To Yuri Alexandrovich Bahturin

REPRESENTATIONS OF LIE ALGEBRA OF VECTOR FIELDS ON A TORUS AND CHIRAL DE RHAM COMPLEX

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Abstract. The goal of this paper is to study the representation theory of a classical infinite-dimensional Lie algebra – the Lie algebra $\mathrm{Vect}\mathbb{T}^N$ of vector fields on an N-dimensional torus for $N > 1$. The case $N = 1$ gives a famous Virasoro algebra (or its centerless version - the Witt algebra). The algebra $\mathrm{Vect}\mathbb{T}^N$ has an important class of tensor modules parametrized by finite-dimensional modules of gl_N. Tensor modules can be used in turn to construct bounded irreducible modules for $\mathrm{Vect}\mathbb{T}^{N+1}$ (induced from $\mathrm{Vect}\mathbb{T}^N$), which are the central objects of our study. We solve two problems regarding these induced modules: we construct their free field realizations and determine their characters. To solve these problems we analyze the structure of the irreducible $\Omega^1\left(\mathbb{T}^{N+1}\right)/d\Omega^0\left(\mathbb{T}^{N+1}\right) \rtimes \mathrm{Vect}\mathbb{T}^{N+1}$-modules constructed in [2]. These modules remain irreducible when restricted to the subalgebra $\mathrm{Vect}\mathbb{T}^{N+1}$, unless they belongs to the *chiral de Rham complex*, introduced by Malikov-Schechtman-Vaintrob [20].

1. Introduction.

In this paper we study the representation theory of a classical infinite-dimensional Lie algebra – the Lie algebra $\mathrm{Vect}\mathbb{T}^N$ of vector fields on a torus. This algebra has a class of representations of a geometric nature – tensor modules, since vector fields act on tensor fields of any given type via Lie derivative. Tensor modules are parametrized by finite-dimensional representations of gl_N, with the fiber of a tensor bundle being a gl_N-module.

Irreducible gl_N-modules yield tensor modules that are irreducible over $\mathrm{Vect}\mathbb{T}^N$, with exception of the modules of differential k-forms. In the latter case, the gl_N-module is irreducible, yet the modules of k-forms are reducible, which follows from the fact that the differential of the de Rham complex is a homomorphism of $\mathrm{Vect}\mathbb{T}^N$-modules. In the present paper we give a vertex algebra analogue of this result.

In case of a circle, a conjecture of Kac, proved by Mathieu [21], states that for the Lie algebra of vector fields on a circle an irreducible weight module with finite-dimensional weight spaces is either a tensor module or a highest/lowest weight module. There is a generalization of this conjecture to an arbitrary N due to Eswara Rao [7]. The analogues of the highest weight modules in this case are defined using the technique introduced by Berman-Billig [1]. These modules are bounded with respect to one of the variables. It follows from a general result of [1] that irreducible bounded modules for the Lie algebra of vector fields on a torus have finite-dimensional weight spaces, however the method of [1] yields no information on the dimensions of the weight spaces. This is the question that we solve in the present paper – we find explicit realizations of the irreducible bounded modules, using which the dimensions of the weight spaces may be readily determined.

A partial solution of this problem for the 2-dimensional torus was given by Billig-Molev-Zhang [3] using non-commutative differential equations in vertex algebras. The algebra of vector fields on $\mathbb{T}^2$ contains the loop algebra $\widetilde{sl}_2 = \mathbb{C}[t_0, t_0^{-1}] \otimes sl_2$. This subalgebra plays an important role in representation theory of $\mathrm{Vect}\mathbb{T}^2$. According to the results of [3], some of the bounded modules for $\mathrm{Vect}\mathbb{T}^2$ remain irreducible when restricted to the subalgebra $\widetilde{sl}_2$. Futorny classified in [11] irreducible generalized Verma modules for $\widetilde{sl}_2$. Such generalized Verma modules admit the action of the much larger algebra $\mathrm{Vect}\mathbb{T}^2$.

This relation between representations of $\widetilde{sl}_2$ and $\mathrm{Vect}\mathbb{T}^2$ suggests that for the Lie algebra of vector fields on $\mathbb{T}^N$, an important role is played by its subalgebra $\widetilde{sl}_N$. An unexpected twist here is that it is not the generalized Verma modules for $\widetilde{sl}_N$ that admit the action of $\mathrm{Vect}\mathbb{T}^N$ for $N > 2$, but rather the generalized Wakimoto modules. The generalized Wakimoto modules are

$\widetilde{sl}_N$-modules that have the same character as the generalized Verma modules, but need not to be isomorphic to them.

The generalized Wakimoto modules for $\widetilde{sl}_N$ that we use here were constructed in [2] in the context of the representation theory of toroidal Lie algebras, however their special properties with respect to the loop subalgebra $\widetilde{sl}_N$ were not previously recognized.

Let us outline the result of [2] that we use here. Since one of the variables plays a special role, it will be more convenient to work with an $(N+1)$-dimensional torus. To construct a full toroidal algebra, one begins with the algebra of $\dot{\mathfrak{g}}$-valued functions on $\mathbb{T}^{N+1}$:

$$\mathrm{Map}(\mathbb{T}^{N+1}, \dot{\mathfrak{g}}) \cong \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \dots, t_N^{\pm 1}] \otimes \dot{\mathfrak{g}},$$

where $\dot{\mathfrak{g}}$ is a finite-dimensional simple Lie algebra. Next we take the universal central extension of this multiloop algebra, with the center realized as the quotient of 1-forms on the torus by differentials of functions [14]:

$$\mathcal{K} = \Omega^1\left(\mathbb{T}^{N+1}\right)/d\Omega^0\left(\mathbb{T}^{N+1}\right).$$

Finally, one adds the Lie algebra of vector fields on the torus:

$$\left(\mathbb{C}[t_0^{\pm 1}, \dots, t_N^{\pm 1}] \otimes \dot{\mathfrak{g}} \oplus \mathcal{K}\right) \rtimes \mathrm{Vect}\mathbb{T}^{N+1}.$$

Irreducible bounded modules for this Lie algebra were constructed in [2] using vertex algebra methods. Note that the results of [2] admit a specialization to $\dot{\mathfrak{g}} = (0)$. The multiloop algebra then disappears, leaving behind, like the smile of the Cheshire Cat, the space of its central extension:

$$\mathcal{K} \rtimes \mathrm{Vect}\mathbb{T}^{N+1}.$$

It turns out that it is easier to study representations of this Lie algebra, rather than of vector fields alone, because of the duality between vector fields and 1-forms. Representation theory of this larger Lie algebra is controlled by a tensor product of three vertex algebras:

$$V^+_{Hyp} \otimes V_{gl_N} \otimes V_{Vir},$$

a subalgebra of a hyperbolic lattice vertex algebra, the affine $\widehat{gl}_N$ vertex algebra at level 1 and the Virasoro vertex algebra of rank 0. The tensor product of the first two components, $V^+_{Hyp} \otimes V_{gl_N}$ is one of the bounded modules for $\mathcal{K} \rtimes \mathrm{Vect}\mathbb{T}^{N+1}$, and in fact it is a generalized Wakimoto module for the subalgebra $\widetilde{sl}_{N+1} \subset \mathrm{Vect}\mathbb{T}^{N+1}$. Then the results of [3] suggest that there is a chance that $\mathcal{K} \rtimes \mathrm{Vect}\mathbb{T}^{N+1}$-modules constructed in [2] remain irreducible when restricted to $\mathrm{Vect}\mathbb{T}^{N+1}$. The study of this question is the main part of the present paper. The answer that we get is remarkably parallel to the classical picture with the tensor modules. We prove that a bounded irreducible $\mathcal{K} \rtimes \mathrm{Vect}\mathbb{T}^{N+1}$-module remains irreducible when restricted to the subalgebra of vector fields, unless it belongs to the *chiral de Rham complex*, introduced by Malikov-Schechtman-Vaintrob [20] (for arbitrary manifolds).

It is only in very special situations an irreducible module remains irreducible when restricted to a subalgebra. A prime example of this is the basic module for an affine Kac-Moody algebra, which remains irreducible when restricted to the principal Heisenberg subalgebra. This exceptional property of the basic module leads to its vertex operator realization and is at heart of several spectacular applications of this theory.

The space of the chiral de Rham complex is the vertex superalgebra

$$V^+_{Hyp} \otimes V_{\mathbb{Z}^N},$$

where $V_{\mathbb{Z}^N}$ is the lattice vertex superalgebra associated with the standard euclidean lattice $\mathbb{Z}^N$. The vertex superalgebra $V_{\mathbb{Z}^N}$ is graded by fermionic degree:

$$V_{\mathbb{Z}^N} = \bigoplus_{k \in \mathbb{Z}} V^k_{\mathbb{Z}^N},$$

and the components

$$V^+_{Hyp} \otimes V^k_{\mathbb{Z}^N}$$

are irreducible $\mathcal{K} \rtimes \mathrm{Vect}\mathbb{T}^{N+1}$-modules. Yet for these modules the restriction to the Lie algebra of vector fields is no longer irreducible since the differential of the chiral de Rham complex

$$\mathbf{d}: \ V^+_{Hyp} \otimes V^k_{\mathbb{Z}^N} \to V^+_{Hyp} \otimes V^{k+1}_{\mathbb{Z}^N}$$

is a homomorphism of $\mathrm{Vect}\mathbb{T}^{N+1}$-modules.

In fact it was noted in [18] that these components admit the action of the Lie algebra $\mathbb{C}[t_0, t_0^{-1}] \otimes \mathrm{Vect}\mathbb{T}^N$, but here we prove a much stronger result.

In conclusion, we make two curious observations. The Lie algebra of vector fields on a torus has a trivial center, yet its representation theory is described in terms of vertex algebras V^+_{Hyp} and V_{gl_N} that involve non-trivial central extensions. The central charges of these tensor factors cancel out to give a vertex algebra of total rank 0.

The final remark is that the chiral de Rham complex is an essentially super object, whereas the Lie algebra of vector fields we started with, is classical.

The structure of the paper is as follows. We introduce the main objects of our study in Sections 2, 3 and 4. We discuss vertex algebras and their applications to the representation theory of $\mathrm{Vect}\mathbb{T}^{N+1}$ in Sections 5 and 6. In Section 7 we introduce the generalized Wakimoto modules for the loop algebra $\widetilde{sl}_N$. We construct a non-degenerate pairing for the bounded modules in Section 8. We prove the main result on irreducibility in Section 9 and make a connection with the chiral de Rham complex in the final section of the paper.

2. Lie algebra of vector fields and its tensor modules

We begin with the algebra of Fourier polynomials on an N-dimensional torus $\mathbb{T}^N$. Introducing the variables $t_j = e^{ix_j}$, $j = 1, \ldots, N$, we realize the algebra of functions as Laurent polynomials $\mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}]$. The Lie algebra of vector fields on a torus is

$$\mathrm{Vect}\mathbb{T}^N = \mathrm{Der}\,\mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] = \mathop{\oplus}_{p=1}^{N} \mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] \frac{\partial}{\partial t_p}.$$

It will be more convenient for us to work with the degree derivations $d_p = t_p \frac{\partial}{\partial t_p}$ as the free generators of $\mathrm{Vect}\mathbb{T}^N$ as a $\mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}]$-module:

$$\mathrm{Vect}\mathbb{T}^N = \mathop{\oplus}_{p=1}^{N} \mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] d_p.$$

The Lie bracket in $\mathrm{Vect}\mathbb{T}^N$ is then written as

$$[t^r d_a, t^m d_b] = m_a t^{r+m} d_b - r_b t^{r+m} d_a, \quad a, b = 1, \ldots, N.$$

Here we are using the multi-index notations $t^r = t_1^{r_1} \ldots t_N^{r_N}$ for $r = (r_1, \ldots, r_N) \in \mathbb{Z}^N$.

The Cartan subalgebra $\langle d_1, \ldots, d_N \rangle$ acts on $\mathrm{Vect}\mathbb{T}^N$ diagonally and induces on it a $\mathbb{Z}^N$-grading.

The Lie algebra of vector fields (on any manifold) has a class of representations of a geometric nature. Vector fields act via Lie derivative on the space of tensor fields of a given type. The resulting tensor modules are parametrized by representations of gl_N. Let us describe the construction of tensor modules in case of a torus $\mathbb{T}^N$.

Fix a finite-dimensional gl_N-module W. In case when W is irreducible, the identity matrix acts as multiplication by a scalar $\alpha \in \mathbb{C}$. Let $\gamma \in \mathbb{C}^N$. We define the tensor module $T = T(W, \gamma)$ to be the vector space

$$T = q^\gamma \mathbb{C}[q_1^{\pm 1}, \ldots, q_N^{\pm 1}] \otimes W$$

with the action given by

$$t^r d_a (q^\mu \otimes w) = \mu_a q^{\mu + r} \otimes w + \sum_{p=1}^{N} r_p q^{\mu + r} \otimes E^{pa} w, \tag{2.1}$$

where $r \in \mathbb{Z}^N, \mu \in \gamma + \mathbb{Z}^N$, $a = 1, \ldots, N$ and E^{pa} is the matrix with 1 in (p, a)-position and zeros elsewhere.

Theorem 2.1. *[[6], cf. [26]] Let W be an irreducible finite-dimensional gl_N-module. The tensor module $T(W,\gamma)$ is an irreducible* $\mathrm{Vect}\mathbb{T}^N$*-module, unless it appears in the de Rham complex of differential forms*

$$q^\gamma\Omega^0(\mathbb{T}^N) \xrightarrow{\mathbf{d}} q^\gamma\Omega^1(\mathbb{T}^N) \xrightarrow{\mathbf{d}} \ \dots\ \xrightarrow{\mathbf{d}} q^\gamma\Omega^N(\mathbb{T}^N).$$

The middle terms in this complex are reducible $\mathrm{Vect}\mathbb{T}^N$*-modules, while the terms $q^\gamma\Omega^0(\mathbb{T}^N)$ and $q^\gamma\Omega^N(\mathbb{T}^N)$ are reducible whenever $\gamma \in \mathbb{Z}^N$.*

Note that de Rham differential $\mathbf{d}$ is a homomorphism of $\mathrm{Vect}\mathbb{T}^N$-modules. Let us specify irreducible gl_N-modules that correspond to the tensor modules in de Rham complex. The modules of functions Ω^0 and the module of differential N-forms Ω^N correspond to 1-dimensional gl_N-modules W on which the identity matrix acts as multiplication by $\alpha = 0$ and $\alpha = N$ respectively. The remaining modules Ω^k, $k = 1, \dots, N-1$, are the highest weight modules for sl_N with the fundamental highest weights ω_k and $\alpha = k$ (see e.g. [6]). Even though they correspond to irreducible gl_N-modules, tensor modules of differential forms are reducible since the kernels and images of the differential $\mathbf{d}$ are obviously the submodules in Ω^k.

3. Bounded modules

Our goal is to generalize to an arbitrary N the category of the highest weight modules over $\mathrm{Vect}\mathbb{T}^N$. In our constructions one of the coordinates will play a special role. From now on, we will be working with the $N+1$-dimensional torus and will index our coordinates as $t_0, t_1, \dots, t_N$, where t_0 is the "special variable". We would like to construct modules for the Lie algebra $\mathcal{D} = \mathrm{Vect}\mathbb{T}^{N+1}$ in which the "energy operator" $-d_0$ has spectrum bounded from below.

Let us consider a $\mathbb{Z}$-grading of $\mathcal{D}$ by degrees in t_0. This $\mathbb{Z}$-grading induces a decomposition

$$\mathcal{D} = \mathcal{D}_- \oplus \mathcal{D}_0 \oplus \mathcal{D}_+$$

into subalgebras of positive, zero and negative degrees in t_0. The degree zero part is

$$\mathcal{D}_0 = \mathop{\oplus}_{p=0}^{N} \mathbb{C}[t_1^{\pm 1}, \dots, t_N^{\pm 1}] d_p.$$

In particular, $\mathcal{D}_0$ is a semi-direct product of the Lie algebra of vector fields on $\mathbb{T}^N$ with an abelian ideal $\mathbb{C}[t_1^{\pm 1}, \dots, t_N^{\pm 1}] d_0$.

We begin the construction of a bounded module by taking a tensor module for $\mathcal{D}_0$. Fix a finite-dimensional irreducible gl_N-module W, $\beta \in \mathbb{C}$ and $\gamma \in \mathbb{C}^N$. We define a $\mathcal{D}_0$-module T as a space

$$T = q^\gamma \mathbb{C}[q_1^{\pm 1}, \dots, q_N^{\pm 1}] \otimes W$$

with the tensor module action (2.1) of the subalgebra $\mathrm{Vect}\mathbb{T}^N \subset \mathcal{D}_0$ and with $\mathbb{C}[t_1^{\pm 1}, \dots, t_N^{\pm 1}] d_0$ acting by shifts

$$t^r d_0 (q^\mu \otimes w) = \beta\, q^{\mu + r} \otimes w.$$

Next we let $\mathcal{D}_+$ act on T trivially and define $M(T)$ as the induced module

$$M(T) = \mathrm{Ind}_{\mathcal{D}_0 \oplus \mathcal{D}_+}^{\mathcal{D}} T \cong U(\mathcal{D}_-) \otimes T.$$

The module $M(T)$ has a weight decomposition with respect to the Cartan subalgebra $\langle d_0, \dots, d_N \rangle$ and the (real part of) spectrum of $-d_0$ on $M(T)$ is bounded from below. However the weight spaces of $M(T)$ that lie below T are all infinite-dimensional.

It turns out that the situation improves dramatically when we pass to the irreducible quotient of $M(T)$. One can immediately see that the Lie algebra $\mathcal{D}$ belongs to the class of Lie algebras with polynomial multiplication (as defined in [1]), whereas tensor modules belong to the class of modules with polynomial action. A general theorem of [1] (see also [4]) yields in this particular situation the following

Theorem 3.1. *([1]) (i) The module $M(T)$ has a unique maximal submodule M^{rad}.*

(ii) The irreducible quotient $L(T) = M(T)/M^{rad}$ has finite-dimensional weight spaces.

This leads to the following natural questions:

Problem 1. Determine the character of $L(T)$.
Problem 2. Find a realization of $L(T)$.

In [3] these problems were solved for some of the modules $L(T)$ in case of a 2-dimensional torus ($N = 1$). In the present paper we will give a solution in full generality for any N.

4. Toroidal Lie algebras

For a finite-dimensional simple Lie algebra $\dot{\mathfrak{g}}$ we consider a multiloop algebra $\mathbb{C}[t_0^{\pm 1}, \dots, t_N^{\pm 1}] \otimes \dot{\mathfrak{g}}$. Its universal central extension has a realization with center $\mathcal{K}$ identified as the quotient space of 1-forms by differentials of functions [14],

$$\mathcal{K} = \Omega^1(\mathbb{T}^{N+1})/d\Omega^0(\mathbb{T}^{N+1}).$$

The Lie bracket in

$$\mathbb{C}[t_0^{\pm 1}, \dots, t_N^{\pm 1}] \otimes \dot{\mathfrak{g}} \oplus \mathcal{K}$$

is given by

$$[f_1(t) \otimes g_1, f_2(t) \otimes g_2] = f_1 f_2 \otimes [g_1, g_2] + (g_1|g_2)\overline{f_2 df_1},$$

where $g_1, g_2 \in \dot{\mathfrak{g}}$, $f_1, f_2 \in \mathbb{C}[t_0^{\pm 1}, \dots, t_N^{\pm 1}]$, $(\cdot|\cdot)$ is the Killing form on $\dot{\mathfrak{g}}$ and $\overline{}$ denotes the projection $\Omega^1 \to \Omega^1/d\Omega^0$.

We will set 1-forms $k_a = t_a^{-1} dt_a$, $a = 0, \dots, N$ as generators of $\Omega^1(\mathbb{T}^{N+1})$ as a free $\mathbb{C}[t_0^{\pm 1}, \dots, t_N^{\pm 1}]$-module. We will use the same notations for their images in $\mathcal{K}$.

The Lie algebra $\mathcal{D} = \mathrm{Vect}\mathbb{T}^{N+1}$ acts on the universal central extension of the multiloop algebra with the natural action on $\mathbb{C}[t_0^{\pm 1}, \dots, t_N^{\pm 1}] \otimes \dot{\mathfrak{g}}$, and the action on $\mathcal{K}$ induced from the Lie derivative action of vector fields on Ω^1:

$$f_1(t) d_a(f_2(t) k_b) = f_1 d_a(f_2) k_b + \delta_{ab} f_2 d(f_1), \quad a, b = 0, \dots, N.$$

The full toroidal Lie algebra is a semi-direct product

$$\mathfrak{g} = \left(\mathbb{C}[t_0^{\pm 1}, \dots, t_N^{\pm 1}] \otimes \dot{\mathfrak{g}} \oplus \mathcal{K}\right) \rtimes \mathcal{D}.$$

In fact [2] treats a more general family of Lie algebras, where the Lie bracket in $\mathfrak{g}$ is twisted with a 2-cocycle $\tau \in H^2(\mathcal{D}, \Omega^1/d\Omega^0)$. However for the purposes of the present work we need to consider only the semi-direct product, i.e., set $\tau = 0$.

A category of bounded modules for the full toroidal Lie algebra is studied in [2] and realizations of irreducible modules in this category are given. The constructions of [2] admit a specialization $\dot{\mathfrak{g}} = (0)$, which yields representations of the semi-direct product

$$\mathcal{D} \ltimes \mathcal{K}.$$

The approach of the present paper is to look at the representations of this semidirect product, constructed in [2], and to study their reductions to the subalgebra $\mathcal{D}$ of vector fields on $\mathbb{T}^{N+1}$. Surprisingly, as we shall see below, most of the irreducible modules for $\mathcal{D} \ltimes \mathcal{K}$ remain irreducible when restricted to $\mathcal{D}$.

In order to describe here the results of [2], we will need to present a background material on vertex algebras.

5. Vertex superalgebras: definitions and notations

Let us recall the basic notions of the theory of the vertex operator (super) algebras. Here we are following [12] and [17].

Definition 5.1. A vertex superalgebra is a $\mathbb{Z}_2$-graded vector space V with a distinguished vector $\mathbb{1}$ (vacuum vector) in V, a parity-preserving operator D (infinitesimal translation) on the space V, and a linear map Y (state-field correspondence)

$$Y(\cdot,z): \quad V \to (\mathrm{End}V)[[z,z^{-1}]],$$

$$a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \quad (\text{where } \ a_{(n)} \in \mathrm{End}V),$$

such that the following axioms hold:

(V1) For any $a, b \in V$, $\quad a_{(n)}b = 0$ for n sufficiently large;

(V2) $[D, Y(a,z)] = Y(D(a),z) = \frac{d}{dz}Y(a,z)$ for any $a \in V$;

(V3) $Y(\mathbb{1},z) = \mathrm{Id}_V\, z^0$;

(V4) $Y(a,z)\mathbb{1} \in V[[z]]$ and $Y(a,z)\mathbb{1}|_{z=0} = a$ for any $a \in V$ (self-replication);

(V5) For any $a, b \in V$, the fields $Y(a,z)$ and $Y(b,z)$ are mutually local, that is,

$$(z-w)^n\,[Y(a,z), Y(b,w)] = 0, \quad \text{for } \ n \text{ sufficiently large.}$$

A vertex superalgebra V is called a vertex operator superalgebra (VOA) if, in addition, V contains a vector ω (Virasoro element) such that

(V6) The components $L_n = \omega_{(n+1)}$ of the field

$$Y(\omega,z) = \sum_{n\in\mathbb{Z}} \omega_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

satisfy the Virasoro algebra relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}C_{Vir}, \tag{5.1}$$

where C_{Vir} acts on V by scalar, called the *rank* of V.

(V7) $D = L_{-1}$;

(V8) Operator L_0 is diagonalizable on V.

This completes the definition of a VOA.

As a consequence of the axioms of the vertex superalgebra we have the following important commutator formula:

$$[Y(a,z_1), Y(b,z_2)] = \sum_{n\geq 0} \frac{1}{n!} Y(a_{(n)}b, z_2) \left[z_1^{-1} \left(\frac{\partial}{\partial z_2}\right)^n \delta\left(\frac{z_2}{z_1}\right)\right]. \tag{5.2}$$

As usual, the delta function is

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n.$$

By (V1), the sum in the left hand side of the commutator formula is actually finite. The commutator in the left hand side of (5.2) is of course the supercommutator.

Let us recall the definition of a normally ordered product of two fields. For a formal field $a(z) = \sum\limits_{j\in\mathbb{Z}} a_{(j)} z^{-j-1}$ define its positive and negative parts as follows:

$$a(z)_- = \sum_{j=0}^{\infty} a_{(j)} z^{-j-1}, \quad a(z)_+ = \sum_{j=-1}^{-\infty} a_{(j)} z^{-j-1}.$$

Then the normally ordered product of two formal fields $a(z), b(z)$ of parities $p(a), p(b) \in \{0,1\}$ respectively, is defined as

$$: a(z)b(z) := a(z)_+ b(z) + (-1)^{p(a)p(b)} b(z) a(z)_- \ .$$

The following property of vertex superalgebras will be used extensively in this paper:

$$Y(a_{(-1)}b, z) = \,: Y(a,z)Y(b,z):, \quad \text{for all } \ a, b \in V.$$

6. Vertex Lie superalgebras

An important source of the vertex superalgebras is provided by the vertex Lie superalgebras. In presenting this construction we will be following [5] (see also [8], [12], [24], [25]).

Let $\mathcal{L}$ be a Lie superalgebra with the basis $\{u_{(n)}, c_{(-1)} | u \in \mathcal{U}, c \in \mathcal{C}, n \in \mathbb{Z}\}$ ($\mathcal{U}$, $\mathcal{C}$ are some index sets). Define the corresponding fields in $\mathcal{L}[[z, z^{-1}]]$:

$$u(z) = \sum_{n\in\mathbb{Z}} u_{(n)} z^{-n-1}, \quad c(z) = c_{(-1)} z^0, \quad u \in \mathcal{U}, c \in \mathcal{C}.$$

Let $\mathcal{F}$ be a subspace in $\mathcal{L}[[z, z^{-1}]]$ spanned by all the fields $u(z), c(z)$ and their derivatives of all orders.

Definition 6.1. A Lie superalgebra $\mathcal{L}$ with the basis as above is called a vertex Lie superalgebra if the following two conditions hold:

(VL1) for all $x, y \in \mathcal{U}$,

$$[x(z_1), y(z_2)] = \sum_{j=0}^{n} f_j(z_2) \left[z_1^{-1} \left(\frac{\partial}{\partial z_2} \right)^j \delta\left(\frac{z_2}{z_1} \right) \right], \tag{6.1}$$

where $f_j(z) \in \mathcal{F}, n \geq 0$ and depend on x, y,

(VL2) for all $c \in \mathcal{C}$, the elements $c_{(-1)}$ are central in $\mathcal{L}$.

Let $\mathcal{L}_{(+)}$ be a subspace in $\mathcal{L}$ with the basis $\{u_{(n)} \big| u \in \mathcal{U}, n \geq 0\}$ and let $\mathcal{L}_{(-)}$ be a subspace with the basis $\{u_{(n)}, c_{(-1)} \big| u \in \mathcal{U}, c \in \mathcal{C}, n < 0\}$. Then $\mathcal{L} = \mathcal{L}_{(+)} \oplus \mathcal{L}_{(-)}$ and $\mathcal{L}_{(+)}, \mathcal{L}_{(-)}$ are in fact subalgebras in $\mathcal{L}$.

The universal enveloping vertex algebra $V_{\mathcal{L}}$ of a vertex Lie superalgebra $\mathcal{L}$ is defined as the induced module

$$V_{\mathcal{L}} = \mathrm{Ind}_{\mathcal{L}_{(+)}}^{\mathcal{L}} (\mathbb{C}\mathbb{1}) = U(\mathcal{L}_{(-)}) \otimes \mathbb{1},$$

where $\mathbb{C}\mathbb{1}$ is a trivial 1-dimensional $\mathcal{L}_{(+)}$ module.

Theorem 6.2. *[[5], Theorem 4.8] Let $\mathcal{L}$ be a vertex Lie superalgebra. Then*

(a) $V_{\mathcal{L}}$ has a structure of a vertex superalgebra with the vacuum vector $\mathbb{1}$, infinitesimal translation D being a natural extension of the derivation of $\mathcal{L}$ given by $D(u_{(n)}) = -n u_{(n-1)}$, $D(c_{(-1)}) = 0$, $u \in \mathcal{U}$, $c \in \mathcal{C}$, and the state-field correspondence map Y defined by the formula:

$$Y\left(a^1_{(-1-n_1)} \dots a^{k-1}_{(-1-n_{k-1})} a^k_{(-1-n_k)} \mathbb{1}, z\right)$$

$$= : \left(\frac{1}{n_1!} \left(\frac{\partial}{\partial z} \right)^{n_1} a^1(z) \right) \dots : \left(\frac{1}{n_{k-1}!} \left(\frac{\partial}{\partial z} \right)^{n_{k-1}} a^{k-1}(z) \right) \left(\frac{1}{n_k!} \left(\frac{\partial}{\partial z} \right)^{n_k} a^k(z) \right) : \dots : \quad , \tag{6.2}$$

where $a^j \in \mathcal{U}, n_j \geq 0$ or $a^j \in \mathcal{C}, n_j = 0$.

(b) Any bounded $\mathcal{L}$-module is a vertex superalgebra module for $V_{\mathcal{L}}$.

(c) For an arbitrary character $\chi : \mathcal{C} \to \mathbb{C}$, the quotient module

$$V_{\mathcal{L}}(\chi) = U(\mathcal{L}_{(-)})\mathbb{1} / U(\mathcal{L}_{(-)}) \big\langle (c_{(-1)} - \chi(c))\mathbb{1} \big\rangle_{c \in \mathcal{C}}$$

is a quotient vertex superalgebra.

(d) Any bounded $\mathcal{L}$-module in which $c_{(-1)}$ act as $\chi(c)\mathrm{Id}$, for all $c \in \mathcal{C}$, is a vertex superalgebra module for $V_{\mathcal{L}}(\chi)$.

The value $\chi(c)$ is referred to as *central charge* or *level*.

The vertex algebra that controls representation theory of $\mathcal{D} \ltimes \mathcal{K}$ is the tensor product of three VOAs: a subalgebra V^+_{Hyp} of a hyperbolic lattice vertex algebra, an affine $\widehat{gl}_N$ vertex algebra V_{gl_N} at level 1, and the Virasoro vertex algebra V_{Vir} of rank 0. In order to apply the results of [2] to representation theory of $\mathcal{D} \ltimes \mathcal{K}$, we use specializations $\dot{\mathfrak{g}} = (0)$ and $\tau = 0$. In this specialization one has to fix the following values for the various central charges appearing in ([2], Theorems 5.3 and 6.4):

$$c = 1, \ \ c_{sl_N} = 1; \ \ c_{Hei} = N,$$
$$c_{VH} = \frac{N}{2}, \ \ c'_{Vir} = 0.$$

Let us briefly review the constructions of these three vertex algebras.

Hyperbolic lattice VOA. Consider a hyperbolic lattice Hyp, which is a free abelian group on $2N$ generators $\{u^a, v^a | a = 1, \dots, N\}$ with the symmetric bilinear form

$$(\cdot|\cdot) : \ \mathrm{Hyp} \times \mathrm{Hyp} \to \mathbb{Z},$$

defined by

$$(u^a|v^b) = \delta_{ab}, \ \ (u^a|u^b) = (v^a|v^b) = 0, \ \ a, b = 1, \dots, N.$$

We complexify Hyp to get a $2N$-dimensional vector space

$$\mathcal{H} = \mathrm{Hyp} \otimes_{\mathbb{Z}} \mathbb{C}$$

and extend the bilinear form by linearity on $\mathcal{H}$. Next, we affinize $\mathcal{H}$ by defining a Heisenberg Lie algebra

$$\widehat{\mathcal{H}} = \mathbb{C}[t, t^{-1}] \otimes \mathcal{H} \oplus \mathbb{C}K$$

with the Lie bracket

$$[x_{(n)}, y_{(m)}] = n(x|y)\delta_{n,-m}K, \ \ x, y \in \mathcal{H}, \tag{6.3}$$
$$[K, \widehat{\mathcal{H}}] = 0.$$

Here and in what follows, we are using the notation $x_{(n)} = t^n \otimes x$.

The algebra $\widehat{\mathcal{H}}$ has a triangular decomposition $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_- \oplus \widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_+$, where $\widehat{\mathcal{H}}_0 = 1 \otimes \mathcal{H} \oplus \mathbb{C}K$, and $\widehat{\mathcal{H}}_\pm = t^{\pm 1}\mathbb{C}[t^{\pm 1}] \otimes \mathcal{H}$.

Let Hyp^+ be an isotropic sublattice of Hyp generated by $\{u^a | a = 1, \dots, N\}$. We consider its group algebra $\mathbb{C}[\mathrm{Hyp}^+] = \mathbb{C}[e^{\pm u^1}, \dots, e^{\pm u^N}]$ and define the action of $\widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_+$ on $\mathbb{C}[\mathrm{Hyp}^+]$ by

$$x_{(0)} e^y = (x|y) e^y, \ \ K e^y = e^y, \ \ \widehat{\mathcal{H}}_+ e^y = 0.$$

To be consistent with our previous notations, we set $q_a = e^{u^a}$, $a = 1, \dots, N$.

Finally, let V^+_{Hyp} be the induced module

$$V^+_{Hyp} = \mathrm{Ind}^{\widehat{\mathcal{H}}}_{\widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_+} \left(\mathbb{C}[\mathrm{Hyp}^+] \right).$$

We coordinatize V^+_{Hyp} as a Fock space over $\widehat{\mathcal{H}}$:

$$V^+_{Hyp} = \mathbb{C}[q_1^{\pm 1}, \dots, q_N^{\pm 1}] \otimes \mathbb{C}[u_{pj}, v_{pj} |_{j=1,2,\dots}^{p=1,\dots,N}],$$

where $\widehat{\mathcal{H}}$ acts by operators of multiplication and differentiation:

$$u^p_{(-j)} = j u_{pj}, \ \ u^p_{(j)} = \frac{\partial}{\partial v_{pj}}, \ \ u^p_{(0)} = 0,$$

$$v^p_{(-j)} = j v_{pj}, \ \ v^p_{(j)} = \frac{\partial}{\partial u_{pj}}, \ \ v^p_{(0)} = q_p \frac{\partial}{\partial q_p},$$

for $p = 1, \dots, N, \ j = 1, 2, \dots$.

The space V^+_{Hyp} has the structure of a vertex algebra - it is a vertex subalgebra in the vertex algebra corresponding to lattice Hyp. We give here the values of the state-field correspondence map on the generators of this vertex algebra:

$$Y(u_{p1}, z) = u^p(z) = \sum_{j \in \mathbb{Z}} u^p_{(j)} z^{-j-1},$$

$$Y(v_{p1}, z) = v^p(z) = \sum_{j \in \mathbb{Z}} v^p_{(j)} z^{-j-1}, \ \ p = 1, \dots, N,$$

$$Y(q^r, z) = q^r exp\left(\sum_{p=1}^N r_p \sum_{j=1}^\infty \frac{z^j}{j} u^p_{(-j)}\right) exp\left(-\sum_{p=1}^N r_p \sum_{j=1}^\infty \frac{z^{-j}}{j} u^p_{(j)}\right).$$

The Virasoro element of V^+_{Hyp} is

$$\omega^{Hyp} = \sum_{p=1}^N u_{p1} v_{p1}$$

and the Virasoro field is

$$Y(\omega^{Hyp}, z) = \sum_{p=1}^N : u^p(z) v^p(z) : \ .$$

The rank of V^+_{Hyp} is $2N$.

Fix $\gamma \in \mathbb{C}^N$. The space

$$M_{Hyp}(\gamma) = q^\gamma \mathbb{C}[q_1^{\pm 1}, \dots, q_N^{\pm 1}] \otimes \mathbb{C}[u_{pj}, v_{pj}|^{p=1,\dots,N}_{j=1,2,\dots}]$$

has a natural structure of a simple module for V^+_{Hyp} (see e.g. [1] for details).

Affine $\widehat{gl}_N$ VOA. The second vertex algebra that we will need is the affine $\widehat{gl}_N$ vertex algebra at level 1. Since gl_N is reductive, but not simple, it has more than one affinization. Here we consider a particular version of $\widehat{gl}_N$:

$$\widehat{gl}_N = \mathbb{C}[t, t^{-1}] \otimes gl_N \oplus \mathbb{C}C$$

with the Lie bracket

$$[t^n \otimes X, t^m \otimes Y] = t^{n+m} \otimes [X, Y] + n\delta_{n,-m} Tr(XY)C, \quad X, Y \in gl_N. \tag{6.4}$$

We note that $\widehat{gl}_N$ is a vertex Lie algebra and consider its universal enveloping vertex algebra V_{gl_N} at level 1 (i.e., $\chi(C) = 1$).

Let us give the value of the state-field correspondence map on the generators of this affine vertex algebra:

$$Y(X_{(-1)}\mathbb{1}, z) = X(z) = \sum_{j\in\mathbb{Z}} X_{(j)} z^{-j-1}, \quad \text{for} \ \ X \in gl_N.$$

Since gl_N has a decomposition $gl_N = sl_N \oplus \mathbb{C}I$, where I is the identity $N \times N$ matrix, the affine $\widehat{gl}_N$ vertex algebra is the tensor product of the affine $\widehat{sl}_N$ vertex algebra and a Heisenberg vertex algebra. The Virasoro element ω^{gl_N} of V_{gl_N} can be thus written as a sum of the Virasoro elements ω^{sl_N} for the affine $\widehat{sl}_N$ vertex algebra and ω^{Hei} for the Heisenberg vertex algebra. The usual formula for the Virasoro element in affine vertex algebra gives the following explicit expression:

$$\omega^{sl_N} = \frac{1}{2(N+1)}\left(\sum_{i,j=1}^N E^{ij}_{(-1)} E^{ji}_{(-1)}\mathbb{1} - \frac{1}{N} I_{(-1)} I_{(-1)}\mathbb{1}\right). \tag{6.5}$$

The rank of the affine $\widehat{sl}_N$ vertex algebra at level 1 is $N-1$.

For the Heisenberg vertex algebra we choose a non-standard Virasoro element (see [2], (4.33)):

$$\omega^{Hei} = \frac{1}{2N} I_{(-1)} I_{(-1)}\mathbb{1} + \frac{1}{2} I_{(-2)}\mathbb{1}. \tag{6.6}$$

The rank of this Heisenberg VOA is $1 - 3N$.

Adding the two Virasoro elements, we get the Virasoro element for V_{gl_N}:

$$\omega^{gl_N} = \frac{1}{2(N+1)}\left(\sum_{i,j=1}^N E^{ij}_{(-1)} E^{ji}_{(-1)}\mathbb{1} + I_{(-1)} I_{(-1)}\mathbb{1}\right) + \frac{1}{2} I_{(-2)}\mathbb{1}. \tag{6.7}$$

The corresponding Virasoro field is

$$Y(\omega^{gl_N}, z) = \frac{1}{2(N+1)}\left(\sum_{i,j=1}^N : E^{ij}(z) E^{ji}(z) : + : I(z) I(z) :\right) + \frac{1}{2}\frac{d}{dz} I(z). \tag{6.8}$$

The rank of V_{gl_N} is $-2N$.

Let W be a finite-dimensional simple module for gl_N. Let C act on W as the identity operator and set $(t\mathbb{C}[t] \otimes gl_N)\, W = 0$. Construct the generalized Verma module for the Lie algebra $\widehat{gl}_N$ as the induced module from W, and consider its irreducible quotient $L_{gl_N}(W)$. Then $L_{gl_N}(W)$ is a simple module for the vertex algebra V_{gl_N}.

Virasoro VOA. The last vertex algebra that we need to introduce is the Virasoro vertex algebra V_{Vir} of rank 0. The Virasoro Lie algebra (5.1) is a vertex Lie algebra with $\mathcal{U} = \{\omega^{Vir}\}$ and $\mathcal{C} = \{C_{Vir}\}$, where

$$\omega^{Vir}(z) = \sum_{j\in\mathbb{Z}} \omega^{Vir}{}_{(j)} z^{-j-1} = \sum_{j\in\mathbb{Z}} L_j z^{-j-2}.$$

Let V_{Vir} be its universal enveloping vertex algebra of zero central charge, $\chi(C_{Vir}) = 0$.

Let $L_{Vir}(h)$ be the irreducible highest weight module for the Virasoro Lie algebra with central charge 0 with the highest weight vector v_h, satisfying $L_0 \mathrm{v}_h = h \mathrm{v}_h$.

The vertex algebra that controls representation theory of $\mathcal{D} \ltimes \mathcal{K}$ is the tensor product of the sub-VOA V^+_{Hyp} of the hyperbolic lattice vertex algebra, affine $\widehat{gl}_N$ vertex algebra V_{gl_N} at level 1, and the Virasoro VOA V_{Vir} of rank 0

$$V^+_{Hyp} \otimes V_{gl_N} \otimes V_{Vir}$$

with the Virasoro element

$$\omega = \omega^{Hyp} + \omega^{gl_N} + \omega^{Vir}.$$

The rank of this VOA is $2N - 2N + 0 = 0$. Now we are ready to present a result of [2] (Theorems 5.3 and 6.4):

Theorem 6.3. *([2]) (i) Let M_{Hyp}, M_{gl_N}, M_{Vir} be modules for V^+_{Hyp}, V_{gl_N} and V_{Vir} respectively. Then the tensor product*

$$\mathcal{M} = M_{Hyp} \otimes M_{gl_N} \otimes M_{Vir}$$

is a module for the Lie algebra $\mathcal{D} \ltimes \mathcal{K}$ with the action given as follows:

$$\sum_{j\in\mathbb{Z}} t_0^j t^r k_0 z^{-j} = k_0(r,z) \mapsto Y(q^r, z), \tag{6.9}$$

$$\sum_{j\in\mathbb{Z}} t_0^j t^r k_a z^{-j-1} = k_a(r,z) \mapsto u^a(z) Y(q^r, z), \tag{6.10}$$

$$\sum_{j\in\mathbb{Z}} t_0^j t^r d_a z^{-j-1} = d_a(r,z) \mapsto : v^a(z) Y(q^r, z) : + \sum_{p=1}^{N} r_p E^{pa}(z) Y(q^r, z), \tag{6.11}$$

$$\begin{aligned}\sum_{j\in\mathbb{Z}} t_0^j t^r d_0 z^{-j-2} = d_0(r,z) \mapsto - : Y(\omega, z) Y(q^r, z) : - \sum_{i,j=1}^{N} r_i u^j(z) E^{ij}(z) Y(q^r, z) \\ + \sum_{p=1}^{N} r_p \left(\frac{d}{dz} u^p(z)\right) Y(q^r, z),\end{aligned} \tag{6.12}$$

for $a = 1, \ldots, N$.

(ii) The module

$$L(W, \gamma, h) = M_{Hyp}(\gamma) \otimes L_{gl_N}(W) \otimes L_{Vir}(h)$$

is an irreducible module over the Lie algebra $\mathcal{D} \ltimes \mathcal{K}$.

7. Generalized Wakimoto modules.

In [3] the structure of irreducible modules $L(T)$ over the Lie algebra of vector fields was determined in case of a 2-dimensional torus ($N = 1$). It turned out that the situation was analogous to the case of a basic module for an affine Kac-Moody algebra, which remains irreducible when restricted to the principal Heisenberg subalgebra [16], [13]. For the Lie algebra of vector fields on $\mathbb{T}^2$ this role is played by its loop subalgebra $\widetilde{sl}_2 = \mathbb{C}[t_0, t_0^{-1}] \otimes sl_2$. Indeed we have sl_2 embedded into $\mathrm{Vect}\mathbb{T}^1$:

$$sl_2 \cong \left\langle t_1^{-1} d_1, d_1, t_1 d_1 \right\rangle \subset \mathbb{C}[t_1, t_1^{-1}] d_1.$$

This extends to an embedding

$$\widetilde{sl}_2 \cong \mathbb{C}[t_0, t_0^{-1}] \otimes \left\langle t_1^{-1} d_1, d_1, t_1 d_1 \right\rangle \subset \mathrm{Vect}\mathbb{T}^2.$$

The following theorem was proved in [3]:

Theorem 7.1. *([3]) Let W be the 1-dimensional gl_1-module in which the identity matrix acts as multiplication by $\alpha \in \mathbb{C}$. Assume $\alpha \notin \mathbb{Q}$, $\beta = \frac{\alpha(\alpha-1)}{2}$, and let $\gamma \in \mathbb{C}$. Then the module $L(T) = L(\alpha, \beta, \gamma)$ over the Lie algebra $\mathrm{Vect}\mathbb{T}^2$ remains irreducible when restricted to subalgebra $\widetilde{sl}_2$.*

Note that [3] uses a different convention for the sign of α.

The loop algebra $\widetilde{sl}_2 = \mathbb{C}[t_0, t_0^{-1}] \otimes sl_2$ is $\mathbb{Z}$-graded by degree in t_0. This gives its decomposition $\widetilde{sl}_2 = \widetilde{sl}_2^+ \oplus \widetilde{sl}_2^0 \oplus \widetilde{sl}_2^-$. The zero part $\widetilde{sl}_2^0 \cong sl_2$ is a subalgebra in $\mathbb{C}[t_1, t_1^{-1}] d_1$ and thus acts on T. The positive part $\widetilde{sl}_2^+$ acts on T trivially. We can form the generalized Verma module over $\widetilde{sl}_2$:

$$\mathrm{Ind}_{\widetilde{sl}_2^0 \oplus \widetilde{sl}_2^+}^{\widetilde{sl}_2} T(\alpha, \beta, \gamma) \cong U(\widetilde{sl}_2^-) \otimes T(\alpha, \beta, \gamma).$$

By the results of [11], this generalized Verma module is irreducible over $\widetilde{sl}_2$ if and only if $\alpha \notin \frac{1}{2}\mathbb{Z}$. This gives the following

Corollary 7.2. *([3]) Let $\alpha \in \mathbb{C}, \alpha \notin \mathbb{Q}$, $\beta = \frac{\alpha(\alpha-1)}{2}$. Then the $\mathrm{Vect}\mathbb{T}^2$-module $L(\alpha, \beta, \gamma)$ when restricted to $\widetilde{sl}_2$ is isomorphic to the generalized Verma module over $\widetilde{sl}_2$:*

$$L(\alpha, \beta, \gamma) \cong U(\widetilde{sl}_2^-) \otimes T(\alpha, \beta, \gamma).$$

These results show that the loop subalgebra $\widetilde{sl}_2$ plays a crucial role in representation theory of the Lie algebra of vector fields on $\mathbb{T}^2$. It is natural to conjecture that in the representation theory of $\mathcal{D} = \mathrm{Vect}\mathbb{T}^{N+1}$ such a role is played by the loop algebra $\widetilde{sl}_{N+1}$. Indeed, $\mathcal{D}_0$ has $\mathrm{Vect}\mathbb{T}^N$ as a subalgebra and $\mathrm{Vect}\mathbb{T}^N$ contains sl_{N+1} (see e.g. [22]). Thus $\mathrm{Vect}\mathbb{T}^{N+1}$ contains the loop algebra $\widetilde{sl}_{N+1} = \mathbb{C}[t_0, t_0^{-1}] \otimes sl_{N+1}$. The modules $T(W, \beta, \gamma)$ may be viewed as sl_{N+1}-modules, and we can form the generalized Verma module over $\widetilde{sl}_{N+1}$:

$$U(\widetilde{sl}_{N+1}^-) \otimes T(W, \beta, \gamma).$$

It turns out, however, that in general, the action of $\widetilde{sl}_{N+1}$ on the generalized Verma module can not be extended to the action of the bigger algebra $\mathrm{Vect}\mathbb{T}^{N+1}$. Instead one should use certain generalized Wakimoto modules. We define the generalized Wakimoto modules in the following way:

Definition 7.3. Let T be an sl_{N+1}-module. A *generalized Wakimoto module* $\mathcal{M}$ with top T is an $\widetilde{sl}_{N+1}$-module that contains T as an sl_{N+1}-submodule with $\widetilde{sl}_{N+1}^+$ acting on T trivially and having the property that the character of $\mathcal{M}$ coincides with the character of the generalized Verma module for $\widetilde{sl}_{N+1}$:

$$char\, \mathcal{M} = char\, \left(U(\widetilde{sl}_{N+1}^-) \right) \times char\, T.$$

The generalized Verma module is by definition a generalized Wakimoto module. In case when the generalized Verma module is irreducible, it is the only generalized Wakimoto module with the given top T. As we mentioned above, for $\widetilde{sl}_2$ this happens for the tops $T(\alpha, \beta, \gamma)$ with $\alpha \notin \frac{1}{2}\mathbb{Z}$ [11]. For $N > 1$ and any top $T(W, \beta, \gamma)$ with a finite-dimensional gl_N-module W, the generalized Verma module over $\widetilde{sl}_{N+1}$ is always reducible.

We shall now see that Theorem 6.3 yields a construction of a generalized Wakimoto module for $\widetilde{sl}_{N+1}$. These modules admit the action of the whole algebra of vector fields on $\mathbb{T}^{N+1}$.

Proposition 7.4. *Let $M_{gl_N}(W)$ be the generalized Verma module for $\widehat{gl}_N$ at level 1, induced from an irreducible finite-dimensional gl_N-module W. Then the module*

$$\mathcal{M} = q^\gamma \mathbb{C}[q_1^{\pm 1}, \ldots, q_N^{\pm 1}] \otimes \mathbb{C}[u_{pj}, v_{pj}] \otimes M_{gl_N}(W)$$

is a generalized Wakimoto module for the loop algebra $\widetilde{sl}_{N+1}$.

Proof. Applying Theorem 6.3 with a trivial 1-dimensional Virasoro module, we see that $\mathcal{M}$ as a module for the Lie algebra $\mathrm{Vect}\mathbb{T}^{N+1}$. By restriction, view $\mathcal{M}$ as an $\widetilde{sl}_{N+1}$-module. The top of the module $\mathcal{M}$ is the tensor module

$$T(W, \gamma) = q^\gamma \mathbb{C}[q_1^{\pm 1}, \ldots, q_N^{\pm 1}] \otimes W.$$

To show that $\mathcal{M}$ is a generalized Wakimoto module for $\widetilde{sl}_{N+1}$, we need to compare the characters of $\mathbb{C}[u_{pj}, v_{pj}] \otimes U(\widehat{gl}_N^-)$ and $U(\widetilde{sl}_{N+1}^-)$. We have

$$char\, \mathbb{C}[u_{pj}, v_{pj}] \times char\, U(\widehat{gl}_N^-) = \prod_{k=1}^{\infty} (1 - s^k)^{-2N} \times \prod_{k=1}^{\infty} (1 - s^k)^{-N^2} = char\, U(\widetilde{sl}_{N+1}^-).$$

This completes the proof of the proposition. □

Theorem 6.3 describes irreducible $\mathcal{D} \ltimes \mathcal{K}$-modules. We would like to study their reductions to subalgebra $\mathcal{D}$. In general, when reduced to a subalgebra, modules become reducible. Here, however, the link with generalized Wakimoto modules for $\widetilde{sl}_{N+1}$ and the result of [3] for $N = 1$, give us hope that the situation may be better than one would expect a priory.

8. Duality for modules over the Lie algebra of vector fields

In this section we will establish a duality for the class of modules described in Theorem 6.3 (ii), that will be useful for the study of their irreducibility as modules over Lie algebra $\mathcal{D}$. We begin by looking at this question in a general setup.

Let $\mathcal{L}$ be a $\mathbb{Z}$-graded Lie algebra $\mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$ with an anti-involution σ such that $\sigma(\mathcal{L}_n) = \mathcal{L}_{-n}$. Extend σ to the universal enveloping algebra $U(\mathcal{L})$ by $\sigma(ab) = \sigma(b)\sigma(a)$. Let $\mathcal{L}_\pm = \bigoplus_{n > 0} \mathcal{L}_{\pm n}$. Suppose T_1, T_2 be two $\mathcal{L}_0$-modules with a bilinear pairing

$$T_1 \times T_2 \to \mathbb{C} \tag{8.1}$$

such that

$$\langle x w_1, w_2 \rangle = \langle w_1, \sigma(x) w_2 \rangle$$

for all $x \in \mathcal{L}_0$, $w_1 \in T_1$, $w_2 \in T_2$.

For an $\mathcal{L}_0$-module T we let $\mathcal{L}_+$ act on T trivially and construct the generalized Verma modules for $\mathcal{L}$: $M(T) = U(\mathcal{L}_-) \otimes T$.

The generalized Verma module $M(T)$ inherits the $\mathbb{Z}$-grading (assuming degree of T to be zero). Define the *radical* of a generalized Verma module $M(T)$ as the maximal homogeneous submodule trivially intersecting with the top T. If T is an irreducible $\mathcal{L}_0$-module then the quotient $L(T)$ of $M(T)$ by its radical is an irreducible module for $\mathcal{L}$.

Consider the Shapovalov projection

$$S : U(\mathcal{L}) \to U(\mathcal{L}_0)$$

with kernel $\mathcal{L}_- U(\mathcal{L}) + U(\mathcal{L}) \mathcal{L}_+$.

Define a bilinear pairing

$$M(T_1) \times M(T_2) \to \mathbb{C},$$

defined by

$$\langle a w_1, b w_2 \rangle = \langle w_1, S(\sigma(a) b) w_2 \rangle \tag{8.2}$$

for $a, b \in U(\mathcal{L}_-)$, $w_1 \in T_1$, $w_2 \in T_2$.

Proposition 8.1. *(i) The pairing (8.2) is contragredient, i.e.,*

$$\langle xu, v\rangle = \langle u, \sigma(x)v\rangle$$

for all $x \in \mathcal{L}$, $u \in M(T_1)$, $v \in M(T_2)$.

(ii) If $n \neq k$ then $\langle M(T_1)_n, M(T_2)_k\rangle = 0$.

(iii) The radicals of $M(T_1)$, $M(T_2)$ are in the kernel of the pairing.

(iv) Assume that T_1, T_2 are irreducible $\mathcal{L}_0$-modules and the pairing (8.1) is non-zero. Then the contragredient pairing factors to the simple modules

$$L(T_1) \times L(T_2) \to \mathbb{C},$$

on which it is non-degenerate.

This proposition is standard (cf., [23], Proposition 2.8.1) and its proof is left to the reader as an exercise.

Next we will apply this proposition to establish the duality for the bounded modules described in Theorem 6.3 (ii).

We consider the following anti-involution on $\mathcal{D} \ltimes \mathcal{K}$:

$$\sigma(t_0^j t^r d_a) = t_0^{-j} t^{-r} d_a, \quad \sigma(t_0^j t^r k_a) = t_0^{-j} t^{-r} k_a, \quad a = 0, \dots, N.$$

For a finite-dimensional simple gl_N-module W, on which the identity matrix I acts as multiplication by $\alpha \in \mathbb{C}$, let W^* be the dual space to W with sl_N-module structure of the dual module, but with I acting as scalar $N - \alpha$. The natural pairing between W and W^* satisfies

$$\langle E^{ab} w | w^*\rangle = \langle w | - E^{ab} w^* + \delta_{ab} w^*\rangle.$$

Theorem 8.2. *There exists a non-degenerate contragredient pairing of simple $\mathcal{D} \ltimes \mathcal{K}$-modules defined in Theorem 6.3 (ii):*

$$L(W, \gamma, h) \times L(W^*, \gamma, h) \to \mathbb{C}, \tag{8.3}$$

satisfying

$$\langle xu, v\rangle = \langle u, \sigma(x)v\rangle,$$

for all $x \in \mathcal{D} \ltimes \mathcal{K}$, $u \in L(W, \gamma, h)$, $v \in L(W^*, \gamma, h)$.

For the proof of this theorem we will use an alternative construction of the simple $\mathcal{D} \ltimes \mathcal{K}$-module $L(W, \gamma, h)$, which is discussed in [2]. These modules may be abstractly defined using approach of Theorem 3.1. The *top* of the module $L(W, \gamma, h)$ is the space

$$T = T(W, \gamma, h) = q^\gamma \mathbb{C}[q_1^{\pm 1}, \dots, q_N^{\pm 1}] \otimes W \otimes \mathrm{v}_h,$$

which is a module for the zero degree component $\mathcal{D}_0 \ltimes \mathcal{K}_0$ of $\mathcal{D} \ltimes \mathcal{K}$ with respect to its $\mathbb{Z}$-grading by degree in t_0. The positive part of $\mathcal{D} \ltimes \mathcal{K}$ acts on $T(W, \gamma, h)$ trivially, and we can consider the induced $\mathcal{D} \ltimes \mathcal{K}$-module $M(T)$. The induced module has a unique irreducible quotient, which is isomorphic to $L(W, \gamma, h)$.

The action of $\mathcal{D}_0 \ltimes \mathcal{K}_0$ on $T(W, \gamma, h)$ can be derived from Theorem 6.3 (i) (see Theorem 6.4 in [2] for details) and T is essentially a tensor module that we discussed above:

$$t^r k_0(q^\mu \otimes w \otimes \mathrm{v}_h) = q^{\mu + r} \otimes w \otimes \mathrm{v}_h, \tag{8.4}$$

$$t^r k_a(q^\mu \otimes w \otimes \mathrm{v}_h) = 0, \tag{8.5}$$

$$t^r d_a(q^\mu \otimes w \otimes \mathrm{v}_h) = \mu_a q^{\mu + r} \otimes w \otimes \mathrm{v}_h + \sum_{p=1}^N r_p q^{\mu + r} \otimes E^{pa} w \otimes \mathrm{v}_h, \tag{8.6}$$

$$t^r d_0(q^\mu \otimes w \otimes \mathrm{v}_h) = \beta\, q^{\mu + r} \otimes w \otimes \mathrm{v}_h, \tag{8.7}$$

where

$$\beta = -h - \frac{\Omega_W}{2(N+1)} - \frac{\alpha(\alpha - N)}{2N}, \tag{8.8}$$

$a = 1, \ldots, N$, $r \in \mathbb{Z}^N$, $\mu \in \gamma + \mathbb{Z}^N$. Here Ω_W is the scalar with which the Casimir operator of sl_N acts on W.

The claim of Theorem 8.2 follows from Proposition 8.1 and the following lemma.

Lemma 8.3. *Let W be a simple finite-dimensional gl_N-module on which the identity matrix acts as scalar α, and let W^* be a gl_n-module which is dual to W as a sl_N-module, and on which the identity matrix acts as scalar $N - \alpha$. Then the pairing*

$$T(W, \gamma, h) \times T(W^*, \gamma, h) \to \mathbb{C},$$

given by

$$\langle q^\mu \otimes w \otimes \mathrm{v}_h | q^\eta \otimes w^* \otimes \mathrm{v}_h \rangle = \delta_{\mu,\eta} \langle w | w^* \rangle$$

is a non-degenerate contragredient pairing of $\mathcal{D}_0 \ltimes \mathcal{K}_0$-modules.

Proof. Using (8.4) we get

$$\langle t^r k_0 (q^\mu \otimes w \otimes \mathrm{v}_h) | q^\eta \otimes w^* \otimes \mathrm{v}_h \rangle = \delta_{\mu+r,\eta} \langle w | w^* \rangle$$
$$= \delta_{\mu,\eta-r} \langle w | w^* \rangle = \langle q^\mu \otimes w \otimes \mathrm{v}_h | \sigma(t^r k_0)(q^\eta \otimes w^* \otimes \mathrm{v}_h) \rangle.$$

In case of $t^r k_a$, $a = 1, \ldots, N$, both left and right hand sides are zero. Let us verify the contragredient property for the action of $t^r d_a$, $a = 1, \ldots, N$:

$$\langle t^r d_a (q^\mu \otimes w \otimes \mathrm{v}_h) | q^\eta \otimes w^* \otimes \mathrm{v}_h \rangle$$
$$= \langle \mu_a q^{\mu+r} \otimes w \otimes \mathrm{v}_h + \sum_{p=1}^N r_p q^{\mu+r} \otimes E^{pa} w \otimes \mathrm{v}_h | q^\eta \otimes w^* \otimes \mathrm{v}_h \rangle$$
$$= \delta_{\mu+r,\eta} \left(\mu_a \langle w | w^* \rangle + \sum_{p=1}^N r_p \langle E^{pa} w | w^* \rangle \right)$$
$$= \delta_{\mu,\eta-r} \left(\mu_a \langle w | w^* \rangle + \sum_{p=1}^N r_p \langle w | -E^{pa} w^* + \delta_{pa} w^* \rangle \right)$$
$$= \delta_{\mu,\eta-r} \left((\eta_a - r_a) \langle w | w^* \rangle - \sum_{p=1}^N r_p \langle w | E^{pa} w^* \rangle + r_a \langle w | w^* \rangle \right)$$
$$= \delta_{\mu,\eta-r} \left(\eta_a \langle w | w^* \rangle - \sum_{p=1}^N r_p \langle w | E^{pa} w^* \rangle \right)$$
$$= \langle q^\mu \otimes w \otimes \mathrm{v}_h | \sigma(t^r d_a)(q^\eta \otimes w^* \otimes \mathrm{v}_h) \rangle.$$

Finally, to check the case of $t^r d_0$, we note that the constant β in (8.8) is the same for $T(W, \gamma, h)$ and $T(W^*, \gamma, h)$. This follows from the fact that the Casimir operator for sl_N acts with the same scalar on W and W^*, while the last term in (8.8) is invariant under the substitution $\alpha \mapsto N - \alpha$. Thus the computation in the case of $t^r d_0$ is analogous to the case of $t^r k_0$. This completes the proof of the lemma. □

Remark 8.4. The pairing (8.3) is in fact a product of contragredient pairings of tensor factors

$$M_{Hyp}(\gamma) \times M_{Hyp}(\gamma) \to \mathbb{C}, \quad L_{gl_N}(W) \times L_{gl_N}(W^*) \to \mathbb{C}, \quad L_{Vir}(h) \times L_{Vir}(h) \to \mathbb{C},$$

with respect to appropriate anti-involutions of corresponding Lie algebras.

Remark 8.5. The duality of Theorem 8.2 can be alternatively constructed via vertex algebra approach, using the definition of the contragredient module over a vertex algebra (see section 5.2 in [10]).

One of the goals of this paper is to analyze which of the modules defined in Theorem 6.3 (ii) remain irreducible after restriction to $\mathcal{D}$. First of all, let us look at the question of irreducibility of the top $T(W, \gamma, h)$ as a module over $\mathcal{D}_0$.

Lemma 8.6. *Let W be an irreducible finite-dimensional gl_N-module. The module $T(W,\gamma,h)$ is reducible as a $\mathcal{D}_0$-module if and only if it is reducible as a* $\mathrm{Vect}\mathbb{T}^N$*-module (see Theorem 2.1) and $h = 0$.*

Proof. Clearly, if $T(W,\gamma,h)$ is reducible as a $\mathcal{D}_0$-module, it must also be reducible as a $\mathrm{Vect}\mathbb{T}^N$-module. By Theorem 2.1, all such modules appear in the de Rham complex. Note that

$$\mathcal{D}_0 = \mathrm{Vect}\mathbb{T}^N \oplus \mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}]d_0,$$

and by (8.7), $t^r d_0$ acts on $T(W,\gamma,h)$ as multiplication by βq^r. It is well known that in the modules of differential forms there are no proper submodules that are $\mathbb{C}[q_1^{\pm 1}, \ldots, q_N^{\pm 1}]$-invariant. Thus for $T(W,\gamma,h)$ to be reducible as a $\mathcal{D}_0$-module it is necessary and sufficient that it is reducible as a $\mathrm{Vect}\mathbb{T}^N$-module and the value of β given by (8.8) is zero. Let us analyze the values of β for the modules in the de Rham complex. For the modules $\Omega^0(\mathbb{T}^N)$ and $\Omega^N(\mathbb{T}^N)$ the sl_N-module W is trivial, so the Casimir operator acts with constant $\Omega_W = 0$, while the identity matrix acts on W with scalars $\alpha = 0$ and $\alpha = N$ respectively. Simplifying the expression in (8.8) we get in this case that $\beta = -h$. In case of the modules of k-forms $\Omega^k(\mathbb{T}^N)$, $k = 1, \ldots, N-1$, the highest weight of the corresponding sl_N-module W is the fundamental weight ω_k. A standard computation shows that in this case the Casimir operator acts with the scalar

$$\Omega_W = (\omega_k | \omega_k + 2\rho) = k(N-k)\frac{N+1}{N}. \tag{8.9}$$

Since the identity matrix acts with the scalar $\alpha = k$, the formula (8.8) again simplifies to $\beta = -h$. This implies the claim of the Lemma.

□

Consider now an irreducible $\mathcal{D} \ltimes \mathcal{K}$ module $L(W,\gamma,h)$ described in Theorem 6.3 (ii), and assume that its top $T(W,\gamma,h)$ is irreducible as a $\mathcal{D}_0$-module. To show that $L(W,\gamma,h)$ remains irreducible as a module over $\mathcal{D}$, it is sufficient to establish two properties:

(C) Every critical vector of $L(W,\gamma,h)$ (i.e., annihilated by $\mathcal{D}_+$) belongs to its top $T(W,\gamma,h)$.

(G) $L(W,\gamma,h)$ is generated by its top $T(W,\gamma,h)$ as a module over $\mathcal{D}_-$.

The following standard observation will be quite useful:

Lemma 8.7. *Condition (C) holds for the module $L(W,\gamma,h)$ if and only if condition (G) holds for $L(W^*,\gamma,h)$.*

Proof. We use the existence of a non-degenerate contragredient pairing of $\mathcal{D} \ltimes \mathcal{K}$-modules:

$$L(W,\gamma,h) \times L(W^*,\gamma,h) \to \mathbb{C}.$$

If $L(W,\gamma,h)$ has a vector annihilated by $\mathcal{D}_+$, which does not belong to the top, it also has a homogeneous vector with this property. Suppose u is a critical vector of degree $s \in \mathbb{Z}^{N+1}$. Since the pairing is non-degenerate, there exists a vector v in $L(W^*,\gamma,h)$ of the same degree, such that $\langle u|v\rangle \neq 0$. If property (G) holds for $L(W^*,\gamma,h)$, v can be written as $v = \sum_i x_i v_i$, where $x_i \in \mathcal{D}_-$, $v_i \in L(W^*,\gamma,h)$. Applying the contragredient property, we get

$$\langle u|v\rangle = \langle u|\sum_i x_i v_i\rangle = \sum_i \langle \sigma(x_i)u|v_i\rangle.$$

The last expression is zero since $\sigma(x_i) \in \mathcal{D}_+$ and u is a critical vector. This gives a contradiction, which implies that property (G) does not hold for $L(W^*,\gamma,h)$.

To prove the converse, assume that the component of degree s in $L(W^*,\gamma,h)$ is not generated by $\mathcal{D}_-$ acting on the top. Let V be the intersection of that homogeneous component with the space $U(\mathcal{D}_-)T(W^*,\gamma,h)$. Since the pairing is non-degenerate, we can find a non-zero vector u in the degree s component of $L(W,\gamma,h)$, such that $\langle u|V\rangle = 0$. If property (C) holds for the module $L(W,\gamma,h)$, there exists a homogeneous $y \in U(\mathcal{D}_+)$, such that $z = yu$ is a non-zero vector in $T(W,\gamma,h)$. Let $z' \in T(W,\gamma,h)$ be such that $\langle z|z'\rangle \neq 0$. Then

$$\langle z|z'\rangle = \langle yu|z'\rangle = \langle u|\sigma(y)z'\rangle.$$

But $\sigma(y) \in U(\mathcal{D}_-)$, thus $\sigma(y)z' \in V$, which leads to a contradiction. The lemma is now proved. $\square$

Lemma 8.7 reduces the question of irreducibility of the family of modules $L(W,\gamma,h)$ to the question of existence of critical vectors. If both $L(W,\gamma,h)$ and $L(W^*,\gamma,h)$ satisfy condition (C), then they are both irreducible as modules over $\mathcal{D}$.

9. Critical vectors

In this section we will establish a necessary condition for the existence of non-trivial critical vectors in the modules $L(W,\gamma,h)$, which together with Lemma 8.7 will give a sufficient condition for the irreducibility of such modules.

Theorem 9.1. *Let W be an irreducible finite-dimensional gl_N-module, $\gamma \in \mathbb{C}^N, h \in \mathbb{C}$. Every critical vector (i.e., annihilated by $\mathcal{D}_+$) in the module $L(W,\gamma,h)$ belongs to its top, unless $h = 0$ and W is either a trivial sl_N-module with identity matrix acting with scalar $\alpha = N - mN$, $m = 1, 2, \dots$, or the highest weight of W is a fundamental weight ω_k, $k = 1, \dots, N-1$, with identity matrix acting by scalar $\alpha = k - mN$, $m = 1, 2, \dots$.*

We will call a module $L(W,\gamma,h)$ *exceptional* if $h = 0$, the identity matrix acts on W by an integer $k \in \mathbb{Z}$ and W is a trivial one-dimensional sl_N-module when $k = 0 \mod N$ or has a fundamental highest weight $\omega_{k'}$ with $1 \le k' \le N-1$ and $k = k' \mod N$.

Theorem 9.2. *Let W be an irreducible finite-dimensional gl_N-module, $\gamma \in \mathbb{C}^N, h \in \mathbb{C}$. Every non-exceptional module $L(W,\gamma,h)$ is irreducible as a $\mathrm{Vect}\mathbb{T}^N$-module.*

Theorem 9.2 is an immediate consequence of Theorem 9.1 and Lemma 8.7. The proof of Theorem 9.1 will be split into a sequence of lemmas.

Lemma 9.3. *Let $g \in L(W,\gamma,h)$ be a critical vector. Then g does not depend on variables $\{u_{pj}\}$, i.e., g belongs to the subspace*

$$q^\gamma \mathbb{C}[q_1^{\pm 1}, \dots, q_N^{\pm 1}] \otimes \mathbb{C}[v_{pj}|_{j=1,2,\dots}^{p=1,\dots,N}] \otimes L_{gl_N}(W) \otimes L_{Vir}(h). \tag{9.1}$$

Proof. The algebra $\mathcal{D}_+$ contains the elements $t_0^j d_p$ with $p = 1, \dots, N$, $j \ge 1$, which act as $\frac{\partial}{\partial u_{pj}}$. The condition $(t_0^j d_p) g = 0$ implies the claim of the lemma. $\square$

For a formal series $a(z)$ we denote by $a(z)_-$ its part that only involves negative powers of z. Recalling that $d_a(r,z) = \sum_{j\in\mathbb{Z}} t_0^j t^r d_a z^{-j-1}$, we have $(z d_a(r,z))_- g = 0$. Using (6.11) for the action of $d_a(r,z)$ and taking into account that g does not depend on $\{u_{pj}\}$, we get

$$q^r \left(\exp\left(\sum_{p=1}^N r_p \sum_{j=1}^\infty u_{pj} z^j \right) \left(\sum_{i=1}^\infty i v_{ai} z^i + q_a \frac{\partial}{\partial q_a} + \sum_{p=1}^N r_p \sum_{k\in\mathbb{Z}} E_{(k)}^{pa} z^{-k} \right) \right.$$
$$\left. \times \exp\left(-\sum_{p=1}^N r_p \sum_{j=1}^\infty \frac{z^{-j}}{j} \frac{\partial}{\partial v_{pj}} \right) \right)_- g = 0.$$

Let us project to the subspace (9.1), setting $u_{pj} = 0$ in the above equality. Also, since the operator of multiplication by q^r is invertible, we can drop it. We then get

$$P_a(r,z)_- g = 0, \quad a = 1, \dots, N, \tag{9.2}$$

where

$$P_a(r,z) = \left(\sum_{i=1}^\infty i v_{ai} z^i + q_a \frac{\partial}{\partial q_a} + \sum_{p=1}^N r_p \sum_{k\in\mathbb{Z}} E_{(k)}^{pa} z^{-k} \right) \exp\left(-\sum_{p=1}^N r_p \sum_{j=1}^\infty \frac{z^{-j}}{j} \frac{\partial}{\partial v_{pj}} \right).$$

At this point we find it convenient to make a change of variables $x_{aj} = j v_{aj}$. In these notations $P_a(r,z)$ takes form

$$P_a(r,z) = \left(\sum_{i=1}^{\infty} x_{ai} z^i + q_a \frac{\partial}{\partial q_a} + \sum_{p=1}^{N} r_p \sum_{k\in\mathbb{Z}} E^{pa}_{(k)} z^{-k} \right) \exp\left(- \sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_{pj}} \right). \tag{9.3}$$

Let us expand $P_a(r,z)$ in a formal series in variables $r = (r_1, \dots, r_N)$:

$$P_a(r,z) = \sum_{s\in\mathbb{Z}_+^N} r^s P_{as}(z).$$

It is easy to see that for any $j \in \mathbb{Z}$ and any vector g', there are only finitely many $s \in \mathbb{Z}_+^N$ such that the coefficient at z^j in $P_{as}(z) g'$ is non-zero. Thus the coefficient at z^j in $\sum\limits_{s\in\mathbb{Z}_+^N} r^s P_{as}(z) g$ is a polynomial in r. Since for each $j < 0$ these polynomials vanish for all $r \in \mathbb{Z}^N$, we conclude that for all $s \in \mathbb{Z}_+^N$, $a = 1, \dots, N$,

$$P_{as}(z)_- g = 0.$$

Note that for $s = 0$ this equation is trivial. Let us consider the case $s \in \mathbb{Z}_+^N$, with $s_p = 1$ and $s_i = 0$ for $i \neq p$. This gives us an equation

$$\begin{aligned} \sum_{k=1}^{\infty} z^{-k} E^{pa}_{(k)} g &= \left(\left(\sum_{i=1}^{\infty} x_{ai} z^i + q_a \frac{\partial}{\partial q_a} \right) \left(\sum_{k=1}^{\infty} z^{-k} \frac{\partial}{\partial x_{pk}} \right) \right)_- g \\ &= \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} x_{ai} z^i + q_a \frac{\partial}{\partial q_a} \right) z^{-k} \frac{\partial}{\partial x_{pk}} \right) g. \end{aligned} \tag{9.4}$$

Substituting (9.4) into (9.3) we get

$$P'_a(r,z)_- g = 0, \tag{9.5}$$

where

$$\begin{aligned} P'_a(r,z) = &\left(\sum_{i=1}^{\infty} x_{ai} z^i + q_a \frac{\partial}{\partial q_a} + \sum_{p=1}^{N} r_p \sum_{k=0}^{\infty} E^{pa}_{(-k)} z^k \right) \exp\left(- \sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_{pj}} \right) \\ &+ \exp\left(- \sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_{pj}} \right) \left(\sum_{p=1}^{N} r_p \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} x_{ai} z^i + q_a \frac{\partial}{\partial q_a} \right) z^{-k} \frac{\partial}{\partial x_{pk}} \right). \end{aligned}$$

Our module is $\mathbb{Z}^{N+1}$-graded via the action of operators $d_0, \dots, d_N$, and without loss of generality we may assume that g is homogeneous relative to $\mathbb{Z}^{N+1}$-grading. We will call the eigenvalue of $\beta \,\mathrm{Id} - d_0$ the *degree* of g. We use the negative sign here to make the degree non-negative. In fact the $\mathbb{Z}$-grading by degree may be defined on each of the tensor factors $\mathbb{C}[x_{pj}|^{p=1,\dots,N}_{j=1,2,\dots}]$, L_{gl_N} and L_{Vir} by

$$\deg(x_{pj}) = \deg(E^{ab}_{(-j)}) = \deg(L_{(-j)}) = j,$$
$$\deg(W) = \deg(\mathrm{v}_h) = 0.$$

On the space $\mathbb{C}[x_{pj}|^{p=1,\dots,N}_{j=1,2,\dots}]$ we will also consider a refinment of $\mathbb{Z}$-grading by degree, where we will compute the degree in each of the N families of variables. For each $a = 1, \dots, N$, define

$$\deg_a(x_{pj}) = j\delta_{ap}.$$

Then for a monomial $y \in \mathbb{C}[x_{pj}|^{p=1,\dots,N}_{j=1,2,\dots}]$ we have

$$\deg(y) = \sum_{a=1}^{N} \deg_a(y).$$

In addition to the degree of monomials, we will consider another $\mathbb{Z}^N$-grading by *length*, where

$$\mathrm{len}_a(x_{pj}) = \delta_{ap}$$

and define the *total length* to be

$$\mathrm{len}(y) = \sum_{a=1}^{N} \mathrm{len}_a(y).$$

Let us fix homogeneous bases $\{y'_i\}$, $\{y''_j\}$, $\{y'''_k\}$ in the spaces $\mathbb{C}[x_{pj}|_{j=1,2,\dots}^{p=1,\dots,N}]$, L_{gl_N} and L_{Vir} respectively. Then we can expand g into a finite sum

$$g = \sum_{ijk} \alpha_{ijk} q^\mu y'_i \otimes y''_j \otimes y'''_k. \tag{9.7}$$

Note that in the above decomposition $\deg(g) = \deg(y'_i) + \deg(y''_j) + \deg(y'''_k)$. Since equations (9.2), (9.4) and (9.5) do not involve any operators acting on the component L_{Vir}, we conclude that these must be satisfied not only by g, but also by each of the components

$$g_k = \sum_{ij} \alpha_{ijk} q^\mu y'_i \otimes y''_j \otimes y'''_k.$$

Lemma 9.4. *Let g be a homogeneous non-zero critical vector. Then in the decomposition (9.7) there exist $y''_j \in W$ and $y'''_k \in \mathbb{C}\mathrm{v}_h$ with $\alpha_{ijk} \neq 0$ for some i.*

Proof. Let us rewrite (9.7) as

$$g = \sum_{ij} q^\mu y'_i \otimes y''_j \otimes \left(\sum_k \alpha_{ijk} y'''_k \right).$$

Consider the smallest degree n_0 of y'''_k for which $\alpha_{ijk} \neq 0$ for some i, j. We claim that $n_0 = 0$. Otherwise, since L_{Vir} is irreducible, there exists a raising operator $\omega^{Vir}_{(n)}$, $n \geq 2$, such that

$$\omega^{Vir}_{(n)} \sum_{\substack{k \\ \deg(y'''_k) = n_0}} \alpha_{ijk} y'''_k \neq 0.$$

Consider now the Virasoro operator acting on all three factors of the tensor product: $\omega_{(n)} = \omega^{Hyp}_{(n)} + \omega^{gl_N}_{(n)} + \omega^{Vir}_{(n)}$. The part of $\omega_{(n)} g$ involving the terms of the smallest degree in the component L_{Vir} will be

$$\sum_{ij} q^\mu y'_i \otimes y''_j \otimes \omega^{Vir}_{(n)} \sum_{\substack{k \\ \deg(y'''_k) = n_0}} \alpha_{ijk} y'''_k \neq 0.$$

Thus $\omega_{(n)} g \neq 0$. However operator $\omega_{(n)}$ represents $-t_0^{n-1} d_0 \in \mathcal{D}_+$ and must annihilate g since g is a critical vector. This is a contradiction, which implies that $n_0 = 0$.

Let $\widetilde{g} = q^\mu \sum_{ij} \alpha_{ij} y'_i \otimes y''_j \otimes \mathrm{v}_h$ be the projection of g to the space

$$q^\mu \mathbb{C}[x_{pj}|_{j=1,2,\dots}^{p=1,\dots,N}] \otimes L_{gl_N}(W) \otimes \mathrm{v}_h.$$

We just proved that $\widetilde{g} \neq 0$, and it was noted earlier that it satisfies equation (9.4). Let n_1 be the smallest degree of y''_j such that $\alpha_{ij} \neq 0$ for some i. To complete the proof of the lemma, we need to show that $n_1 = 0$. If $n_1 > 0$ using the same argument as above we see that there exists a raising operator $E^{pa}_{(n)}$, $n \geq 1$ such that $E^{pa}_{(n)} \widetilde{g}$ will have a non-zero component with terms in $L_{gl_N}(W)$ of degree $n_1 - n$. However the equation (9.4) implies that all factors from $L_{gl_N}(W)$ that appear in $E^{pa}_{(n)} \widetilde{g}$ have degrees at least n_1. This contradiction implies $n_1 = 0$, and the lemma is proved. □

Let $\overline{g}$ be the projection of the critical vector g to the space

$$q^\mu \mathbb{C}[x_{pj}|_{j=1,2,\dots}^{p=1,\dots,N}] \otimes W \otimes \mathrm{v}_h. \tag{9.8}$$

By the above Lemma, $\overline{g} \neq 0$. Let us take the projection of the equation (9.5) to the space (9.8). This yields

$$P''_a(r, z)_- \overline{g} = 0, \tag{9.6}$$

where

$$P''_a(r,z) = \left(\sum_{i=1}^{\infty} x_{ai} z^i + q_a \frac{\partial}{\partial q_a} + \sum_{p=1}^{N} r_p E_{(0)}^{pa}\right) \exp\left(-\sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_{pj}}\right)$$
$$+ \exp\left(-\sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_{pj}}\right) \left(\sum_{p=1}^{N} r_p \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} x_{ai} z^i + q_a \frac{\partial}{\partial q_a}\right) z^{-k} \frac{\partial}{\partial x_{pk}}\right).$$

Again we decompose $P''_a(r,z)$ in a formal power series in r, $P''_a(r,z) = \sum\limits_{s\in\mathbb{Z}_+^N} r^s P''_{as}(z)$, and we have $P''_{as}(z)_{-}\overline{g} = 0$ for all $s \in \mathbb{Z}_+^N$.

Next we will consider the grading of $\overline{g}$ by (total) length. Note that the operator $P''_{as}(z)$ has two homogeneous components with respect to the length grading – one that decreases the length by $s_1 + \ldots + s_N - 1$ and the other (that contains terms involving $q_a \frac{\partial}{\partial q_a}$) by $s_1 + \ldots + s_N$. Let f be the maximum length component of $\overline{g}$ and suppose $\mathrm{len}(f) = \ell$. Denote by $Q_{as}(z)$ the component of $P''_{as}(z)$ that reduces the length by $s_1 + \ldots + s_N - 1$. Then the component of length $\ell + 1 - s_1 - \ldots - s_N$ in $P''_{as}(z)_{-}\overline{g}$ is $Q_{as}(z)_{-} f$. Thus

$$Q_{as}(z)_{-} f = 0. \tag{9.9}$$

Assembling back the generating series $Q_a(r,z) = \sum\limits_{s\in\mathbb{Z}_+^N} r^s Q_{as}(z)$, we get

$$Q_a(r,z) = \left(\sum_{i=1}^{\infty} x_{ai} z^i + \sum_{p=1}^{N} r_p E_{(0)}^{pa}\right) \exp\left(-\sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_{pj}}\right)$$
$$+ \exp\left(-\sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_{pj}}\right) \left(\sum_{p=1}^{N} r_p \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} x_{ai} z^i\right) z^{-k} \frac{\partial}{\partial x_{pk}}\right)$$
$$= \left(\sum_{i=1}^{\infty} x_{ai} z^i + \sum_{p=1}^{N} r_p E_{(0)}^{pa}\right) \exp\left(-\sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_{pj}}\right)$$
$$+ \left(\sum_{p=1}^{N} r_p \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} (x_{ai} - r_a z^{-i}) z^i\right) z^{-k} \frac{\partial}{\partial x_{pk}}\right) \exp\left(-\sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_{pj}}\right)$$
$$= \left(\sum_{i=1}^{\infty} x_{ai} z^i\right) \exp\left(-\sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_{pj}}\right)$$
$$+ \left(\sum_{p=1}^{N} r_p \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} x_{ai} z^i\right) z^{-k} \frac{\partial}{\partial x_{pk}}\right) \exp\left(-\sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_{pj}}\right)$$
$$+ \left(\sum_{p=1}^{N} r_p E_{(0)}^{pa} - r_a \sum_{k=1}^{\infty} \sum_{p=1}^{N} (k-1) r_p z^{-k} \frac{\partial}{\partial x_{pk}}\right) \exp\left(-\sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_{pj}}\right). \tag{9.10}$$

Our goal is to solve the system of equations (9.9) in the space $\mathbb{C}[x_{pj}|_{j=1,2,\ldots}^{p=1,\ldots,N}] \otimes W$. The solutions we are looking for are homogeneous in both degree and length. Let $\deg(f) = m$, $\mathrm{len}(f) = \ell$. The equation (9.9) has trivial solutions with $\ell = 0$. These correspond to the critical vectors in the top of the module. We are going to show that non-trivial solutions of (9.9) must have length $\ell = 1$.

To establish this claim we will first analyze the equation (9.9) in cases $N = 1$ and $N = 2$. The case of the general N will follow from the following simple observation. Consider a proper subset $S \subset \{1, \ldots, N\}$. Let us take a solution f of (9.9) and specialize all variables x_{pj} with $p \notin S$ to scalars, we will get a solution for (9.9) with a smaller N. To see this, set in (9.9) $r_p = 0$ for all $p \notin S$ and restrict a to the set S. The information about the solutions of (9.9) with $N = 1$ and $N = 2$ may be used to establish properties of solutions of this equation for a general N.

Lemma 9.5. *Let $N = 1$, and let W be a 1-dimensional gl_1-module with identity matrix $I = E^{11}$ acting by scalar $\alpha \in \mathbb{C}$. Let f be a non-constant homogeneous (in both length and degree) solution of (9.9) with $N = 1$. Then* $\mathrm{len}(f) = 1$ *and* $\alpha = 1 - \deg(f)$.

Proof. First of all, let us rewrite (9.10) for the case $N = 1$. To simplify notations we will drop redundant indices and write x_i instead of x_{pi}, etc.,

$$Q(r,z) = \left(\sum_{i=1}^{\infty} x_i z^i + r \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} x_i z^i \right) z^{-k} \frac{\partial}{\partial x_k} + r\alpha - r^2 \sum_{k=1}^{\infty} (k-1) z^{-k} \frac{\partial}{\partial x_k} \right) \exp \left(-r \sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_j} \right). \tag{9.11}$$

Let f satisfy $Q(r,z)_- f = 0$. We may view f as a polynomial of length $\ell > 0$ and degree $m \geq \ell$ in $\mathbb{C}[x_1, x_2, \ldots]$. Choose a natural number s such that

$$\left(\sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_j} \right)^{s+1} f = 0 \tag{9.12}$$

but

$$R(z) = \left(\sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_j} \right)^{s} f \neq 0. \tag{9.13}$$

Clearly $1 \leq s \leq \ell$.

Let us consider the coefficient at r^{s+1} in the equation $Q(r,z)_- f = 0$:

$$\frac{(-1)^s}{s!} \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} x_i z^i \right) z^{-k} \frac{\partial}{\partial x_k} \right) R(z) + \frac{(-1)^s}{s!} \alpha R(z)$$

$$- \frac{(-1)^{s-1}}{(s-1)!} \left(\sum_{k=1}^{\infty} (k-1) z^{-k} \frac{\partial}{\partial x_k} \right) \left(\sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_j} \right)^{s-1} f = 0.$$

Thus

$$\left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} x_i z^i \right) z^{-k} \frac{\partial}{\partial x_k} \right) R(z) + \alpha R(z) - z \frac{d}{dz} R(z) - s R(z) = 0. \tag{9.14}$$

Consider an expansion $R(z) = R_n z^{-n} + R_{n+1} z^{-n-1} + \ldots + R_m z^{-m}$ with $R_n \neq 0$. Let us look at the coefficient at z^{-n} in the above equation:

$$\alpha R_n + n R_n - s R_n = 0,$$

which implies $\alpha = s - n \in \mathbb{Z}_-$.

Applying the operator $z \frac{d}{dz}$ to (9.12) we get

$$\left(\sum_{k=1}^{\infty} k z^{-k} \frac{\partial}{\partial x_k} \right) R(z) = 0.$$

Let us prove by induction that for all $j \geq 0$

$$\left(\sum_{k=1}^{\infty} k^j z^{-k} \frac{\partial}{\partial x_k} \right) R(z) = 0. \tag{9.15}$$

Suppose (9.15) holds for all $j' \leq j$, $j \geq 1$. Applying the operator $\sum_{k=1}^{\infty} k^j z^{-k} \frac{\partial}{\partial x_k}$ to (9.14) and using the induction assumption, we get

$$0 = \left(\sum_{p=1}^{\infty} p^j z^{-p} \frac{\partial}{\partial x_p} \right) \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} x_i z^i \right) z^{-k} \frac{\partial}{\partial x_k} \right) R(z) - \left(\sum_{k=1}^{\infty} k^j z^{-k} \frac{\partial}{\partial x_k} \right) z \frac{d}{dz} R(z)$$

$$= \left(\sum_{k=1}^{\infty} \left(\sum_{p=1}^{k-1} p^j \right) z^{-k} \frac{\partial}{\partial x_k} \right) R(z) - \left(\sum_{k=1}^{\infty} k^{j+1} z^{-k} \frac{\partial}{\partial x_k} \right) R(z)$$

$$= \left(\frac{1}{j+1} - 1 \right) \left(\sum_{k=1}^{\infty} k^{j+1} z^{-k} \frac{\partial}{\partial x_k} \right) R(z),$$

which establishes the inductive step. In the above calculation we used the fact that $\sum\limits_{p=1}^{k-1} p^j$ is a polynomial in k with the leading term $\frac{k^{j+1}}{j+1}$.

Using the Vandermonde determinant argument we conclude from (9.15) that

$$\frac{\partial}{\partial x_k} R(z) = 0$$

for all k. This implies that all R_i are scalars, but since they can't have equal degrees, we conclude that R_n is a non-zero scalar, while all other coefficients are zero. Without the loss of generality we may thus assume

$$R(z) = \left(\sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_j} \right)^s f = z^{-n}.$$

Thus $s = \mathrm{len}(f) = \ell$ and $n = \deg(f) = m$, and then $\alpha = \mathrm{len}(f) - \deg(f)$. It remains to prove that $\mathrm{len}(f) = 1$. We will reason by contradiction. Let us suppose that $\ell = \mathrm{len}(f) > 1$ and consider

$$S(z) = \left(\sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_j} \right)^{\ell-1} f.$$

Since every term in $S(z)$ has length 1, we can write

$$S(z) = \beta_1 x_1 z^{-m+1} + \beta_2 x_2 z^{-m+2} + \ldots + \beta_{m-\ell+1} x_{m-\ell+1} z^{-\ell+1}.$$

Now we look at the coefficient at r^ℓ in the equation $Q(r,z)_- f = 0$:

$$\begin{aligned} 0 = {} & \frac{(-1)^\ell}{\ell!} \left(\left(\sum_{i=1}^{\infty} x_i z^i \right) \left(\sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_j} \right)^\ell f \right)_- \\ & + \frac{(-1)^{\ell-1}}{(\ell-1)!} \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} x_i z^i \right) z^{-k} \frac{\partial}{\partial x_k} \right) \left(\sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_j} \right)^{\ell-1} f \\ & - \frac{(-1)^{\ell-2}}{(\ell-2)!} \left(\sum_{k=1}^{\infty} (k-1) z^{-k} \frac{\partial}{\partial x_k} \right) \left(\sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_j} \right)^{\ell-2} f \\ & + \frac{(-1)^{\ell-1}}{(\ell-1)!} \alpha \left(\sum_{j=1}^{\infty} z^{-j} \frac{\partial}{\partial x_j} \right)^{\ell-1} f. \end{aligned}$$

Taking out the factor of $\frac{(-1)^\ell}{\ell!}$ we get

$$\begin{aligned} 0 &= \sum_{i=1}^{m-1} x_i z^{-m+i} - \ell \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k-1} x_i z^i \right) z^{-k} \frac{\partial}{\partial x_k} \right) S(z) \\ & \quad + \ell z \frac{d}{dz} S(z) + \ell(\ell-1) S(z) + \ell(m-\ell) S(z) \\ &= \sum_{i=1}^{m-1} x_i z^{-m+i} - \ell \sum_{k=1}^{m-\ell+1} \left(\sum_{i=1}^{k-1} x_i z^i \right) \beta_k z^{-m} \\ & \quad + \ell \sum_{k=1}^{m-\ell+1} (-m+k) \beta_k x_k z^{-m+k} + \ell(m-1) \sum_{k=1}^{m-\ell+1} \beta_k x_k z^{-m+k} \end{aligned}$$

$$= \sum_{i=1}^{m-1} x_i z^{-m+i} - \ell \sum_{i=1}^{m-\ell} \left(\sum_{k=i+1}^{m-\ell+1} \beta_k \right) x_i z^{-m+i} + \ell \sum_{k=1}^{m-\ell+1} (k-1)\beta_k x_k z^{-m+k}.$$

We stress that the above calculation is only valid when $\ell > 1$. If $\ell > 2$ we immediately get a contradiction since the coefficient at z^{-1} yields $x_{m-1} = 0$. It only remains to rule out the case $\ell = 2$. In the latter case the equation simplifies to the following:

$$\sum_{k=1}^{m-1} x_k z^{-m+k} - 2 \sum_{k=1}^{m-1} \left(\sum_{i=1}^{k-1} x_i z^i \right) \beta_k z^{-m} + 2 \sum_{k=1}^{m-1} (k-1)\beta_k x_k z^{-m+k} = 0.$$

Let us specialize this equation to $x_1 = \ldots = x_{m-1} = 1$, $z = 1$. Then we get

$$(m-1) - 2 \sum_{k=1}^{m-1} (k-1)\beta_k + 2 \sum_{k=1}^{m-1} (k-1)\beta_k = 0,$$

which implies $m = 1$, which is impossible since $m = \deg(f)$ can not be less than $\ell = \mathrm{len}(f)$. Thus $\ell = \mathrm{len}(f) = 1$ and the lemma is proved.

□

Next we are going to show that for a general N, a homogeneous non-trivial solution of (9.9) must have total length 1. The previous lemma implies that such a solution may have only two components with respect to len_a grading for each a, where len_a may be either 0 or 1. To prove the general case, it is sufficient to consider $N = 2$, since if a monomial has $\mathrm{len}_a + \mathrm{len}_b$ at most 1 for any pair of distinct indices, then its total length does not exceed 1.

Lemma 9.6. *Let $N = 2$ and let W be a finite-dimensional gl_2-module. Then any homogeneous (in both length and degree) non-constant solution f of (9.9) has total length* 1.

Proof. It is sufficient to consider the case of W being irreducible since the equation (9.9) is compatible with the gl_N-module homomorphisms. Let us fix a basis $\{w_n, w_{n-1}, \ldots, w_{-n}\}$ of W, where $n \in \frac{1}{2}\mathbb{Z}_+$ and $(E^{11} - E^{22})w_i = 2iw_i$. Assuming that the identity matrix acts on W by scalar α, we get

$$E^{11} w_i = \left(\frac{\alpha}{2} + i \right) w_i \text{ and } E^{22} w_i = \left(\frac{\alpha}{2} - i \right) w_i.$$

It follows from the previous lemma that every monomial in the decomposition of f has length at most 1 with respect to each of the two indexes. Thus we only need to prove that f can not have total length 2. We will reason by contradiction. If $\mathrm{len}(f) = 2$ then for each monomial in f both len_1 and len_2 are 1. Suppose $\deg(f) = m$. Let us write

$$f = \sum_{i=-n}^{n} f_i \otimes w_i.$$

By Lemma 9.5 we have

$$\deg_1(f_i) = 1 - \left(\frac{\alpha}{2} + i \right), \quad \deg_2(f_i) = 1 - \left(\frac{\alpha}{2} - i \right),$$

so $m = \deg(f) = 2 - \alpha$. Let $b_i = \deg_1(f_i) = \frac{m}{2} - i$ and $c_i = \deg_2(f_i) = \frac{m}{2} + i$. Then f may be written as

$$f = \sum_{i=-n}^{n} \beta_i x_{1,b_i} x_{2,c_i} \otimes w_i, \tag{9.16}$$

where we set $\beta_i = 0$ whenever $b_i \leq 0$ or $c_i \leq 0$.

Let us take the equation derived from (9.9) by taking the coefficient at $r_1 r_2$ with $a = 1$ and substitute (9.16) in it. We get

$$0 = \sum_{i=-n}^{n} \beta_i \left(\sum_{j=1}^{m-1} z^{-m+j} x_{1,j} \right) \otimes w_i - \sum_{i=-n}^{n} \beta_i \left(\sum_{j=1}^{b_i - 1} z^{-m+j} x_{1,j} \right) \otimes w_i$$

$$- \sum_{i=-n}^{n} \beta_i \left(\sum_{j=1}^{c_i-1} z^{-m+j} x_{1,j} \right) \otimes w_i - \sum_{i=-n}^{n} \beta_i (c_i - 1) z^{-m+b_i} x_{1,b_i} \otimes w_i$$

$$- \sum_{i=-n}^{n} \beta_i (1 - b_i) z^{-m+b_i} x_{1,b_i} \otimes w_i - \sum_{i=-n}^{n} \beta_i z^{-m+c_i} x_{2,c_i} \otimes E^{21} w_i. \tag{9.17}$$

Note that only the last sum contains variables $x_{2,j}$. By equating this sum to zero, we conclude that $\beta_i E^{21} w_i = 0$ for all $i = -n, \dots, n$. It follows that $\beta_i = 0$ for all $i \neq -n$. Similarly, taking the same equation with $a = 2$, we will get that $\beta_i = 0$ for all $i \neq n$. Thus the only possibility for a non-zero solution is when $n = 0$, which means that W is 1-dimensional and m is even. In this case $b_0 = c_0 = \frac{m}{2}$ and $f = x_{1,\frac{m}{2}} x_{2,\frac{m}{2}} \otimes w$, and the equation (9.17) becomes

$$\sum_{j=1}^{m-1} x_{1,j} z^{-m+j} \otimes w - 2 \sum_{j=1}^{\frac{m}{2}-1} z^{-m+j} x_{1,j} \otimes w = 0,$$

which gives a contradiction. Thus the total length of f must be 1 and the lemma is proved. □

Now we return to the general case. We proved that a non-trivial homogeneous solution f of (9.3) must have length 1. Suppose $\deg(f) = m$. Then f can be written as

$$f = \sum_{p=1}^{N} x_{pm} \otimes w_p, \quad w_p \in W.$$

The equation (9.9) then simplifies as follows:

$$0 = \sum_{p=1}^{N} r_p \sum_{b=1}^{N} r_b E^{pa} w_b + r_a (m-1) \sum_{b=1}^{N} r_b w_b$$

$$= \sum_{p=1}^{N} \sum_{b=1}^{N} r_p r_b \left(E^{pa} + \delta_{pa} (m-1) \right) w_b. \tag{9.18}$$

Consider a new action ρ' of gl_N on W:

$$\rho'(E^{pa}) w = E^{pa} w + (m-1) \delta_{ap} w, \quad w \in W.$$

This gives the same structure of W as an sl_N-module, but now the identity matrix acts with scalar $\alpha' = \alpha + (m-1)N$. Then (9.18) is equivalent to the system of equations

$$\rho'(E^{ca}) w_b + \rho'(E^{ba}) w_c = 0 \tag{9.19}$$

where $a, b, c = 1, \dots, N$.

We will also use a third gl_N-action ρ'' on W:

$$\rho''(E^{pa}) w = \rho'(E^{pa}) w + \delta_{ap} w = E^{pa} w + m \delta_{ap} w, \quad w \in W.$$

The identity matrix here acts with scalar $\alpha'' = \alpha + mN$. For this action the equation (9.19) may be written as

$$\rho''(E^{ca}) w_b + \rho''(E^{ba}) w_c = \delta_{ca} w_b + \delta_{ba} w_c. \tag{9.20}$$

We also have

$$\sum_{p=1}^{N} \sum_{b=1}^{N} r_p r_b \rho''(E^{pa}) w_b = r_a \sum_{b=1}^{N} r_b w_b. \tag{9.21}$$

We are going to classify gl_N-modules W for which the system (9.19) has non-trivial solutions. We will do this indirectly, linking this system with reducibility of tensor modules.

Lemma 9.7. *Let (W, ρ'') be a finite-dimensional irreducible gl_N-module. Let $\mathcal{P}$ be the set of all solutions $(w_1, \dots, w_N) \in W \times \dots \times W$ of the system of equations (9.20). Then the subspace*

$$\widetilde{\mathcal{P}} = \left\{ \underset{r\in\mathbb{Z}^N}{\oplus} q^r \otimes (r_1 w_1 + \dots r_N w_N) \big| (w_1, \dots, w_N) \in \mathcal{P} \right\}$$

is a $\mathrm{Vect}\mathbb{T}^N$*-submodule in the tensor module* $T(W) = \mathbb{C}[q_1^{\pm 1}, \dots, q_N^{\pm 1}] \otimes W$*, associated with* (W, ρ'').

Proof. Let $(w_1, \dots, w_N) \in \mathcal{P}$. Then using the tensor module action and (9.21) we get

$$t^s d_a \left(q^r \otimes \sum_{b=1}^N r_b w_b \right) = r_a q^{r+s} \otimes \sum_{b=1}^N r_b w_b + \sum_{p=1}^N \sum_{b=1}^N s_p r_b q^{r+s} \otimes \rho''(E^{pa}) w_b$$
$$= \sum_{b=1}^N r_b q^{r+s} \otimes \sum_{p=1}^N (r_p + s_p) \rho''(E^{pa}) w_b.$$

Fix $1 \leq a, i \leq N$. Set $\widetilde{w}_p = \rho''(E^{pa}) w_i$, $p = 1, \dots, N$. To complete the proof of the lemma, it is sufficient to show that $(\widetilde{w}_1, \dots, \widetilde{w}_N) \in \mathcal{P}$. Instead of working with (9.20), it will be easier to check an equivalent condition (9.19). Note that $\widetilde{w}_p = \rho'(E^{pa}) w_i + \delta_{pa} w_i$. Then using the fact that $(w_1, \dots, w_N)$ satisfies (9.19), we obtain

$$\rho'(E^{cd}) \widetilde{w}_b + \rho'(E^{bd}) \widetilde{w}_c$$
$$= \rho'(E^{cd}) \rho'(E^{ba}) w_i + \delta_{ab} \rho'(E^{cd}) w_i + \rho'(E^{bd}) \rho'(E^{ca}) w_i + \delta_{ac} \rho'(E^{bd}) w_i$$
$$= -\rho'(E^{cd}) \rho'(E^{ia}) w_b + \delta_{ab} \rho'(E^{cd}) w_i - \rho'(E^{bd}) \rho'(E^{ia}) w_c + \delta_{ac} \rho'(E^{bd}) w_i$$
$$= -\rho'(E^{ia}) \rho'(E^{cd}) w_b - \delta_{id} \rho'(E^{ca}) w_b + \delta_{ac} \rho'(E^{id}) w_b + \delta_{ab} \rho'(E^{cd}) w_i$$
$$- \rho'(E^{ia}) \rho'(E^{bd}) w_c - \delta_{id} \rho'(E^{ba}) w_c + \delta_{ab} \rho'(E^{id}) w_c + \delta_{ac} \rho'(E^{bd}) w_i = 0.$$

Thus $(\widetilde{w}_1, \dots, \widetilde{w}_N) \in \mathcal{P}$. Lemma is now proved.

□

Corollary 9.8. *Let (W, ρ) be a finite-dimensional irreducible gl_N-module. If $L(W, \gamma, h)$ has a critical vector of degree $m \geq 1$ then either W has a fundamental highest weight ω_k, $1 \leq k \leq N-1$, with respect to sl_N-action, with identity matrix acting with scalar $\alpha = k - mN$, or W is a 1-dimensional module with identity matrix acting with scalar $\alpha = N - mN$.*

Proof. If the system (9.20) has a non-trivial solution then the submodule $\widetilde{\mathcal{P}}$ in the tensor module $T(W)$ corresponding to (W, ρ'') is non-zero. It is a proper submodule since its component at q^0 is trivial. Using the classification of reducible tensor modules (Theorem 2.1), we conclude that $T(W)$ is one of the de Rham modules $\Omega^k(\mathbb{T}^N)$, $k = 1, \dots, N$. Taking into account the relation $\alpha = \alpha'' - mN$, we obtain the claim of the corollary.

□

To complete the proof of Theorem 9.1 it remains to establish the following

Lemma 9.9. *If $L(W, \gamma, h)$ has a critical vector that does not belong the top then $h = 0$.*

Proof. Our strategy will be the same as in derivation of equation (9.9). A critical vector g is annihilated by $t_0^j t^r d_0$ for $j \geq 0$. Thus

$$(z^2 d_0(r, z))_- g = 0.$$

We will project this equation to the subspace (9.8) in order to derive an equation on f. Finally, we will take r_a-component of the resulting equation. The action of $d_0(r, z)$ is given by (6.12), which has three summands. We will analyze the contribution of each summand in $z^2 d_0(r, z)$ separately.

Consider the first summand

$$-\left(z^2 \left(\sum_{j=1}^\infty \omega_{(-j)} z^{j-1} \right) Y(q^r, z) + z^2 Y(q^r, z) \left(\sum_{j=0}^\infty \omega_{(j)} z^{-j-1} \right) \right)_- g. \tag{9.22}$$

We have

$$\omega_{(j)} g = 0 \quad \text{for} \quad j \geq 2,$$

since $-\omega_{(j)}$ represents $t_0^{j-1} d_0$. Thus the corresponding terms in the above expression may be dropped. We also recall that

$$\omega_{(j)} = \omega_{(j)}^{Hyp} + \omega_{(j)}^{gl_N} + \omega_{(j)}^{Vir}.$$

We further split (9.22) into three summands corresponding to this decomposition. For the case of the Virasoro field of the hyperbolic lattice component we have

$$\omega^{Hyp}(z) = \sum_{p=1}^{N} \left(\sum_{j=1}^{\infty} j u_{pj} z^{j-1} \right) \left(\sum_{k=1}^{\infty} k v_{pk} z^{k-1} + z^{-1} q_p \frac{\partial}{\partial q_p} + \sum_{k=1}^{\infty} \frac{\partial}{\partial u_{pk}} z^{-k-1} \right)$$
$$+ \sum_{p=1}^{N} \left(\sum_{k=1}^{\infty} k v_{pk} z^{k-1} + z^{-1} q_p \frac{\partial}{\partial q_p} + \sum_{k=1}^{\infty} \frac{\partial}{\partial u_{pk}} z^{-k-1} \right) \left(\sum_{j=1}^{\infty} \frac{\partial}{\partial v_{pj}} z^{-j-1} \right). \tag{9.23}$$

The first summand in (9.23) does not contribute to the projection to (9.8) since it contains multiplications by u_{pj}, while $Y(q^r, z)$ does not involve differentiations in these variables. Note that we are only interested in powers z^j in $\omega^{Hyp}(z)$ with $j \geq -2$. Thus the only terms that will contribute are:

$$\sum_{p=1}^{N} \left(\sum_{k=1}^{\infty} k v_{pk} z^{k-1} \right) \left(\sum_{j=1}^{\infty} \frac{\partial}{\partial v_{pj}} z^{-j-1} \right).$$

In operator $Y(q^r, z)$ we may then drop the factors containing u_{pj} when taking the projection to (9.8). The contribution that we get will be

$$- \left(z^2 \sum_{p=1}^{N} \left(\sum_{k=3}^{\infty} \sum_{j=1}^{k-2} k v_{pk} \frac{\partial}{\partial v_{pj}} z^{k-j-2} \right) \exp \left(- \sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} \frac{z^{-j}}{j} \frac{\partial}{\partial v_{pj}} \right) \right)_{-} f$$
$$- \left(\exp \left(- \sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} \frac{z^{-j}}{j} \frac{\partial}{\partial v_{pj}} \right) \sum_{p=1}^{N} \left(z \sum_{j=1}^{\infty} (j+1) v_{p,j+1} \frac{\partial}{\partial v_{pj}} + \sum_{j=1}^{\infty} j v_{pj} \frac{\partial}{\partial v_{pj}} \right) \right)_{-} f.$$

Let us now take the r_a-coefficient of the expansion in powers of r:

$$\left(\sum_{p=1}^{N} \left(\sum_{k=3}^{\infty} \sum_{j=1}^{k-2} k v_{pk} \frac{\partial}{\partial v_{pj}} z^{k-j} \right) \left(\sum_{j=1}^{\infty} \frac{z^{-j}}{j} \frac{\partial}{\partial v_{aj}} \right) f \right)_{-}$$
$$+ \left(\left(\sum_{j=1}^{\infty} \frac{z^{-j}}{j} \frac{\partial}{\partial v_{aj}} \right) \sum_{p=1}^{N} \left(z \sum_{j=1}^{\infty} (j+1) v_{p,j+1} \frac{\partial}{\partial v_{pj}} + \sum_{j=1}^{\infty} j v_{pj} \frac{\partial}{\partial v_{pj}} \right) f \right)_{-}. \tag{9.24}$$

Since f is linear in v_{pm},

$$f = \sum_{p=1}^{N} v_{pm} \otimes w_p \otimes \mathrm{v}_h, \quad w_p \in W,$$

the first summand in (9.24) vanishes, while the second simplifies to

$$\left(z^{-m} \sum_{j=1}^{\infty} \frac{\partial}{\partial v_{aj}} + z^{-m} \sum_{j=1}^{\infty} \frac{\partial}{\partial v_{aj}} \right) f$$
$$= 2 z^{-m} 1 \otimes w_a \otimes \mathrm{v}_h. \tag{9.25}$$

Next, let us consider the contribution of the Virasoro field of V_{gl_N}:

$$- \left(z^2 \left(\sum_{j=1}^{\infty} \omega_{(-j)}^{gl_N} z^{j-1} \right) Y(q^r, z) + z^2 Y(q^r, z) \left(z^{-1} \omega_{(0)}^{gl_N} + z^{-2} \omega_{(1)}^{gl_N} \right) \right)_{-} g. \tag{9.26}$$

The operators $\omega^{gl_N}_{(j)}$ with $j \leq 0$ increase the degree in the component L_{gl_N} and thus do not contribute to the projection to the space (9.8), and only the term with $\omega^{gl_N}_{(1)}$ will contribute. The Virasoro field of $\widehat{gl}_N$ is a sum of the Virasoro fields of $\widehat{sl}_N$ (6.5) and the Heisenberg algebra (6.6).

Using (6.5) we can write

$$\omega^{sl_N}_{(1)} = \frac{1}{2(N+1)} \left(\sum_{k=1}^{\infty} \sum_{i,j=1}^{N} E^{ij}_{(-k)} E^{ji}_{(k)} + \sum_{k=0}^{\infty} \sum_{i,j=1}^{N} E^{ji}_{(-k)} E^{ij}_{(k)} \right.$$

$$\left. - \frac{1}{N} \sum_{k=1}^{\infty} I_{(-k)} I_{(k)} - \frac{1}{N} \sum_{k=0}^{\infty} I_{(-k)} I_{(k)} \right),$$

and the terms that contribute to the projection are

$$\frac{1}{2(N+1)} \left(\sum_{i,j=1}^{N} E^{ij}_{(0)} E^{ji}_{(0)} - \frac{1}{N} I_{(0)} I_{(0)} \right),$$

which is a multiple of the Casimir operator for sl_N. If W corresponds to the tensor module of k-forms, $k = 1, \dots, N$, this operator will act on the space (9.8) with scalar $\frac{k(N-k)}{2N}$ (see (8.9), this also includes the case of a trivial sl_N module W when $k = N$).

Analogously, for the Virasoro field (6.6) of the Heisenberg algebra $\omega^{Hei}_{(1)}$ will be acting on f with the scalar

$$\frac{1}{2N} I_{(0)} I_{(0)} - \frac{1}{2} I_{(0)} = \frac{(k-mN)^2}{2N} - \frac{k-mN}{2}.$$

Going back to (9.26), we get the contribution of the Virasoro field in V_{gl_N}.

$$-\left(\frac{k(N-k)}{2N} + \frac{(k-mN)^2}{2N} - \frac{k-mN}{2} \right) \exp\left(-\sum_{p=1}^{N} r_p \sum_{j=1}^{\infty} \frac{z^{-j}}{j} \frac{\partial}{\partial v_{pj}} \right) f,$$

and its r_a-term will yield

$$\frac{z^{-m}}{m} \left(\frac{k(N-k)}{2N} + \frac{(k-mN)^2}{2N} - \frac{k-mN}{2} \right) 1 \otimes w_a \otimes \mathrm{v}_h. \tag{9.27}$$

Now let us deal with the Virasoro field of L_{Vir}. The corresponding term is

$$-\left(z^2 \left(\sum_{j=1}^{\infty} \omega^{Vir}_{(-j)} z^{j-1} \right) Y(q^r, z) + z^2 Y(q^r, z) \left(z^{-1} \omega^{Vir}_{(0)} + z^{-2} \omega^{Vir}_{(1)} \right) \right)_{-} g. \tag{9.28}$$

Since the operators $\omega^{Vir}_{(j)}$ with $j \leq 0$ increase the degree in the component L_{Vir}, the only term that contributes to the projection to (9.8) is $\omega^{Vir}_{(1)}$, which acts on (9.8) with scalar h. Thus the r_a-term of (9.28) gives the contribution

$$\left(\sum_{j=1}^{\infty} \frac{z^{-j}}{j} \frac{\partial}{\partial v_{aj}} \right) \omega^{Vir}_{(1)} f = \frac{z^{-m}}{m} h \, 1 \otimes w_a \otimes \mathrm{v}_h. \tag{9.29}$$

Next we shall look at the the summand $- \sum\limits_{a,b=1}^{N} r_a u^b(z) E^{ab}(z) Y(q^r, z)$ in (6.12). Its r_a-term is

$$-\sum_{p=1}^{N} \left(z^2 \left(\sum_{j=1}^{\infty} j u_{pj} z^{j-1} + \sum_{j=1}^{\infty} \frac{\partial}{\partial v_{pj}} z^{-j-1} \right) \times \left(\sum_{k \in \mathbb{Z}} E^{ap}_{(k)} z^{-k-1} \right) \right)_{-} g.$$

When we consider the projection to (9.8), we can drop terms with multiplications by u_{pj} and $E^{ap}_{(k)}$ with $k \leq -1$, while for $E^{ap}_{(k)}$ with $k \geq 1$ we may use (9.4), which yields

$$-\sum_{p=1}^{N}\left(\sum_{j=1}^{\infty}\frac{\partial}{\partial v_{pj}}z^{-j}\right)E^{ap}_{(0)}f$$

$$-\sum_{p=1}^{N}\left(\sum_{j=1}^{\infty}\frac{\partial}{\partial v_{pj}}z^{-j}\right)\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k-1}iv_{pi}z^{i}+q_p\frac{\partial}{\partial q_p}\right)\frac{z^{-k}}{k}\frac{\partial}{\partial v_{ak}}\right)f.$$

Taking into account that f is linear in v_{pm}, this can be simplified to the following

$$-\sum_{p=1}^{N}\left(\sum_{j=1}^{\infty}\frac{\partial}{\partial v_{pj}}z^{-j}\right)E^{ap}_{(0)}f-N\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k-1}i\right)\frac{z^{-k}}{k}\frac{\partial}{\partial v_{ak}}\right)f$$

$$=-z^{-m}\sum_{p=1}^{N}1\otimes E^{ap}w_p\otimes \mathrm{v}_h-z^{-m}N\frac{m-1}{2}1\otimes w_a\otimes \mathrm{v}_h. \tag{9.30}$$

Lemma 9.7 provides a relation between components $(w_1,\ldots,w_N)$ and submodules in the tensor modules $\Omega^k(\mathbb{T}^N)$. Using computations in the tensor module of k-forms one can show that

$$\sum_{p=1}^{N}E^{ap}w_p=(N-k-m+1)w_a,$$

thus (9.30) reduces to

$$-z^{-m}\left((N-k-m+1)+N\frac{m-1}{2}\right)1\otimes w_a\otimes \mathrm{v}_h. \tag{9.31}$$

The r_a-coefficient of the last summand

$$\sum_{p=1}^{N}\left(z^2\left(\frac{d}{dz}u^p(z)\right)Y(q^r,z)\right)_{-}g$$

is

$$\left(\sum_{j=1}^{\infty}j(j-1)u_{aj}z^{j}+\sum_{j=1}^{\infty}(-j-1)\frac{\partial}{\partial v_{aj}}z^{-j}\right)_{-}g,$$

and its projection to (9.8) yields

$$\sum_{j=1}^{\infty}(-j-1)\frac{\partial}{\partial v_{aj}}z^{-j}f=-(m+1)z^{-m}1\otimes w_a\otimes \mathrm{v}_h \tag{9.32}$$

Finally, collecting (9.25), (9.27), (9.29), (9.31) and (9.32) together, we get

$$0=\left(2+\frac{1}{m}\left(\frac{k(N-k)}{2N}+\frac{(k-mN)^2}{2N}-\frac{k-mN}{2}\right)+\frac{h}{m}-(N-k-m+1)\right.$$
$$\left.-N\frac{m-1}{2}-(m+1)\right)w_a=\frac{h}{m}w_a.$$

Since a is arbitrary, we can choose it so that $w_a\neq 0$. Thus $h=0$, which was to be demonstrated. $\square$

10. Chiral de Rham complex

Chiral de Rham complex was introduced by Malikov et al. in [20]. In case a of torus $\mathbb{T}^N$ the space of this differential complex is a tensor product of two vertex (super) algebras

$$V_{Hyp}^{+} \otimes V_{\mathbb{Z}^N}.$$

Here $V_{\mathbb{Z}^N}$ is the lattice vertex superalgebra of the standard euclidean lattice $\mathbb{Z}^N$. Before we define the differential of this complex, let us review the structure of $V_{\mathbb{Z}^N}$. The vertex superalgebra $V_{\mathbb{Z}^N}$ has two main realizations – the bosonic realization and the fermionic one, with boson-fermion correspondence being an isomorphism between the two models. For our purposes it will be more convenient to use the fermionic realization of $V_{\mathbb{Z}^N}$.

Consider the Clifford Lie superalgebra Cl_N of "charged free fermions" with basis

$$\{\varphi^p_{(j)}, \psi^p_{(j)} | p = 1, \ldots, N, j \in \mathbb{Z}\}$$

of its odd part and a 1-dimensional even part spanned by a central element K. The Lie bracket in Cl_N is given by

$$[\varphi^a_{(m)}, \psi^b_{(n)}] = \delta_{ab}\delta_{m,-n-1}K, \quad [\varphi^a_{(m)}, \varphi^b_{(n)}] = [\psi^a_{(m)}, \psi^b_{(n)}] = 0.$$

Define formal fields

$$\varphi^a(z) = \sum_{j\in\mathbb{Z}} \varphi^a_{(j)} z^{-j}, \qquad \psi^a(z) = \sum_{j\in\mathbb{Z}} \psi^a_{(j)} z^{-j-1}, \qquad K(z) = Kz^0.$$

With this choice of fields Cl_N becomes a vertex Lie superalgebra since the only non-trivial relation between these fields is

$$\left[\varphi^a(z_1), \psi^b(z_2)\right] = \delta_{ab} K(z_2) \left[z_1^{-1} \delta\left(\frac{z_2}{z_1}\right)\right].$$

The lattice vertex superalgebra $V_{\mathbb{Z}^N}$ is isomorphic to the universal enveloping vertex algebra of Cl_N at level 1. As a vector space it is the unique Cl_N-module generated by vacuum vector $\mathbb{1}$, satisfying

$$K\mathbb{1} = \mathbb{1}, \quad \varphi^p_{(j)}\mathbb{1} = \psi^p_{(j)}\mathbb{1} = 0 \text{ for } \ j \geq 0, \ p = 1, \ldots, N.$$

In its fermionic realization $V_{\mathbb{Z}^N}$ is the exterior algebra with generators $\{\varphi^p_{(j)}, \psi^p_{(j)}|^{p=1,\ldots,N}_{j\leq -1}\}$ and is irreducible as a module over Cl_N. The state-field correspondence map Y is given by the standard formula (6.1).

We fix the Virasoro element in $V_{\mathbb{Z}^N}$:

$$\omega^{fer} = \sum_{p=1}^{N} \varphi^p_{(-2)}\psi^p_{(-1)}\mathbb{1}.$$

The rank of this VOA is $-2N$.

It is well-known that vertex superalgebra $V_{\mathbb{Z}^N}$ contains a level 1 simple $\widehat{gl}_N$ vertex algebra. The fields generating this subalgebra are

$$E^{ab}(z) =: \varphi^a(z)\psi^b(z) : .$$

It is easy to check that these satisfy relations (6.4) and the central element of $\widehat{gl}_N$ acts as identity operator. It is also straightforward to verify that the Virasoro element (6.7) in the $\widehat{gl}_N$ vertex algebra maps to ω^{fer} under this embedding.

Let us define two $\mathbb{Z}$-gradings on $V_{\mathbb{Z}^N}$. The *fermionic* degree is defined by

$$\deg_{fer}(\varphi^p_{(j)}) = 1, \quad \deg_{fer}(\psi^p_{(j)}) = -1, \quad \deg_{fer}(K) = \deg_{fer}(\mathbb{1}) = 0.$$

The *bosonic* grading is defined as follows:

$$\deg_{bos}(\varphi^p_{(j)}) = -j-1, \quad \deg_{bos}(\psi^p_{(j)}) = -j, \quad \deg_{bos}(K) = \deg_{bos}(\mathbb{1}) = 0.$$

Let $V^k_{\mathbb{Z}^N}$ be the subspace of the elements of fermionic degree k. We have a decomposition

$$V_{\mathbb{Z}^N} = \underset{k\in\mathbb{Z}}{\oplus} V^k_{\mathbb{Z}^N}.$$

Note that each subspace $V^k_{\mathbb{Z}^N}$ is a $\widehat{gl}_N$-submodule, which is graded by the bosonic degree. Its structure is described by the following well-known result (see e.g. [9] or [15]):

Theorem 10.1. *For each $k \in \mathbb{Z}$, $V^k_{\mathbb{Z}^N}$ is an irreducible $\widehat{gl}_N$-module at level 1. Let $\widetilde{V}^k_{\mathbb{Z}^N}$ be the non-trivial component of $V^k_{\mathbb{Z}^N}$ of the lowest bosonic degree. If $k = 0 \mod N$ then $\widetilde{V}^k_{\mathbb{Z}^N}$ is 1-dimensional. If $k = k' \mod N$ with $1 \leq k' < N$, then as an sl_N-module $\widetilde{V}^k_{\mathbb{Z}^N}$ has the fundamental highest weight $\omega_{k'}$. The identity matrix of gl_N acts on $\widetilde{V}^k_{\mathbb{Z}^N}$ as $k\,\mathrm{Id}$.*

Combining this result with Theorem 6.3, we get

Corollary 10.2. *The space*

$$M_{Hyp}(\gamma) \otimes V^k_{\mathbb{Z}^N}$$

has a structure of a module for the Lie algebra $\mathrm{Vect}\mathbb{T}^{N+1}$ *of vector fields.*

For these modules the Virasoro tensor factor $L_{Vir}(h)$ is 1-dimensional ($h = 0$). The modules in this family are precisely the exceptional modules $L(W, \gamma, h)$ for which Theorem 9.2 does not claim irreducibility. We are going to see below that these modules are in fact reducible.

Let us express the action (6.11), (6.12) of the Lie algebra $\mathrm{Vect}\mathbb{T}^{N+1}$ on $M_{Hyp}(\gamma) \otimes V^k_{\mathbb{Z}^N}$ using the fermionic realization:

$$d_a(r, z) \mapsto Y(d_a(r), z), \quad d_0(r, z) \mapsto Y(d_0(r), z),$$

where

$$d_a(r) = v^a_{(-1)} q^r + \sum_{p=1}^{N} r_p \varphi^p_{(-1)} \psi^a_{(-1)} q^r,$$

$$\begin{aligned} d_0(r) &= -\left(\omega^{Hyp}_{(-1)} q^r + \omega^{fer}_{(-1)} q^r + \sum_{a,b=1}^{N} r_a u^b_{(-1)} \varphi^a_{(-1)} \psi^b_{(-1)} q^r - \sum_{p=1}^{N} r_p u^p_{(-2)} q^r \right) \\ &= -\left(\sum_{p=1}^{N} u^p_{(-1)} v^p_{(-1)} q^r + \sum_{p=1}^{N} \varphi^p_{(-2)} \psi^p_{(-1)} q^r + \sum_{a,b=1}^{N} r_a u^b_{(-1)} \varphi^a_{(-1)} \psi^b_{(-1)} q^r \right). \end{aligned}$$

Here we used the relation

$$\omega^{Hyp}_{(-1)} q^r = \sum_{p=1}^{N} u^p_{(-1)} v^p_{(-1)} q^r + \sum_{p=1}^{N} r_p u^p_{(-2)} q^r.$$

Following [20], let us now introduce the differential

$$\ldots \xrightarrow{\mathbf{d}} M_{Hyp}(\gamma) \otimes V^k_{\mathbb{Z}^N} \xrightarrow{\mathbf{d}} M_{Hyp}(\gamma) \otimes V^{k+1}_{\mathbb{Z}^N} \xrightarrow{\mathbf{d}} \ldots$$

of the chiral de Rham complex.

Let

$$Q = \sum_{p=1}^{N} v^p_{(-1)} \varphi^p_{(-1)} \mathbb{1}$$

and set $\mathbf{d} = Q_{(0)}$, i.e., $\mathbf{d}$ is a coefficient at z^{-1} in $Y(Q, z) = \sum\limits_{p=1}^{N} v^p(z) \varphi^p(z)$. Vanishing of the supercommutator

$$[Y(Q, z_1), Y(Q, z_2)] = 0$$

implies $\mathbf{d} \circ \mathbf{d} = 0$.

Theorem 10.3. *The map*

$$\mathbf{d} : \; M_{Hyp}(\gamma) \otimes V^k_{\mathbb{Z}^N} \to M_{Hyp}(\gamma) \otimes V^{k+1}_{\mathbb{Z}^N}$$

is a homomorphism of $\mathrm{Vect}\mathbb{T}^{N+1}$*-modules.*

The statement of this theorem is equivalent to the claim that the following operators on $M_{Hyp}(\gamma) \otimes V_{\mathbb{Z}^N}$ commute:

$$[\mathbf{d}, d_a(r,z)] = 0,$$
$$[\mathbf{d}, d_0(r,z)] = 0.$$

The proof of these relations will be based on the following simple observation:

Lemma 10.4. *Let V be a vertex superalgebra and let $a, b \in V$. Suppose that $a_{(0)}b = 0$. Then*

$$\left[a_{(0)}, Y(b,z)\right] = 0.$$

Proof. Since $a_{(0)}b = 0$, the commutator formula (5.2) yields

$$[Y(a,z_1), Y(b,z_2)] = \sum_{j\geq 1} \frac{1}{j!} Y(a_{(j)}b, z_2) \left[z_1^{-1} \left(\frac{\partial}{\partial z_2} \right)^j \delta\left(\frac{z_2}{z_1} \right) \right].$$

However the right hand side does not contain terms with z_1^{-1} and the claim of the lemma follows. □

Let us continue with the proof of the theorem. We need to show that

$$Q_{(0)} d_a(r) = 0 \quad \text{and} \quad Q_{(0)} d_0(r) = 0.$$

Since $Y(Q,z) = \sum\limits_{i=1}^N : v^i(z)\varphi^i(z) :$, we have

$$Q_{(0)} = \sum_{i=1}^N \left(\sum_{j=0}^\infty \varphi^i_{(-j-1)} v^i_{(j)} + \sum_{j=1}^\infty v^i_{(-j)} \varphi^i_{(j-1)} \right).$$

It is easy to see that $v^i_{(j)} d_a(r) = 0$ for $j \geq 1$ and $\varphi^i_{(j)} d_a(r) = 0$ for $j \geq 1$. Thus

$$Q_{(0)} d_a(r) = \sum_{i=1}^N \left(\varphi^i_{(-1)} v^i_{(0)} + v^i_{(-1)} \varphi^i_{(0)} \right) d_a(r)$$
$$= \sum_{i=1}^N r_i \varphi^i_{(-1)} v^a_{(-1)} q^r - \sum_{i=1}^N \sum_{p=1}^N r_p v^i_{(-1)} \varphi^p_{(-1)} \varphi^i_{(0)} \psi^a_{(-1)} q^r$$
$$= \sum_{i=1}^N r_i \varphi^i_{(-1)} v^a_{(-1)} q^r - \sum_{p=1}^N r_p v^a_{(-1)} \varphi^p_{(-1)} q^r = 0.$$

Let us now show that $Q_{(0)} d_0(r) = 0$. Since $v^i_{(j)} d_0(r) = 0$ for $j \geq 2$ and $\varphi^i_{(j)} d_0(r) = 0$ for $j \geq 1$, we get

$$-Q_{(0)} d_0(r) = \sum_{i=1}^N \left(\varphi^i_{(-1)} v^i_{(0)} + \varphi^i_{(-2)} v^i_{(1)} + v^i_{(-1)} \varphi^i_{(0)} \right) (-d_0(r)).$$

Let us compute each of three terms in the right hand side separately:

$$\left(\sum_{i=1}^N \varphi^i_{(-1)} v^i_{(0)} \right) (-d_0(r))$$
$$= \sum_{i=1}^N \sum_{p=1}^N \varphi^i_{(-1)} v^i_{(0)} u^p_{(-1)} v^p_{(-1)} q^r + \sum_{i=1}^N \sum_{p=1}^N \varphi^i_{(-1)} v^i_{(0)} \varphi^p_{(-2)} \psi^p_{(-1)} q^r$$
$$+ \sum_{i=1}^N \sum_{a,b=1}^N r_a \varphi^i_{(-1)} v^i_{(0)} u^b_{(-1)} \varphi^a_{(-1)} \psi^b_{(-1)} q^r$$
$$= \sum_{i=1}^N \sum_{p=1}^N r_i \varphi^i_{(-1)} u^p_{(-1)} v^p_{(-1)} q^r + \sum_{i=1}^N \sum_{p=1}^N r_i \varphi^i_{(-1)} \varphi^p_{(-2)} \psi^p_{(-1)} q^r$$

$$+\sum_{i=1}^{N}\sum_{a=1}^{N}\sum_{b=1}^{N} r_a r_i u^b_{(-1)}\varphi^i_{(-1)}\varphi^a_{(-1)}\psi^b_{(-1)}q^r. \tag{10.1}$$

The last summand in (10.1) vanishes since it is antisymmetric in $\{a, i\}$. Next,

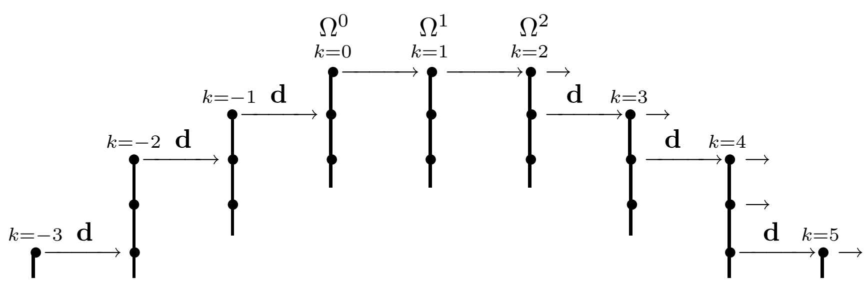
$$\begin{aligned}
&\left(\sum_{i=1}^{N}\varphi^i_{(-2)}v^i_{(1)}\right)(-d_0(r))\\
&=\sum_{i=1}^{N}\sum_{p=1}^{N}\varphi^i_{(-2)}v^i_{(1)}u^p_{(-1)}v^p_{(-1)}q^r+\sum_{i=1}^{N}\sum_{p=1}^{N}\varphi^i_{(-2)}v^i_{(1)}\varphi^p_{(-2)}\psi^p_{(-1)}q^r\\
&\quad+\sum_{i=1}^{N}\sum_{a,b=1}^{N} r_a\varphi^i_{(-2)}v^i_{(1)}u^b_{(-1)}\varphi^a_{(-1)}\psi^b_{(-1)}q^r\\
&=\sum_{p=1}^{N}\varphi^p_{(-2)}v^p_{(-1)}q^r+\sum_{i=1}^{N}\sum_{a=1}^{N} r_a\varphi^i_{(-2)}\varphi^a_{(-1)}\psi^i_{(-1)}q^r.
\end{aligned}\tag{10.2}$$

And finally,

$$\begin{aligned}
&\left(\sum_{i=1}^{N}v^i_{(-1)}\varphi^i_{(0)}\right)(-d_0(r))\\
&=\sum_{i=1}^{N}\sum_{p=1}^{N}v^i_{(-1)}\varphi^i_{(0)}u^p_{(-1)}v^p_{(-1)}q^r+\sum_{i=1}^{N}\sum_{p=1}^{N}v^i_{(-1)}\varphi^i_{(0)}\varphi^p_{(-2)}\psi^p_{(-1)}q^r\\
&\quad+\sum_{i=1}^{N}\sum_{a,b=1}^{N} r_a v^i_{(-1)}\varphi^i_{(0)}u^b_{(-1)}\varphi^a_{(-1)}\psi^b_{(-1)}q^r\\
&=-\sum_{i=1}^{N}\sum_{p=1}^{N}v^i_{(-1)}\varphi^p_{(-2)}\varphi^i_{(0)}\psi^p_{(-1)}q^r-\sum_{i=1}^{N}\sum_{a,b=1}^{N} r_a v^i_{(-1)}u^b_{(-1)}\varphi^a_{(-1)}\varphi^i_{(0)}\psi^b_{(-1)}q^r\\
&=-\sum_{p=1}^{N}v^p_{(-1)}\varphi^p_{(-2)}q^r-\sum_{i=1}^{N}\sum_{a=1}^{N} r_a v^i_{(-1)}u^i_{(-1)}\varphi^a_{(-1)}q^r.
\end{aligned}\tag{10.3}$$

Combining (10.1), (10.2) and (10.3) we get $Q_{(0)}(-d_0(r)) = 0$, and the theorem is proved.

Let us present here a diagram of the Chiral de Rham complex for $N = 2$. On the diagram, the fermionic degree increases in the horizontal direction and bosonic in vertical.



The tops of the modules $M_{Hyp}(\gamma)\otimes V^k_{\mathbb{Z}^N}$ with $0 \le k \le N$ are the spaces $q^\gamma\Omega^k(\mathbb{T}^N)$ of differential k-forms that form the classical de Rham complex. Non-trivial Vect$\mathbb{T}^N$-submodules in these tops generate non-trivial Vect$\mathbb{T}^{N+1}$ submodules in corresponding modules $M_{Hyp}(\gamma)\otimes V^k_{\mathbb{Z}^N}$.

It was proved in [20] that the cohomology of the chiral de Rham complex coincides with the classical de Rham cohomology. This implies, in particular, that for $k < 0$ or $k > N$ the short sequences

$$M_{Hyp}(\gamma)\otimes V^{k-1}_{\mathbb{Z}^N}\xrightarrow{\mathbf{d}} M_{Hyp}(\gamma)\otimes V^{k}_{\mathbb{Z}^N}\xrightarrow{\mathbf{d}} M_{Hyp}(\gamma)\otimes V^{k+1}_{\mathbb{Z}^N}$$

are exact. Using this fact, we get

Corollary 10.5. *(i) For $k \leq 0$, $\mathrm{Vect}\mathbb{T}^{N+1}$-modules $M_{Hyp}(\gamma) \otimes V^k_{\mathbb{Z}^N}$ have non-trivial critical vectors.*

(ii) For $k \geq N$, $\mathrm{Vect}\mathbb{T}^{N+1}$-modules $M_{Hyp}(\gamma) \otimes V^k_{\mathbb{Z}^N}$ are not generated by their top spaces.

Proof. We can see from the above diagram that for $k < 0$ the images of the top vectors in $M_{Hyp}(\gamma) \otimes V^k_{\mathbb{Z}^N}$ are non-trivial critical vectors in $M_{Hyp}(\gamma) \otimes V^{k+1}_{\mathbb{Z}^N}$. For $k \geq N$, the top spaces of $M_{Hyp}(\gamma) \otimes V^k_{\mathbb{Z}^N}$ are in the kernel of $\mathbf{d}$. Thus the submodules generated by the tops are annihilated by $\mathbf{d}$ as well. Since the map $\mathbf{d}$ is non-zero, these submodules are proper. □

As a result we see that all modules that belong to the chiral de Rham complex are reducible. The claim of Corollary 10.5 is consistent with the existence of the contragredient pairing given by Theorem 8.2:

$$\left(M_{Hyp}(\gamma) \otimes V^k_{\mathbb{Z}^N}\right) \times \left(M_{Hyp}(\gamma) \otimes V^{N-k}_{\mathbb{Z}^N}\right) \to \mathbb{C}.$$

For the chiral de Rham complex this duality was constructed in [19].

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