Decompositions of Bruhat type for the Kac-Moody groups.

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ABSTRACT. In this article we construct new Bruhat type decompositions for the Kac-Moody groups, and present canonical forms for the elements of a group with respect to these decompositions. We also give some partial results on closure and intersection patterns in the corresponding decompositions of the flag variety.

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Introduction. There are several versions of the decompositions of Bruhat type for the Kac-Moody groups; the most important of them are the Bruhat decomposition and the Birkhoff decomposition [2], [13], [18]. These decompositions are powerful tools for the study of the Kac-Moody groups from both algebraic and geometric points of view.

Here we construct a new family of Bruhat type decompositions, including classical Bruhat and Birkhoff decompositions as special cases, and study the closure patterns of the double cosets involved. The decompositions are associated with (non-standard) positive root systems.

Positive root systems were studied in [9] and [10] , where the highest weight representation theory with respect to a positive root system is developed. It is known that in the finite-dimensional case all positive root systems are conjugate under the action of the Weyl group, so all decompositions are equivalent to classical Bruhat decomposition. In the affine case all positive root systems are classified. There exist, up to conjugation, a finite number of them, so we get a finite number of corresponding non-equivalent decompositions. For the Kac-Moody groups of indefinite type, the number of non-equivalent decompositions may be infinite.

The idea of the proof of the generalized Bruhat decomposition is the same as in Kac-Peterson paper [13], where the Birkhoff decomposition is proven. Victor Kac wrote us that this generalization was known to him.

Finite and cofinite Schubert varieties in the flag variety of a Kac-Moody group have been studied extensively (see e.g. [14, 15]). The closure patterns are described by Chevalley (Bruhat) order. We show here that closure patterns for the decompositions of flag varieties arising from the generalized Bruhat decompositions are described by the "twisted orders"on the Weyl group studied in [6,7].

The main results of the paper are as follows. Let G be a Kac-Moody group with standard Borel subgroup B. In Theorem 1 we prove that G decomposes as the product of its subgroups:

$$G = BWQ,$$

where Q is an arbitrary subgroup of G that contains "one half of all the real roots". Theorem 2 states that the natural map from W to the set of double cosets $B\backslash G/Q$ is a bijection. Theorem 3 provides a canonical form of such a presentation of elements of G. Theorem 4 describes the closure patterns of the double cosets QwB in the "Zariski" topology considered by Kac-Peterson [14], and Theorem 5 gives results similar to those of Curtis [4] in this context.

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Definitions and notations. It will be convenient for us to use the version of the Kac-Moody groups studied in [14, 15] over fields of characteristic zero. However, most results not involving the "Zariski topology" defined there can be proved by essentially identical arguments for some similar versions of the groups (e.g. those constructed over arbitrary fields in [11] using one of Tits' $\mathbb{Z}$-forms for enveloping algebras of Kac-Moody Lie algebras).

Let A be a symmetrizable generalized Cartan matrix and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A over a field $\mathbb{F}$ of characteristic zero, (symmetrizability is required only for some of our results involving the Zariski topology). One then has the corresponding Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ over $\mathbb{F}$ generated by $\mathfrak{h}$ and symbols e_α, f_α for $\alpha \in \Pi$ with the usual relations [15, (1.1), (1.2)]. One has the root space decomposition $\mathfrak{g} = \oplus_{\alpha \in h^*} \mathfrak{g}_\alpha$, and we denote the roots (resp. positive roots) by Δ (resp., Δ_+).

Let W denote the subgroup of $\mathrm{Aut}_{\mathbb{F}}(\mathfrak{h})$ generated by the "simple reflections"

$v \mapsto v - \langle v, \alpha^\vee \rangle \alpha$ for $\alpha \in \Pi$. The set $\{ w\alpha \mid w \in W,\ \alpha \in \Pi \}$ of real roots will be denoted Δ^{re}. Define positive and negative real roots as usual by $\Delta^{re}_{\pm} = \Delta^{re} \cap \pm\Delta_{+}$.

A $\mathfrak{g}'$-module V, or (V, π) where $\pi\colon \mathfrak{g}' \to \mathrm{End}_{\mathbb{F}}(V)$, is said to be integrable if for all $\alpha \in \Delta^{re}$ and $v \in V$, there exists N with $\pi(\mathfrak{g}_\alpha)^N(v) = 0$. One associates a group G to $\mathfrak{g}(A)$ as in [14, 15]; here we recall only that G is the quotient $G = G^*/N$ of the free product G^* of the root subgroups $\mathfrak{g}_\alpha$ (for real roots α) by the largest normal subgroup N of G^* acting trivially on all integrable $\mathfrak{g}'$-modules V (where $\mathfrak{g}_\alpha \subset G^*$ acts on each such (V, π) by $(e, v) \mapsto \exp(\pi(e))v$). For real roots α, one has the canonical injection (also denoted exp) $\mathfrak{g}_\alpha \to G^* \to G$ with image the one-parameter subgroup U_α. Then G is generated by the U_α for real roots α, and denoting the natural action of G on an integrable $\mathfrak{g}'$-module (V, π) also by π, one has $\pi(\exp\, e) = \exp\, \pi(e)$ for $e \in \mathfrak{g}_\alpha$.

We abbreviate $x_\alpha(t) := \exp(te_\alpha)$ and $x_{-\alpha}(t) := \exp(tf_\alpha)$ for $\alpha \in \Pi$, $t \in \mathbb{F}$. For $\alpha \in \Pi$, there is a unique homomorphism $\phi_\alpha : SL_2(\mathbb{F}) \to G$ with $\phi_\alpha\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) = x_\alpha(t)$ and $\phi_\alpha\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right) = x_{-\alpha}(t)$; ϕ_α is actually an isomorphism onto its image G_α. Let $H_\alpha(t) = \phi_\alpha(\mathrm{diag}(t, t^{-1})$ for $t \in \mathbb{F}^*$ and $s_\alpha = \phi_\alpha\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, so one has

$$x_\alpha(\lambda) x_{-\alpha}(-\lambda^{-1}) x_\alpha(\lambda) = H_\alpha(\lambda) s_\alpha. \tag{1}$$

Let H be the (abelian) subgroup of G generated by the images of the H_α for $\alpha \in \Pi$, and N be the sugroup generated by H and the s_α for $\alpha \in \Pi$. Then N normalizes H, and one may identify W and the group N/H so that for $\alpha \in \Pi$, the simple reflection $v \to v - \langle v, \alpha^\vee \rangle \alpha$ identifies with the coset $s_\alpha H \in N/H$ (we often denote a coset representative for $w \in W$ still by w). We may also identify W with the contragredient subgroup of $\mathrm{Aut}_{\mathbb{F}}(\mathfrak{h}^*)$, where $\mathfrak{h}^*$ is the dual space. Then the bijection $\Pi \to \Pi^\vee$ given by $\alpha \mapsto \alpha^\vee$ extends to a W-invariant bijection $\Delta^{re} = W\Pi \to W\Pi^\vee$ (from the real roots to the real coroots) which we still denote by $\alpha \mapsto \alpha^\vee$. For $\alpha \in \Delta^{re}$, we have the corresponding reflection $v \mapsto v - \langle v, \alpha^\vee \rangle \alpha$, for $v \in \mathfrak{h}$, in W. We let S denote the set of simple reflections of W, and $l : W \to \mathbb{N}$ denote the usual length function of the Coxeter system (W, S).

The following relation holds for all $w \in W, \gamma \in \Delta^{re}$:

$$wU_\gamma w^{-1} = U_{w(\gamma)}. \tag{2}$$

The subgroup of G generated by all U_α , where $\alpha \in \Delta_+^{re}$ (resp. $\alpha \in \Delta_-^{re}$), is denoted by U_+ (resp. U_-). The subgroup H normalizes each U_α for $\alpha \in \Delta^{re}$, hence also U_+ and U_- . Define the Borel subgroup B (resp. B_-) as the product $B := HU_+ = U_+H$ (resp. $B_- := HU_- = U_-H$).

The Bruhat decomposition for a Kac-Moody group is a presentation of G as a product of the subgroups:

$$G = BWB. \tag{3}$$

However, the presentation of $g \in G$ in the form $g = b_1 w b_2;\ b_1, b_2 \in B$ is not unique. The following version of Bruhat decomposition gives unique presentation:

$$G = \bigcup_{w \in W} BwU_w\ , \tag{4}$$

where $U_w = U_+ \cap (w^{-1}U_- w)$.

Denote by U'_w the subgroup $U_+ \cap (w^{-1}U_+ w)$. The group U_+ is the product of these subgroups:

$$U_+ = U_w U'_w = U'_w U_w.$$

The group U_w is nilpotent and may be presented as a product of those groups U_α that are contained in U_w . Precisely, let $w = s_n s_{n-1} \ldots s_2 s_1$ be the presentation of w as a product of the elements from S of minimal length: $l(w) = n$. Then

$$U_w = U_{\beta_n} \ldots U_{\beta_2} U_{\beta_1}, \tag{5}$$

with uniqueness of expression, where $\beta_1 = \alpha_1$, $\beta_k = s_1 \dots s_{k-1}(\alpha_k)$, for $k = 2, \dots, n$.

Bruhat-type Decompositions.

Theorem 1. *Let Q be a subgroup of a Kac-Moody group G, such that for all $\alpha \in \Delta^{re}$ either $U_\alpha \subset Q$ or $U_{-\alpha} \subset Q$. Then*

$$G = BWQ.$$

Proof. It is sufficient to prove that

$$BwU_wWQ \subset BWQ. \tag{6}$$

Then, using the Bruhat decomposition (4) we get:

$$G = \bigcup_{w \in W} BwU_w \subset \bigcup_{w \in W} BwU_wWQ \subset BWQ. \tag{7}$$

We shall prove (6) by induction on $l(w)$.

If $l(w) = 0$, then $w = e$ and $U_w = \{e\}$, so

$$BwU_wWQ = BWQ.$$

Suppose $l(w) = n$ and the induction assumption holds for all elements of the Weyl group of smaller length.

Let $u \in U_w$ and $w_1 \in W$. We shall prove that $Bwuw_1Q \subset BWQ$.

We use (5) to present u as a product: $u = u_n \dots u_2 u_1$, where $u_k \in U_{\beta_k}$.

Now we will consider three cases:

1) $u_1 = e$;

2) $u_1 \neq e, U_{w_1^{-1}(\alpha_1)} \subset Q$;

3) $u_1 \neq e, U_{-w_1^{-1}(\alpha_1)} \subset Q$.

Case 1. If $u_1 = e$ then $u \in U_{\beta_n} \dots U_{\beta_2}$.

Let $w' = ws_1 = s_n \dots s_2$; $l(w') = n - 1$.

Then

$$U_{\beta_n} \dots U_{\beta_2} = s_1 U_{s_1(\beta_n)} \dots U_{s_1(\beta_2)} s_1 = s_1 U_{w'} s_1.$$

Hence,

$$Bwuw_1Q \subset Bws_1U_{w'}s_1w_1Q = Bw'U_{w'}s_1w_1Q.$$

As $l(w') < l(w)$, we may apply the induction assumption.

Case 2. Let $U_{w_1^{-1}(\alpha_1)} \subset Q$. Then

$$u_1w_1Q \subset U_{\alpha_1}w_1Q = w_1(w_1^{-1}U_{\alpha_1}w_1)Q = w_1U_{w_1^{-1}(\alpha_1)}Q = w_1Q.$$

Consequently,

$$Bwuw_1Q = Bwu_n \dots u_2u_1w_1Q \subset Bwu_n \dots u_2w_1Q.$$

So, this case reduces to Case 1.

Case 3. Let $U_{-w_1^{-1}(\alpha_1)} \subset Q$ and $u_1 \neq e$. Then $u_1 = x_{\alpha_1}(\lambda)$, for some $\lambda \neq 0$. Then (2) yields:

$$w_1^{-1}x_{-\alpha_1}(-\lambda^{-1})w_1 \in w_1^{-1}U_{-\alpha_1}w_1 = U_{-w_1^{-1}(\alpha_1)} \subset Q,$$

hence,

$$w_1Q = w_1(w_1^{-1}x_{-\alpha_1}(-\lambda^{-1})w_1)Q = x_{-\alpha_1}(-\lambda^{-1})w_1Q.$$

Thus, applying (1) we get:

$$Bwuw_1Q =$$

$$= Bwu_n \dots u_2x_{\alpha_1}(\lambda)x_{-\alpha_1}(-\lambda^{-1})w_1Q =$$

$$= Bwu_n \dots u_2 H_{\alpha_1}(\lambda) s_1 x_{\alpha_1}(-\lambda) w_1 Q \subset$$

$$\subset BHwU_{\beta_n} \dots U_{\beta_2} s_1 U_{\alpha_1} w_1 Q = Bws_1 s_1 U_{\beta_n} \dots U_{\beta_2} s_1 U_{\alpha_1} w_1 Q =$$

$$= Bw' U_{w'} U_{\alpha_1} w_1 Q \subset Bw' U_+ w_1 Q \subset Bw' U_{w'} w_1 Q \ .$$

As $l(w') < l(w)$, we are able to apply the induction assumption. This completes the proof.

Corollary 1. (of the proof). *If $s = s_\alpha \in S$, where $\alpha \in \Pi$, and $w \in W$, then*

$$BsBwQ = \begin{cases} BwQ \cup BswQ & \text{if } U_{-w^{-1}(\alpha)} \subset Q \\ BswQ & \text{if } U_{w^{-1}(\alpha)} \subset Q. \end{cases}$$

Corollary 2. *Let Q satisfy the condition of Theorem 1, and let P be a parabolic subgroup of G containing B. Suppose $P \cap W = W_1$, $Q \cap W = W_2$. Then*

$$G = \bigcup_{w \in W_1 \backslash W / W_2} PwQ.$$

Examples.

1. If $Q = B$ then the decomposition (6)

$$G = BWQ = BWB$$

is the Bruhat decomposition (3) for G.

2. If $Q = B_-$ then (6) gives the Birkhoff decomposition for G:

$$G = BWB_- \ .$$

3. Let G be an affine Kac-Moody group. Consider the realization of the corresponding Kac-Moody algebra as a subalgebra in $\mathfrak{g}^0 \otimes \mathbb{F}[t, t^{-1}] \oplus \mathbb{F}c \oplus \mathbb{F}d$,

where $\mathfrak{g}^0$ is a simple finite-dimensional Lie algebra. Define Ψ to be the set of real roots of the subalgebra $\mathfrak{n}_+^0 \otimes \mathbb{F}[t,t^{-1}]$.

Consider in the corresponding group G the subgroup Q generated by all U_α such that $\alpha \in \Psi$. The decomposition (6) we obtain in this way is known as the Iwasawa decomposition.

Now suppose that A is the extended Cartan matrix of SL_2. The affine Weyl group W has the corresponding finite Weyl group as parabolic subgroup in the usual way; taking $P \supset B$ to be the corresponding parabolic subgroup of G, the decomposition of $G/\mathbb{F}^* \cong SL_2(\mathbb{F}[t,t^{-1}])$ from Corollary 2 is:

$$SL_2(\ \mathbb{F}[t,t^{-1}]\ ) = \bigcup_{k\in\mathbb{Z}} SL_2(\ \mathbb{F}[t]\ ) \begin{pmatrix} t^k & 0 \\ 0 & t^{-k} \end{pmatrix} \begin{pmatrix} 1 & \mathbb{F}[t,t^{-1}] \\ 0 & 1 \end{pmatrix} =$$

$$= \bigcup_{k\in\mathbb{Z}} SL_2(\ \mathbb{F}[t]\ ) \begin{pmatrix} 1 & t^{-1}\mathbb{F}[t^{-1}] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^k & 0 \\ 0 & t^{-k} \end{pmatrix} .$$

Definition. A (non-standard) set of positive roots Ψ is defined as an arbitrary subset in Δ with the following properties:

a) $\Psi \cup -\Psi = \Delta \backslash \{0\}$.

b) $\Psi \cap -\Psi = \emptyset$.

c) Ψ is closed, i.e. if $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Psi$.

Consider for some ordered basis over the reals of the real span of the roots, the set Ψ of all roots whose first non-zero component in that basis is positive (lexicographically positive roots with respect to this basis). Then Ψ is a set of positive roots. Let us prove that all positive root systems arise this way.

For the following results, it is convenient to identify the root lattice with $\mathbb{Z}^n$ so the simple roots coorespond to the standard coordinate vectors.

Proposition 1. *If Ψ is a positive root system in Δ then there exists a basis B in $\mathbb{R}^n$ such that Ψ is the set of roots that are lexicographically positive with respect to B.*

To prove this proposition we need the following lemma.

Lemma 1. *Let $\Psi_1 = \{\sum_{i=1}^{k} n_i\mu_i \mid k \geq 1\ ,\ 0 < n_i \in \mathbb{N},\ \mu_i \in \Psi\}$ be the semigroup in $\mathbb{Z}^n$ generated by Ψ. Then $\Psi_1 \cap (-\Psi_1) = \emptyset$.*

Proof. If $\sum_{i=1}^{k} n_i\mu_i = \sum_{j=1}^{s} m_j\beta_j$, where $\mu_i \in \Psi$, $\beta_j \in -\Psi$, then $\sum_{i=1}^{k} n_i\mu_i + \sum_{j=1}^{s} m_j(-\beta_j) = 0$, with $(-\beta_j) \in \Psi$. So, it is sufficient to consider the case when $\sum_{i=1}^{k} n_i\mu_i = 0$ and we may assume that this sum has the minimal possible positive $\sum_{i=1}^{k} n_i$.

Suppose that one of the roots in this sum is real, let say $\mu_1 \in \Delta^{re}$. Then

$$0 = \langle\sum_{i=1}^{k} n_i\mu_i\ ,\ \mu_1^\vee\rangle = n_1\langle\mu_1, \mu_1^\vee\rangle + \sum_{i=2}^{k} n_i\langle\mu_i, \mu_1^\vee\rangle.$$

As $\langle\mu_1, \mu_1^\vee\rangle > 0$ then for some i we have $\langle\mu_i, \mu_1^\vee\rangle < 0$, hence $\mu_i + \mu_1$ is a root and $\mu_i + \mu_1 \in \Psi$. This gives us a sum with lesser $\sum_{i=1}^{k} n_i$. Consequently, all μ_i should be imaginary and hence all n_i are equal to 1 as $n\mu_i \in \Delta$ for $\mu_i \in \Delta^{\mathrm{im}}$.

Now we may present our expression in the form:

$$\sum_{i=1}^{k} \beta_i = \sum_{j=1}^{s} \gamma_j, \quad \text{where} \ \ \beta_i \in \Psi \cap \Delta_+^{\mathrm{im}}, \quad \gamma_j \in (-\Psi) \cap \Delta_+^{\mathrm{im}} \tag{1}$$

and we assume that $k + s$ is the minimal possible.

Let us consider two cases:

I. Suppose that $\sum_{i=1}^{k} \beta_i$ is not a root. Note that for all $w \in W$, $w(\sum_{i=1}^{k} \beta_i) \in \mathbb{N}^n$. Choosing w so this element has minimal height, one has $w(\sum_{i=1}^{k} \beta_i) \in -C^\vee$. As $w(\sum_{i=1}^{k} \beta_i) \notin \Delta$, then the support of $w(\sum_{i=1}^{k} \beta_i)$ decomposes into several connected components. Choosing those β_i and γ_j for which the support of $w(\beta_i)$ and $w(\gamma_j)$ belongs to one of the connected components we get the equality of type (1) with lesser $k+s$, which is a contradiction.

II. Suppose that $\sum_{i=1}^{k} \beta_i$ is a root. Then $k = 1$ or $s = 1$ because of the minimality of $k+s$. Without a loss of generality we can assume that

$$\sum_{i=1}^{k} \beta_i = \delta, \qquad \beta_i \in \Psi \cap \Delta_+^{\mathrm{im}}, \quad \delta \in (-\Psi) \cap \Delta_+^{\mathrm{im}}.$$

It follows from the axioms of the positive root system that $k > 2$. As k is the minimal possible then no subsum in $\sum_{i=1}^{k} \beta_i$ with more than one summand may be equal to a root.

Following a similar argument as in (I) we conclude that there exists $w \in W$ such that $w\beta_2, \dots, w\beta_{k-1}, w\beta_k$ belong to $-C^\vee$ and their supports are pairwise non-connected. Consequently, $\mathrm{supp}(w\beta_1)$ connects all of them. In the same way there exists $w_1 \in W$ such that $w_1\beta_1, w_1\beta_2 \in -C^\vee$ and their supports are non-connected. Since $C^\vee$ is a fundamental domain for W on the (dual) Tits' cone, $w\beta_2 = w_1\beta_2$, hence $ww_1^{-1} \in \mathrm{Stab}_W(w\beta_2)$ which is generated by reflections in W that stabilize $w\beta_2$. As $w\beta_2 \in -C^\vee$ then $\mathrm{Stab}_W(w\beta_2)$ is generated by reflections with respect to the simple roots that are non-connected to the $\mathrm{supp}(w\beta_2)$ or belong to the $\mathrm{supp}(w\beta_2)$. Consequently, the support of $w\beta_1 = ww_1^{-1}w_1\beta_1$ is non-connected to the support of $w\beta_2$, this is a contradiction.

Corollary 3. *Let* $\Psi_2 = \{\sum_{i=1}^{k} q_i\beta_i \mid k \geq 1, q_i \in \mathbb{Q}_+, \beta_i \in \Psi\}$. *Then* $\Psi_2 \cap (-\Psi_2) = \emptyset$.

Proof of the proposition. Let's prove by induction on n that for any semigroup $\Psi_1 \subset \mathbb{Z}^n$ such that $\Psi_1 \cap (-\Psi_1) = \emptyset$ there exists a basis B in $\mathbb{R}^n$ with respect to which Ψ_1 is lexicographically positive.

If $n = 1$ then this statement is evident.

Without the loss of generality we may assume that Ψ_1 spans $\mathbb{R}^n$.

Consider

$$\Omega = \{\sum_{i=1}^{m} r_i\alpha_i \mid r_i \in \mathbb{R}_+, \ \alpha_i \in \Psi_1, \ \langle\alpha_1, \dots, \alpha_m\rangle \ \text{ span } \mathbb{R}^n\}.$$

Note that Ω is an open convex cone in $\mathbb{R}^n$ and $\Psi_1 \subset \bar{\Omega}$. It can be easily seen that $\Omega \cap (-\Omega) \neq \emptyset$ implies $\Psi_2 \cap (-\Psi_2) \neq \emptyset$ which contradicts the previous corollary. Consequently, $\Omega \cap (-\Omega) = \emptyset$.

Now we use one of the basic theorems of convex analysis which states that if Ω is an open cone in $\mathbb{R}^n$ such that $\Omega \cap (-\Omega) = \emptyset$ then Ω and $-\Omega$ can be separated by some hyperplane H. Choose as the first vector for basis B the vector normal to H in the direction of Ω. Due to the induction assumption there exists a basis in H such that $H \cap \Psi_1$ consists of lexicographically positive vectors with respect to this basis. This completes the proof of the proposition.

For any set of positive roots Ψ we let Q_Ψ denote the subgroup of G generated by all U_α such that $\alpha \in \Psi \cap \Delta^{re}$. Then $Q_{w\Psi} = wQ_\Psi w^{-1}$, and we get the decomposition for G :

$$G = BWQ_\Psi \ .$$

Lemma 2. *Let* Ψ *be a set of positive roots. Then* $BQ_\Psi \cap W = \{e\}$.

Proof. This follows immediately from representation theory. Consider an irreducible right highest weight $\mathfrak{g}$-module $L(\Lambda)$ with integral strictly dominant highest weight Λ. This $\mathfrak{g}$-module is integrable. Suppose that $w = bq$, where $w \in W$, $b \in B$, $q \in Q_\Psi$. Then for the highest weight vector v_Λ we have $v_\Lambda b \in \mathbb{F}^* v_\Lambda$ and consequently $v_\Lambda bq$ has a non-zero Λ-component as according to Lemma 1 no linear combination of elements of Ψ with positive coefficients equals zero. However, $v_\Lambda w \in (L(\Lambda))_{w(\Lambda)}$, hence $v_\Lambda w$ has a non-zero Λ-component only if $w = e$.

Theorem 2. *Let Ψ be a set of positive roots, and $Q = Q_\Psi$ be the corresponding subgroup of G. Then the map $w \mapsto BwQ$ is a bijection of W onto the double cosets $B\backslash G/Q$.*

The proof of this theorem may be easily derived now as a modification of the corresponding proof from [1] for the Bruhat decomposition applying Corollary 1 and the previous lemma.

We now investigate the structure of Q. Let $Q_+ = Q \cap U_+$, $Q_- = Q \cap U_-$.

Lemma 3. *Let α be a simple root.*

i) *If $U_\alpha \subset Q$, then $Q_+ = U_\alpha(U'_{s_\alpha} \cap Q)$.*

ii) *If $U_\alpha \not\subset Q$, then $Q_+ = (U'_{s_\alpha} \cap Q)$.*

Proof. As $U_+ = U_\alpha U'_{s_\alpha}$, then i) is evident.

Let us prove ii). Consider a lowest weight module $L(\Lambda)$, where Λ is a strictly antidominant integral weight. Consider the action of the subgroup $G_\alpha \cong SL_2(\mathbb{F})$ on $L(\Lambda)$ and let V be the G_α-submodule

$$V = \bigoplus_{n \in \mathbb{Z}_+} L(\Lambda)_{\Lambda + n\alpha},$$

of $L(\Lambda)$. There is a natural projection $p: L(\Lambda) \to V$, which is an identity on the appropriate weight spaces and zero on the others.

Consider the action of the subgroups $U_+ = U_\alpha U'_{s_\alpha}$ and Q on the vector v_Λ. Let $u \in U_+$. Note that $u \in U'_{s_\alpha}$ if and only if $p(uv_\Lambda) = p(v_\Lambda)$. At the same time if a weight γ belongs to the support of qv_Λ, then $\gamma - \Lambda$ can be represented as a linear combination of elements of Ψ with non-negative coefficients. As $\alpha \notin \Psi$, then $\pi(qv_\Lambda) = \pi(v_\Lambda)$ for all $q \in Q$. Consequently, $U_+ \cap Q = U'_{s_\alpha} \cap Q$.

Corollary 4. $U_+ \cap Q = (U_w \cap Q)(U'_w \cap Q)$.

Theorem 3. *Let $Q = Q_\Psi$ be associated to a set of positive roots Ψ . Then*

$$Q = Q_+ Q_- \quad .$$

Corollary 5. *An element $g \in G$ may be uniquely represented in the form $g = Bwq$, where $B \in B$, $w \in W$ and $q \in Q \cap (w^{-1}U_- w)$.*

Proof of the Corollary. By Theorem 1 an element $g \in G$ may be presented in the form $g = Bwq = B(wqw^{-1})w$. Let us consider the positive root system $w\Psi w^{-1}$. By Theorem 3, $wQ_\Psi w^{-1}$ is the product of subgroups:

$$wQ_\Psi w^{-1} = (U_+ \cap wQ_\Psi w^{-1})(U_- \cap wQ_\Psi w^{-1}),$$

so $wqw^{-1} = q_+ q_-$, where $q_\pm \in U_\pm \cap (wQ_\Psi w^{-1})$.

Hence, $g = (Bq_+)w(w^{-1}q_- w)$. Remark that $w^{-1}q_- w \in Q \cap (w^{-1}U_- w)$. This proves the existence of this presentation. Let us prove uniqueness. Suppose $g = b_1 w_1 q_1 = b_2 w_2 q_2$. By Theorem 2 we have $w_1 = w_2$. Consequently, $b_1(w_1 q_1 w_1^{-1}) = b_2(w_1 q_2 w_1^{-1})$ and $b_1^{-1} b_2 = w_1 q_1 q_2^{-1} w_1^{-1}$. But $b_1^{-1} b_2 \in B$ and $w_1 q_1 q_2^{-1} w_1^{-1} \in U_-$. As $B \cap U_- = \{e\}$, we get $b_1^{-1} b_2 = q_1^{-1} q_2 = e$. This completes the proof of the corollary.

Proof of Theorem 3. Let $q \in Q$. Consider the Bruhat decomposition for q:

$$q = b_0 w_0' u_0, \quad \text{where} \quad u_0 \in U_{w_0'}.$$

Let us prove that $q \in Q_+ Q_-$ by induction on $l(w_0')$.

If $l(w_0') = 0$, then $w_0' = u_0 = e$, hence $q \in B \cap Q = U_+ \cap Q = Q_+$.

Let $l(w_0') = n > 0$, $w_0' = s_n s_{n-1} \dots s_1$. Consider the procedure of reduction of $b_0 w_0' u_0$ to the form $b_n w_n q_n$, described in the proof of the Theorem 1, using at each step the following presentations :

$$q = b_i w_i' u_i w_i q_i \quad , \tag{8}$$

by gradual elimination of $w_i' = s_n \dots s_{i+1}$. Theorem 2 implies that we shall eventually get $w_n = e$, as $q \in Q$.

If we use Cases 1 or 2 for the reduction of (8) then $q_{i+1} q_i^{-1} \in U_{w_i}$ and $w_{i+1} = s_{i+1} w_i$, hence $w_{i+1}' w_{i+1} = w_i' w_i$. Note that if Case 3 works then $w_{i+1} = w_i$. So, if this process involves Cases 1 or 2 only then $w_n = w_n' w_n = = \dots = w_0' w_0 = w_0'$, which is impossible as $w_n = e$.

Consequently, Case 3 is involved in this process. Suppose that Case 3 occurs in (8) for the first time. Then in (8) we have $w_i = s_i s_{i-1} \dots s_1, \quad q_i \in Q \cap U_{w_i}$.

Let α be a simple root, corresponding to s_{i+1} . Note that $w_i^{-1} U_{-\alpha} w_i \subset Q$. We have $w_i' = w_{i+1}' s_{i+1}$ and $u_i = u_i' u_\alpha$, where $u_\alpha \in U_\alpha$ and $u_i' \in s_{i+1} U_{w_{i+1}'} s_{i+1}$.

Hence, $q = b_i w_{i+1}' (s_{i+1} u_i' s_{i+1}) s_{i+1} u_\alpha w_i q_i$.

Following the arguments the proof of Theorem 1 we find $u_{-\alpha} \in U_{-\alpha}$ such that $s_{i+1} u_\alpha u_{-\alpha} = H_\alpha(\lambda) u_\alpha^{-1}$.

Consequently,

$$q = b_i w_{i+1}' (s_{i+1} u_i' s_{i+1}) s_{i+1} u_\alpha u_{-\alpha} u_{-\alpha}^{-1} w_i q_i =$$

$$= b_i w'_{i+1}(s_{i+1}u'_i s_{i+1})H_\alpha(\lambda)u_\alpha^{-1}w_i(w_i^{-1}u_{-\alpha}^{-1}w_i)q_i \ .$$

Hence, $q_{i+1} = (w_i^{-1}u_{-\alpha}^{-1}w_i)q_i \in (w_i^{-1}U_{-\alpha}w_i)(U_{w_i}\cap Q) \subset$

$$\subset w_i^{-1}(U_- \cap (w_i Q w_i^{-1}))w_i.$$

Applying Corollary 4 to U_-, w_i^{-1} and $w_i Q w_i^{-1}$ we get

$$U_- \cap (w_i Q w_i^{-1}) =$$

$$= (U_- \cap (w_i U_+ w_i^{-1}) \cap (w_i Q w_i^{-1}))(U_- \cap (w_i U_- w_i^{-1}) \cap (w_i Q w_i^{-1})).$$

Consequently,

$$w_i^{-1}(U_- \cap (w_i Q w_i^{-1}))w_i \subset$$

$$\subset (w_i^{-1}U_- w_i \cap U_+ \cap Q)(w_i^{-1}U_- w_i \cap U_- \cap Q) \subset$$

$$\subset U_{w_i}(U_- \cap Q) = U_{w_i}Q_-.$$

Thus, $q \in Bw'_{i+1}U_{w'_{i+1}}w_i U_{w_i}Q_-$. Using Corollary 1 for the Bruhat decomposition we conclude that $q \in BwU_wQ_-$ for some $w \in W$ with $l(w) \leq l(w'_{i+1}) + l(w_i) < l(w'_0)$.

Let $q = bwuq_-$, where $b \in B$, $u \in U_w$ and $q_- \in Q_-$. We have $bwu = qq_-^{-1} \in Q$. As $l(w) < l(w'_0)$ we may apply the induction assumption. Hence $qq_-^{-1} \in Q_+Q_-$, consequently $q \in Q_+Q_-$, as was to be proven.

Remark. If we replace both Borel subgroups with Q in the Bruhat decomposition (3) we do not in general obtain a decomposition for G. In particular, for the affine case if we take Q as in Example 3 then $G \neq QWQ$.

Closure patterns. We fix a system of positive roots Ψ and let $\leq_\Psi$ denote the partial order on W generated by the relations $s_\gamma w <_\Psi w$ for $w \in W$ and $\gamma \in \Delta_+^{re} \setminus w\Psi$; this is the order denoted $\leq_A$ in [6, 1.5], where A is the "initial section” $A = \{ s_\alpha \mid \Delta_+^{re} \setminus \Psi \}$ of the reflections of W. An alternative description of $\leq_\Psi$ may sometimes be useful. Define a (non-standard) length function $l_\Psi\colon W \to \mathbb{Z}$ by setting

$$l_\Psi(w) = l(w) - 2\sharp\big((\Delta_+^{re} \setminus \Psi) \cap w^{-1}(\Delta_-^{re})\big)$$

for $w \in W$. Then by [6, 1.7], one has $v \leq_\Psi w$ in W iff there is a sequence $v = v_0, v_1, \dots, v_n = w$ of elements of W with $v_i v_{i-1}^{-1}$ a reflection in W (i.e. a conjugate of a simple reflection) and $l_\Psi(v_i) = l_\Psi(v) + i$ for $i = 1, \dots, n$. If $\Psi = \Delta_+$ (resp., $\Psi = \Delta_-$) then $\leq_\Psi$ is the usual Chevalley (Bruhat) order on W (resp., reverse Chevalley order).

We may now restate Corollary 1 more familiarly as

Corollary 1′. *Let $Q = Q_\Psi$. If $s = s_\alpha \in S$ where $\alpha \in \Pi$ and $w \in W$, then*

$$BsBwQ = \begin{cases} BwQ \cup BswQ & \text{if } sw <_\Psi w \\ BswQ & \text{if } sw >_\Psi w. \end{cases}$$

Proof. One has $w^{-1}(\alpha) \in \Psi$ iff $\alpha \in w\Psi \cap \Delta_+^{re}$ i.e. iff $sw >_\Psi w$.

Example. For Ψ as in example 3, the order $\leq_\Psi$ on W is isomorphic to the order on the alcoves of an affine Weyl group considered by Lusztig in [17] (by [7]).

We now wish to extend to the orders $\leq_\Psi$ the usual interpretation of Chevalley (Bruhat) order in terms of closure patterns (of Schubert cells, or (B, B)-double cosets). To this end, we introduce on G the Zariski topology defined by strongly regular functions as in [15, 2E]. We recall here that Zariski topology is G- biinvariant, that $\phi_\alpha\colon SL_2(\mathbb{F}) \to G_\alpha$ is a homeomorphism (where SL_2 has the usual Zariski topology), and that $U_w \cong \mathbb{F}^{l(w)}$. We now prove

Theorem 4. *Let $Q = Q_\Psi$. Then for any $w \in W$,*

$$\overline{QwB} = \bigcup_{v : v^{-1} \leq_\Psi w^{-1}} QvB.$$

Note that $v^{-1} \leq_\Psi w^{-1}$ is not equivalent to $v \leq_\Psi w$ in general. The proof of Theorem 4 is essentially the same as that of [14, 3.4], using integrable highest weight modules. We fix a strictly dominant integral weight Λ, and consider the corresponding integrable highest weight (left) $\mathfrak{g}$-module $L(\Lambda)$, with highest weight vector $v_\Lambda \in L(\Lambda)_\Lambda$. We endow $L(\Lambda)$ with the Zariski topology defined by strongly regular functions [15, 3A].

Introduce the partial order $\leq_\Psi$ on the W-orbit $W\Lambda$ of Λ, generated by $\mu \leq_\Psi \lambda$ if $\mu - \lambda \in \mathbb{Q}_{\geq 0}\Psi$ and $\mu = s_\alpha(\lambda)$ for some $\alpha \in \Delta^{re}$. The map $W \to W\Lambda$ given by $w \mapsto w\Lambda$ is a bijection, and we claim that

$$w\Lambda \leq_\Psi v\Lambda \text{ iff } w^{-1} \leq_\Psi v^{-1}. \tag{9}$$

To check (9), one may assume by definition of the orders $\leq_\Psi$ that $w = vs_\alpha$ for some $\alpha \in \Delta_+^{re}$. Then $w\Lambda \leq_\Psi v\Lambda$ iff $-\langle \Lambda, \alpha^\vee \rangle v\alpha \in \mathbb{Q}_{\geq 0}\Psi$ i.e. iff $\alpha \in \Delta_+^{re} \setminus v^{-1}\Psi$, as needed.

Let $\mathcal{V} = G(\mathbb{F}v_\Lambda)$; as shown in [14], this is a Zariski closed subset of $L(\Lambda)$ (defined by quadratic "Plucker polynomials"), and the group $\mathbb{F}^* \times U_-$ acts simply transitively on $\mathcal{V} \setminus \{0\}$. We recall an important fact concerning $\mathcal{V}$ from [14]. For $v \in L(\Lambda)$, write $v = \sum_\lambda v_\lambda$ with $v_\lambda \in L(\Lambda)_\lambda$, put $\text{supp}(v) = \{\lambda \mid v_\lambda \neq 0\}$ and let $S(v)$ denote the convex hull of $\text{supp}(v)$. Then

for $v \in \mathcal{V}$, the vertices of the polyhedron $S(v)$ lie in the W-orbit of λ, and the edges of $S(v)$ are parallel to real roots. (10)

For $\lambda \in W\Lambda$, set

$$\mathcal{V}(\lambda)_\Psi = \{\, v \in \mathcal{V} \mid \lambda \in \text{supp}(v), \quad \mathit{supp}(v) - \lambda \subset \mathbb{Q}_{\geq 0}\Psi \,\}. \tag{11}$$

Now we have the following analogue of [14, Theorem 1].

Proposition 2. i) $\mathcal{V} \setminus \{0\}$ *is the disjoint union of the* $\mathcal{V}(\lambda)_\Psi$ *for* $\lambda \in W\Lambda$

ii) *for* $\lambda = w\Lambda \in W\Lambda$, *the group* $Q_\Psi \cap wU_-w^{-1}$ *acts simply transitively on* $\mathcal{V}(\lambda)_\Psi$.

iii) *for* $\lambda \in W\Lambda$, *one has* $\overline{\mathcal{V}(\lambda)_\Psi} \setminus \{0\} = \cup_{\mu \leq_\Psi \lambda} \mathcal{V}(\mu)_\Psi$.

Proof. Write Q for Q_Ψ. The assertion i) follows from (10) above; note also the $\mathcal{V}(\lambda)_\Psi$ are Q-invariant. For ii), observe one has a decomposition

$$Q = (Q \cap wU_-w^{-1})(Q \cap wU_+w^{-1})$$

using Theorem 3, so it suffices to show that

$$\mathbb{F}^* \times Q \ \text{ acts transitively on } \mathcal{V}(\lambda)_\Psi. \tag{12}$$

This is proved as in [14] by "killing edges of $S(v)$." More precisely, for $v \in \mathcal{V}(\lambda)_\Psi$, set $\Phi(\lambda) = \{\, \alpha \in \Psi \mid s_\alpha(\lambda) <_\Psi \lambda) \,\}$,

$$\Phi'(v) = \{\, \alpha \in \Phi(\lambda) \mid [\lambda, s_\alpha(\lambda)] \text{ is an edge of } S(v) \,\}$$

and $\Phi(v) = \Phi(\lambda) \cap \mathbb{Q}_{\geq 0}(S(v) - \lambda)$. If $\alpha \in \Phi'(v)$, the argument of loc. cit. shows that there exists $t \in \mathbb{F}$ so $\alpha \notin \Phi(x_\alpha(t)v) \subset \Phi(v)$. Using this repeatedly, one finds $u \in Q$ so $\Phi'(uv) = \emptyset$, so $uv \in L(\Lambda)_\lambda$ by (10), proving (12). Finally, iii) is proved from i), ii) in exactly the same way as Theorem 1(c) of [14].

Proof of Theorem 4. The map $\phi : G \to \mathcal{V}$ with $g \mapsto gv_\Lambda$ is Zariski continuous, with $\phi^{-1}(Q(L(\Lambda)_{w(\Lambda)})) = QwB$ by Theorems 1 and 2. Taking (9) into account, part (iii) of Proposition 2 gives

$$\overline{QwB} \subset \bigcup_{v : v^{-1} \leq_\Psi w^{-1}} QvB.$$

To prove the reverse inclusion, its sufficient to show for $w \in W$ and $\alpha \in \Delta_+^{re} \setminus w^{-1}(\Psi)$, that $\overline{QwB} \supset Qws_\alpha B$. Since $Q_{w^{-1}\Psi} = w^{-1}Qw$ and the Zariski topology is G-biinvariant, one may even assume in addition that $w = 1$. Write $\alpha = w(\beta)$ for some $w \in W$ and $\beta \in \Pi$. Then $U_{\pm\alpha} = wU_{\pm\beta}w^{-1}$ and we set $H_\alpha = wH_\alpha w^{-1}$. Recall $\phi_\beta\colon SL_2(\mathbb{F}) \to G_\beta$ is a Zariski homeomorphism. Now $Q \supset U_{-\alpha}$ and $B \supset H_\alpha U_\alpha$, so one has by biinvariance again that

$$\overline{QB} \supset Q\overline{U_{-\alpha}H_\alpha U_\alpha}B \supset Qs_\alpha B.$$

Intersection Patterns. In this section, Ψ is a fixed set of positive roots. For $A \subset G$, write $A \cdot B := \{\, gB \mid g \in A \,\} \subset G/B$. Here, we study intersection patterns amongst the sets $By \cdot B$, $Q_\Psi w \cdot B$ and $Q_{-\Psi}x \cdot B$ for $x, y, w \in W$. The basic result is Theorem 5, which is similar to [4] (cf. also [5], [11]). The proof here is very similar to that in [5], but we give the details involving the ordering $\leq_\Psi$.

Fix $y \in W$ and a reduced expression $y = s_1 \ldots s_k$. Let D_y be the set of sequences $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_k) \in W^{k+1}$ satisfying the conditions (a)–(c) below;

(a) $\sigma_0 = e$, the identity element of W

(b) $\sigma_j \in \{\sigma_{j-1}, s_j\sigma_{j-1}\}$ for $j = 1, \ldots, k$

(c) if $s_j\sigma_{j-1} >_\Psi \sigma_{j-1}$, then $\sigma_j = s_j\sigma_{j-1}$.

For $\sigma \in D$, we set $m(\sigma) = \sharp\{j \mid \sigma_j >_\psi \sigma_{j-1}\}$, $n(\sigma) = \sharp\{j \mid \sigma_j = \sigma_{j-1}\}$ and $\pi(\sigma) = \sigma_k^{-1} \in W$.

Now we define a map $\eta : U_{y^{-1}} \to D_y$ by $\eta(u_1) = (\sigma_0, \ldots, \sigma_k)$ where the $\sigma_j \in W$ are determined by $u_1 s_1 \ldots s_j \in Q_\Psi \sigma_j^{-1} B$ (the conditions (a)–(c) are easily verified, using Corollary $1'$ for (c)).

Theorem 5. i) *For any $\sigma \in D_y$, the set $\eta^{-1}(\sigma)$ is a locally closed subset of $U_{y^{-1}}$ homeomorphic to $\mathbb{F}^{m(\sigma)} \times (\mathbb{F}^*)^{n(\sigma)}$.*

ii) *For any $x \in W$,*

$$By \cdot B \cap Q_\Psi x^{-1} \cdot B = \bigcup_{\sigma \in D : \pi(\sigma) = x} \eta^{-1}(\sigma) \cdot B.$$

Corollary 6. *For $x, y \in W$, one has*

i) *$By \cdot B \cap Q_\Psi x \cdot B = \emptyset$ unless $y^{-1} \leq_\Psi x^{-1}$ and $x \leq y$, where $\leq$ denotes Chevalley (Bruhat) order on W*

ii)

$$Q_{-\Psi} y \cdot B \cap Q_\Psi x \cdot B \subset \bigcup_{v \in I(y,x)} Bv \cdot B$$

where $I(y,x) = \{\, v \in W \mid y^{-1} \leq_\Psi v^{-1} \leq_\Psi x^{-1} \,\}$.

Proof of the Corollary. Part ii) follows immediately from i) on noting that $\leq_{-\Psi}$ is the reverse of the order $\leq_\Psi$. For one part of i), one shows

$$ByB \subseteq \bigcup_{x \leq y} Q_\Psi x B$$

by induction on $l(y)$ using Corollary 1. For the remaining part, suppose that the intersection in i) is non-empty. Then Theorem 5 implies that there exists $\sigma = (\sigma_0, \ldots, \sigma_k) \in D_y$ with $\pi(\sigma) = \sigma_k^{-1} = x$. Set $y_j = s_j \ldots s_k$ for $j = 1, \ldots, k+1$, with $y_{k+1} = e$. For $j = 0, \ldots, k$, let $t_j = y_{j+1}^{-1} \sigma_j$. Then $t_{j-1} = t_j$ unless $\sigma_{j-1} = \sigma_j$, in which case $t_{j-1} = y_{j+1}^{-1} s_j \sigma_j <_\Psi y_{j+1}^{-1} \sigma_j = t_j$ since $s_j \sigma_{j-1} <_\Psi \sigma_{j-1}$ and $s_j y_{j+1} > y_{j+1}$. This gives, as needed,

$$y^{-1} = t_0 \leq_\Psi t_1 \leq_\Psi \ldots \leq_\Psi t_k = x^{-1}.$$

For the proof of the theorem, we keep the notation $y_j = s_j \ldots s_k$ from above, and also abbreviate $U_{y_j^{-1}}$ as U_j and U'_{y_j} as U'_j, for $j = 1, \ldots, k+1$, so one has $U_+ = U_j y_j U'_j y_j^{-1}$ with uniqueness of expression. For $w \in W$, $s \in S$ with $l(sw) > l(w)$, one has $U_w \subset U_{sw}$ by (5), and then Corollary 4 with $Q = w^{-1} s^{-1} U_+ sw$ gives $U'_{sw} \subset U'_w$. Hence $U'_1 \subset U'_2 \subset \ldots$.

Lemma 4. *Fix $\sigma \in D_y$ and an integer j with $1 \leq j \leq k$. Set $\Omega(\sigma, j)$ equal to $\mathbb{F}$, $\{0\} \subset \mathbb{F}$ or $\mathbb{F}^*$ according as whether $\sigma_j >_\Psi \sigma_{j-1}$, $\sigma_j <_\Psi \sigma_{j-1}$ or*

$\sigma_j = \sigma_{j-1}$. *Then there is an injective map* $f_j : \Omega(\sigma, j) \times U_{j+1} \to U_j$ *with the following properties:*

(a) *the image of* f_j *is either* U_j, $s_j U_{j+1} s_j^{-1}$ *or* $U_j \setminus s_j U_{j+1} s_j^{-1}$ *according as* $\sigma_j >_\Psi \sigma_{j-1}$, $\sigma_j <_\Psi \sigma_{j-1}$ *or* $\sigma_j = \sigma_{j-1}$, *and* f_j *is a homeomorphism onto its image*

(b) *one has* $\sigma_{j-1}^{-1} f_j(t, u_{j+1}) y_j = b_j \sigma_j^{-1} u_{j+1} y_{j+1} v_{j+1}$ *for some* $b_j \in B_\Psi := HQ_\Psi$ *and* $v_{j+1} \in U'_{j+1}$.

Proof of the Lemma. Write $s_j = s_\alpha$, $\alpha \in \Pi$.

First, suppose that $\sigma_j >_\Psi \sigma_{j-1}$, so $\sigma_j = s_j \sigma_{j-1}$ and $\alpha \notin \sigma_{j-1}\Psi$. Define $f_j(t, u_{j+1}) = x_\alpha(t) s_j u_{j+1} s_j^{-1}$. The required properties of f_j follow immediately on noting that $x_\alpha(t) \in Q_{\sigma_{j-1}\Psi} = \sigma_{j-1} Q_\Psi \sigma_{j-1}^{-1}$.

The second case is $\sigma_j <_\Psi \sigma_{j-1}$; one again has $\sigma_j = s_j \sigma_{j-1}$. The map f_j defined by $f_j(0, u_{j+1}) = s_j u_{j+1} s_j^{-1}$ has the required properties.

The remaining case is that $\sigma_j = \sigma_{j-1}$; here, $s_j \sigma_{j-1} <_\Psi s_{j-1}$, so $\alpha \notin \sigma_{j-1}\Psi$. Write $x_\alpha(t) u_{j+1} = u'_{j+1} y_{j+1} v_{j+1}^{-1} y_{j+1}^{-1}$ for some (uniquely determined) $u'_{j+1} \in U_{j+1}$ and $v_{j+1} \in U'_{j+1}$, and set $f_j(t, u_{j+1}) = x_\alpha(t^{-1}) s_j u'_{j+1} s_j^{-1} \in U_j$. As in the proof of (3.2) in [5], f_j is a homeomorphism onto its image $U_j \setminus s_j U_{j+1} s_j^{-1}$. Now we compute

$$\begin{aligned}\sigma_{j-1}^{-1} f_j(t, u_{j+1}) y_j &= \sigma_{j-1}^{-1} x_\alpha(t^{-1}) s_j u'_{j+1} y_{j+1} \\ &= \sigma_{j-1}^{-1} x_\alpha(t^{-1}) s_j x_\alpha(t) u_{j+1} y_{j+1} v_{j+1}.\end{aligned}$$

We may write $x_\alpha(t^{-1}) s_j x_\alpha(t) = h s_j x_\alpha(-t^{-1}) s_j^{-1}$ for some $h \in H$. Setting

$$b_j = \sigma_{j-1}^{-1} h s_j x_\alpha(-t^{-1}) s_j^{-1} \sigma_j,$$

we have

$$\sigma_{j-1}^{-1} f_j(t, u_{j+1}) y_j = b_j \sigma_j^{-1} u_{j+1} y_{j+1} v_{j+1}.$$

Finally, observe that $b_j \in B_\Psi$ since $-\alpha \in \sigma_{j-1}\Psi$ implies

$$\sigma_{j-1} b_j \sigma_{j-1}^{-1} \in HU_{-\alpha} \subset B_{\sigma_{j-1}\Psi} = \sigma_{j-1} B_\Psi s_{j-1}^{-1}.$$

Proof of Theorem 5. The map $U_{y^{-1}} = U_1 \to By \cdot B$ given by $u \to u \cdot B$ is bijective, and clearly maps $\eta^{-1}(\sigma)$ into $Q_\Psi \pi(\sigma) \cdot B$ by definition of η and π. Since U_1 is the disjoint union of the sets $\eta^{-1}(\sigma)$ for $\sigma \in D_y$, we need only prove (1).

Fix $\sigma \in D$ and define subsets $A_j \subset U_j$ for $k+1 \geq j \geq 1$ by setting $A_{k+1} = \{e\}$ and $A_j = f_j(\Omega(\sigma, j) \times A_{j+1})$ for $k \geq j \geq 1$. The lemma implies that A_j is a locally closed subset of U_j homeomorphic to $\mathbb{F}^{m(j,\sigma)} \times (\mathbb{F}^*)^{n(j,\sigma)}$ where

$$m(j, \sigma) = \sharp\{\, p \mid j \leq p \leq k \text{ and } \Omega(\sigma, p) = \mathbb{F}\,\},$$

$$n(j, \sigma) = \sharp\{\, p \mid j \leq p \leq k \text{ and } \Omega(\sigma, p) = \mathbb{F}^*\,\}.$$

Since $m(\sigma) = m(1, \sigma)$ and $n(\sigma) = n(1, \sigma)$, the theorem will be proved if we show that $A_1 = \eta^{-1}(\sigma)$. Fix $u_1 \in U_1$.

Suppose for $j = 1, \ldots, p$ we have $u_{j+1} \in U_{j+1}$ and $t_j \in \Omega(\sigma, j)$ with $u_j = f_j(t_j, u_{j+1})$. Choosing $b_j \in B_\Psi$ and $v_j \in U'_j$ as in the lemma, it follows immediately by induction on j that for $1 \leq j \leq p$,

$$u_1 s_1 s_2 \ldots s_j = b_1 b_2 \ldots b_j \sigma_j^{-1} u_{j+1} y_{j+1} v_{j+1} \ldots v_3 v_2 y_{j+1}^{-1}. \tag{13}$$

Recalling $U'_1 \subset U'_2 \subset \ldots$, the right hand side of (13) is an element of $Q_\Psi \sigma_j^{-1} B$. In particular, if $u_1 \in A_1$, one can take $p = k$ in the above and deduce that $\eta(u_1) = \sigma$.

Conversely, suppose that $u_1 \in \eta^{-1}(\sigma)$. We prove we have u_{j+1} and t_j as in the previous paragraph, for $j = 1, \ldots, p$, by induction on p. Suppose inductively this is true for $p-1$, so in particular (13) holds if $j = p-1$. Choose $\delta \in W$ so that $u_p s_p \in Q_{\sigma_{p-1}\Psi} \delta^{-1} B = \sigma_{p-1} Q_\Psi \sigma_{p-1}^{-1} \delta^{-1} B$. Then, recalling $U'_j \subset U'_{j+1}$, one has

$u_1 s_1 \dots s_{p-1} s_p \in Q_\Psi \sigma_{p-1}^{-1} \delta^{-1} B$ so $\delta\sigma_{p-1} = \sigma_p$. To complete the proof, we just need to show that u_p is in the image of f_p, which is given in the Lemma. Write $s_p = s_\alpha$ for $\alpha \in \Pi$. We must show that

i) if $\delta = s_p$ and $\alpha \notin \sigma_{p-1}\Psi$, then $u_p \in s_p U_{p+1} s_p^{-1}$ and

ii) if $\delta = 1$ then $u_p \in U_p \setminus s_p U_{p+1} s_p^{-1}$

(the other case being trivial).

Consider the situation i). We have $s_p^{-1} u_p s_p \in Q_{s_p \sigma_{p-1}\Psi} B = Q_{\sigma_p \Psi} B$. Since $u_p \in U_p$, we may write $s_p^{-1} u_p s_p = x_{-\alpha}(t) v$ for some $t \in \mathbb{F}$ and $v \in U_{p+1}$, and we must show $t = 0$. But if $t \neq 0$, we get

$$s_j B = x_\alpha(-t^{-1})(x_{-\alpha}(t)v)v^{-1} x_\alpha(-t^{-1}) B \subset Q_{\sigma_p \Psi} B$$

(since $\alpha \in \sigma_p \Psi \cap \Delta_+^{re}$), contrary to Theorem 2. In the other situation ii), one has $u_p s_p \in Q_{\sigma_{p-1}\Psi} B$ so $s_p^{-1} u_p s_p \in Q_{s_p \sigma_{p-1} \Psi} s_p B$. Write $s_p^{-1} u_p s_p = x_{-\alpha}(t) v$ as in case i). Then $x_{-\alpha}(t) \in Q_{s_p \sigma_{p-1}\Psi} s_p B$ so Theorem 2 implies $t \neq 0$, $s_p^{-1} u_p s_p \notin U_{p+1}$ as required. This completes the proof.

Additional Remarks. We make some remarks concerning non-standard Schubert-type decompositions of G/B. Fix a strictly dominant integral weight Λ, and endow $L(\Lambda)$ with the Zariski topology. Recall the notations $\mathcal{V}$, $\mathcal{V}(\lambda)_\Psi$ from the proof of Theorem 4, and note that G/B injects naturally into the set $\mathbb{P}(L(\Lambda))$ of lines of $L(\Lambda)$ as in [14].

For any $\lambda \in W\Lambda$, one has the closed subset $\overline{C}_{\lambda,\Psi} := \mathbb{P}(\overline{\mathcal{V}(\lambda)_\Psi})$ of the (infinite-dimensional) projective space $\mathbb{P}L(\Lambda)$. For $\Psi_0 = \Delta_+$, $\overline{C}_{\lambda,\Psi_0}$ is the the finite Schubert variety $\overline{C}_\lambda$ and $\overline{C}_{\lambda,-\Psi_0}$ is the the cofinite Schubert variety $\overline{C}^\lambda$ of [14, 15]. In general, $\overline{C}_{\lambda,\Psi}$ is the directed union of its intersections with the finite Schubert varieties, these intersections being finite-dimensional projective varieties.

For certain Ψ of particular interest for the representation theory of $\mathfrak{g}$, every interval in the order $\leq_\Psi$ on W is finite (see [6, 6.4 and 6.7]). We suppose henceforward for simplicity that Ψ has this property. Since the ordinary Chevalley order is directed, Corollary 6(b) implies that each (closed) intersection

$$\overline{C}_{\lambda,\mu,\Psi} := \overline{C}_{\lambda,\Psi} \cap \overline{C}_{\mu,-\Psi} \tag{15}$$

will be contained in some finite Schubert variety, and hence aquires a natural structure of finite-dimensional projective variety (essentially independent of the choice of Λ, since this is known for the finite Schubert varieties). One has that $\overline{C}_{\lambda,\mu,\Psi}$ is non-empty precisely when $\mu \leq_\Psi \lambda$ (but see the example following).

By [6,7] the (non-empty, finite) open intervals (μ, λ) in the order $\leq_\Psi$ on $W\Lambda$ (or $\leq_\Psi$ on W) are of two types; spherical (i.e the order complex of the open interval is a combinatorial sphere) or non-spherical (the order complex is a combinatorial ball); many very interesting constructions (e.g. Kazhdan-Lusztig polynomials, see [6]) can be extended to the former, but either fail or give pathological results for non-spherical intervals. It can be shown (cf. [8]) that for a non-empty spherical interval (v, w) in W, one has

$$\sharp\{\, \alpha \in \Delta_+^{re} \mid v \leq_\psi s_\alpha u \leq_\Psi w \,\} \geq l_\Psi(w) - l_\Psi(v) \tag{16}$$

for all $v \leq_\Psi u \leq_\Psi w$ in W, but that this result need not hold for a non-spherical interval. Now (16) has been established for ordinary Chevalley order [3] by studying H-invariant curves in the intersections of finite and cofinite Schubert varieties,, and one might therefore expect similar results to apply to $\overline{C}_{\lambda,\mu,\Psi}$ for spherical intervals (μ, λ) in $W\Lambda$ in the order $\leq \Psi$. It seems likely that (also only for spherical intervals) there should be a decomposition of $\overline{C}_{\lambda,\mu,\Psi}$ into locally closed subsets (parametrized by the data in [6, 3.1] used to construct the R-polynomial) similar to that in Theorem 5.

Example. Here we show that, in contrast to the classical situation (where $\Psi = \Delta_+^{re}$), one can have $C_{\lambda,\mu,\Psi} := \mathcal{V}(\lambda)_\Psi \cap \mathcal{V}(\mu)_{-\Psi} = \emptyset$ even though $\mu \leq_\Psi \lambda$ (notation as in the above remark). First, note that by (2) in the proof of Theorem 4 and the definition of $\leq_\Psi$, one has that

$$C_{\lambda,\mu,\Psi} = \{\, v \in \mathcal{V} \mid \{\mu, \lambda\} \subset \operatorname{supp}(v) \subset [\mu, \lambda]\,\}$$

where $I = [\mu, \lambda] = \{\, \nu \in W\Lambda \mid \mu \leq_\Psi \nu \leq_\Psi \lambda\,\}$; also, for v in $C_{\lambda,\mu,Psi}$, the edges of $\operatorname{supp}(v)$ are parallel to real roots. Suppose now that the poset I is a chain of cardinality three (by [6] any finite non-spherical interval in an order $\leq_\Psi$ contains such a chain as a subinterval). Then $\mu - \lambda$ is a linear combination with strictly positive coefficients of two "adjacent" real roots lying on a plane in $\mathfrak{h}$, so it is not a multiple of a real root, and hence $C_{\lambda,\mu,\Psi} = \emptyset$. As a specific example, for Ψ as in Example 3 in the case of SL_2, every length two subinterval of $\leq_\Psi$ is a chain of cardinality three.

To conclude these remarks, we make an observation on a relationship of the decompositions here to certain Hecke algebra modules associated to the orders $\leq_\Psi$ on W.

The results of this paper excepting Theorems 4 and Proposition 2 can be proved by essentially identical arguments for the version of G in [11]. (For the topological statements in Theorem 5, one should take the field algebraically closed; then the sets $\eta^{-1}(\sigma)$ are locally closed subvarieties of $U_{y^{-1}}$ in a natural structure of unipotent algebraic group. We haven't pursued rationality questions over other fields.) For the following remark, consider the group G from [11] for $\mathbb{F}$ a finite field of q elements.

Let $\mathcal{F}$ denote the set of complex-valued functions on G/B. For $x \in W$, define a $\mathbb{C}$-linear map $T_x: \mathcal{F} \to \mathcal{F}$ by the formula

$$(T_x f)(g \cdot B) = \sum_{z \in g(Bx \cdot B)} f(z)$$

for $g \in G$.

It is easily checked (cf. [12]) that the T_x span over $\mathbb{C}$ a copy $\mathcal{H}$ of the Iwahori-Hecke algebra of W, with parameter q. In fact, for $s \in S$ and $x \in W$ one has

$$T_s T_x = \begin{cases} T_{sx} & \text{if } sx > x \\ qT_{sx} + (q-1)T_x & \text{if } sx < x \end{cases}$$

where $\leq$ denotes Chevalley order.

Setting $t_w = q^{l_\Psi(w)} \chi_{A_w}$ for $w \in W$, where χ_{A_w} is the characteristic function of the subset $A_w := Q_\Psi w^{-1} \cdot B$ of G/B, one can show similarly (or deduce from Theorem 5) that for $s \in S$,

$$T_s t_w = \begin{cases} t_{sw} & \text{if } sw >_\Psi w \\ qt_{sw} + (q-1)t_w & \text{if } sw <_\Psi w. \end{cases}$$

In [6], a module $\mathcal{H}_A$ for the generic Iwahori-Hecke algebra of W was associated to the initial section $A = \{\, s_\alpha \mid \alpha \in \Delta_+^{re} \setminus \Psi\}$. Specializing the indeterminate there to $q \in \mathbb{C}$, the resulting $\mathcal{H}$-module is a natural "completion" of the $\mathcal{H}$-submodule of $\mathcal{F}$ spanned here by the t_w for $w \in W$.

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