FREE GROUPS OF LIE TYPE

Y. Billig and A. Pianzola

Dept. of Mathematics & Statistics
University of New Brunswick
Fredericton, New Brunswick
Canada E3B 5A3

Dept. of Mathematical Sciences
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

Abstract

Groups are constructed which afford rings of polynomial functions. Using invariant derivations of these rings along classical lines, yields Lie algebras which are completions of free Lie algebras.

Throughout this report I will be a finite set † and $\mathbb{K}$ a field of characteristic 0.

For each $i \in I$ we consider a multiplicative copy $E_i := \{E_i(\lambda),\, \lambda \in \mathbb{K}\}$ of $(\mathbb{K}, +)$. Thus $E_i(\lambda)E_i(\mu) = E_i(\lambda + \mu)$. Our group F is the free product $F = *_{i \in I} E_i$.

Let W be the free associative monoid (words) on I. Then for $\boldsymbol{w} \in W$, $\boldsymbol{w} \neq 1$, one has $\boldsymbol{w} = \boldsymbol{w}_1 \cdots \boldsymbol{w}_w$ with $w > 0$ and $\boldsymbol{w}_1, \cdots, \boldsymbol{w}_w$ in I unique (so we use bold face characters for words, and the corresponding unbold characters for their lengths). There is also a unique expression, called *reduced*, of the form $\boldsymbol{w} = \overline{\boldsymbol{w}}_1^{n_1} \cdots \overline{\boldsymbol{w}}_{\overline{w}}^{n_{\overline{w}}}$ where $\overline{\boldsymbol{w}}_n \neq \overline{\boldsymbol{w}}_{n+1}$. We set $\overline{\boldsymbol{w}} := \overline{\boldsymbol{w}}_1 \cdots \overline{\boldsymbol{w}}_{\overline{w}}$ and $\overline{W} = \{\overline{w} : w \in W\}$ (reduced words).

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† This assumption is not essential. It is made in this report for ease of exposition, to rend some of the notation and concepts used more managable.

To each $\boldsymbol{w} \in W$ we attach a function $\pi_{\boldsymbol{w}} : \mathbb{K}^w \to F$ by

$$\pi_{\boldsymbol{w}} : (\lambda_1, \ldots, \lambda_w) \mapsto E_{\boldsymbol{w}_1}(\lambda_1) \cdots E_{\boldsymbol{w}_w}(\lambda_w).$$

A function $f : F \to \mathbb{K}$ is *polynomial* if for all $\boldsymbol{w} \in W$ the function $f_{\boldsymbol{w}} := f \circ \pi_{\boldsymbol{w}}$ is polynomial. The ring of all such functions we denote by $\mathrm{Pol}\,(F)$.

Let $\mathcal{A}$ be the free associative $\mathbb{K}$-algebra on the set of symbols $\{x_i\}_{i \in I}$. To each $\boldsymbol{w} \in W$ we attach the element $x^{\boldsymbol{w}} := \prod_{i=1}^{w} x_{\boldsymbol{w}_i} \in \mathcal{A}$. Consider the completion $\overline{\mathcal{A}}$ of $\mathcal{A}$ and corresponding Magnus group

$$M := \Big\{ 1 + \sum_{\boldsymbol{w} \in W, \boldsymbol{w} \neq \emptyset} c_{\boldsymbol{w}} x^{\boldsymbol{w}},\ c_{\boldsymbol{w}} \in \mathbb{K} \Big\} \subset \overline{\mathcal{A}}.$$

For each $i \in I$ there is a group homomorphism $\varepsilon_i : \mathbb{K} \to M$ given by

$$\varepsilon_i : \lambda \mapsto \sum_{n \geq 0} \frac{\lambda^n x^n}{n!}\ .$$

By universal nonsense these yield a (unique) group homomorphism $\varepsilon : F \to M$.

For each $\boldsymbol{a} \in W$ define $X^{(\boldsymbol{a})} \in \overline{\mathcal{A}}^*$ (dual space) by $\langle X^{(\boldsymbol{a})}, \sum_{\boldsymbol{b} \in W} c_{\boldsymbol{b}} x_{\boldsymbol{b}} \rangle = \delta_{\boldsymbol{a},\boldsymbol{b}} c_{\boldsymbol{b}}$ (Kronecker δ). This yields a function

$$X^{\boldsymbol{a}} := X^{(\boldsymbol{a})} \circ \varepsilon : F \to \mathbb{K}.$$

These functions are polynomial and allow us to describe $\mathrm{Pol}\,(F)$: When can a formal expression $\sum_{\boldsymbol{s} \in W} c_{\boldsymbol{s}} X^{\boldsymbol{s}}$ be thought of as a function on F? The answer is that the set $S := \{\boldsymbol{s} \in W : c_{\boldsymbol{s}} \neq 0\}$ must have the property that the set $S_L := \{\boldsymbol{s} \in S : \overline{s} \leq L\}$ be finite for all $L \in \mathbb{N}$. We call such sets *summable*.

Proposition 1. *Every polynomial function on* F *can uniquely be written in the form* $\sum_{\boldsymbol{s} \in W} c_{\boldsymbol{s}} X^{\boldsymbol{s}}$; $c_{\boldsymbol{s}} \in \mathbb{K}$, *where the set* $S := \{\boldsymbol{s} \in W : c_{\boldsymbol{s}} \neq 0\}$ *is summable.* $\square$

In view of this result the multiplicative structure of $\mathrm{Pol}\,(F)$ is completely determined by

Proposition 2. *Let* $\boldsymbol{a}, \boldsymbol{b} \in W$. *Then*

$$X^{\boldsymbol{a}} X^{\boldsymbol{b}} = \sum_{\boldsymbol{s} \in \boldsymbol{a} \leftrightsquigarrow \boldsymbol{b}} X^{\boldsymbol{s}}$$

where $\leftrightsquigarrow$ *is the shuffle product.* $\square$

We now begin to look at the derivations of $\mathrm{Pol}\,(F)$. Recall that $\mathrm{Pol}\,(F)$ has a natural right F-module structure via $(x \cdot f)(y) = f(yx^{-1})$. Recall also that $\partial \in \mathrm{End}\,\mathrm{Pol}\,(F)$ (the $\mathbb{K}$ algebra of $\mathbb{K}$-linear endomorphisms of $\mathrm{Pol}\,(F)$) is *right invariant* if $(x \cdot \partial)(f) = \partial(x \cdot f)$ for all $x \in F$ and $f \in \mathrm{Pol}\,(F)$.

Let $\partial \in \mathrm{End}\,\mathrm{Pol}\,(F)$ and for all $\boldsymbol{s} \in W$ write $\partial(X^{\boldsymbol{s}}) = \sum_{\boldsymbol{w} \in W} \partial^{\boldsymbol{s}}_{\boldsymbol{w}} X^{\boldsymbol{w}}$. Assume that for all summable sets S we have

CONT 1: For all $\boldsymbol{w} \in W$ the set $\{\boldsymbol{s} \in S : \partial^{\boldsymbol{s}}_{\boldsymbol{w}} \neq 0\}$ is finite.

CONT 2: The set $\{\boldsymbol{w} \in W : \exists\, \boldsymbol{s} \in S$ with $\partial^{\boldsymbol{s}}_{\boldsymbol{w}} \neq 0\}$ is summable.

Then if $f = \sum c_{\boldsymbol{s}} X^{\boldsymbol{s}} \in \mathrm{Pol}\,(F)$ we have $\partial(f) = \sum c_{\boldsymbol{s}} \partial(X^{\boldsymbol{s}})$. Such endomorphisms we call *continuous*.

For $\boldsymbol{w} \in W$ define $d(\boldsymbol{w}) \in \mathbb{Z}^I$ by $d(\boldsymbol{w})(i) := \mathrm{Card}\,\{n : \boldsymbol{w}_n = i\}$.

Proposition 3. *Let* $\partial \in \mathrm{End}\,\mathrm{Pol}\,(F)$. *For* $\omega \in \mathbb{Z}^I$ *and* $\boldsymbol{s} \in W$ *write* $\partial_\omega(X^{\boldsymbol{s}}) = \sum_{d(\boldsymbol{w}) = \omega + d(\boldsymbol{s})} \partial^{\boldsymbol{s}}_{\boldsymbol{w}} X^{\boldsymbol{w}}$ *(a finite sum). Assume* ∂ *is continuous. Then*

(i) ∂_ω *can uniquely be extended to a continuous operator on* $\mathrm{Pol}\,(F)$. *Moreover* $\partial = \sum_{\omega \in \mathbb{Z}^I} \partial_\omega$ *(this last sum with the obvious meaning).*

(ii) *If* ∂ *is a derivation then so is* ∂_ω.

(iii) *If* ∂ *is right invariant then so is* ∂_ω. *Furthermore* $\partial_\omega = 0$ *unless* $\omega \in \mathbb{Z}^I_-$, *(i.e.* $\omega(i) \leq 0$ *for all* $i \in I$*).* $\square$

For each $i \in I$, and $\boldsymbol{s} \in W$ with reduced expression $\boldsymbol{s} = \overline{\boldsymbol{s}}_1^{n_1} \cdots \overline{\boldsymbol{s}}_{\overline{s}}^{n_{\overline{s}}}$, define

$$\partial_i(X^{\boldsymbol{s}}) = \begin{cases} n_1 X^{\overline{\boldsymbol{s}}_1^{(n_1-1)} \overline{\boldsymbol{s}}_2^{n_2} \cdots \overline{\boldsymbol{s}}_{\overline{s}}^{n_{\overline{s}}}} & \text{if} \quad i = \overline{\boldsymbol{s}}_1 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that ∂_i extends uniquely to a continuous operator on $\mathrm{Pol}\,(F)$ which is also a right invariant derivation of $\mathrm{Pol}\,(F)$.

For each $\boldsymbol{w} \in W$ let $\partial^{\boldsymbol{w}} := \partial_{\boldsymbol{w}_1} \cdots \partial_{\boldsymbol{w}_w}$. By acting on the $X^{\boldsymbol{s}}$'s one sees that the $\partial^{\boldsymbol{w}}$'s are linearly independent, and hence that the associative subalgebra A of End Pol (F) generated by the ∂_i's is free. This together with Proposition 3 and Friedrichs' theorem yields:

Theorem 1. *Let Lie (F) be the Lie algebra of right invariant continuous derivations of* $\mathrm{Pol}\,(F)$. *Then*

(i) *The ∂_i's generate a subalgebra L of* $\mathrm{Lie}\,(F)$ *which is free.*

(ii) *If $\partial \in$* $\mathrm{Lie}\,(F)$ *and we write $\partial = \sum_{\omega \in \mathbb{Z}_-^I} \partial_\omega$ then each ∂_ω belongs to L.*

(iii) $\mathrm{Lie}\,(F)$ *is a completion of the free Lie algebra L. More precisely*

$$\mathrm{Lie}\,(F) = \Big(\prod_{\omega \in \mathbb{Z}_-^I} L_\omega \Big) \cap \Big(\bigoplus_{\boldsymbol{r} \in \overline{W}} \Big(\prod_{\substack{\boldsymbol{w} \in W \\ \overline{\boldsymbol{w}} = \boldsymbol{r}}} \mathbb{K} \partial^w \Big) \Big). \qquad \square$$

References

D. Peterson and V. Kac, *Infinite flag varieties and conjugacy theorems*, Proc. Nat'l. Acad. Sci. USA **80** (1983), 1778-1782.