in \mathbb{N}} q_j z^j \right) \exp \left(-\frac{1}{2} \sum_{j \in \mathbb{N}} p_j z^{-j} \right).$$

Here the correspondence between two formal series is the correspondence between their respective components.

Note that the components of the vertex operators belong to the algebra $B = \mathbb{C}[u, u^{-1}] \otimes \overline{\mathcal{W}}$ and act on B by left multiplication. The symbol $z^{u \frac{\partial}{\partial u}}$ is interpreted as $z^{u \frac{\partial}{\partial u}}(u^m) = z^m u^m$.

The only difference between the statement above and Theorem 14.10 in [14] is that here we consider the action of these vertex operators by left multiplication on $\mathbb{C}[u, u^{-1}] \otimes \overline{\mathcal{W}}$ and not on $\mathbb{C}[u, u^{-1}] \otimes F$ as differential operators. Proposition 1.1 is thus an immediate corollary.

The obvious relations between the operator of multiplication by u and the fields $\psi(z), \psi^*(z)$

$$\psi(z)u = zu\psi(z), \quad \psi^*(z)u = z^{-1}u\psi^*(z)$$

can be rewritten for their components as

$$\psi_k u = u\psi_{k+1}, \quad \psi_k^* u = u\psi_{k+1}^*. \tag{1.6}$$

The classical matrix Lie algebras can be embedded into Clifford algebras and Weyl algebras. It is well-known that this can be also done for the affine Kac-Moody algebras [8] and, in particular, for the affine algebra a_∞ of infinite rank.

The algebra a_∞ is a non-trivial one-dimensional central extension of the Lie algebra \overline{a}_∞ of infinite matrices with finitely many non-zero diagonals:

$$\overline{a}_\infty = \left\{ \sum_{i,j \in \mathbb{Z}} a_{ij} E_{ij} \mid \exists n \in \mathbb{N} \quad |i - j| > n \Rightarrow a_{ij} = 0 \right\},$$

$$a_\infty = \overline{a}_\infty \oplus \mathbb{C}1.$$

The action of a_∞ on $\mathbb{C}[u, u^{-1}] \otimes \overline{\mathcal{W}}$ is determined by the formula:

$$E_{ij} \mapsto \begin{cases} \psi_i \psi_j^* & \text{if } i \neq j \text{ or } i = j > 0, \\ -\psi_j^* \psi_i & \text{if } i = j \leq 0. \end{cases}$$

Using (1.5) we can write a generating series for this action ([14], 14.10.9):

$$\sum_{i,j \in \mathbb{Z}} E_{ij} z_1^i z_2^{-j} \mapsto \frac{1}{1 - \frac{z_2}{z_1}} \left(\left(\frac{z_1}{z_2} \right)^{u \frac{\partial}{\partial u}} \Gamma(z_1, z_2) - 1 \right),$$

where

$$\Gamma(z_1, z_2) = \exp \left(\frac{1}{2} \sum_{j=1}^{\infty} (z_1^j - z_2^j) q_j \right) \exp \left(\frac{1}{2} \sum_{j=1}^{\infty} (z_1^{-j} - z_2^{-j}) p_j \right).$$

In order to get a connection with the differential equations in Hirota form, we should also consider the tensor square $B \otimes B$ of the a_∞ -module $B = \mathbb{C}[u, u^{-1}] \otimes \overline{\mathcal{W}}$.

The key link is the Casimir operator ([14], 14.11.2)

$$\Omega = \sum_{k \in \mathbb{Z}} \psi_k \otimes \psi_k^*$$

which acts on a completion $\overline{B \otimes B}$ and commutes with the a_∞ -action ([14], 14.11). To construct this completion we view $\mathcal{W} \otimes \mathcal{W}$ as a Weyl algebra in twice as many generators and take its completion as in (1.3), (1.4). The algebra $\overline{B \otimes B}$ is the tensor product of $\overline{\mathcal{W} \otimes \mathcal{W}}$ with two copies of $\mathbb{C}[u, u^{-1}]$.

2. Skew Casimir operators

It is well-known that affine Kac-Moody algebra $\widehat{sl}_2 = sl_2(\mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}K$ may be embedded into a_∞ by

$$H_{2j+1} \mapsto \sum_{i \in \mathbb{Z}} E_{i, i+2j+1}, \quad A_j \mapsto \sum_{i \in \mathbb{Z}} (-1)^{i+j} E_{i, i+j}, \quad K \mapsto 1, \quad j \in \mathbb{Z}, \quad (2.1)$$

where

$$H_{2j+1} = \begin{pmatrix} 0 & t^j \\ t^{j+1} & 0 \end{pmatrix}, \quad A_{2j} = \begin{pmatrix} -t^j & 0 \\ 0 & t^j \end{pmatrix}, \quad A_{2j+1} = \begin{pmatrix} 0 & t^j \\ -t^{j+1} & 0 \end{pmatrix}.$$

The restriction of the representation of a_∞ on B to this subalgebra yields (cf. [14], 14.13):

$$H_k \mapsto -\frac{k}{2} p_k, \quad H_{-k} \mapsto \frac{k}{2} q_k, \quad k \in \mathbb{N}_{\text{odd}}, \quad (2.2)$$

$$\sum_{j \in \mathbb{Z}} A_j z^{-j} \mapsto \frac{1}{2} \left((-1)^{u \frac{\partial}{\partial u}} \Gamma(z) - 1 \right), \quad (2.3)$$

where

$$\Gamma(z) = \exp \left(\sum_{j \in \mathbb{N}_{\text{odd}}} q_j z^j \right) \exp \left(\sum_{j \in \mathbb{N}_{\text{odd}}} p_j z^{-j} \right). \quad (2.4)$$

Lie algebra $\mathfrak{g} = \widehat{sl}_2$ has a \mathbb{Z}_2 -grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where \mathfrak{g}_0 is the principal Heisenberg subalgebra spanned by H_j , $j \in \mathbb{Z}_{\text{odd}}$ and K , while \mathfrak{g}_1 is spanned by $A_j + \frac{1}{2} \delta_{j,0} K$, $j \in \mathbb{Z}$. The following lemma is an immediate consequence of the formulas (2.2)-(2.4) above:

Lemma 2.1. (i) $xu = ux$ for $x \in \mathfrak{g}_0$,

(ii) $xu = -ux$ for $x \in \mathfrak{g}_1$.

This result can also be seen from the realization of elements of \mathfrak{g} as infinite matrices. The conjugation by u is a shift operator: $u^{-1} E_{ij} u = E_{i+1, j+1} - \delta_{i,0} \delta_{j,0} 1$. It follows from (2.1) that the elements in \mathfrak{g}_0 are represented by matrices invariant under this shift, while elements of \mathfrak{g}_1 are represented by anti-invariant matrices.

The subspace $1 \otimes \overline{\mathcal{W}}$ in B is clearly invariant under a_∞ and \widehat{sl}_2 -actions. While the Casimir operator Ω does not leave the space $\overline{\mathcal{W}} \otimes \overline{\mathcal{W}}$ invariant, it can be easily modified into an operator Λ_0 , for which $\overline{\mathcal{W}} \otimes \overline{\mathcal{W}}$ is invariant:

$$\Lambda_0 = (1 \otimes u) \Omega (u^{-1} \otimes 1).$$

Moreover using the conjugation by $1 \otimes u^k$ we obtain a whole family of skew Casimir operators:

$$\Lambda_k = (1 \otimes u^{k+1}) \Omega (u^{-1} \otimes u^{-k}), \quad k \in \mathbb{Z}.$$

Note that the conjugation by elements $u^k \otimes 1$ produces the same family because Ω commutes with $u \otimes u$.

It is easy to see that

$$\sum_{k \in \mathbb{Z}} \Lambda_k z^{-k} = (\psi(z)u^{-1}) \otimes (u\psi^*(z)).$$

Restricted to $\overline{\mathcal{W} \otimes \mathcal{W}}$, the generating series $\Lambda(z) = \sum_{k \in \mathbb{Z}} \Lambda_k z^{-k}$ can be explicitly written as

$$\Lambda(z) = \exp\left(\frac{1}{2} \sum_{j \in \mathbb{N}} q_j z^j\right) \exp\left(\frac{1}{2} \sum_{j \in \mathbb{N}} p_j z^{-j}\right) \otimes \exp\left(-\frac{1}{2} \sum_{j \in \mathbb{N}} q_j z^j\right) \exp\left(-\frac{1}{2} \sum_{j \in \mathbb{N}} p_j z^{-j}\right).$$

The skew Casimir operators Λ_k do not anymore commute with the action of a_∞ , but as we see in the following Proposition, behave nicely relatively to the subalgebra \widehat{sl}_2 in a_∞ .

Proposition 2.2. (a) For $x \in \mathfrak{g}_0$

$$(x \otimes 1 + 1 \otimes x)\Lambda_k = \Lambda_k(x \otimes 1 + 1 \otimes x), \quad k \in \mathbb{Z}.$$

(b) For $x \in \mathfrak{g}_1$

$$(x \otimes 1 + 1 \otimes x)\Lambda_k = -\Lambda_k(x \otimes 1 + 1 \otimes x), \quad k \in \mathbb{Z}_{\text{odd}},$$

$$(x \otimes 1 - 1 \otimes x)\Lambda_k = -\Lambda_k(x \otimes 1 - 1 \otimes x), \quad k \in \mathbb{Z}_{\text{ev}}.$$

Proof. Recall that $\Lambda_k = (1 \otimes u^{k+1})\Omega(u^{-1} \otimes u^{-k})$. Part (a) is obvious because $xu = ux$ for $x \in \mathfrak{g}_0$ by Lemma 2.1 and $x \otimes 1 + 1 \otimes x$ commutes with the Casimir operator Ω .

Verification of (b) is also straightforward. Suppose that $x \in \mathfrak{g}_1$ and k is odd. By Lemma 2.1, x commutes with the even powers of u and anticommutes with the odd powers of u . Thus

$$\begin{aligned} & (x \otimes 1 + 1 \otimes x)(1 \otimes u^{k+1})\Omega(u^{-1} \otimes u^{-k}) \\ &= (1 \otimes u^{k+1})(x \otimes 1 + 1 \otimes x)\Omega(u^{-1} \otimes u^{-k}) \\ &= -(1 \otimes u^{k+1})\Omega(u^{-1} \otimes u^{-k})(x \otimes 1 + 1 \otimes x). \end{aligned}$$

If k is even then

$$\begin{aligned} & (x \otimes 1 - 1 \otimes x)(1 \otimes u^{k+1})\Omega(u^{-1} \otimes u^{-k}) \\ &= (1 \otimes u^{k+1})(x \otimes 1 + 1 \otimes x)\Omega(u^{-1} \otimes u^{-k}) \end{aligned}$$

$$= -(1 \otimes u^{k+1})\Omega(u^{-1} \otimes u^{-k})(x \otimes 1 - 1 \otimes x).$$

Let $\hat{\tau} = \hat{\tau}(z_1, z_2, \dots)$ be a formal Laurent series in finitely many formal real variables z_1, z_2, \dots with coefficients in $\overline{\mathcal{W}}$. Let $\hat{\tau}^*(z_1, z_2, \dots)$ be the complex conjugate of $\hat{\tau}$.

Proposition 2.3. The sets of solutions $\hat{\tau}(z_1, z_2, \dots)$ of equations

$$(\hat{\tau} \otimes \hat{\tau})\Lambda_k = \Lambda_k(\hat{\tau}^* \otimes \hat{\tau}^*), \quad k \in \mathbb{Z}_{\text{odd}}, \quad (2.5)$$

and

$$(\hat{\tau} \otimes \hat{\tau}^*)\Lambda_k = \Lambda_k(\hat{\tau}^* \otimes \hat{\tau}), \quad k \in \mathbb{Z}_{\text{ev}}, \quad (2.6)$$

are invariant under the transformation

$$\hat{\tau}(z_1, z_2, \dots) \mapsto \hat{\tau}'(z_1, z_2, \dots, z) = \hat{\tau}(z_1, z_2, \dots) \exp(\alpha i \Gamma(z)),$$

where $\alpha \in \mathbb{R}$ and z is a formal real variable.

The equations (2.5) and (2.6) are understood here as equalities of the corresponding $\overline{\mathcal{W}} \otimes \mathcal{W}$ -valued coefficients in the formal Laurent series.

Proof. Since the components of $\Gamma(z)$ represent the elements of \mathfrak{g}_1 , we get from Proposition 2.2 that for $k \in \mathbb{Z}_{\text{odd}}$

$$\begin{aligned} \exp(\alpha i \Gamma(z)) \otimes \exp(\alpha i \Gamma(z)) \Lambda_k &= \exp(\alpha i (\Gamma(z) \otimes 1 + 1 \otimes \Gamma(z))) \Lambda_k \\ &= \Lambda_k \exp(-\alpha i (\Gamma(z) \otimes 1 + 1 \otimes \Gamma(z))) = \Lambda_k \exp(-\alpha i \Gamma(z)) \otimes \exp(-\alpha i \Gamma(z)), \end{aligned}$$

and in a similar way for $k \in \mathbb{Z}_{\text{ev}}$:

$$\exp(\alpha i \Gamma(z)) \otimes \exp(-\alpha i \Gamma(z)) \Lambda_k = \Lambda_k \exp(-\alpha i \Gamma(z)) \otimes \exp(\alpha i \Gamma(z)).$$

Suppose that $\hat{\tau}$ is a solution of the equation (2.5)

$$(\hat{\tau} \otimes \hat{\tau})\Lambda_k = \Lambda_k(\hat{\tau}^* \otimes \hat{\tau}^*), \quad k \in \mathbb{Z}_{\text{odd}}.$$

Then for $\hat{\tau}' = \hat{\tau} \exp(\alpha i \Gamma(z))$ we have

$$\begin{aligned} (\hat{\tau}' \otimes \hat{\tau}')\Lambda_k &= (\hat{\tau} \otimes \hat{\tau}) (\exp(\alpha i \Gamma(z)) \otimes \exp(\alpha i \Gamma(z))) \Lambda_k \\ &= (\hat{\tau} \otimes \hat{\tau})\Lambda_k (\exp(-\alpha i \Gamma(z)) \otimes \exp(-\alpha i \Gamma(z))) \\ &= \Lambda_k(\hat{\tau}^* \otimes \hat{\tau}^*) (\exp(-\alpha i \Gamma(z)) \otimes \exp(-\alpha i \Gamma(z))) = \Lambda_k((\hat{\tau}')^* \otimes (\hat{\tau}')^*). \end{aligned}$$

Thus $\hat{\tau}'$ is also a solution of (2.5). The case of (2.6) is completely analogous.

Corollary 2.4. The series $\hat{\tau} = (1 + \alpha_1 i\Gamma(z_1)) \dots (1 + \alpha_N i\Gamma(z_N))$ in formal real variables z_1, \dots, z_N with $\alpha_1, \dots, \alpha_N \in \mathbb{R}$ is a solution for both (2.5) and (2.6).

Proof. Trivially $\hat{\tau} = 1$ is a solution for both equations. Thus by the above proposition, $\hat{\tau} = \exp(\alpha_1 i\Gamma(z_1)) \dots \exp(\alpha_N i\Gamma(z_N))$ satisfies (2.5) and (2.6). Finally, $\exp(\alpha i\Gamma(z)) = 1 + \alpha i\Gamma(z)$ because $\Gamma^2(z) = 0$ ([14], 14.11.15), and the claim of the Corollary follows.

Consider the real and the imaginary parts of $\hat{\tau}$: $\hat{\tau}_0(z_1, z_2, \dots) = \text{Re}(\hat{\tau})$, $\hat{\tau}_1(z_1, z_2, \dots) = \text{Im}(\hat{\tau})$. The equations (2.5) and (2.6) can be rewritten as the systems

$$\begin{cases} (\hat{\tau}_0 \otimes \hat{\tau}_0 - \hat{\tau}_1 \otimes \hat{\tau}_1)\Lambda_k - \Lambda_k(\hat{\tau}_0 \otimes \hat{\tau}_0 - \hat{\tau}_1 \otimes \hat{\tau}_1) = 0 \\ (\hat{\tau}_0 \otimes \hat{\tau}_1 + \hat{\tau}_1 \otimes \hat{\tau}_0)\Lambda_k + \Lambda_k(\hat{\tau}_0 \otimes \hat{\tau}_1 + \hat{\tau}_1 \otimes \hat{\tau}_0) = 0 \end{cases} \quad k \in \mathbb{Z}_{\text{odd}} \quad (2.7)$$

and

$$\begin{cases} (\hat{\tau}_0 \otimes \hat{\tau}_0 + \hat{\tau}_1 \otimes \hat{\tau}_1)\Lambda_k - \Lambda_k(\hat{\tau}_0 \otimes \hat{\tau}_0 + \hat{\tau}_1 \otimes \hat{\tau}_1) = 0 \\ (\hat{\tau}_0 \otimes \hat{\tau}_1 - \hat{\tau}_1 \otimes \hat{\tau}_0)\Lambda_k + \Lambda_k(\hat{\tau}_0 \otimes \hat{\tau}_1 - \hat{\tau}_1 \otimes \hat{\tau}_0) = 0 \end{cases} \quad k \in \mathbb{Z}_{\text{ev}}. \quad (2.8)$$

The requirement that α and z are real may be dropped if we appropriately adjust the statement of Proposition 2.3:

Proposition 2.5. The sets of solutions $(\hat{\tau}_0, \hat{\tau}_1)$ of the systems (2.7) and (2.8) are invariant under the transformation

$$(\hat{\tau}_0, \hat{\tau}_1) \mapsto (\hat{\tau}'_0, \hat{\tau}'_1) = (\hat{\tau}_0, \hat{\tau}_1) \begin{pmatrix} 1 & \alpha\Gamma(z) \\ -\alpha\Gamma(z) & 1 \end{pmatrix}$$

where z is a formal variable and $\alpha \in \mathbb{C}$.

The proof parallels the one of Proposition 2.3.

3. Hirota bilinear equations and their solutions

The differential nature of the Weyl algebra is encoded in its defining relations (1.1). In this section we shall see how to convert purely algebraic equations

$$(\hat{\tau} \otimes \hat{\tau})\Lambda_k = \Lambda_k(\hat{\tau}^* \otimes \hat{\tau}^*), \quad k \in \mathbb{Z}_{\text{odd}},$$

$$(\hat{\tau} \otimes \hat{\tau}^*)\Lambda_k = \Lambda_k(\hat{\tau}^* \otimes \hat{\tau}), \quad k \in \mathbb{Z}_{\text{ev}},$$

into a hierarchy of differential equations in Hirota form on the “dequantized” functions $\tau_0 = \pi(\hat{\tau}_0)$, $\tau_1 = \pi(\hat{\tau}_1)$.

When a product of two “functions” in \mathcal{W} is written in the normally ordered form, expressions involving partial derivatives (1.2) appear. One special case of this procedure is recorded in the following Lemma and will be later used in our calculations.

Lemma 3.1. Let p, q be generators of the Weyl algebra satisfying $pq - qp = c \cdot 1$. Then for $\hat{\tau} = \hat{\tau}(q, p)$ we have

$$\exp(zp)\hat{\tau} = \left\{ \exp\left(cz\frac{\partial}{\partial q}\right)\hat{\tau} \right\} \exp(zp)$$

and

$$\hat{\tau}\exp(zq) = \exp(zq)\left\{ \exp\left(cz\frac{\partial}{\partial p}\right)\hat{\tau} \right\}.$$

Here z is a formal variable and the above equalities are interpreted as equalities of formal power series.

Proof. We shall prove the second identity, the first will then follow by applying the automorphism $p \mapsto q, q \mapsto -p$. From (1.2) we get that

$$\hat{\tau}(q, p)q = q\hat{\tau}(q, p) + c\frac{\partial\hat{\tau}}{\partial p} = \left(q + c\frac{\partial}{\partial p}\right)\hat{\tau}(q, p).$$

By induction we obtain

$$\hat{\tau}(q, p)q^n = \left(q + c\frac{\partial}{\partial p}\right)^n \hat{\tau}(q, p).$$

To complete the proof, we apply the above equality to the Taylor expansion of $\exp(zq)$:

$$\begin{aligned} \hat{\tau}(q, p)\exp(zq) &= \exp\left(zq + cz\frac{\partial}{\partial p}\right)\hat{\tau}(q, p) \\ &= \exp(zq)\left\{ \exp\left(cz\frac{\partial}{\partial p}\right)\hat{\tau}(q, p) \right\}. \end{aligned}$$

At this point we are ready to transform (2.7) and (2.8) into a hierarchy of Hirota bilinear equations. Note that the terms involved in these equations are of the form $(f \otimes g)\Lambda_k \pm \Lambda_k(f \otimes g)$. We shall work with such expressions using the generating series $(f \otimes g)\Lambda(z) \pm \Lambda(z)(f \otimes g)$, $f \otimes g \in \overline{\mathcal{W}} \otimes \overline{\mathcal{W}}$. Denote the variables in the first copy of $\overline{\mathcal{W}}$ by q'_i, p'_i and in the second copy of $\overline{\mathcal{W}}$ by q''_i, p''_i . Then

$$\begin{aligned} &(f \otimes g)\Lambda(z) \pm \Lambda(z)(f \otimes g) = \\ &= f(\mathbf{q}', \mathbf{p}')g(\mathbf{q}'', \mathbf{p}'') \exp\left(\frac{1}{2}\sum_{j=1}^{\infty}(q'_j - q''_j)z^j\right) \exp\left(\frac{1}{2}\sum_{j=1}^{\infty}(p'_j - p''_j)z^{-j}\right) \\ &\pm \exp\left(\frac{1}{2}\sum_{j=1}^{\infty}(q'_j - q''_j)z^j\right) \exp\left(\frac{1}{2}\sum_{j=1}^{\infty}(p'_j - p''_j)z^{-j}\right) f(\mathbf{q}', \mathbf{p}')g(\mathbf{q}'', \mathbf{p}''). \end{aligned}$$

We shall assume that both $f(\mathbf{q}', \mathbf{p}')$ and $g(\mathbf{q}'', \mathbf{p}'')$ are in the normal form. Moreover since p'_i commutes with q'_j , we can replace $f(\mathbf{q}', \mathbf{p}')g(\mathbf{q}'', \mathbf{p}'')$ with the normally ordered product $: f(\mathbf{q}', \mathbf{p}')g(\mathbf{q}'', \mathbf{p}'') :$, i.e. move all q 's to the left of all p 's.

Making the change of variables $p_i = \frac{1}{2}(p'_i + p''_i)$, $\tilde{p}_i = \frac{1}{2}(p'_i - p''_i)$, $q_i = \frac{1}{2}(q'_i + q''_i)$, $\tilde{q}_i = \frac{1}{2}(q'_i - q''_i)$, we rewrite the above expression as

$$\begin{aligned} & : f(\mathbf{q} + \tilde{\mathbf{q}}, \mathbf{p} + \tilde{\mathbf{p}})g(\mathbf{q} - \tilde{\mathbf{q}}, \mathbf{p} - \tilde{\mathbf{p}}) : \exp\left(\sum_{j=1}^{\infty} \tilde{q}_j z^j\right) \exp\left(\sum_{j=1}^{\infty} \tilde{p}_j z^{-j}\right) \\ & \pm \exp\left(\sum_{j=1}^{\infty} \tilde{q}_j z^j\right) \exp\left(\sum_{j=1}^{\infty} \tilde{p}_j z^{-j}\right) : f(\mathbf{q} + \tilde{\mathbf{q}}, \mathbf{p} + \tilde{\mathbf{p}})g(\mathbf{q} - \tilde{\mathbf{q}}, \mathbf{p} - \tilde{\mathbf{p}}) : . \end{aligned}$$

Collecting \tilde{q} 's on the left and \tilde{p} 's on the right using Lemma 3.1 (note that $[\tilde{p}_i, \tilde{q}_j] = -\frac{2}{j}\delta_{ij} \cdot 1$), we get

$$\begin{aligned} & \exp\left(\sum_{j=1}^{\infty} \tilde{q}_j z^j\right) \times \\ & \left\{ \left(\exp\left(-2 \sum_{j=1}^{\infty} \frac{z^j}{j} \frac{\partial}{\partial \tilde{p}_j}\right) \pm \exp\left(-2 \sum_{j=1}^{\infty} \frac{z^{-j}}{j} \frac{\partial}{\partial \tilde{q}_j}\right) \right) : f(\mathbf{q} + \tilde{\mathbf{q}}, \mathbf{p} + \tilde{\mathbf{p}})g(\mathbf{q} - \tilde{\mathbf{q}}, \mathbf{p} - \tilde{\mathbf{p}}) : \right\} \\ & \times \exp\left(\sum_{j=1}^{\infty} \tilde{p}_j z^{-j}\right). \quad (3.1) \end{aligned}$$

We can convert this expression into the Hirota form. Recall that the Hirota bilinear differentiation is defined as

$$\begin{aligned} & P(D_x, D_y, \dots) [f(x, y, \dots) \circ g(x, y, \dots)] := \\ & P\left(\frac{\partial}{\partial \tilde{x}}, \frac{\partial}{\partial \tilde{y}}, \dots\right) f(x + \tilde{x}, y + \tilde{y}, \dots)g(x - \tilde{x}, y - \tilde{y}, \dots)|_{\tilde{x}=0, \tilde{y}=0, \dots}. \end{aligned}$$

Using the Taylor formula we get that (see [14], 14.11.8)

$$\begin{aligned} & P\left(\frac{\partial}{\partial \tilde{x}}, \frac{\partial}{\partial \tilde{y}}, \dots\right) f(x + \tilde{x}, y + \tilde{y}, \dots)g(x - \tilde{x}, y - \tilde{y}, \dots) = \\ & P(D_x, D_y, \dots) \exp(\tilde{x}D_x + \tilde{y}D_y + \dots) [f(x, y, \dots) \circ g(x, y, \dots)]. \end{aligned}$$

Applying this identity we transform (3.1) into

$$\exp\left(\sum_{j=1}^{\infty} \tilde{q}_j z^j\right) \left\{ \exp\left(\sum_{j=1}^{\infty} \tilde{q}_j D_{q_j}\right) \left(\exp\left(-2 \sum_{j=1}^{\infty} \frac{z^j}{j} D_{p_j}\right) \pm \exp\left(-2 \sum_{j=1}^{\infty} \frac{z^{-j}}{j} D_{q_j}\right) \right) \right. \\ \left. \exp\left(\sum_{j=1}^{\infty} \tilde{p}_j D_{p_j}\right) [: f(\mathbf{q}, \mathbf{p}) \circ g(\mathbf{q}, \mathbf{p}) :] \right\} \exp\left(\sum_{j=1}^{\infty} \tilde{p}_j z^{-j}\right). \quad (3.2)$$

We use the reduction to the subalgebra \widehat{sl}_2 in a_{∞} to construct solutions of (2.5) and (2.6). We see from (2.2)-(2.4) that this reduction involves only variables p_j, q_j with $j \in \mathbb{N}_{\text{odd}}$, so the solutions we get in Corollary 2.4 do not depend on the even-indexed variables. Thus in (3.2) we may set $D_{p_j} = 0, D_{q_j} = 0$ for $j \in \mathbb{N}_{\text{ev}}$. In (3.2) all q 's are collected to the left of all p 's and all \tilde{q} 's are to the left of all \tilde{p} 's. Because of that we can easily evaluate the image of this expression under the map π , which gives us:

$$R_{\pm}(z)[\pi(f)(\mathbf{x}) \circ \pi(g)(\mathbf{x})],$$

where

$$R_{\pm}(z) = \exp\left(\sum_{j \in \mathbb{Z} \setminus \{0\}} \tilde{x}_j z^j\right) \exp\left(\sum_{j \in \mathbb{Z}_{\text{odd}}} \tilde{x}_j D_{x_j}\right) \\ \times \left\{ \exp\left(-2 \sum_{j \in \mathbb{N}_{\text{odd}}} \frac{z^j}{j} D_{x_{-j}}\right) \pm \exp\left(-2 \sum_{j \in \mathbb{N}_{\text{odd}}} \frac{z^{-j}}{j} D_{x_j}\right) \right\}$$

and $\mathbf{x} = (\dots, x_{-3}, x_{-1}, x_1, x_3, \dots)$. Here $R_{\pm}(z)$ is a Laurent series in z and a Taylor series in \tilde{x}_j . The coefficients of this series are Hirota differential operators.

Now we can rewrite (2.7) and (2.8) as the generating series for the hierarchy of Hirota bilinear equations on $\tau_0 = \pi(\hat{\tau}_0)$ and $\tau_1 = \pi(\hat{\tau}_1)$:

$$\text{Res}\left(z^j R_{-}(z)\right) (\tau_0 \circ \tau_0 - \tau_1 \circ \tau_1) = 0, \quad j \in \mathbb{Z}_{\text{ev}}, \quad (3.3)$$

$$\text{Res}\left(z^j R_{+}(z)\right) (\tau_0 \circ \tau_1 + \tau_1 \circ \tau_0) = 0, \quad j \in \mathbb{Z}_{\text{ev}}, \quad (3.4)$$

$$\text{Res}\left(z^j R_{-}(z)\right) (\tau_0 \circ \tau_0 + \tau_1 \circ \tau_1) = 0, \quad j \in \mathbb{Z}_{\text{odd}}, \quad (3.5)$$

$$\text{Res} (z^j R_+(z)) (\tau_0 \circ \tau_1 - \tau_1 \circ \tau_0) = 0, \quad j \in \mathbb{Z}_{\text{odd}}. \quad (3.6)$$

As usual, the residue denotes the coefficient at z^{-1} of a formal Laurent series. The coefficient at each monomial in \hat{x} 's in the formal equations above is a Hirota bilinear equation on τ_0 and τ_1 . We shall study the Hirota equations that occur here in the next section. Now we turn to describing solutions of this hierarchy.

From Corollary 2.4 it follows that

$$\tau_0 + i\tau_1 = \pi ((1 + \alpha_1 i\Gamma(z_1)) \dots (1 + \alpha_N i\Gamma(z_N)))$$

is a solution of (3.3) - (3.6). From [14] we get that the right hand side converges to

$$1 + \sum_{k=1}^N \sum_{1 \leq j_1 < \dots < j_k \leq N} \alpha_{j_1} \dots \alpha_{j_k} i^k \prod_{1 \leq r < s \leq k} \left(\frac{z_{j_r} - z_{j_s}}{z_{j_r} + z_{j_s}} \right)^2 \exp \left(\sum_{m \in \mathbb{Z}_{\text{odd}}} \left(\sum_{s=1}^k z_{j_s}^m \right) x_m \right) \quad (3.7)$$

when $|z_1| > |z_2| > \dots > |z_N|$. By analytic continuation we conclude that

$$\begin{aligned} \tau_0 = 1 + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} (-1)^k \sum_{1 \leq j_1 < \dots < j_{2k} \leq N} \alpha_{j_1} \dots \alpha_{j_{2k}} \prod_{1 \leq r < s \leq 2k} \left(\frac{z_{j_r} - z_{j_s}}{z_{j_r} + z_{j_s}} \right)^2 \\ \times \exp \left(\sum_{m \in \mathbb{Z}_{\text{odd}}} \left(\sum_{s=1}^{2k} z_{j_s}^m \right) x_m \right), \end{aligned} \quad (3.8)$$

$$\begin{aligned} \tau_1 = \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^k \sum_{1 \leq j_1 < \dots < j_{2k+1} \leq N} \alpha_{j_1} \dots \alpha_{j_{2k+1}} \prod_{1 \leq r < s \leq 2k+1} \left(\frac{z_{j_r} - z_{j_s}}{z_{j_r} + z_{j_s}} \right)^2 \\ \times \exp \left(\sum_{m \in \mathbb{Z}_{\text{odd}}} \left(\sum_{s=1}^{2k+1} z_{j_s}^m \right) x_m \right) \end{aligned} \quad (3.9)$$

is a solution of the hierarchy (3.3)-(3.6) for all z_1, \dots, z_N .

As we shall see in the next section, this hierarchy contains the sine-Gordon equation, as well as two copies of the Korteweg - de Vries hierarchy.

4. The KdV – sine-Gordon – KdV hierarchy

In this section we transform the Hirota equations (3.3)-(3.6) into non-linear PDEs. We show that among other equations, this hierarchy contains the sine-Gordon equation, the KdV and the modified KdV equations. We begin by listing some of the Hirota equations that are coefficients at monomials in (3.3)-(3.6) (we multiply them by appropriate constants to avoid fractional coefficients). For simplicity of notations, we write D_j for D_{x_j} .

From the series (3.3):

$$D_{-1}D_1(\tau_0 \circ \tau_0 - \tau_1 \circ \tau_1) = 0 \quad (\text{at } x_1z), \quad (4.1)$$

$$(D_1^4 - D_1D_3)(\tau_0 \circ \tau_0 - \tau_1 \circ \tau_1) = 0 \quad (\text{at } x_1z^{-3}), \quad (4.2)$$

$$(D_{-1}D_3 + 2D_{-1}D_1^3 - 3D_1^2)(\tau_0 \circ \tau_0 - \tau_1 \circ \tau_1) = 0 \quad (\text{at } x_{-1}z^{-3}), \quad (4.3)$$

$$(D_{-1}D_3 + D_1^2)(\tau_0 \circ \tau_0 - \tau_1 \circ \tau_1) = 0 \quad (\text{at } x_3z). \quad (4.4)$$

From the series (3.4):

$$(D_{-1}D_1 - 1)(\tau_1 \circ \tau_0) = 0 \quad (\text{at } x_1z), \quad (4.5)$$

$$(D_1^4 - D_1D_3)(\tau_1 \circ \tau_0) = 0 \quad (\text{at } x_1z^{-3}), \quad (4.6)$$

$$(D_1^2 - D_{-1}D_3)(\tau_1 \circ \tau_0) = 0 \quad (\text{at } x_3z), \quad (4.7)$$

$$(D_1^2 - D_{-1}D_1^3)(\tau_1 \circ \tau_0) = 0 \quad (\text{at } x_1^3z). \quad (4.8)$$

From the series (3.5):

$$D_1^2(\tau_0 \circ \tau_0 + \tau_1 \circ \tau_1) = 0 \quad (\text{at } x_1^2z^0), \quad (4.9)$$

$$(D_1^4 + 2D_1D_3)(\tau_0 \circ \tau_0 + \tau_1 \circ \tau_1) = 0 \quad (\text{at } x_1x_3z^0), \quad (4.10)$$

$$(D_{-1}D_1^3 - D_{-1}D_3)(\tau_0 \circ \tau_0 + \tau_1 \circ \tau_1) = 0 \quad (\text{at } x_{-1}x_1z^{-2}). \quad (4.11)$$

And finally from the series (3.6):

$$(D_1^3 - D_3)(\tau_1 \circ \tau_0) = 0 \quad (\text{at } x_1z^{-2}), \quad (4.12)$$

$$(D_{-1}D_1^2 - D_1)(\tau_1 \circ \tau_0) = 0 \quad (\text{at } x_{-1}z^{-2}), \quad (4.13)$$

$$(D_1^5 - D_5)(\tau_1 \circ \tau_0) = 0 \quad (\text{at } x_1z^{-4}), \quad (4.14)$$

$$(2D_1^5 - 5D_1^2D_3 + 3D_5)(\tau_1 \circ \tau_0) = 0 \quad (\text{at } x_3z^{-2}). \quad (4.15)$$

One of the transformations that we consider below is $u = 4 \arctan\left(\frac{\tau_1}{\tau_0}\right)$ or, equivalently, $\frac{\tau_1}{\tau_0} = \tan\left(\frac{u}{4}\right)$. Let more generally $\frac{\tau_1}{\tau_0} = \varphi(u)$, hence $\tau_1 = \varphi\tau_0$.

In order to rewrite systems of Hirota equations as non-linear PDEs in function u , we need the Leibnitz rule Lemma for the Hirota differential operators. For a multi-index $\beta = (\beta_1, \dots, \beta_n)$ let $D^\beta = D_{x_1}^{\beta_1} \dots D_{x_n}^{\beta_n}$ be the Hirota differential operator and let $\partial^\beta =$

$\partial_{x_1}^{\beta_1} \dots \partial_{x_n}^{\beta_n}$ be the usual differential operator. For a pair of multi-indices β and γ with

$$0 \leq \gamma_j \leq \beta_j \text{ let } \binom{\beta}{\gamma} = \binom{\beta_1}{\gamma_1} \dots \binom{\beta_n}{\gamma_n}.$$

Lemma 4.1.

$$(a) D^\beta(\varphi f \circ g) = \sum_{\gamma} \binom{\beta}{\gamma} \partial^\gamma(\varphi) D^{\beta-\gamma}(f \circ g).$$

$$(b) D^\beta(\varphi_1 f \circ \varphi_2 g) = \sum_{\gamma} \binom{\beta}{\gamma} D^\gamma(\varphi_1 \circ \varphi_2) D^{\beta-\gamma}(f \circ g).$$

The proof of this Lemma is straightforward.

Sine-Gordon equation. Consider (4.1) and (4.5):

$$\begin{cases} D_{-1}D_1(\tau_0 \circ \tau_0 - \tau_1 \circ \tau_1) = 0 \\ (D_{-1}D_1 - 1)(\tau_1 \circ \tau_0) = 0 \end{cases} \quad (4.16)$$

and set $\tau_1 = \varphi\tau_0$. Applying Lemma 4.1, we transform this system into

$$\begin{cases} (1 - \varphi^2)D_{-1}D_1(\tau_0 \circ \tau_0) - D_{-1}D_1(\varphi \circ \varphi)\tau_0^2 = 0 \\ \varphi D_{-1}D_1(\tau_0 \circ \tau_0) + (\partial_{-1}\partial_1(\varphi) - \varphi)\tau_0^2 = 0. \end{cases}$$

Interpreting this as a system of linear equations on $D_{-1}D_1(\tau_0 \circ \tau_0)$ and τ_0^2 we get that

$$\det \begin{pmatrix} 1 - \varphi^2 & -D_{-1}D_1(\varphi \circ \varphi) \\ \varphi & \partial_{-1}\partial_1(\varphi) - \varphi \end{pmatrix} = 0 \quad (4.17)$$

since this system has a non-trivial solution.

If we now substitute $\varphi = \tan\left(\frac{u}{4}\right)$ and multiply (4.17) by $\cos^4\left(\frac{u}{4}\right)$ then we obtain the sine-Gordon equation:

$$\frac{\partial^2 u}{\partial x_{-1}\partial x_1} = \sin(u). \quad (4.18)$$

If we let parameters α_j in (3.8), (3.9) be purely imaginary then τ_1 will become purely imaginary, while τ_0 will remain real. In this case u will be purely imaginary, $u = iU$, $\frac{\tau_1}{i\tau_0} = \tanh\left(\frac{U}{4}\right)$, and the function U satisfies the sinh-Gordon equation:

$$\frac{\partial^2 U}{\partial x_{-1}\partial x_1} = \sinh(U). \quad (4.19)$$

Modified KdV equation. The same technique applied to (4.9) and (4.12)

$$\begin{cases} D_1^2(\tau_0 \circ \tau_0 + \tau_1 \circ \tau_1) = 0 \\ (D_1^3 - D_3)(\tau_1 \circ \tau_0) = 0 \end{cases} \quad (4.20)$$

yields

$$\begin{cases} (1 + \varphi^2)D_1^2(\tau_0 \circ \tau_0) + D_1^2(\varphi \circ \varphi)\tau_0^2 = 0 \\ 3\partial_1(\varphi)D_1^2(\tau_0 \circ \tau_0) + (\partial_1^3(\varphi) - \partial_3(\varphi))(\varphi \circ \varphi)\tau_0^2 = 0. \end{cases} \quad (4.21)$$

After the same substitution as above, $\varphi = \tan\left(\frac{u}{4}\right)$, multiplying the determinant of this system by $\cos^4\left(\frac{u}{4}\right)$, we get the mKdV equation:

$$\frac{\partial u}{\partial x_3} = \frac{1}{2} \left(\frac{\partial u}{\partial x_1} \right)^3 + \frac{\partial^3 u}{\partial x_1^3}, \quad (4.22)$$

or for $v = \frac{\partial u}{\partial x_1}$:

$$\frac{\partial v}{\partial x_3} = \frac{3}{2} v^2 \frac{\partial v}{\partial x_1} + \frac{\partial^3 v}{\partial x_1^3}. \quad (4.23)$$

Mixed sine-Gordon – mKdV equation. The system of three equations (4.4), (4.7) and (4.9)

$$\begin{cases} (D_1^2 + D_{-1}D_3)(\tau_0 \circ \tau_0 - \tau_1 \circ \tau_1) = 0 \\ (D_1^2 - D_{-1}D_3)(\tau_1 \circ \tau_0) = 0 \\ D_1^2(\tau_0 \circ \tau_0 + \tau_1 \circ \tau_1) = 0 \end{cases} \quad (4.24)$$

becomes a system of three linear equations on $D_1^2(\tau_0 \circ \tau_0)$, $D_{-1}D_3(\tau_0 \circ \tau_0)$, τ_0^2 and from its determinant

$$\det \begin{pmatrix} 1 - \varphi^2 & 1 - \varphi^2 & (D_1^2 + D_{-1}D_3)(\varphi \circ \varphi) \\ \varphi & -\varphi & \partial_1^2(\varphi) - \partial_{-1}\partial_3(\varphi) \\ 1 + \varphi^2 & 0 & D_1^2(\varphi \circ \varphi) \end{pmatrix} = 0 \quad (4.25)$$

after substitution $\varphi = \tan\left(\frac{u}{4}\right)$ we obtain the equation

$$\frac{\partial^2 u}{\partial x_{-1}\partial x_3} = \frac{1}{2} \sin(u) \left(\frac{\partial u}{\partial x_1} \right)^2 + \cos(u) \frac{\partial^2 u}{\partial x_1^2}. \quad (4.26)$$

One can show that this equation follows from the sine-Gordon and mKdV equations (4.18), (4.22) taken together, however (4.26) may have an independent physical meaning.

The second mKdV equation. The system of Hirota equations (4.9), (4.10), (4.14) and (4.15)

$$\begin{cases} (D_1^5 - D_5)(\tau_1 \circ \tau_0) = 0 \\ (2D_1^5 - 5D_1^2D_3 + 3D_5)(\tau_1 \circ \tau_0) = 0 \\ D_1^2(\tau_0 \circ \tau_0 + \tau_1 \circ \tau_1) = 0 \\ (D_1^4 + 2D_1D_3)(\tau_0 \circ \tau_0 + \tau_1 \circ \tau_1) = 0 \end{cases} \quad (4.27)$$

applying the same method as above and eliminating x_3 using (4.23), is transformed into the following equation for $v = 4 \frac{\partial}{\partial x_1} \arctan\left(\frac{\tau_1}{\tau_0}\right)$:

$$\frac{\partial v}{\partial x_5} = \frac{\partial}{\partial x_1} \left(\frac{3}{8} v^5 + \frac{5}{2} v^2 \frac{\partial^2 v}{\partial x_1^2} + \frac{5}{2} v \left(\frac{\partial v}{\partial x_1} \right)^2 + \frac{\partial^4 v}{\partial x_1^4} \right), \quad (4.28)$$

which is essentially the second equation in the mKdV hierarchy constructed by the AKNS method [1].

The double KdV hierarchy. It can be easily seen that the hierarchy (3.3)-(3.6) is invariant under the transformation $z \mapsto z^{-1}$, $x_j \mapsto x_{-j}$, i.e., it is symmetric in the positive and negative directions.

If in (3.3) and (3.4) we set $\tilde{x}_k = 0$ for $k < 0$ and consider non-negative values of j , then

$$\text{Res} (z^j R_{\pm}(z)) = \pm \text{Res} (z^j P(z)),$$

where

$$P(z) = \exp \left(\sum_{j=1}^{\infty} \tilde{x}_j z^j \right) \exp \left(\sum_{j \in \mathbb{N}_{\text{odd}}} \tilde{x}_j D_{x_j} \right) \exp \left(-2 \sum_{j \in \mathbb{N}_{\text{odd}}} \frac{z^{-j}}{j} D_{x_j} \right). \quad (4.29)$$

In this case the Hirota polynomials appearing in (3.3) and (3.4) differ only by sign. Combining (3.3) and (3.4) together we get that $\tau = \tau_0 + i\tau_1$ satisfies

$$\text{Res} (z^j P(z)) (\tau \circ \tau) = 0 \quad \text{for } j \in \mathbb{N}_{\text{ev}} \cup \{0\}. \quad (4.30)$$

The Hirota equation

$$(D_1^4 - D_1 D_3)(\tau \circ \tau) = 0 \quad (4.31)$$

(cf. (4.2), (4.6)) after the substitution $f = \frac{\partial^2}{\partial x_1^2} \ln(\tau)$ becomes the Korteweg - de Vries equation

$$\frac{\partial f}{\partial x_3} = 12f \frac{\partial f}{\partial x_1} + \frac{\partial^3 f}{\partial x_1^3}. \quad (4.32)$$

The series (4.30) is the KdV hierarchy [4], [14] in variables x_1, x_3, x_5, \dots . By symmetry, the second copy of the KdV hierarchy extends in the negative direction x_{-1}, x_{-3}, \dots . These two KdV hierarchies are linked together by the sine-Gordon equation (4.18) which involves variables x_1 and x_{-1} . The soliton solutions for the double KdV hierarchy are given by (3.7) with purely imaginary values of $\alpha_1, \dots, \alpha_N$.

Remark 4.2. Another way to construct this subhierarchy is to consider the Casimir operator Ω_{aff} for affine Kac-Moody algebra \widehat{sl}_2 (see [14]) acting on $\overline{\mathcal{W}} \otimes \overline{\mathcal{W}}$. The double KdV hierarchy then arises from the equation

$$(\hat{\tau} \otimes \hat{\tau}) \Omega_{\text{aff}} = \Omega_{\text{aff}}(\hat{\tau} \otimes \hat{\tau}).$$

The double KdV hierarchy that we obtain here contains more than just two copies of the KdV hierarchy. It also contains equations that involve both positive and negative variables. The simplest such an equation on the function $\tau = \tau_0 + i\tau_1$ is

$$(D_{-1} D_3 - 4D_{-1} D_1^3 + 3D_1^2) (\tau \circ \tau) = 0, \quad (4.33)$$

which follows from (4.3), (4.4), (4.7) and (4.8). A similar equation (without the last term) appears in the $D_4^{(1)}$ -hierarchy [16], but the solutions constructed here and in [16] are inequivalent.

After the substitution $g = \frac{\partial}{\partial x_1} \ln(\tau)$ we get

$$\frac{\partial^2 g}{\partial x_{-1} \partial x_3} = \frac{\partial}{\partial x_1} \left(4 \frac{\partial^3 g}{\partial x_{-1} \partial x_1^2} + 24 \frac{\partial g}{\partial x_{-1}} \frac{\partial g}{\partial x_1} - 3 \frac{\partial g}{\partial x_1} \right), \quad (4.34)$$

which is a generalization of the KdV with two spatial variables, but different from the Kadomtsev-Petviashvili equation.

It is interesting to note that the same equation appears in our hierarchy in a different way. If we combine (4.3), (4.4) with (4.9) and (4.11) then we get

$$(D_{-1} D_3 - D_{-1} D_1^3 + 3 D_1^2) (\tau_0 \circ \tau_0) = 0 \quad (4.35)$$

and

$$(D_{-1} D_3 - D_{-1} D_1^3 + 3 D_1^2) (\tau_1 \circ \tau_1) = 0. \quad (4.36)$$

These can be transformed into (4.33) by rescaling the variables $x_j = \frac{1}{2} x'_j$. Thus in addition to the solutions $g = \frac{\partial}{\partial x_1} \ln(\tau)$ of (4.34) with τ given by (3.7), we also get solutions

$$\begin{aligned} \tau = 1 + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} (-1)^k \sum_{1 \leq j_1 < \dots < j_{2k} \leq N} \alpha_{j_1} \dots \alpha_{j_{2k}} \prod_{1 \leq r < s \leq 2k} \left(\frac{z_{j_r} - z_{j_s}}{z_{j_r} + z_{j_s}} \right)^2 \\ \times \exp \left(\frac{1}{2} \left(\sum_{s=1}^{2k} z_{j_s}^{-1} \right) x_{-1} + \frac{1}{2} \left(\sum_{s=1}^{2k} z_{j_s} \right) x_1 + \frac{1}{2} \left(\sum_{s=1}^{2k} z_{j_s}^3 \right) x_3 \right) \end{aligned} \quad (4.37)$$

and

$$\begin{aligned} \tau = \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^k \sum_{1 \leq j_1 < \dots < j_{2k+1} \leq N} \alpha_{j_1} \dots \alpha_{j_{2k+1}} \prod_{1 \leq r < s \leq 2k+1} \left(\frac{z_{j_r} - z_{j_s}}{z_{j_r} + z_{j_s}} \right)^2 \\ \times \exp \left(\frac{1}{2} \left(\sum_{s=1}^{2k+1} z_{j_s}^{-1} \right) x_{-1} + \frac{1}{2} \left(\sum_{s=1}^{2k+1} z_{j_s} \right) x_1 + \frac{1}{2} \left(\sum_{s=1}^{2k+1} z_{j_s}^3 \right) x_3 \right). \end{aligned} \quad (4.38)$$

Miura transformation. It is well-known that solutions of the mKdV equation

$$\frac{\partial V}{\partial x_3} = -\frac{3}{2} V^2 \frac{\partial V}{\partial x_1} + \frac{\partial^3 V}{\partial x_1^3} \quad (4.39)$$

with $V = \frac{1}{i} v$ (cf. (4.23)) can be transformed into solutions of the KdV equation

$$\frac{\partial f}{\partial x_3} = 12f \frac{\partial f}{\partial x_1} + \frac{\partial^3 f}{\partial x_1^3} \quad (4.40)$$

by the Miura substitution

$$f = \frac{1}{4} \frac{\partial V}{\partial x_1} - \frac{1}{8} V^2. \quad (4.41)$$

The soliton solutions of (4.39) and (4.40) are given by $V = 4 \frac{\partial}{\partial x_1} \operatorname{arctanh} \left(\frac{1}{i} \frac{\tau_1}{\tau_0} \right) = 2 \frac{\partial}{\partial x_1} \ln \left(\frac{\tau_0 + i\tau_1}{\tau_0 - i\tau_1} \right)$ and $f = \frac{\partial^2}{\partial x_1^2} \ln(\tau_0 + i\tau_1)$, where τ_0 and τ_1 are given by (3.8) and (3.9) with purely imaginary parameters α_j . Note that $\tau_0 \pm i\tau_1$ is real in this case.

A natural question to ask is whether for a given pair (τ_0, τ_1) , the functions V and f are linked by the Miura transform (4.41). The answer to this question is positive, and as we see from the following Lemma, the Miura transform is a feature of the whole hierarchy and not specific just to the pair of the KdV and mKdV equations.

Lemma 4.3. The functions $V = 2 \frac{\partial}{\partial x_1} \ln \left(\frac{\tau_0 + i\tau_1}{\tau_0 - i\tau_1} \right)$ and $f = \frac{\partial^2}{\partial x_1^2} \ln(\tau_0 + i\tau_1)$ are related by the Miura transform (4.41) if and only if the Hirota equation (4.10) holds:

$$D_1^2(\tau_0 \circ \tau_0 + \tau_1 \circ \tau_1) = 0.$$

The proof of this Lemma is straightforward.

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