STURM-LIOUVILLE PROBLEM ON FOCK SPACE

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Abstract. We consider a second order differential operator on the Fock space and compute its eigenvalues, which turn out to be non-negative integers.

Consider the following second order differential operator

$$T = \frac{1}{2} \sum_{\substack{a+b=p+q \\ a,b,p,q \geq 1}} x_a x_b \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_q}$$

acting on the the Fock space $\mathcal{F} = \mathbb{R}[x_1, x_2, x_3, \ldots]$.

The Fock space $\mathcal{F}$ has two natural $\mathbb{Z}$-gradings, by degree, with $\deg(x_k) = k$, and by length, with $\mathrm{len}(x_k) = 1$. We can decompose $\mathcal{F}$ into a direct sum according to these gradings:

$$\mathcal{F} = \bigoplus_{d \geq \ell \geq 0} \mathcal{F}(d, \ell),$$

where $\mathcal{F}(d, \ell)$ is the span of all monomials of degree d and length ℓ.

Since operator T preserves both gradings, the subspaces $\mathcal{F}(d, \ell)$ are T-invariant. This work started with an observation that the eigenvalues of T on $\mathcal{F}(d, \ell)$ appeared to be non-negative integers. For example, the spectrum of T on $\mathcal{F}(12, 4)$ is

$$[1, 3, 3, 5, 6, 7, 7, 10, 10, 10, 13, 15, 17, 19, 30].$$

The goal of this note is to shed light on the pattern of the eigenvalues of T.

We begin by showing that T is diagonalizable.

Proposition 1. *Operator T is diagonalizable on $\mathcal{F}$ with real non-negative eigenvalues.*

Proof. Introduce a positive-definite scalar product on $\mathbb{R}[x]$ with $\langle x^n, x^m \rangle = n! \delta_{n,m}$. It is easy to check that this scalar product satisfies

$$\left\langle \frac{d}{dx} f(x), g(x) \right\rangle = \left\langle f(x), x g(x) \right\rangle.$$

Viewing $\mathcal{F}$ as a tensor product of infinitely many copies of $\mathbb{R}[x]$, $\mathcal{F} = \mathbb{R}[x_1] \otimes \mathbb{R}[x_2] \otimes \ldots$, we obtain a positive-definite scalar product on $\mathcal{F}$ for which $\frac{\partial}{\partial x_k}$ is adjoint to multiplication by

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x_k. Then for $f, g \in \mathcal{F}$

$$\left\langle \sum_{a+b=p+q} x_a x_b \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_q} f, g \right\rangle = \left\langle f, \sum_{a+b=p+q} x_q x_p \frac{\partial}{\partial x_b} \frac{\partial}{\partial x_a} g \right\rangle$$

and hence T is a self-adjoint operator. Thus T is diagonalizable on each invariant subspace $\mathcal{F}(d, \ell)$ with real eigenvalues.

Since

$$\langle Tf, f \rangle = \frac{1}{2} \sum_{n=2}^{\infty} \left\langle \sum_{p+q=n} \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_q} f, \sum_{a+b=n} \frac{\partial}{\partial x_b} \frac{\partial}{\partial x_a} f \right\rangle \geq 0,$$

the eigenvalues of T are non-negative. $\square$

Corollary 2. *There is an orthonormal basis (with respect to the scalar product introduced in the proof of Proposition 1) of $\mathcal{F}$, consisting of the eigenfunctions of T.*

Consider the generating series

$$e(r, z) = \exp\left(r \sum_{j=1}^{\infty} x_j z^j \right) = \sum_{d \geq \ell \geq 0} s(d, \ell) r^\ell z^d.$$

Note that $s(d, \ell) \in \mathcal{F}(d, \ell)$. We shall see below that $s(d, \ell)$ is an eigenfunction for T which corresponds to the dominant eigenvalue on $\mathcal{F}(d, \ell)$.

The dimension of $\mathcal{F}(d, \ell)$ is equal to the number of partitions of d with exactly ℓ parts. Each such partition may be presented as a Young diagram Λ, for example the following diagram represents a partition $28 = 7 + 7 + 5 + 4 + 3 + 2$ with $d = 28$ and $\ell = 6$. Parameter

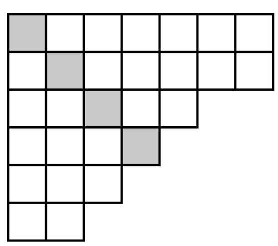
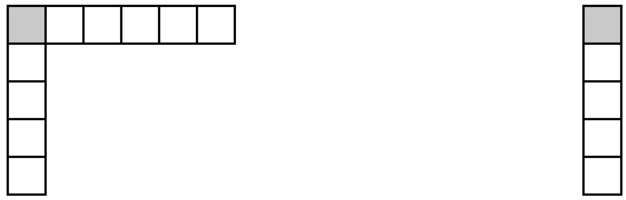


FIGURE 1

ℓ is the number of rows in Λ, while d is the total number of boxes in Λ.

Let k be the number of the diagonal boxes in Λ (shaded boxes in Fig. 1). For each diagonal box, consider its *hook*, the boxes in its row to the right of the diagonal box, the boxes in its column below the diagonal box, together with the diagonal box itself. Also consider the *leg* of a diagonal box, the boxes in its column together with the diagonal box itself.

In Fig. 2 we show the hook and the leg corresponding to the second diagonal box in the above Young diagram Λ.



(A) Hook (B) Leg

Figure 2

The *hook number* d_i and the *leg number* g_i of a diagonal box are the numbers of boxes in its hook and leg respectively. For the Young diagram Λ above, the hook and the leg numbers (d_i, g_i), $i = 1, \ldots, k$, are $(12, 6), (10, 5), (5, 3), (1, 1)$.

Note that our definition of the leg number is not quite standard, usually the diagonal box is not included in its leg.

The hook and leg numbers satisfy $\sum_{i=1}^{k} d_i = d$, $g_1 = \ell$, $d_i - g_i > d_{i+1} - g_{i+1}$ for $i < k$.

To each diagonal box we also assign its *leg increment* $\ell_i = g_i - g_{i+1}$, where g_{k+1} is taken to be 0. For the Young diagram in Fig. 1, $\ell_1 = 1$, $\ell_2 = 2$, $\ell_3 = 2$, $\ell_4 = 1$. Leg increments satisfy

$$\sum_{i=1}^{k} \ell_i = \ell, \quad d_i > d_{i+1} + \ell_i \quad \text{for} \quad i < k, \quad d_k \geq \ell_k, \quad \ell_1, \ldots, \ell_k \geq 1. \tag{1}$$

Note that there is a bijective correspondence between Young diagrams with ℓ rows and sequences $(d_1, \ell_1), \ldots, (d_k, \ell_k)$ satisfying (1).

Theorem 3. *The set $S(d, \ell)$ of polynomials $s(d_1, \ell_1)s(d_2, \ell_2) \ldots s(d_k, \ell_k)$ satisfying conditions*

$$\sum_{i=1}^{k} d_i = d, \quad \sum_{i=1}^{k} \ell_i = \ell, \quad d_i > d_{i+1} + \ell_i \quad \textit{for } i < k, \quad d_k \geq \ell_k, \quad \ell_1, \ldots, \ell_k \geq 1,$$

forms a basis of $\mathcal{F}(d, \ell)$ for $d \geq \ell \geq 1$.

Let $\ell', \ell'' \geq 1$. The products $s(d', \ell')s(d'', \ell'')$ with $d'' + \ell' \geq d' \geq d''$ will be called *irregular*, while the products with $d' > d'' + \ell'$ will be called *regular*. We also consider $s(d, \ell)s(0, 0)$ to be a regular product.

The proof of Theorem 3 will be based on the following

Lemma 4. *Every irregular product $s(d', \ell')s(d'', \ell'')$ with $d'' + \ell' \geq d' \geq d''$, $\ell', \ell'' \geq 1$, is a linear combination of regular products $s(d_1, \ell_1)s(d_2, \ell_2)$, with $d_1 + d_2 = d' + d''$, $\ell_1 + \ell_2 = \ell' + \ell''$, where either $d_1 > d'$ or $d_1 = d'$ and $\ell_1 < \ell'$.*

Proof. We will consider the case when $d = d' + d''$ is odd and $\ell = \ell' + \ell''$ is even, $d = 2n + 1$, $\ell = 2m$. The cases of other parities are analogous. We can write $d' = n + p$, $d'' = n - p + 1$, $2p - 1 \leq \ell' \leq 2m - 1$, $\ell'' = 2m - \ell'$, $1 \leq p \leq m$.

We will use a decreasing induction in p. As a basis of induction we may choose $p = m + 1$, in which case all products are regular and there is nothing to prove. Let us carry out the step of induction. We assume that the claim of the Lemma holds for irregular products $s(d_1,\ell_1)s(d_2,\ell_2)$ with $d_1 > d'$.

Consider the generating function

$$\left[\prod_{i=1}^{2p-1}\left(z\frac{d}{dz}+p-n-i\right)\prod_{\substack{j=2p-1\\ j\neq \ell'}}^{2m-1}\left(r\frac{d}{dr}-j\right)e(r,z)\right]e(-r,z). \tag{2}$$

Since the total number of derivatives is $\ell-1$, (2) is in fact a polynomial in r of degree $\ell-1$. Hence the coefficient at $z^d r^\ell$ in (2) is equal to 0. This yields an identity

$$\sum_{\substack{d_1+d_2=d\\ \ell_1+\ell_2=\ell}}(-1)^{\ell_2}\prod_{i=1}^{2p-1}(d_1+p-n-i)\prod_{\substack{j=2p-1\\ j\neq \ell'}}^{2m-1}(\ell_1-j)\,s(d_1,\ell_1)s(d_2,\ell_2)=0. \tag{3}$$

The terms in (3) with $n-p+1\le d_1\le n+p-1$ vanish, thus the only terms that occur have $d_1\ge d'$ or $d_2>d'$. If we look at the terms in (3) with $d_1=d'$, all such irregular terms will vanish, except for $s(d',\ell')s(d'',\ell'')$. Thus we can use (3) to express $s(d',\ell')s(d'',\ell'')$ as a linear combination of regular products and those irregular products for which the claim of the Lemma holds by the induction assumption. All regular products in the expansion of $s(d',\ell')s(d'',\ell'')$ will have $d_1>d'$ or $d_1=d'$ and $\ell_1<\ell'$. This completes the proof of the Lemma. $\square$

Let us order the set of pairs (d,ℓ) as follows: $(d_1,\ell_1)\succ(d_2,\ell_2)$ if either $d_1>d_2$ or $d_1=d_2$ and $\ell_1<\ell_2$. Consider the set $\widehat{S}(d,\ell)$ of ordered products $s(d_1,\ell_1)s(d_2,\ell_2)\dots s(d_k,\ell_k)$ with $(d_1,\ell_1)\succeq(d_2,\ell_2)\succeq\dots\succeq(d_k,\ell_k)$, $\sum_{i=1}^k d_i=d$, $\sum_{i=1}^k \ell_i=\ell$, $\ell_i\ge 1$. Introduce a lexicographic order on $\widehat{S}(d,\ell)$:

$$s(d_1',\ell_1')s(d_2',\ell_2')\dots s(d_k',\ell_k')\succ s(d_1'',\ell_1'')s(d_2'',\ell_2'')\dots s(d_k'',\ell_k'')$$

if for some m, $(d_i',\ell_i')=(d_i'',\ell_i'')$ for $i=1,\dots,m-1$, and $(d_m',\ell_m')\succ(d_m'',\ell_m'')$.

Now we can prove Theorem 3. The set $\widehat{S}(d,\ell)$ clearly spans the space $\mathcal{F}(d,\ell)$ since $s(p,1)=x_p$ and $\mathcal{F}(d,\ell)$ is spanned by monomials. It follows from Lemma 4 that every product from $\widehat{S}(d,\ell)$ which is not in $S(d,\ell)$ may be expressed as a linear combination of the elements of $\widehat{S}(d,\ell)$ which are greater in the lexicographic order. By induction with respect to this ordering we conclude that

$$\mathcal{F}(d,\ell)=\mathrm{Span}\ \widehat{S}(d,\ell)=\mathrm{Span}\ S(d,\ell).$$

However, elements of $S(d,\ell)$ are parametrized by Young diagrams with d boxes and ℓ rows. Hence $|S(d,\ell)|=\dim\mathcal{F}(d,\ell)$ and $S(d,\ell)$ is a basis of $\mathcal{F}(d,\ell)$. This completes the proof of Theorem 3.

Let us compute the eigenvalues of the differential operator T.

Theorem 5. *Eigenvalues of the operator*

$$T = \frac{1}{2} \sum_{\substack{a+b=p+q \\ a,b,p,q \geq 1}} x_a x_b \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_q}$$

on $\mathcal{F}(d,\ell)$, $\ell \geq 1$, *are parametrized by sequences* $(d_1,\ell_1),(d_2,\ell_2),\dots,(d_k,\ell_k)$ *with*

$$d_i > d_{i+1} + \ell_i,\ i = 1,\dots,k-1, \quad d_k \geq \ell_k, \quad \sum_{i=1}^{k} d_i = d, \quad \sum_{i=1}^{k} \ell_i = \ell, \quad \ell_1,\dots,\ell_k \geq 1.$$

The corresponding eigenvalue is

$$\lambda = \frac{1}{2} \sum_{i=1}^{k} (\ell_i - 1)(2d_i - \ell_i).$$

Proof. We are going to show that the matrix of the operator T is upper-triangular in the basis $S(d,\ell)$ ordered by $\succ$. Then the spectrum of T is given by the diagonal of this matrix.

Consider the generating functions

$$X_i = r_i \sum_{j=1}^{\infty} x_j z_i^j$$

and

$$E = \exp\left(\sum_{i=1}^{\infty} X_i \right) = \exp\left(\sum_{i=1}^{\infty} r_i \sum_{j=1}^{\infty} x_j z_i^j \right).$$

The product $s(d_1,\ell_1)\dots s(d_k,\ell_k)$ is the coefficient at $z_1^{d_1}\dots z_k^{d_k} r_1^{\ell_1}\dots r_k^{\ell_k}$ in E. Let us apply operator T to the generating function E:

$$TE = \frac{1}{2} \sum_{n=2}^{\infty} \left(\sum_{a+b=n} x_a x_b \right) \sum_{p+q=n} \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_q} E$$

$$= \frac{1}{2} \sum_{n=2}^{\infty} \left(\sum_{a+b=n} x_a x_b \right) \sum_{p+q=n} \left(\sum_{i=1}^{\infty} r_i z_i^p \right) \left(\sum_{j=1}^{\infty} r_j z_j^q \right) E$$

$$= \frac{1}{2} \sum_{i=1}^{\infty} r_i^2 \sum_{n=2}^{\infty} \left(\sum_{a+b=n} x_a x_b \right) \sum_{p+q=n} z_i^{p+q} E$$

$$+ \sum_{i<j} r_i r_j \sum_{n=2}^{\infty} \left(\sum_{a+b=n} x_a x_b \right) \sum_{p+q=n} z_i^p z_j^q E$$

$$= \frac{1}{2} \sum_{i=1}^{\infty} r_i^2 \sum_{n=2}^{\infty} \left(\sum_{a+b=n} x_a x_b \right) (n-1) z_i^n E$$

$$+ \sum_{i<j} r_i r_j \sum_{n=2}^{\infty} \left(\sum_{a+b=n} x_a x_b \right) \left(1 - \frac{z_j}{z_i} \right)^{-1} \left(z_j z_i^{n-1} - z_j^n \right) E$$

$$= \frac{1}{2}\sum_{i=1}^{\infty}\left[\left(z_i\frac{d}{dz_i}-1\right)r_i^2\sum_{a,b\geq 1}x_ax_bz_i^{a+b}\right]E$$

$$-\sum_{i<j}\left(1-\frac{z_j}{z_i}\right)^{-1}\frac{r_i}{r_j}r_j^2\sum_{a,b\geq 1}\left(x_ax_bz_j^{a+b}\right)E+\sum_{i<j}\frac{z_j}{z_i}\left(1-\frac{z_j}{z_i}\right)^{-1}\frac{r_j}{r_i}r_i^2\sum_{a,b\geq 1}\left(x_ax_bz_i^{a+b}\right)E$$

$$= \frac{1}{2}\sum_{i=1}^{\infty}\left[\left(z_i\frac{d}{dz_i}-1\right)X_i^2\right]E-\sum_{i<j}\left(1-\frac{z_j}{z_i}\right)^{-1}\frac{r_i}{r_j}X_j^2E+\sum_{i<j}\frac{z_j}{z_i}\left(1-\frac{z_j}{z_i}\right)^{-1}\frac{r_j}{r_i}X_i^2E$$

$$= \sum_{i=1}^{\infty}\left[\left(z_i\frac{d}{dz_i}-\frac{1}{2}\right)X_i\right]X_iE-\sum_{i<j}\left(1-\frac{z_j}{z_i}\right)^{-1}\frac{r_i}{r_j}X_j^2E+\sum_{i<j}\frac{z_j}{z_i}\left(1-\frac{z_j}{z_i}\right)^{-1}\frac{r_j}{r_i}X_i^2E$$

$$= \sum_{i=1}^{\infty}\left[\left(z_i\frac{d}{dz_i}-\frac{1}{2}\right)X_i\right]r_i\frac{d}{dr_i}E$$

$$-\sum_{i<j}\left(1-\frac{z_j}{z_i}\right)^{-1}\frac{r_i}{r_j}X_jr_j\frac{d}{dr_j}E+\sum_{i<j}\frac{z_j}{z_i}\left(1-\frac{z_j}{z_i}\right)^{-1}\frac{r_j}{r_i}X_ir_i\frac{d}{dr_i}E$$

$$= \sum_{i=1}^{\infty}\left(r_i\frac{d}{dr_i}-1\right)\left[z_i\frac{d}{dz_i}X_i-\frac{1}{2}X_i\right]E$$

$$-\sum_{i<j}\left(1-\frac{z_j}{z_i}\right)^{-1}r_j\frac{d}{dr_j}\frac{r_i}{r_j}X_jE+\sum_{i<j}\frac{z_j}{z_i}\left(1-\frac{z_j}{z_i}\right)^{-1}r_i\frac{d}{dr_i}\frac{r_j}{r_i}X_iE$$

$$= \sum_{i=1}^{\infty}\left(r_i\frac{d}{dr_i}-1\right)\left(z_i\frac{d}{dz_i}-\frac{1}{2}r_i\frac{d}{dr_i}\right)E$$

$$-\sum_{i<j}\left(1-\frac{z_j}{z_i}\right)^{-1}r_j\frac{d}{dr_j}\frac{r_i}{r_j}r_j\frac{d}{dr_j}E+\sum_{i<j}\frac{z_j}{z_i}\left(1-\frac{z_j}{z_i}\right)^{-1}r_i\frac{d}{dr_i}\frac{r_j}{r_i}r_i\frac{d}{dr_i}E.$$

To get the formula for the action of T on the elements of $S(d,\ell)$, we extract the coefficient at $z_1^{d_1}\dots z_k^{d_k}r_1^{\ell_1}\dots r_k^{\ell_k}$ in TE:

$$Ts(d_1,\ell_1)s(d_2,\ell_2)\dots s(d_k,\ell_k)$$
$$= \sum_{i=1}^{k}(\ell_i-1)\left(d_i-\frac{\ell_i}{2}\right)s(d_1,\ell_1)s(d_2,\ell_2)\dots s(d_k,\ell_k)$$
$$-\sum_{i<j}\sum_{p=0}^{\infty}\ell_j(\ell_j+1)s(d_1,\ell_1)\dots s(d_i+p,\ell_i-1)\dots s(d_j-p,\ell_j+1)\dots s(d_k,\ell_k)$$
$$+\sum_{i<j}\sum_{p=1}^{\infty}\ell_i(\ell_i+1)s(d_1,\ell_1)\dots s(d_i+p,\ell_i+1)\dots s(d_j-p,\ell_j-1)\dots s(d_k,\ell_k).$$

The first part in the above expression yields the diagonal part of T with the eigenvalue $\lambda = \frac{1}{2}\sum_{i=1}^{k}(\ell_i - 1)(2d_i - \ell_i)$, while the last two sums when expanded in the basis $S(d,\ell)$ applying Lemma 4 whenever necessary, only contain terms that are strictly greater than $s(d_1,\ell_1)s(d_2,\ell_2)\dots s(d_k,\ell_k)$ with respect to the lexicographic order $\succ$. This completes the proof of the Theorem. $\square$

It follows from the proof of Theorem 5 that $s(d,\ell)$ is the eigenfunction for the operator T with the eigenvalue $\lambda = \frac{1}{2}(\ell - 1)(2d - \ell)$, which is the dominant eigenvalue on $\mathcal{F}(d,\ell)$.

Let us point out that 0 is an eigenvalue of T on $\mathcal{F}(d,\ell)$ if and only if $d \geq \ell^2$.

We can obtain the orthogonal basis of eigenfunctions for T in $\mathcal{F}(d,\ell)$ from the ordered basis $S(d,\ell)$ using the Gram-Schmidt procedure.

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