An Extension of the KdV Hierarchy
Arising from a Representation of a Toroidal Lie Algebra

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ABSTRACT. In this article we show how to construct hierarchies of partial differential equations from the vertex operator representations of toroidal Lie algebras. In the smallest example - rank 2 toroidal cover of sl_2 - we obtain an extension of the KdV hierarchy. We use the action of the corresponding infinite-dimensional group to construct solutions for these non-linear PDEs.

0. Introduction.

In this article we show how to construct hierarchies of partial differential equations and their soliton-type solutions from the vertex operator representations of toroidal Lie algebras.

Soliton theory was given a new impetus when it was linked with the representation theory of infinite-dimensional Lie algebras in the works of Sato [S], Date-Jimbo-Kashiwara-Miwa [DJKM] and Drinfeld-Sokolov [DS]. It was discovered that for various partial differential equations the space of soliton solutions has a large group of hidden symmetries. Moreover, for every Kac-Moody algebra one can construct a hierarchy of PDEs whose symmetries form the corresponding Kac-Moody group [KW].

The most famous example occurs in the context of the affine Kac-Moody algebra $A_1^{(1)}$ which is a central extension of the loop algebra $sl_2(\mathbb{C}[t, t^{-1}])$. In this case the hierarchy contains the Korteweg-de Vries equation

$$f_t = f f_x + f_{xxx}$$

as the equation of the lowest degree.

We generalize this approach for the toroidal Lie algebras. The difficulties arise because the toroidal Lie algebras have no triangular decomposition and many methods of the Kac-Moody theory do not work. However, an important class of representations of these algebras was constructed in [MEY], [EM] and [B]. Here we establish a connection between the principal vertex operator realization developed in [B] and non-linear partial differential equations.

We study in detail the case of the smallest toroidal algebra - the universal central extension of $sl_2(\mathbb{C}[t_0, t_0^{-1}, t_1, t_1^{-1}])$. This algebra has affine Kac-Moody algebra $A_1^{(1)}$ as a subalgebra. The hierarchy we obtain has the KdV hierarchy as a proper subset. Equations of low degrees in the extended KdV hierarchy, but not in the KdV subhierarchy are the following:

$$\frac{\partial}{\partial x}\left(f_t - \frac{1}{6}f_{xxy} - f_x f_y\right) = \frac{1}{6}f_{yz}$$

and

$$f_{xxxt} + 6f_{xx}f_{xt} - f_{xxyz} - 4f_{xy}f_{xz} - 2f_{xx}f_{yz} = 0.$$

We use algebraic methods to construct solutions for the extended KdV hierarchy.

The paper is organized as follows. In Section 1 we present the construction of the toroidal algebras. In Section 2 we review the main results of [B] on the principal vertex operator representations. We obtain the extended KdV hierarchy in Section 3 and construct its solutions in Section 4. In the Appendix we discuss the generalized Casimir operators and give a proof of Proposition 1 which is fairly standard but crucial for our derivation.

1. Toroidal Lie algebras.

Throughout this paper we will use the constructions and the notations of [B].

Let $\dot{\mathfrak{g}}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ of type A_ℓ, D_ℓ or E_ℓ (i.e., simply-laced) with the root system $\dot{\Delta}$. The algebra $\dot{\mathfrak{g}}$ possesses a non-degenerate symmetric invariant bilinear form $(\cdot|\cdot)$. The reduction of this form to the Cartan subalgebra $\dot{\mathfrak{h}}$ induces the map $\nu : \dot{\mathfrak{h}} \to \dot{\mathfrak{h}}^*$ and a bilinear form on $\dot{\mathfrak{h}}^*$. We normalize both forms by the condition $(\alpha|\alpha) = 2$ for all nonzero roots $\alpha \in \dot{\Delta}$. Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be the simple roots in $\dot{\Delta}$. Define the height function ht : $\dot{\Delta} \to \mathbb{Z}$ by ht$(\sum_{j=1}^{\ell} k_j \alpha_j) = \sum_{j=1}^{\ell} k_j$. Let $\rho \in \dot{\mathfrak{h}}^*$ be such that $(\rho|\alpha) = \text{ht}(\alpha)$ for $\alpha \in \dot{\Delta}$.

Let h be the Coxeter number of $\dot{\mathfrak{g}}$. Consider the principal $\mathbb{Z}_h$-gradation of $\dot{\mathfrak{g}}$

$$\dot{\mathfrak{g}} = \sum_{j \in \mathbb{Z}_h} \dot{\mathfrak{g}}_j,$$

where $\dot{\mathfrak{g}}_j$ is the direct sum of the root spaces $\dot{\mathfrak{g}}^\alpha$ with ht$(\alpha) = j(\text{mod } h)$.

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The algebra $\dot{\mathfrak{g}}$ possesses a Cartan subalgebra $\dot{\mathfrak{s}}$ which is homogeneous with respect to the principal $\mathbb{Z}_h$-gradation and has a basis $\{T_1, \ldots, T_\ell\}$ such that $T_i \in \dot{\mathfrak{g}}_{m_i}$, where the numbers $1 = m_1 \leq m_2 \leq \ldots \leq m_\ell = h - 1$ are the exponents of $\dot{\mathfrak{g}}$. Define a sequence $\{b_i\}_{i \in \mathbb{N}}$ by $b_{j\ell+i} = jh + m_i$ for $i = 1, \ldots, \ell$ and $j \geq 0$.

The basis of $\dot{\mathfrak{s}}$ can be normalized so that

$$(T_i | T_{\ell+1-j}) = h\delta_{ij}. \tag{1.1}$$

Let $\dot{\Delta}_s$ be the root system of $\dot{\mathfrak{g}}$ with respect to the Cartan subalgebra $\dot{\mathfrak{s}}$. For $\alpha \in \dot{\Delta}_s$ fix a root element

$$A^\alpha = \sum_{j \in \mathbb{Z}_h} A_j^\alpha, \qquad A_j^\alpha \in \dot{\mathfrak{g}}_j.$$

Define constants $\lambda_i^\alpha = \alpha(T_i)$. Then $[T_i, A^\alpha] = \lambda_i^\alpha A^\alpha$.

A central extension of the Lie algebra

$$\sum_{j \in \mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j$$

is the principal realization of untwisted affine Kac-Moody algebra. The $n + 1$-toroidal Lie algebra is the universal central extension of

$$\sum_{j \in \mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j \mathbb{C}[t_1^\pm, \ldots, t_n^\pm].$$

The vertex operator representations of this algebra were studied in [MEY], [EM] and [B]. In the present paper we will use a larger algebra

$$\tilde{\mathfrak{g}} = \sum_{j \in \mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j \mathbb{C}[\mathbb{R}^n].$$

We replace the algebra of Laurent polynomials $\mathbb{C}[t_1^\pm, \ldots, t_n^\pm]$ which is the group algebra of $\mathbb{Z}^n$ with the group algebra of $\mathbb{R}^n$. The algebra $\mathbb{C}[\mathbb{R}^n]$ has a basis of monomials $\mathbf{t}^{\mathbf{r}} = t_1^{r_1} \ldots t_n^{r_n}, \mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$. The multiplication in $\mathbb{C}[\mathbb{R}^n]$ is given by $\mathbf{t}^{\mathbf{r}}\mathbf{t}^{\mathbf{m}} = \mathbf{t}^{\mathbf{r}+\mathbf{m}}$. All the results (and their proofs) from [B] remain valid for this version of the toroidal Lie algebras.

The following description of the universal central extension of $\tilde{\mathfrak{g}}$ is based on the general result of [Kas] (see also [MEY]).

Let $\dot{\mathcal{K}}$ be an $(n + 1)$-dimensional space with the basis $\{K_0, K_1, \ldots, K_n\}$. Consider the space

$$\tilde{\mathcal{K}} = \dot{\mathcal{K}} \otimes \mathbb{C}[s^h, s^{-h}] \otimes \mathbb{C}[\mathbb{R}^n]$$

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and its subspace $d\tilde{\mathcal{K}}$ spanned by the elements

$$r_0 K_0 \otimes s^{r_0 h} \mathbf{t}^{\mathbf{r}} + r_1 K_1 \otimes s^{r_0 h} \mathbf{t}^{\mathbf{r}} + \ldots + r_n K_n \otimes s^{r_0 h} \mathbf{t}^{\mathbf{r}}.$$

The factor space $\mathcal{K} = \tilde{\mathcal{K}}/d\tilde{\mathcal{K}}$ is the space of the universal central extension of $\tilde{\mathfrak{g}}$ and the Lie bracket in the toroidal algebra $\mathfrak{g} = \tilde{\mathfrak{g}} \oplus \mathcal{K}$ is given by

$$[g_1 \otimes f_1(s,\mathbf{t}), g_2 \otimes f_2(s,\mathbf{t})] = [g_1, g_2] \otimes (f_1 f_2) + (g_1|g_2) \left\{ \frac{1}{h} s \frac{\partial f_1}{\partial s} f_2 K_0 + \sum_{p=1}^{n} t_p \frac{\partial f_1}{\partial t_p} f_2 K_p \right\}$$

and

$$[\mathfrak{g}, \mathcal{K}] = 0.$$

From now on we will omit the tensor product sign when writing the elements of $\mathfrak{g}$.

2. Principal vertex operator construction for toroidal Lie algebras.

Now we can describe the representation of $\mathfrak{g}$ constructed in [B]. The space F of this representation (the Fock space) is the tensor product of the group algebra of $\mathbb{R}^n$ (with the basis $\{\mathbf{q}^{\mathbf{r}} | \mathbf{r} \in \mathbb{R}^n\}$) and the algebra of polynomials in infinitely many variables:

$$F = \mathbb{C}[\mathbb{R}^n] \otimes \mathbb{C}[x_{b_i}, u_{pi}, v_{pi}]_{i \in \mathbb{N}}^{p=1,\ldots,n}.$$

Instead of specifying the action of individual elements of $\mathfrak{g}$ on F, we will represent certain generating series (by vertex operators). Let z be a formal variable. We set

$$\sum_{j \in \mathbb{Z}} \varphi(s^{jh} \mathbf{t}^{\mathbf{r}} K_0) z^{-jh} = K_0(z, \mathbf{r}), \quad \text{where} \tag{2.1}$$

$$K_0(z, \mathbf{r}) = \mathbf{q}^{\mathbf{r}} \exp\left(\sum_{p=1}^{n} r_p \sum_{j \geq 1} z^{jh} u_{pj} \right) \exp\left(-\sum_{p=1}^{n} r_p \sum_{j \geq 1} \frac{z^{-jh}}{j} \frac{\partial}{\partial v_{pj}} \right), \tag{2.2}$$

$$\sum_{j \in \mathbb{Z}} \varphi(s^{jh} \mathbf{t}^{\mathbf{r}} K_p) z^{-jh} = K_p(z, \mathbf{r}) = K_p(z) K_0(z, \mathbf{r}), \quad \text{where} \tag{2.3}$$

$$K_p(z) = \sum_{i \geq 1} i z^{ih} u_{pi} + \sum_{i \geq 1} z^{-ih} \frac{\partial}{\partial v_{pi}}, \tag{2.4}$$

$$\sum_{j \in \mathbb{Z}} \varphi(T_i s^{m_i + jh} \mathbf{t}^{\mathbf{r}}) z^{-m_i - jh} = T_i(z, \mathbf{r}) = T_i(z) K_0(z, \mathbf{r}), \quad i = 1, \ldots, \ell, \quad \text{where} \tag{2.5}$$

$$T_i(z) = \sum_{j \geq 1} (jh - m_i) z^{jh - m_i} x_{jh - m_i} + \sum_{j \geq 0} z^{-jh - m_i} \frac{\partial}{\partial x_{jh + m_i}}, \tag{2.6}$$

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$$\sum_{j\in\mathbb{Z}} \varphi(A_j^\alpha s^j \mathbf{t^r}) z^{-j} = A^\alpha(z,\mathbf{r}) = A^\alpha(z) K_0(z,\mathbf{r}), \quad \alpha \in \dot{\Delta}_s, \quad \text{where} \tag{2.7}$$

$$A^\alpha(z) = -\frac{\rho(A_0^\alpha)}{h} \exp\left(\sum_{i\geq 1} \lambda_i^\alpha z^{b_i} x_{b_i}\right) \exp\left(-\sum_{i\geq 1} \lambda_{\ell+1-i}^\alpha \frac{z^{-b_i}}{b_i} \frac{\partial}{\partial x_{b_i}}\right). \tag{2.8}$$

We also represent derivations of $\mathfrak{g}$ as operators on F. Before we introduce these, we need to discuss the operation of the normal ordering.

Consider the algebra of the differential operators $Diff(y_1, y_2, \ldots)$ on the space $\mathbb{C}[y_1, y_2, \ldots]$:

$$Diff(y_1, y_2, \ldots) = \left\{\sum_{\mathbf{n}\in\mathcal{A}} f_{\mathbf{n}}(\mathbf{y}) \left(\frac{\partial}{\partial \mathbf{y}}\right)^{\mathbf{n}} \Big| f_{\mathbf{n}}(\mathbf{y}) \in \mathbb{C}[y_1, y_2, \ldots]\right\},$$

where $\mathcal{A}$ is the set of sequences $\mathbf{n} = (n_1, n_2, \ldots)$ with $n_i \in \mathbb{Z}_+$ where only finitely many terms are nonzero. We use the notations $\mathbf{y} = (y_1, y_2, \ldots)$ and $\left(\frac{\partial}{\partial \mathbf{y}}\right)^{\mathbf{n}} = \left(\frac{\partial}{\partial y_1}\right)^{n_1} \left(\frac{\partial}{\partial y_2}\right)^{n_2} \ldots$. A differential operator $P \in Diff(y_1, y_2, \ldots)$ can be viewed as a linear map

$$P: \quad \mathbb{C}[y_1, y_2, \ldots] \to \mathbb{C}[y_1, y_2, \ldots].$$

The space $Diff(y_1, y_2, \ldots)$ has a structure of an associative algebra with respect to the composition of operators. This algebra is not commutative since $\frac{\partial}{\partial y_i}$ and y_i do not commute. The normal ordering $:\ :$ is a new commutative associative product on $Diff(y_1, y_2, \ldots)$ defined by

$$:\left(\sum_{\mathbf{n}\in\mathcal{A}} f_{\mathbf{n}}(\mathbf{y}) \left(\frac{\partial}{\partial \mathbf{y}}\right)^{\mathbf{n}}\right) \left(\sum_{\mathbf{n}\in\mathcal{A}} g_{\mathbf{m}}(\mathbf{y}) \left(\frac{\partial}{\partial \mathbf{y}}\right)^{\mathbf{m}}\right): \ = \sum_{\mathbf{n+m}\in\mathcal{A}} f_{\mathbf{n}}(\mathbf{y}) g_{\mathbf{m}}(\mathbf{y}) \left(\frac{\partial}{\partial \mathbf{y}}\right)^{\mathbf{n+m}}.$$

Now consider the following operators on F:

$$D_p(z,\mathbf{r}) = :D_p(z) K_0(z,\mathbf{r}):, \quad p = 1, \ldots, n, \quad, \quad \text{where} \tag{2.9}$$

$$D_p(z) = \sum_{i\geq 1} i z^{ih} v_{pi} + q_p \frac{\partial}{\partial q_p} + \sum_{i\geq 1} z^{-ih} \frac{\partial}{\partial u_{pi}}, \quad \text{and} \tag{2.10}$$

$$D_s(z,\mathbf{r}) = :D_s(z) K_0(z,\mathbf{r}):, \quad \text{where} \tag{2.11}$$

$$D_s(z) = -\frac{1}{2} \sum_{i=1}^{\ell} :T_i(z) T_{\ell+1-i}(z): \ - h \sum_{p=1}^{n} :D_p(z) K_p(z):. \tag{2.12}$$

Expanding the generating series $D_p(z,\mathbf{r})$ and $D_s(z,\mathbf{r})$, we obtain operators $s^{jh}\mathbf{t^r} D_p$ and $s^{jh}\mathbf{t^r} D_s$:

$$D_p(z,\mathbf{r}) = \sum_{j\in\mathbb{Z}} s^{jh}\mathbf{t^r} D_p z^{-jh} \quad, \quad D_s(z,\mathbf{r}) = \sum_{j\in\mathbb{Z}} s^{jh}\mathbf{t^r} D_s z^{-jh}. \tag{2.13}$$

We summarize below the main results (Theorem 5 and Proposition 8) of [B].

Theorem A. (a). *The formulas (2.1)-(2.8) define a representation of the toroidal Lie algebra $\mathfrak{g}$ on the Fock space F.*

(b). *The operators $s^{r_0 h} \mathbf{t^r} D_p$ and $s^{r_0 h} \mathbf{t^r} D_s$ defined by (2.9)-(2.13) act on $\mathfrak{g}$ as derivations:*

$$\left[s^{r_0 h} \mathbf{t^r} D_p, \varphi \left(A_j^\alpha s^j \mathbf{t^m} \right) \right] = m_p \varphi \left(A_j^\alpha s^{r_0 h + j} \mathbf{t^{r+m}} \right),$$

$$\left[s^{r_0 h} \mathbf{t^r} D_s, \varphi \left(A_j^\alpha s^j \mathbf{t^m} \right) \right] = j \varphi \left(A_j^\alpha s^{r_0 h + j} \mathbf{t^{r+m}} \right),$$

$$\left[s^{r_0 h} \mathbf{t^r} D_p, \varphi \left(T_i s^{b_i} \mathbf{t^m} \right) \right] = m_p \varphi \left(T_i s^{r_0 h + b_i} \mathbf{t^{r+m}} \right),$$

$$\left[s^{r_0 h} \mathbf{t^r} D_s, \varphi \left(T_i s^{b_i} \mathbf{t^m} \right) \right] = b_i \varphi \left(T_i s^{r_0 h + b_i} \mathbf{t^{r+m}} \right).$$

Remark. Though the operators $s^{r_0 h} \mathbf{t^r} D_p$ and $s^{r_0 h} \mathbf{t^r} D_s$ act on $\varphi(\mathfrak{g})$ as derivations of $\mathbb{C}[s^h, s^{-h}] \otimes \mathbb{C}[\mathbb{R}^n]$, the algebra they generate together with $\varphi(\mathfrak{g})$ is not isomorphic to the semidirect product of $\mathfrak{g}$ with $\mathcal{D} = \mathrm{Der}\left(\mathbb{C}[s^h, s^{-h}] \otimes \mathbb{C}[\mathbb{R}^n] \right)$. A direct computation shows that the span of $s^{r_0 h} \mathbf{t^r} D_p$ and $s^{r_0 h} \mathbf{t^r} D_s$ is not closed under the Lie bracket. The commutators contain extra terms that commute with $\varphi(\mathfrak{g})$.

Our main tool for the construction of the hierarchies of partial differential equations will be the generalized Casimir operators. We introduce these by the following generating series (see Appendix for details):

$$\Omega(z) = \sum_{j \in \mathbb{Z}} \Omega_j z^{-jh} =$$

$$= :\left\{ \frac{1}{h} \sum_{i=1}^{\ell} T_i(z) \otimes T_{\ell+1-i}(z) + \sum_{\alpha \in \mathring{\Delta}_s} \frac{1}{(A^\alpha | A^{-\alpha})} A^\alpha(z) \otimes A^{-\alpha}(z) - \frac{\ell(h+1)}{12h} \right.$$

$$\left. + \frac{1}{h} D_s(z) \otimes 1 + \frac{1}{h} 1 \otimes D_s(z) + \sum_{p=1}^{n} D_p(z) \otimes K_p(z) + \sum_{p=1}^{n} K_p(z) \otimes D_p(z) \right\} \times$$

$$\times \sum_{\mathbf{r} \in \mathbb{R}} K_0(z, \mathbf{r}) \otimes K_0(z, -\mathbf{r}): . \tag{2.14}$$

Both $\mathfrak{g}$ and its module F are graded by $\mathbb{Z} \times \mathbb{R}^n$. Thus the $U(\mathfrak{g}) \otimes U(\mathfrak{g})$-module $F \otimes F$ is graded by $(\mathbb{Z} \times \mathbb{R}^n) \times (\mathbb{Z} \times \mathbb{R}^n)$ and we can consider its completion $\overline{F \otimes F}$ with respect to this grading. The operators Ω_k act from $F \otimes F$ to $\overline{F \otimes F}$. Also note that both $F \otimes F$ and $\overline{F \otimes F}$ have the $\mathfrak{g}$-module structure.

Proposition 1. *The operators*

$$\Omega_k: \quad F \otimes F \to \overline{F \otimes F}$$

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commute with the action of $\mathfrak{g}$.

The proof of this proposition is given in the Appendix.

3. Extended KdV hierarchy.

As in the affine case, the equation $\Omega_k(\tau \otimes \tau) = 0$ decomposes in the hierarchy of partial differential equations in the Hirota form. In this section we study the hierarchy that corresponds to the smallest toroidal algebra with $\dot{\mathfrak{g}} = sl_2(\mathbb{C})$ and $n = 1$. The Coxeter number of $sl_2(\mathbb{C})$ is $h = 2$ and its only exponent is $m_1 = 1$.

The representation space is

$$F = \mathbb{C}[\mathbb{R}] \otimes \mathbb{C}[x_1, x_3, x_5, \ldots] \otimes \mathbb{C}[u_1, u_2, \ldots] \otimes \mathbb{C}[v_1, v_2, \ldots],$$

where $\mathbb{C}[\mathbb{R}]$ is the group algebra of $(\mathbb{R}, +)$ with the basis $\{q^r | r \in \mathbb{R}\}$. The action (2.1)-(2.8) of $\mathfrak{g}$ on F can be written as follows (cf. [Kac], Sec. 14.13):

$$K_0(z, r) = q^r \exp\left(r \sum_{i \in \mathbb{N}} z^{2i} u_i \right) \exp\left(-r \sum_{i \in \mathbb{N}} \frac{z^{-2i}}{i} \frac{\partial}{\partial v_i} \right),$$

$$K_1(z) = \sum_{i \in \mathbb{N}} i z^{2i} u_i + \sum_{i \in \mathbb{N}} z^{-2i} \frac{\partial}{\partial v_i},$$

$$T(z) = \sum_{j \in \mathbb{N}_{\mathrm{odd}}} j z^j x_j + \sum_{j \in \mathbb{N}_{\mathrm{odd}}} z^{-j} \frac{\partial}{\partial x_j},$$

$$A^{\pm \alpha}(z) = \frac{1}{2} \exp\left(\pm 2 \sum_{j \in \mathbb{N}_{\mathrm{odd}}} z^j x_j \right) \exp\left(\mp 2 \sum_{j \in \mathbb{N}_{\mathrm{odd}}} \frac{z^{-j}}{j} \frac{\partial}{\partial x_j} \right),$$

$$T(z, r) = T(z) K_0(z, r), \quad A^{\pm \alpha}(z, r) = A^{\pm \alpha}(z) K_0(z, r), \quad K_1(z, r) = K_1(z) K_0(z, r).$$

For the positive root α of $sl_2(\mathbb{C})$, we will denote $A^\alpha(z, r)$ simply by $A(z, r)$. Then $A^{-\alpha}(z, r) = A(-z, r)$.

The derivations of $\mathfrak{g}$ are represented by

$$D_1(z) = \sum_{i \in \mathbb{N}} i z^{2i} v_i + q \frac{\partial}{\partial q} + \sum_{i \in \mathbb{N}} z^{-2i} \frac{\partial}{\partial u_i},$$

$$D_s(z) = -\frac{1}{2} :T(z) T(z): \; - \, 2 :D_1(z) K_1(z): \, ,$$

$$D_1(z, r) = \; :D_1(z) K_0(z, r): \, , \qquad D_s(z, r) = \; :D_s(z) K_0(z, r): \, .$$

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The Casimir generating series (2.14) can be written as

$$\Omega(z) = \; :\left\{\frac{1}{4}A(z)\otimes A(-z) + \frac{1}{4}A(-z)\otimes A(z) - \frac{1}{8} - \frac{1}{4}:(T(z)\otimes 1 - 1\otimes T(z))^2:\right.$$

$$\left. - :(D_1(z)\otimes 1 - 1\otimes D_1(z))\,(K_1(z)\otimes 1 - 1\otimes K_1(z)): \right\} \times \sum_{r\in\mathbb{R}} K_0(z,r)\otimes K_0(z,-r): \; .$$

We consider the system of equations

$$\Omega_k(\tau\otimes\tau) = 0, \qquad k\geq -1,$$

on a function $\tau \in F$. In the case of affine Kac-Moody algebra $A_1^{(1)}$ (i.e., $\dot{\mathfrak{g}} = sl_2(\mathbb{C}), n = 0$), this system is equivalent to the KdV hierarchy of partial differential equations ([Kac]). In the toroidal case considered here, we obtain a hierarchy that contains the KdV hierarchy as a proper subset.

The tensor square $F\otimes F$ of a polynomial algebra F is again a polynomial algebra in twice as many variables. Denoting the variables in the first copy of F by q', x'_j, u'_i, v'_i and in the second copy of F by q'', x''_j, u''_i, v''_i, we obtain the following representation for $\Omega(z)$:

$$\Omega(z) = \; :\left\{\frac{1}{16}\exp\left(2\sum_{j\in\mathbb{N}_{\mathrm{odd}}} z^j(x'_j - x''_j)\right)\exp\left(-2\sum_{j\in\mathbb{N}_{\mathrm{odd}}}\frac{z^{-j}}{j}\left(\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j}\right)\right)\right.$$

$$+\frac{1}{16}\exp\left(-2\sum_{j\in\mathbb{N}_{\mathrm{odd}}} z^j(x'_j - x''_j)\right)\exp\left(2\sum_{j\in\mathbb{N}_{\mathrm{odd}}}\frac{z^{-j}}{j}\left(\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j}\right)\right) - \frac{1}{8}$$

$$-\frac{1}{4}:\left(\sum_{j\in\mathbb{N}_{\mathrm{odd}}} jz^j(x'_j - x''_j) + \sum_{j\in\mathbb{N}_{\mathrm{odd}}} z^{-j}\left(\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j}\right)\right)^2:$$

$$-:\left(\sum_{i\geq 1} iz^{2i}(v'_i - v''_i) + \left(q'\frac{\partial}{\partial q'} - q''\frac{\partial}{\partial q''}\right) + \sum_{i\geq 1} z^{-2i}\left(\frac{\partial}{\partial u'_i} - \frac{\partial}{\partial u''_i}\right)\right) \times$$

$$\times\left(\sum_{i\geq 1} iz^{2i}(u'_i - u''_i) + \sum_{i\geq 1} z^{-2i}\left(\frac{\partial}{\partial v'_i} - \frac{\partial}{\partial v''_i}\right)\right): \left.\right\} \times$$

$$\times\sum_{r\in\mathbb{R}}\left(\frac{q'}{q''}\right)^r \exp\left(r\sum_{i\geq 1} z^{2i}(u'_i - u''_i)\right)\exp\left(-r\sum_{i\geq 1}\frac{z^{-2i}}{i}\left(\frac{\partial}{\partial v'_i} - \frac{\partial}{\partial v''_i}\right)\right): \; .$$

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We perform the change of variables:

$$w = \frac{1}{2}\ln\left(\frac{q'}{q''}\right), \qquad \tilde{w} = \frac{1}{2}\ln\left(q'q''\right), \qquad x_j = \frac{1}{2}(x_j' - x_j''), \qquad \tilde{x}_j = \frac{1}{2}(x_j' + x_j''),$$

$$u_i = \frac{1}{2}(u_i' - u_i''), \qquad \tilde{u}_i = \frac{1}{2}(u_i' + u_i''), \qquad v_i = \frac{1}{2}(v_i' - v_i''), \qquad \tilde{v}_i = \frac{1}{2}(v_i' + v_i'').$$

Then

$$\frac{\partial}{\partial w} = q'\frac{\partial}{\partial q'} - q''\frac{\partial}{\partial q''}, \qquad\qquad \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j'} - \frac{\partial}{\partial x_j''},$$

$$\frac{\partial}{\partial u_i} = \frac{\partial}{\partial u_i'} - \frac{\partial}{\partial u_i''}, \qquad\qquad \frac{\partial}{\partial v_i} = \frac{\partial}{\partial v_i'} - \frac{\partial}{\partial v_i''}.$$

The expression $\Omega(z)(\tau \otimes \tau)$ transforms as follows:

$$\Omega(z)\tau(q',\mathbf{x}',\mathbf{u}',\mathbf{v}')\tau(q'',\mathbf{x}'',\mathbf{u}'',\mathbf{v}'') =$$

$$= :\left\{ \frac{1}{16}\exp\left(4\sum_{j\in\mathbb{N}_{\mathrm{odd}}} z^j x_j \right)\exp\left(-2\sum_{j\in\mathbb{N}_{\mathrm{odd}}} \frac{z^{-j}}{j}\frac{\partial}{\partial x_j} \right) \right.$$

$$+ \frac{1}{16}\exp\left(-4\sum_{j\in\mathbb{N}_{\mathrm{odd}}} z^j x_j \right)\exp\left(2\sum_{j\in\mathbb{N}_{\mathrm{odd}}} \frac{z^{-j}}{j}\frac{\partial}{\partial x_j} \right) - \frac{1}{8}$$

$$- \frac{1}{4}:\left(\sum_{j\in\mathbb{N}_{\mathrm{odd}}} 2jz^j x_j + \sum_{j\in\mathbb{N}_{\mathrm{odd}}} z^{-j}\frac{\partial}{\partial x_j} \right)^2:$$

$$- :\left(\sum_{i\geq 1} 2iz^{2i}v_i + \frac{\partial}{\partial w} + \sum_{i\geq 1} z^{-2i}\frac{\partial}{\partial u_i} \right)\left(\sum_{i\geq 1} 2iz^{2i}u_i + \sum_{i\geq 1} z^{-2i}\frac{\partial}{\partial v_i} \right): \left.\right\} \times$$

$$\times \sum_{r\in\mathbb{R}} e^{2rw}\exp\left(2r\sum_{i\geq 1} z^{2i}u_i \right)\exp\left(-r\sum_{i\geq 1} \frac{z^{-2i}}{i}\frac{\partial}{\partial v_i} \right):$$

$$\tau(e^{\tilde{w}+w},\tilde{\mathbf{x}}+\mathbf{x},\tilde{\mathbf{u}}+\mathbf{u},\tilde{\mathbf{v}}+\mathbf{v})\tau(e^{\tilde{w}-w},\tilde{\mathbf{x}}-\mathbf{x},\tilde{\mathbf{u}}-\mathbf{u},\tilde{\mathbf{v}}-\mathbf{v}). \qquad (3.1)$$

Consider the completion of the group algebra $\mathbb{C}[\mathbb{R}]$: $\overline{\mathbb{C}[\mathbb{R}]} = \prod_{\mathbf{r}\in\mathbb{R}} \mathbb{C}e^{rw}$. The element

$$\delta(w) = \sum_{r\in\mathbb{R}} e^{rw}$$

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is a formal analog of the δ-function (cf. [FLM], Sec. 2.2). For any $X(w) \in \mathbb{C}[\mathbb{R}]$ we have

$$\delta(w)X(w) = \delta(w)X(0). \tag{3.2}$$

To establish this identity, it is sufficient to verify it for the basis elements $X(w) = e^{aw}$:

$$\delta(w)e^{aw} = \sum_{r \in \mathbb{R}} e^{(r+a)w} = \delta(w) \cdot 1.$$

Proposition 2. *Let $X(w) \in \mathbb{C}[\mathbb{R}]$. Then*

$$\left[\left(\frac{\partial}{\partial w}\right)^n \delta(w)\right] X(w) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left[\left(\frac{\partial}{\partial w}\right)^k \delta(w)\right] \left[\left(\frac{\partial}{\partial w}\right)^{n-k} X(w)\Big|_{w=0}\right].$$

Proof by induction. The basis of induction is (3.2). To make the inductive step, we use the Leibnitz rule:

$$\left[\left(\frac{\partial}{\partial w}\right)^{n+1} \delta(w)\right] X(w) = \frac{\partial}{\partial w}\left(\left[\left(\frac{\partial}{\partial w}\right)^n \delta(w)\right] X(w)\right) - \left[\left(\frac{\partial}{\partial w}\right)^n \delta(w)\right] \frac{\partial X}{\partial w} =$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left[\left(\frac{\partial}{\partial w}\right)^{k+1} \delta(w)\right] \left[\left(\frac{\partial}{\partial w}\right)^{n-k} X(w)\Big|_{w=0}\right]$$

$$+ \sum_{k=0}^{n} (-1)^{n-k+1} \binom{n}{k} \left[\left(\frac{\partial}{\partial w}\right)^k \delta(w)\right] \left[\left(\frac{\partial}{\partial w}\right)^{n-k+1} X(w)\Big|_{w=0}\right] =$$

$$= \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} \left[\left(\frac{\partial}{\partial w}\right)^k \delta(w)\right] \left[\left(\frac{\partial}{\partial w}\right)^{n+1-k} X(w)\Big|_{w=0}\right].$$

Proposition 3. *Let $P(r) = \sum_{n \geq 0} r^n P_n$, where $P_n \in \text{Diff}(w, y_1, y_2, \ldots)$ are differential operators that may depend on $\frac{\partial}{\partial w}$ but not on w. Suppose that for every $f(w, \mathbf{y}) = \sum_{i=1}^{N} e^{r_i w} f_i(\mathbf{y}) \in \mathbb{C}[\mathbb{R}] \otimes \mathbb{C}[y_1, y_2, \ldots]$ we have $P_n f = 0$ for all but finitely many n. If*

$$\sum_{r \in \mathbb{R}} e^{rw} P(r) g(w, \mathbf{y}) = 0$$

for some $g(w, \mathbf{y}) \in \mathbb{C}[\mathbb{R}] \otimes \mathbb{C}[y_1, y_2, \ldots]$ then

$$P\left(\epsilon - \frac{\partial}{\partial w}\right) g(w, \mathbf{y})\Big|_{w=0} = 0 \qquad \text{as a polynomial in } \epsilon.$$

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Proof. We have

$$0 = \sum_{r \in \mathbb{R}} e^{rw} P(r) g(w, \mathbf{y}) = \sum_{n \geq 0} \sum_{r \in \mathbb{R}} r^n e^{rw} P_n g(w, \mathbf{y}) =$$

$$= \sum_{n \geq 0} \left[\left(\frac{\partial}{\partial w} \right)^n \delta(w) \right] P_n g(w, \mathbf{y}) =$$

$$= \sum_{n \geq 0} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left[\left(\frac{\partial}{\partial w} \right)^k \delta(w) \right] \left[\left(\frac{\partial}{\partial w} \right)^{n-k} P_n g(w, \mathbf{y}) \Big|_{w=0} \right] =$$

$$= \sum_{k \geq 0} \left[\left(\frac{\partial}{\partial w} \right)^k \delta(w) \right] \sum_{m=n-k \geq 0} (-1)^m \binom{m+k}{k} \left[\left(\frac{\partial}{\partial w} \right)^m P_{m+k} g(w, \mathbf{y}) \Big|_{w=0} \right].$$

The last expression is a finite linear combination of $\left\{ \left(\frac{\partial}{\partial w} \right)^k \delta(w) \right\}$. Since these are linearly independent then for every k

$$\sum_{m \geq 0} (-1)^m \binom{m+k}{k} \left[\left(\frac{\partial}{\partial w} \right)^m P_{m+k} g(w, \mathbf{y}) \Big|_{w=0} \right] = 0.$$

Hence the following polynomial in ϵ is zero:

$$0 = \sum_{k \geq 0} \epsilon^k \sum_{m \geq 0} (-1)^m \binom{m+k}{k} \left[\left(\frac{\partial}{\partial w} \right)^m P_{m+k} g(w, \mathbf{y}) \Big|_{w=0} \right] =$$

$$= P\left(\epsilon - \frac{\partial}{\partial w} \right) g(w, \mathbf{y}) \Big|_{w=0}.$$

We will use this proposition in order to derive Hirota bilinear equations from the Casimir operator equation

$$\Omega_k (\tau \otimes \tau) = 0.$$

Let us recall the definition of the Hirota bilinear equations.

For a polynomial $R(y_1, y_2, \ldots)$ and a function $\tau(\tilde{y}_1, \tilde{y}_2, \ldots)$ we denote by $R(H_{\tilde{y}_1}, H_{\tilde{y}_2}, \ldots) \circ \tau(\tilde{y}_1, \tilde{y}_2, \ldots) \tau(\tilde{y}_1, \tilde{y}_2, \ldots)$ the expression

$$R\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots \right) \tau(\tilde{y}_1 + y_1, \tilde{y}_2 + y_2, \ldots) \tau(\tilde{y}_1 - y_1, \tilde{y}_2 - y_2, \ldots) \Big|_{y_i = 0}.$$

Equation

$$R(H_{\tilde{y}_1}, H_{\tilde{y}_2}, \ldots) \circ \tau(\tilde{\mathbf{y}}) \tau(\tilde{\mathbf{y}}) = 0$$

is called a Hirota bilinear equation.

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Using the same technics as in [Kac], we can transform (3.1) in the Hirota form with respect to the variables $\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}$. This is based on the following observation:

$$R(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots)\tau(\tilde{\mathbf{y}} + \mathbf{y})\tau(\tilde{\mathbf{y}} - \mathbf{y}) =$$

$$= R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots)\tau(\tilde{\mathbf{y}} + (\mathbf{x} + \mathbf{y}))\tau(\tilde{\mathbf{y}} - (\mathbf{x} + \mathbf{y}))\Big|_{\mathbf{x}=0} =$$

$$= R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots)\exp\left(\sum_{i \geq 1} y_i \frac{\partial}{\partial x_i}\right)\tau(\tilde{\mathbf{y}} + \mathbf{x})\tau(\tilde{\mathbf{y}} - \mathbf{x})\Big|_{\mathbf{x}=0} =$$

$$= R(H_{\tilde{y}_1}, H_{\tilde{y}_2}, \ldots)\exp\left(\sum_{i \geq 1} y_i H_{\tilde{y}_i}\right) \circ \tau(\tilde{\mathbf{y}})\tau(\tilde{\mathbf{y}}). \tag{3.3}$$

Note that the operator $\Omega_k = \Omega_k(r)$ in (3.1) satisfies the conditions of Proposition 3. The normal ordering in (3.1) guarantees that all the differentiations are performed before the multiplications by x_j, u_i, v_i. Thus applying (3.3) and Proposition 3 we obtain that $\Omega(z)\tau \otimes \tau$ can be written in the Hirota form as follows:

$$\Omega(z)\tau \otimes \tau =$$

$$= \left\{\frac{1}{16}\exp\left(4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j\right)\exp\left(-2 \sum_{j \in \mathbb{N}_{\text{odd}}} \frac{z^{-j}}{j} H_{\tilde{x}_j}\right)\right.$$

$$+\frac{1}{16}\exp\left(-4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j\right)\exp\left(2 \sum_{j \in \mathbb{N}_{\text{odd}}} \frac{z^{-j}}{j} H_{\tilde{x}_j}\right) - \frac{1}{8}$$

$$-\frac{1}{4}\left(\sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^j x_j + \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} H_{\tilde{x}_j}\right)^2$$

$$-\left(\sum_{i \geq 1} 2i z^{2i} v_i + H_{\tilde{w}} + \sum_{i \geq 1} z^{-2i} H_{\tilde{u}_i}\right)\left(\sum_{i \geq 1} 2i z^{2i} u_i + \sum_{i \geq 1} z^{-2i} H_{\tilde{v}_i}\right)\right\} \times$$

$$\times \exp\left((\epsilon - H_{\tilde{w}}) \sum_{i \geq 1} z^{2i} u_i\right)\exp\left(-\frac{1}{2}(\epsilon - H_{\tilde{w}}) \sum_{i \geq 1} \frac{z^{-2i}}{i} H_{\tilde{v}_i}\right) \times$$

$$\times \exp\left(\sum_{j \in \mathbb{N}_{\text{odd}}} x_j H_{\tilde{x}_j}\right)\exp\left(\sum_{i \geq 1} u_i H_{\tilde{u}_i}\right)\exp\left(\sum_{i \geq 1} v_i H_{\tilde{v}_i}\right) \circ$$

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$$\tau(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})\tau(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}). \tag{3.4}$$

The solutions for $\Omega_k \tau \otimes \tau = 0$ that will be constructed in the next section do not depend on $\mathbf{v}$, so we can make a reduction $H_{\tilde{v}_i} = 0$. Also, to simplify the notations we will denote $H_{\tilde{w}}$ by H_0, $H_{\tilde{x}_{2i+1}}$ by H_{2i+1} and $H_{\tilde{u}_i}$ by H_{2i}. We get

$$\sum_{k \in \mathbb{Z}} \Omega_k z^{-2k} \tau \otimes \tau =$$

$$= \left\{ \frac{1}{16} \exp\left(4 \sum_{j \in \mathbb{N}_{odd}} z^j x_j \right) \exp\left(-2 \sum_{j \in \mathbb{N}_{odd}} \frac{z^{-j}}{j} H_j \right) \right.$$

$$+ \frac{1}{16} \exp\left(-4 \sum_{j \in \mathbb{N}_{odd}} z^j x_j \right) \exp\left(2 \sum_{j \in \mathbb{N}_{odd}} \frac{z^{-j}}{j} H_j \right) - \frac{1}{8}$$

$$- \frac{1}{4} \left(\sum_{j \in \mathbb{N}_{odd}} 2j z^j x_j + \sum_{j \in \mathbb{N}_{odd}} z^{-j} H_j \right)^2$$

$$- \left(\sum_{i \geq 1} 2i z^{2i} v_i + H_0 + \sum_{i \geq 1} z^{-2i} H_{2i} \right) \left(\sum_{i \geq 1} 2i z^{2i} u_i \right) \right\} \times$$

$$\times \exp\left((\epsilon - H_0) \sum_{i \geq 1} z^{2i} u_i \right) \exp\left(\sum_{j \in \mathbb{N}_{odd}} x_j H_j \right) \exp\left(\sum_{i \geq 1} u_i H_{2i} \right) \circ$$

$$\tau(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}})\tau(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}) \tag{3.5}$$

This can be interpreted as a formal series in independent variables $z^2, \epsilon, x_j, u_i, v_i$ with the coefficients being Hirota polynomials in $H_0, H_1, H_2, \ldots$. We obtain a hierarchy of the Hirota bilinear equations by considering coefficients at various monomials in the equations $\Omega_k \circ \tau(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}})\tau(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}) = 0$, $k \geq -1$. We call this system of Hirota equations the extended KdV hierarchy.

When we consider Hirota equations corresponding to the monomials that depend on $x_1, x_3, \ldots$ only, we recover the KdV hierarchy. Other non-trivial equations of degrees less or equal to 5 are given below.

From the coefficient at $x_1 u_2$:

$$H_0 H_1^3 + 2 H_0 H_3 - 6 H_1 H_2 = 0. \tag{3.6}$$

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From the coefficient at u_2^2:

$$H_0^2 H_1^4 - 4H_0^2 H_1 H_3 + 48H_0 H_4 - 48H_2^2 = 0. \qquad (3.7)$$

From the coefficient at $x_1 u_4$:

$$H_0 H_1^5 + 20H_0 H_1^2 H_3 + 24H_0 H_5 - 120H_1 H_4 = 0. \qquad (3.8)$$

From the coefficient at $\epsilon x_1 u_2^2$:

$$H_0 H_1^5 + 20H_0 H_1^2 H_3 + 24H_0 H_5 - \frac{40}{3} H_1^3 H_2 - \frac{80}{3} H_2 H_3 - 40H_1 H_4 = 0. \qquad (3.9)$$

From the coefficient at $u_2 x_3$:

$$H_0 H_1^5 + 5H_0 H_1^2 H_3 + 24H_0 H_5 - 5H_1^3 H_2 - 40H_2 H_3 = 0. \qquad (3.10)$$

The equations (3.8)-(3.10) form a basis in the space of equations of degree 5 in this hierarchy. The following Hirota equations belong to this space:

$$H_1^3 H_2 - H_0 H_1^2 H_3 = 0, \qquad (3.11)$$

$$H_1^3 H_2 + 2H_2 H_3 - 6H_1 H_4 = 0. \qquad (3.12)$$

Equations (3.6) and (3.12) coincide up to a relabeling of the variables. An equation, equivalent to (3.12) occurs as the second equation (of degree 5) in the Kadomtsev-Petviashvili (KP) hierarchy. The equation (3.11) is apparently new.

4. N-soliton solutions.

The same method as in [Kac] allows us to construct solutions for the extended KdV hierarchy. The idea of this method is based on the following

Proposition 4. *The operator $\big(1 + \lambda A(z,r)\big) \otimes \big(1 + \lambda A(z,r)\big)$ commutes with Ω_k.*

Proof. Using Proposition 3.4.2 in [FLM] we obtain

$$A(z_1, r_1) A(z_2, r_2) = A(z_1) A(z_2) K_0(z_1, r_1) K_0(z_2, r_2) =$$
$$= \left(\frac{z_1 - z_2}{z_1 + z_2} \right)^2 \; :A(z_1) A(z_2): \, K_0(z_1, r_1) K_0(z_2, r_2). \qquad (4.1)$$

By Proposition 1, Ω_k commutes with

$$A(z, r) \otimes 1 + 1 \otimes A(z, r)$$

and thus with

$$\left(A(z_1, r_1) \otimes 1 + 1 \otimes A(z_1, r_1)\right)\left(A(z_2, r_2) \otimes 1 + 1 \otimes A(z_2, r_2)\right).$$

Passing to the limit with $z_1 \to z$, $z_2 \to z$, taking (4.1) into account and setting $r_1 = r_2 = r$, we get that

$$A(z, r) \otimes A(z, r)$$

commutes with Ω_k. But then

$$\left(1 + \lambda A(z, r)\right) \otimes \left(1 + \lambda A(z, r)\right) =$$

$$= 1 \otimes 1 + \lambda \left(A(z, r) \otimes 1 + 1 \otimes A(z, r)\right) + \lambda^2 A(z, r) \otimes A(z, r)$$

commutes with Ω_k and the proposition is proved.

Corollary 5. *If τ is a solution of $\Omega_k(\tau \otimes \tau) = 0$ then $(1 + \lambda A(z, r))\tau$ is also a solution of this equation.*

Proof. Indeed,

$$0 = \left(\left(1 + \lambda A(z, r)\right) \otimes \left(1 + \lambda A(z, r)\right)\right)\Omega_k(\tau \otimes \tau) = \Omega_k\left((1 + \lambda A(z, r))\tau \otimes (1 + \lambda A(z, r))\tau\right).$$

Lemma 6. *The function $\tau = 1$ is a solution of $\Omega_k(\tau \otimes \tau)$ for $k \geq -1$.*

Proof. Recalling (3.1) we obtain

$$\Omega(z)(1 \otimes 1) = \left\{\frac{1}{16}\exp\left(4\sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j\right) + \frac{1}{16}\exp\left(-4\sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j\right) - \frac{1}{8}\right.$$

$$-\frac{1}{4}\left(\sum_{j \in \mathbb{N}_{\text{odd}}} 2jz^j x_j\right)^2 - \left(\sum_{i \geq 1} 2iz^{2i} v_i\right)\left(\sum_{i \geq 1} 2iz^{2i} u_i\right)\left.\right\} \times$$

$$\times \sum_{r \in \mathbb{R}} e^{rw}\exp\left(r\sum_{i \geq 1} z^{2i} u_i\right).$$

We can see that only non-negative powers of z appear in the right hand side. Moreover, the coefficients at z^0 and z^2 vanish, thus $\tau = 1$ is a solution of $\Omega_k(\tau \otimes \tau) = 0$ for $k \geq -1$.

Remark. $\Omega_k(1 \otimes 1)$ for $k < -1$ has an interesting representation-theoretic meaning - it gives a vacuum vector inside $\overline{F \otimes F}$.

Theorem 7. *For $\lambda_1 \ldots \lambda_N, z_1, \ldots, z_N, r_1, \ldots, r_N \in \mathbb{R}$ the function*

$$\tau\left(w, x_1, x_3, x_5, \ldots, u_1, u_2, \ldots\right) =$$

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$$= \sum_{\substack{0 \le k \le N \\ 1 \le i_1 < \ldots < i_k \le N}} \lambda_{i_1} \ldots \lambda_{i_k} \prod_{1 \le \mu < \nu \le k} \left(\frac{z_{i_\mu} - z_{i_\nu}}{z_{i_\mu} + z_{i_\nu}} \right)^2 \times$$

$$\times \exp \left(\sum_{m=1}^{k} r_{i_m} w + 2 \sum_{j \in \mathbb{N}_{odd}} \sum_{m=1}^{k} z_{i_m}^j x_j + \sum_{j \in \mathbb{N}} \sum_{m=1}^{k} r_{i_m} z_{i_m}^{2j} u_j \right)$$

is a solution of the extended KdV hierarchy.

Proof. Using Lemma 6 and Corollary 5, we obtain that

$$\tau = (1 + 2\lambda_1 A(z_1, r_1)) \ldots (1 + 2\lambda_N A(z_N, r_N))1 =$$

$$= \sum_{\substack{0 \le k \le N \\ 1 \le i_1 < \ldots < i_k \le N}} \lambda_{i_1} \ldots \lambda_{i_k} 2^k A(z_{i_1}, r_{i_1}) \ldots A(z_{i_k}, r_{i_k})1 \tag{4.2}$$

is a solution of the extended KdV hierarchy. Using Proposition 3.4.1 of [FLM], we get a generalization of (4.1):

$$A(z_{i_1}, r_{i_1}) \ldots A(z_{i_k}, r_{i_k}) = \prod_{1 \le \mu < \nu \le k} \left(\frac{z_{i_\mu} - z_{i_\nu}}{z_{i_\mu} + z_{i_\nu}} \right)^2 :A(z_{i_1}, r_{i_1}) \ldots A(z_{i_k}, r_{i_k}): . \tag{4.3}$$

Recalling that

$$A(z, r) =$$

$$= \frac{1}{2} \exp \left(rw + 2 \sum_{j \in \mathbb{N}_{odd}} z^j x_j + r \sum_{i \in \mathbb{N}} z^{2i} u_i \right) \exp \left(-2 \sum_{j \in \mathbb{N}_{odd}} \frac{z^{-j}}{j} \frac{\partial}{\partial x_j} - r \sum_{i \in \mathbb{N}} \frac{z^{-2i}}{i} \frac{\partial}{\partial v_i} \right)$$

and using the definition of the normal ordering, we get

$$2^k :A(z_{i_1}, r_{i_1}) \ldots A(z_{i_k}, r_{i_k}): 1 =$$

$$= \exp \left(\sum_{m=1}^{k} r_{i_m} w + 2 \sum_{j \in \mathbb{N}_{odd}} \sum_{m=1}^{k} z_{i_m}^j x_j + \sum_{j \in \mathbb{N}} \sum_{m=1}^{k} r_{i_m} z_{i_m}^{2j} u_j \right). \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we obtain the claim of the theorem.

Let us finish this section by presenting the formulas for the solutions of the partial differential equations corresponding to Hirota equations (3.6) and (3.11). Setting $x = x_1, y = w, z = x_3, t = u_1$, we rewrite (3.6) and (3.11) as

$$H_x^3 H_y + 2H_y H_z - 6H_x H_t = 0, \tag{4.5}$$

$$H_x^3 H_t - H_x^2 H_y H_z = 0. \tag{4.6}$$

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Every bilinear Hirota equation can be written as a partial differential equation via the logarithmic transformation. Introducing $f(t, x, y, z) = \frac{\partial}{\partial x} \ln \tau$, we can write (4.5) as a non-linear PDE:

$$\frac{\partial}{\partial x}(6f_t - f_{xxy} - 6f_x f_y) - f_{yz} = 0. \tag{4.7}$$

Applying the previous theorem and treating all non-effective variables as parameters, we obtain the following solutions of (4.7):

$$f(t, x, y, z) = \frac{\partial}{\partial x} \ln \tau, \tag{4.8}$$

where

$$\tau = \sum_{\substack{0 \le k \le N \\ 1 \le i_1 < \ldots < i_k \le N}} \lambda_{i_1} \ldots \lambda_{i_k} \prod_{1 \le \mu < \nu \le k} \left(\frac{c_{i_\mu} - c_{i_\nu}}{c_{i_\mu} + c_{i_\nu}} \right)^2 \times$$

$$\times \exp\left(\left(\sum_{m=1}^{k} r_{i_m} c_{i_m}^2 \right) t + 2 \left(\sum_{m=1}^{k} c_{i_m} \right) x + \left(\sum_{m=1}^{k} r_{i_m} \right) y + 2 \left(\sum_{m=1}^{k} c_{i_m}^3 \right) z \right). \tag{4.9}$$

The second derivative of $\ln \tau$ exibits the N-soliton behaviour, thus (4.8-4.9) is the N-soliton potential.

Note that these solutions are different from the solutions arising in the context of the KP hierarchy. This may indicate that there is a larger Lie algebra governing the symmetries of this equation.

The transformation $\tau = e^g$, $g = \ln \tau$ allows us to write Hirota equation (4.6) as a PDE:

$$g_{xxxt} + 6g_{xx}g_{xt} - g_{xxyz} - 4g_{xy}g_{xz} - 2g_{xx}g_{yz} = 0. \tag{4.10}$$

The τ-function expression (4.9) provides a family of solutions of (4.10).

5. Appendix. Generalized Casimir operators.

The Casimir operator plays a prominent role in the representation theory of Lie algebras. The main feature of the Casimir operator is that it commutes with the action of the Lie algebra. In order to construct it one needs a non-degenerate invariant bilinear form on the Lie algebra (see Section 2.8 in [Kac]). The semidirect product of $\mathfrak{g}$ with $\mathcal{D}$ possesses such a form, however we deal here with a deformation of this algebra. Nevertheless, the corresponding operator still commutes with the action of $\mathfrak{g}$ (but not $\mathcal{D}$).

First, we introduce a Casimir operator for $\dot{\mathfrak{g}}$. Let $\{e^i\}$ and $\{f^i\}$ be dual bases in $\dot{\mathfrak{g}}$ with respect to the invariant bilinear form. Then for every $X \in \dot{\mathfrak{g}}$

$$\sum_i (X|e^i)f^i = \sum_i (X|f^i)e^i = X.$$

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Since $(\dot{\mathfrak{g}}_{j_1} | \dot{\mathfrak{g}}_{j_2}) = 0$ unless $j_2 = -j_1 (\mathrm{mod}\ h)$ then for $X \in \dot{\mathfrak{g}}_m$ we have

$$\sum_i (X | e^i_{-m}) f^i_m = \sum_i (X | f^i_{-m}) e^i_m = X. \tag{5.1}$$

The Casimir element $\dot{\Omega}$ is defined by

$$\dot{\Omega} = \sum_i e^i \otimes f^i = \sum_i \sum_{j_1, j_2 \in \mathbb{Z}_h} e^i_{j_1} \otimes f^i_{j_2} \in U(\dot{\mathfrak{g}}) \otimes U(\dot{\mathfrak{g}}).$$

The algebra $U(\dot{\mathfrak{g}}) \otimes U(\dot{\mathfrak{g}})$ is graded by the root lattice of $\dot{\mathfrak{g}}$. The projection of the root lattice on $\mathbb{Z}$ by the height function induces a $\mathbb{Z}$-grading of $U(\dot{\mathfrak{g}}) \otimes U(\dot{\mathfrak{g}})$. Since $\dot{\Omega}$ commutes with the action of $\dot{\mathfrak{h}}$, it belongs to the height 0 component of $U(\dot{\mathfrak{g}}) \otimes U(\dot{\mathfrak{g}})$. Hence the same is true for the principal $\mathbb{Z}_h$-grading of $U(\dot{\mathfrak{g}}) \otimes U(\dot{\mathfrak{g}})$, which means that

$$\sum_i e^i_{j_1} \otimes f^i_{j_2} = 0 \qquad \text{if} \qquad j_1 + j_2 \neq 0 (\mathrm{mod}\ h). \tag{5.2}$$

Consequently,

$$\dot{\Omega} = \sum_i \sum_{j \in \mathbb{Z}_h} e^i_j \otimes f^i_{-j}.$$

The Lie algebra $\dot{\mathfrak{g}}$ is embedded in $U(\dot{\mathfrak{g}}) \otimes U(\dot{\mathfrak{g}})$ via the diagonal map:

$$X \mapsto X \otimes 1 + 1 \otimes X.$$

The Casimir element $\dot{\Omega}$ commutes with $\dot{\mathfrak{g}}$:

$$[X, \dot{\Omega}] = \sum_i \sum_{j \in \mathbb{Z}_h} \left([X, e^i_j] \otimes f^i_{-j} + e^i_j \otimes [X, f^i_{-j}] \right) = 0.$$

In case when $X \in \dot{\mathfrak{g}}_m$, taking the $U(\dot{\mathfrak{g}})_{j+m} \otimes U(\dot{\mathfrak{g}})_{-j}$-component of the previous equality, we obtain

$$\sum_i \left([X, e^i_j] \otimes f^i_{-j} + e^i_{j+m} \otimes [X, f^i_{-j-m}] \right) = 0. \tag{5.3}$$

Now consider the generalized Casimir operators Ω_k, $k \in \mathbb{Z}$, for the toroidal algebra $\mathfrak{g}$:

$$\Omega_k = \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} \sum_i \varphi \left(e^i_j s^j \mathbf{t}^\mathbf{r} \otimes f^i_{-j} s^{kh-j} \mathbf{t}^{-\mathbf{r}} \right)$$

$$+ \frac{1}{h} \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} s^{jh} \mathbf{t}^\mathbf{r} D_s \otimes \varphi(s^{(k-j)h} \mathbf{t}^{-\mathbf{r}} K_0)$$

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$$+\frac{1}{h}\sum_{\mathbf{r}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\varphi(s^{jh}\mathbf{t^r}K_0)\otimes s^{(k-j)h}\mathbf{t^{-r}}D_s$$

$$+\sum_{\mathbf{r}\in\mathbb{R}^n}\sum_{p=1}^{n}\sum_{j\in\mathbb{Z}}s^{jh}\mathbf{t^r}D_p\otimes\varphi(s^{(k-j)h}\mathbf{t^{-r}}K_p)$$

$$+\sum_{\mathbf{r}\in\mathbb{R}^n}\sum_{p=1}^{n}\sum_{j\in\mathbb{Z}}\varphi(s^{jh}\mathbf{t^r}K_p)\otimes s^{(k-j)h}\mathbf{t^{-r}}D_p$$

$$-\frac{(\rho|\rho)}{h^2}\sum_{\mathbf{r}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\varphi\left(s^{jh}\mathbf{t^r}K_0\otimes s^{(k-j)h}\mathbf{t^{-r}}K_0\right).$$

Remark. The constant $\frac{(\rho|\rho)}{h^2}$ is chosen to make $\tau=1$ a solution of $\Omega_0(\tau\otimes\tau)=0$. Using the Freudenthal - de Vries "strange" formula ([Kac], (12.1.8)) we obtain that

$$\frac{(\rho|\rho)}{h^2}=\frac{\dim\dot{\mathbf{g}}}{12h}=\frac{\ell(h+1)}{12h}.$$

Proposition 1. *The operators*

$$\Omega_k:\qquad F\otimes F\to\overline{F\otimes F}$$

commute with the action of $\mathbf{g}$.

Proof. Since $\mathbf{g}$ is generated by the elements $Xs^{m_0}\mathbf{t^m}$ with $X\in\dot{\mathbf{g}}_{m_0}$ then it is sufficient to show that $[\varphi(Xs^{m_0}\mathbf{t^m}),\Omega_k]=0$. Indeed,

$$[\varphi(Xs^{m_0}\mathbf{t^m}),\Omega_k]=\sum_{\mathbf{r}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\sum_{i}\varphi\left([Xs^{m_0}\mathbf{t^m},e_j^i s^j\mathbf{t^r}]\otimes f_{-j}^i s^{kh-j}\mathbf{t^{-r}}\right)$$

$$+\sum_{\mathbf{r}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\sum_{i}\varphi\left(e_j^i s^j\mathbf{t^r}\otimes[Xs^{m_0}\mathbf{t^m},f_{-j}^i s^{kh-j}\mathbf{t^{-r}}]\right)$$

$$-\frac{1}{h}\sum_{\mathbf{r}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}}[s^{jh}\mathbf{t^r}D_s,\varphi(Xs^{m_0}\mathbf{t^m})]\otimes\varphi(s^{(k-j)h}\mathbf{t^{-r}}K_0)$$

$$-\frac{1}{h}\sum_{\mathbf{r}\in\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\varphi(s^{jh}\mathbf{t^r}K_0)\otimes[s^{(k-j)h}\mathbf{t^{-r}}D_s,\varphi(Xs^{m_0}\mathbf{t^m})]$$

$$-\sum_{\mathbf{r}\in\mathbb{R}^n}\sum_{p=1}^{n}\sum_{j\in\mathbb{Z}}[s^{jh}\mathbf{t^r}D_p,\varphi(Xs^{m_0}\mathbf{t^m})]\otimes\varphi(s^{(k-j)h}\mathbf{t^{-r}}K_p)$$

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$$- \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{p=1}^{n} \sum_{j \in \mathbb{Z}} \varphi(s^{jh} \mathbf{t^r} K_p) \otimes [s^{(k-j)h} \mathbf{t^{-r}} D_p, \varphi(X s^{m_0} \mathbf{t^m})] =$$

$$= \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} \sum_{i} \varphi\left([X, e_j^i] s^{j+m_0} \mathbf{t^{r+m}} \otimes f_{-j}^i s^{kh-j} \mathbf{t^{-r}}\right)$$

$$+ \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} \sum_{i} \varphi\left(e_{j+m_0}^i s^{j+m_0} \mathbf{t^{r+m}} \otimes [X, f_{-j-m_0}^i] s^{kh-j} \mathbf{t^{-r}}\right)$$

$$+ \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} \varphi\left(\sum_{i} (X|e_{-m_0}^i) \left\{ \frac{m_0}{h} s^{jh} \mathbf{t^{r+m}} K_0 + \sum_{p=1}^{n} m_p s^{jh} \mathbf{t^{r+m}} K_p \right\} \otimes f_{m_0}^i s^{(k-j)h+m_0} \mathbf{t^{-r}}\right)$$

$$+ \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} \varphi\left(\sum_{i} e_{m_0}^i s^{jh+m_0} \mathbf{t^r} \otimes (X|f_{-m_0}^i) \left\{ \frac{m_0}{h} s^{(k-j)h} \mathbf{t^{m-r}} K_0 + \sum_{p=1}^{n} m_p s^{(k-j)h} \mathbf{t^{m-r}} K_p \right\}\right)$$

$$- \frac{m_0}{h} \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} \varphi\left(X s^{jh+m_0} \mathbf{t^{r+m}} \otimes s^{(k-j)h} \mathbf{t^{-r}} K_0\right)$$

$$- \frac{m_0}{h} \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} \varphi\left(s^{jh} \mathbf{t^r} K_0 \otimes X s^{(k-j)h+m_0} \mathbf{t^{m-r}}\right)$$

$$- \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{p=1}^{n} \sum_{j \in \mathbb{Z}} m_p \varphi\left(X s^{jh+m_0} \mathbf{t^{r+m}} \otimes s^{(k-j)h} \mathbf{t^{-r}} K_p\right)$$

$$- \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{p=1}^{n} \sum_{j \in \mathbb{Z}} m_p \varphi\left(s^{jh} \mathbf{t^r} K_p \otimes X s^{(k-j)h+m_0} \mathbf{t^{m-r}}\right).$$

Applying (5.3) we see immediately that the sum of the first two terms is zero while the rest cancel out due to (5.1).

In order to obtain explicit expression (2.14) for $\Omega(z)$, we use the following dual bases in $\mathring{\mathfrak{g}}$ (see Theorem 2.2 in [Kac] and (1.1)):

$$\{T_i, A^\alpha\}_{\alpha \in \dot{\Delta}_s}^{i=1,\dots,\ell} \qquad and \qquad \left\{ \frac{1}{h} T_{\ell+1-i}, \frac{1}{(A^\alpha|A^{-\alpha})} A^{-\alpha} \right\}_{\alpha \in \dot{\Delta}_s}^{i=1,\dots,\ell}.$$

Using these bases we obtain the following generating series for the family of generalized Casimir operators:

$$\Omega(z) = \sum_{k \in \mathbb{Z}} \Omega_k z^{-kh} =$$

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$$= \frac{1}{h} \sum_{i=1}^{\ell} \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varphi \left(T_i s^{m_i + jh} \mathbf{t}^{\mathbf{r}} \otimes T_{\ell+1-i} s^{-m_i + (k-j)h} \mathbf{t}^{-\mathbf{r}} \right) z^{-kh}$$

$$+ \sum_{\alpha \in \dot{\Delta}_s} \frac{1}{(A^\alpha | A^{-\alpha})} \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varphi \left(A_j^\alpha s^j \mathbf{t}^{\mathbf{r}} \otimes A_{-j}^{-\alpha} s^{-j+kh} \mathbf{t}^{-\mathbf{r}} \right) z^{-kh}$$

$$+ \frac{1}{h} \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} s^{jh} \mathbf{t}^{\mathbf{r}} D_s \otimes \varphi(s^{(k-j)h} \mathbf{t}^{-\mathbf{r}} K_0) z^{-kh}$$

$$+ \frac{1}{h} \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varphi(s^{jh} \mathbf{t}^{\mathbf{r}} K_0) \otimes s^{(k-j)h} \mathbf{t}^{-\mathbf{r}} D_s z^{-kh}$$

$$+ \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{p=1}^{n} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} s^{jh} \mathbf{t}^{\mathbf{r}} D_p \otimes \varphi(s^{(k-j)h} \mathbf{t}^{-\mathbf{r}} K_p) z^{-kh}$$

$$+ \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{p=1}^{n} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varphi(s^{jh} \mathbf{t}^{\mathbf{r}} K_p) \otimes s^{(k-j)h} \mathbf{t}^{-\mathbf{r}} D_p z^{-kh}$$

$$- \frac{\ell(h+1)}{12h} \sum_{\mathbf{r} \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varphi \left(s^{jh} \mathbf{t}^{\mathbf{r}} K_0 \otimes s^{(k-j)h} \mathbf{t}^{-\mathbf{r}} K_0 \right) z^{-kh}.$$

Taking (5.2) into account, we get

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varphi \left(A_j^\alpha s^j \mathbf{t}^{\mathbf{r}} \otimes A_{-j}^{-\alpha} s^{-j+kh} \mathbf{t}^{-\mathbf{r}} \right) z^{-kh} = \sum_{j_1 \in \mathbb{Z}} \sum_{j_2 \in \mathbb{Z}} \varphi \left(A_{j_1}^\alpha s^{j_1} \mathbf{t}^{\mathbf{r}} \otimes A_{j_2}^{-\alpha} s^{j_2} \mathbf{t}^{-\mathbf{r}} \right) z^{j_1 + j_2}.$$

Thus

$$\Omega(z) = \sum_{\mathbf{r} \in \mathbb{R}^n} \left(\frac{1}{h} \sum_{i=1}^{\ell} T_i(z, \mathbf{r}) \otimes T_{\ell+1-i}(z, -\mathbf{r}) + \sum_{\alpha \in \dot{\Delta}_s} \frac{1}{(A^\alpha | A^{-\alpha})} A^\alpha(z, \mathbf{r}) \otimes A^{-\alpha}(z, -\mathbf{r}) \right.$$

$$+ \frac{1}{h} D_s(z, \mathbf{r}) \otimes K_0(z, -\mathbf{r}) + \frac{1}{h} K_0(z, \mathbf{r}) \otimes D_s(z, -\mathbf{r})$$

$$+ \sum_{p=1}^{n} D_p(z, \mathbf{r}) \otimes K_p(z, -\mathbf{r}) + \sum_{p=1}^{n} K_p(z, \mathbf{r}) \otimes D_p(z, -\mathbf{r}) - \frac{\ell(h+1)}{12h} K_0(z, \mathbf{r}) \otimes K_0(z, -\mathbf{r}) \right) =$$

$$= : \left\{ \frac{1}{h} \sum_{i=1}^{\ell} T_i(z) \otimes T_{\ell+1-i}(z) + \sum_{\alpha \in \dot{\Delta}_s} \frac{1}{(A^\alpha | A^{-\alpha})} A^\alpha(z) \otimes A^{-\alpha}(z) - \frac{\ell(h+1)}{12h} \right.$$

$$\left. + \frac{1}{h} D_s(z) \otimes 1 + \frac{1}{h} 1 \otimes D_s(z) + \sum_{p=1}^{n} D_p(z) \otimes K_p(z) + \sum_{p=1}^{n} K_p(z) \otimes D_p(z) \right\} \times$$

$$\times \sum_{\mathbf{r} \in \mathbb{R}^n} K_0(z, \mathbf{r}) \otimes K_0(z, -\mathbf{r}) : .$$

References

[B] Y. Billig, *Principal vertex operator representations for toroidal Lie algebras,* preprint hepth/9703002.

[DJKM] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, *Operator approach to the Kadomtsev - Petviashvili equation. Transformation groups for soliton equations III,* J. Phys. Soc. Japan, **50** (1981), 3806-3812.

[DS] V.G. Drinfeld, V.V. Sokolov, *Equations of the Koteweg - de Vries type and simple Lie algebras,* Doklady AN SSSR, **258** (1981), 11-16.

[EM] S. Eswara Rao, R.V. Moody, *Vertex representations for n-toroidal Lie algebras and a generalization of the Virasoro algebra,* Comm. Math. Phys. **159** (1994), 239-264.

[FLM] I. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the Monster,* Academic Press, Boston, 1989.

[Kac] V.G. Kac, *Infinite-dimensional Lie algebras,* 3rd ed., Cambridge University Press, Cambridge 1990.

[KW] V.G. Kac, M. Wakimoto, *Exceptional hierarchies of soliton equations,* Proc. Symposia in Pure Math., **49** (1989), 191-237.

[Kas] C. Kassel, *Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebra,* J. Pure Appl. Algebra **34** (1985), 265-275.

[Kos] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group,* Amer. J. Math. **81** (1959), 973-1032.

[MEY] R.V. Moody, S. Eswara Rao, T. Yokonuma, *Toroidal Lie algebras and vertex representations,* Geom. Ded., **35** (1990), 283-307.

[S] M. Sato, *Soliton equations as dynamical systems on infinite-dimensional Grassmann manifolds,* RIMS Kokyuroku, **439** (1981), 30-46.