Abelian Extensions of the Group of Diffeomorphisms of a Torus.

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Abstract. In this Letter we construct abelian extensions of the group of diffeomorphisms of a torus. We consider the jacobian map, which is a crossed homomorphism from the group of diffeomorphisms into a toroidal gauge group. A pull-back under this map of a central 2-cocycle on a gauge group turns out to be an abelian cocycle on the group of diffeomorphisms. We show that in the case of a circle, the Virasoro-Bott cocycle is a pull-back of the Heisenberg cocycle. We also give an abelian generalization of the Virasoro-Bott cocycle to the case of a manifold with a volume form.

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0. Introduction.

The Lie group that corresponds to the Virasoro Lie algebra is the central extension of the group of diffeomorphisms of a circle with the Virasoro-Bott cocycle (2.1). It is well-known that the Virasoro cocycle on the Lie algebra of vector fields on a circle does not admit a generalization as a central 2-cocycle to the case of higher dimensional manifolds. Nonetheless there exist abelian extensions of the Lie algebra of vector fields on a torus that generalize the Virasoro Lie algebra. These abelian extensions play an important role in the representation theory of toroidal Lie algebras ([EM], [L], [BB], [Bi]).

The goal of this Letter is to construct the corresponding abelian extensions for the group of diffeomorphisms of a torus.

We approach this problem by linking the abelian cocycles on the group of diffeomorphisms of a torus with the central cocycles on the toroidal gauge groups. The central extensions of the toroidal gauge groups were studied in [LMNS] (see also [PS], [FK]).

The connection between the group of diffeomorphisms of a torus and toroidal gauge groups is given by the jacobian map. Let us briefly outline this correspondence. The differential of a diffeomorphism of a torus is a mapping of the tangent bundle into itself. However, using the fact that torus has a trivial tangent bundle, we can globally identify all tangent spaces with $\mathbb{R}^N$. After this identification, the differential of a diffeomorphism of a torus becomes a mapping of a torus into $GL_N(\mathbb{R})$. In the standard coordinates, this mapping is given by the jacobian matrix. In this way we obtain the jacobian map from the group of diffeomorphisms of a torus into a toroidal gauge group:

$$J : Diff(\mathbb{T}^N) \to Map(\mathbb{T}^N, GL_N(\mathbb{R})).$$

In all constructions of this Letter the torus may be replaced with an arbitrary manifold with a trivial tangent bundle.

The jacobian mapping is not a group homomorphism, but what is called a crossed homomorphism (see definition in Section 3). In case $N = 1$, we get a crossed homomorphism from the group of diffeomorphisms of a circle into abelian loop group $Map(S^1, \mathbb{R}^*)$. A central extension of this loop group is an infinite dimensional Heisenberg group. We point out that the Virasoro-Bott cocycle may be interpreted as a pull-back of the Heisenberg cocycle under the jacobian map (Theorem 2.1). This observation motivates our constructions of 2-cocycles on the group of diffeomorphisms in higher dimensions.

First we describe our construction of 2-cocycles by pull-back in a very general set-up. We show (Lemma 3.1) that under a crossed homomorphism of two groups

$$j : D \to M,$$

a pull-back of a D invariant central 2-cocycle on M is an abelian 2-cocycle on D.

We apply this lemma to construct an abelian extension of the group of diffeomorphisms of a manifold with a volume form (Theorem 4.1), generalizing the Virasoro-Bott group, for which the manifold is the circle. This extension (as well as the Virasoro-Bott extension) trivializes on the subgroup of volume preserving diffeomorphisms.

We also construct another extension of the group of diffeomorphisms of a torus (Theorem 3.2) that remains non-trivial when restricted to the subgroup of volume preserving diffeomorphisms.

We would like to mention here that Ovsienko and Roger described abelian extensions of the group of diffeomorphisms of a circle with the modules of tensor densities [OR].

The structure of the Letter is the following. In Section 1 we present a geometric approach to the abelian extensions of the Lie algebra of vector fields on a torus and introduce the jacobian map from the group of diffeomorphisms of a torus into a toroidal gauge group. In Section 2 we give two constructions of the infinite dimensional Heisenberg group and exhibit the link between the Virasoro-Bott cocycle and the Heisenberg cocycle. In Section 3 we describe the general construction of a pull-back of a cocycle under a crossed homomorphism and use this approach to get an abelian extension of the group of diffeomorphisms of a torus. Finally in Section 4 we obtain another, inequivalent abelian cocycle that generalizes the Virasoro-Bott cocycle to the case of a manifold with a volume form.

1. Abelian extensions of the Lie algebra of vector fields of a torus.

It is well-known that the Lie algebra W_N of vector fields on an N-dimensional torus $\mathbb{T}^N$ (Witt algebra) has a non-trivial central extension only when $N = 1$. In that case there exists a unique non-trivial central extension of W_1, which is the Virasoro Lie algebra. The Virasoro cocycle can not be generalized as a central 2-cocycle to higher dimensions. However when $N > 1$, there exist *abelian* extensions of W_N that generalize the Virasoro cocycle. These abelian extensions appeared in the representation theory [EM], [L], where it was discovered that it is easier to construct representations for certain abelian extensions of W_N rather than for the Witt algebra itself.

Let us describe the construction of these abelian extensions of W_N.

Let $\mathcal{F}$ be the algebra of $\mathcal{C}^\infty$ real-valued functions on a torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$. Note that the algebra of complex-valued functions on a torus, $\mathbb{C}\otimes\mathcal{F}$ has a convenient topological basis

$$\left\{ \exp\left(2\pi i \sum_{j=1}^{N} r_j x_j\right) \Big| r_j \in \mathbb{Z} \right\}.$$

Often this basis is used in algebraic setting (considering Fourier polynomials instead of Fourier series). However, the group of diffeomorphisms of a torus can not be realized using Fourier polynomials, so we will be working with $\mathcal{C}^\infty$ functions instead. We define the group $\mathit{Diff}(\mathbb{T}^N)$ as the group of $\mathcal{C}^\infty$ diffeomorphisms of a torus. An element $F \in \mathit{Diff}(\mathbb{T}^N)$ can be viewed as an N-tuple of functions $F = (F_1(\mathbf{x}), \ldots, F_N(\mathbf{x}))$, such that the map

$$\mathbf{x} = (x_1, \ldots, x_N) \mapsto (F_1(\mathbf{x}), \ldots, F_N(\mathbf{x}))$$

is a bijection $\mathbb{T}^N \to \mathbb{T}^N$.

The Lie algebra W_N consists of the vector fields $\mathbf{v} = \sum_{j=1}^{N} v_j(\mathbf{x}) \frac{\partial}{\partial x_j}$ with $v_j(\mathbf{x}) \in \mathcal{F}$.

Consider the space of differential k-forms on $\mathbb{T}^N$:

$$\Omega^k = \left\{ \sum_{1\leq j_1 < \ldots < j_k \leq N} a_{j_1\ldots j_k}(\mathbf{x})\, dx_{j_1} \wedge \ldots \wedge dx_{j_k} \,\Big|\, a_{j_1\ldots j_k}(\mathbf{x}) \in \mathcal{F} \right\}.$$

There is a differential map $d : \Omega^k \to \Omega^{k+1}$:

$$d\left(a(\mathbf{x}) dx_{j_1} \wedge \ldots \wedge dx_{j_k}\right) = \sum_{j=1}^{k} \frac{\partial a}{\partial x_j} dx_j \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_k}.$$

The Lie algebra of vector fields acts in an obvious way on $\mathcal{F}$: $\mathbf{v}\cdot a(\mathbf{x}) = \sum_{j=1}^{N} v_j(\mathbf{x}) \frac{\partial a}{\partial x_j}$.

The algebra of functions $\mathcal{F}$ is also a right module for the group $\mathit{Diff}(\mathbb{T}^N)$

$$a(\mathbf{x})(F) = a(F(\mathbf{x})), \quad a \in \mathcal{F},\ F \in \mathit{Diff}(\mathbb{T}^N),$$

and so is the spaces of differential forms:

$$\left(a(\mathbf{x}) dx_{j_1} \wedge \ldots \wedge dx_{j_k}\right)(F) = a(F(\mathbf{x})) dF_{j_1}(\mathbf{x}) \wedge \ldots \wedge dF_{j_k}(\mathbf{x}). \tag{1.1}$$

It is easy to see that the differential map is a homomorphism of $\mathit{Diff}(\mathbb{T}^N)$ modules.

The action of the group $\mathit{Diff}(\mathbb{T}^N)$ on k-forms gives rise to the action of its Lie algebra W_N:

$$\mathbf{v}\cdot\left(a(\mathbf{x}) dx_{j_1} \wedge \ldots \wedge dx_{j_k}\right) =$$

$$(\mathbf{v}\cdot a(\mathbf{x})) dx_{j_1} \wedge \ldots \wedge dx_{j_k} + \sum_{r=1}^{k} a(\mathbf{x}) dx_{j_1}(\mathbf{x}) \wedge \ldots \wedge dv_{j_r}(\mathbf{x}) \wedge \ldots \wedge dx_{j_k}.$$

This action is called the Lie derivative action.

Next we need to discuss the jacobian of a diffeomorphism of a torus, and the jacobian of a vector field.

Let $F \in \mathit{Diff}(\mathbb{T}^N)$, $F : \mathbb{T}^N \to \mathbb{T}^N$. The differential of F is a map of tangent bundles:

$$dF : T(\mathbb{T}^N) \to T(\mathbb{T}^N),$$

with linear maps

$$T_{\mathbf{x}}(\mathbb{T}^N) \to T_{F(\mathbf{x})}(\mathbb{T}^N).$$

However torus is a flat manifold and its tangent bundle is trivial: $T(\mathbb{T}^N) = \mathbb{T}^N \times \mathbb{R}^N$, so all tangent spaces can be globally identified with $\mathbb{R}^N$, and we define the jacobian of F as a map

$$F^J : \mathbb{T}^N \to Hom(\mathbb{R}^N, \mathbb{R}^N).$$

Since F is an invertible transformation of the torus, F^J is an invertible linear homomorphism, so

$$F^J : \mathbb{T}^N \to GL_N(\mathbb{R}).$$

In coordinates $(x_1, \dots x_N)$, the matrix of the jacobian of a diffeomorphism F is

$$F^J = \left(\frac{\partial F_i}{\partial x_k} \right)_{i,j=1,\dots,N}.$$

The jacobian map $J : F \mapsto F^J$ associates with every element of $\mathit{Diff}(\mathbb{T}^N)$ an element of a gauge group $\mathit{Map}(\mathbb{T}^N, GL_N)$. However the resulting map

$$J : \mathit{Diff}(\mathbb{T}^N) \to \mathit{Map}(\mathbb{T}^N, GL_N) \tag{1.2}$$

is not a group homomorphism, but what is called a *crossed homomorphism* [K]:

$$(FG)^J = F^J(G) \cdot G^J \quad \text{(chain rule).} \tag{1.3}$$

In the above formula we used the action of $\mathit{Diff}(\mathbb{T}^N)$ on the toroidal gauge group $\mathit{Map}(\mathbb{T}^N, GL_N)$ by automorphisms:

$$m(\mathbf{x}) \mapsto m(F(\mathbf{x})), \text{ where } m \in \mathit{Map}(\mathbb{T}^N, GL_N), \quad F \in \mathit{Diff}(\mathbb{T}^N), \quad \mathbf{x} \in \mathbb{T}^N. \tag{1.4}$$

In a similar way, the jacobian of a vector field $\mathbf{v}$ is defined as a matrix

$$\mathbf{v}^J = \left(\frac{\partial v_i}{\partial x_k} \right)_{i,j=1,\dots,N},$$

thus giving a linear map from W_N to the toroidal analog $\mathit{Map}(\mathbb{T}^N, gl_N) \cong gl_N \otimes \mathcal{F}$ of the loop algebra.

Now we can use the jacobians to give a simple descriptions of the abelian extensions of W_N. We will actually consider two possible extension spaces. In the first case, the

2-cocycles will have values in the factor module $\mathcal{K} = \Omega^1/d\Omega^0$. This module is of special interest since it is the space of the universal central extension for the toroidal Lie algebras. Two such $\mathcal{K}$ valued 2-cocycles have appeared in the representation theory ([EM], [L], [BB]):

$$\tau_1(\mathbf{v},\mathbf{w}) = Tr\left(\mathbf{v}^J d\mathbf{w}^J\right)$$

and

$$\tau_2(\mathbf{v},\mathbf{w}) = Tr\left(\mathbf{v}^J\right) Tr\left(d\mathbf{w}^J\right).$$

Viewed as Ω^1 valued expressions these are not 2-cocycles, but modulo $d\Omega^0$ these become skew-symmetric and satisfy the Jacobi identity.

There is a closely related pair of 2-cocycles on W_N, obtained from τ_1, τ_2 by applying the differential map $d : \Omega^1 \to \Omega^2$. Since $d^2 = 0$, this map factors through the factor module $\Omega^1/d\Omega^0$. The differentials $d\tau_1, d\tau_2$ are thus Ω^2 valued 2-cocycles on W_N:

$$d\tau_1(\mathbf{v},\mathbf{w}) = Tr\left(d\mathbf{v}^J \wedge d\mathbf{w}^J\right)$$

and

$$d\tau_2(\mathbf{v},\mathbf{w}) = Tr\left(d\mathbf{v}^J\right) \wedge Tr\left(d\mathbf{w}^J\right).$$

Dzhumadildaev proved that for $N > 1$ the space of Ω^2 valued 2-cocycles $H^2(W_N, \Omega^2)$ is two-dimensional and is spanned by $d\tau_1, d\tau_2$ [D].

These cocycles give 2-parametric abelian extensions of the Lie algebra of vector fields on a torus with extension spaces $\mathcal{K}$ or Ω^2 (see e.g. Section 2 of [BB]).

The main goal of this Letter is to construct the corresponding abelian extensions for the group of diffeomorphisms of a torus.

Remark 1. We can see that linear combinations of τ_1, τ_2 are in 1-1 correspondence with the symmetric invariant bilinear forms on $gl_N(\mathbb{R})$. Since for $N > 1$ the Lie algebra $gl_N(\mathbb{R})$ is not simple, but only reductive, $gl_N(\mathbb{R}) = sl_N(\mathbb{R}) \oplus \mathbb{R}I$, the space of symmetric invariant forms on $gl_N(\mathbb{R})$ is two-dimensional and is spanned by the forms $Tr(AB)$ and $Tr(A)\,Tr(B)$. Thus a linear combination of τ_1 and τ_2 can be written as

$$\tau(\mathbf{v},\mathbf{w}) = (\mathbf{v}^J | d\mathbf{w}^J),$$

where $(\cdot|\cdot)$ is a symmetric invariant bilinear form of $gl_N(\mathbb{R})$.

Remark 2. Cocylcles τ_2 and $d\tau_2$ trivialize on the subalgebra of divergence zero vector fields, because the divergence of a vector field is equal to the trace of the jacobian: $\mathrm{div}(\mathbf{v}) = Tr(\mathbf{v}^J)$.

Remark 3. In coordinates, cocycles τ_1, τ_2 are written as follows (cf. [BB]):

$$\tau_1\left(v(x)\frac{\partial}{\partial x_i}, w(x)\frac{\partial}{\partial x_j}\right) = \frac{\partial v}{\partial x_j} \sum_{k=1}^{N} \frac{\partial^2 w}{\partial x_i \partial x_k} dx_k,$$

$$\tau_2\left(v(x)\frac{\partial}{\partial x_i}, w(x)\frac{\partial}{\partial x_j}\right) = \frac{\partial v}{\partial x_i}\sum_{k=1}^{N}\frac{\partial^2 w}{\partial x_j \partial x_k}dx_k.$$

Remark 4. When $N = 1$, cocycles τ_1 and τ_2 coincide (the forms $Tr(AB)$ and $Tr(A)\,Tr(B)$ are equal in gl_1), and become the Virasoro cocycle on the Witt algebra W_1. Thus for $N > 1$, these cocycles are two distinct abelian generalizations of the Virasoro cocycle.

2. Realization of the Virasoro-Bott cocycle via the Heisenberg group.

The Virasoro-Bott cocycle on the group of diffeomorphisms of S^1 has a rather mysterious form [Bo]:

$$(F,\alpha)(G,\beta) = (FG, \alpha+\beta+B(F,G)),$$

where $F, G \in Diff(S^1)$, $\alpha, \beta \in \mathbb{R}$ and

$$B(F,G) = \int_0^1 \ln|F'(G(t))|\, d\ln|G'(t)|. \tag{2.1}$$

The cocycle $B(F,G)$ seems to have more analytic flavour rather than algebraic, and looks quite different from other cocycles on the infinite dimensional groups, e.g., affine groups. We are now going to elucidate the situation with the unusual form of this cocycle. We will link the Virasoro-Bott cocycle with the Heisenberg cocycle, which is defined in more algebraic terms. The infinite dimensional Heisenberg group is the affine group which corresponds to GL_1, thus the Heisenberg cocycle may be viewed as the simplest case of the affine group cocycle.

The link between $Diff(S^1)$ and the loop group of GL_1 has been seen in fact in (1.2) in the previous section, where the jacobian J maps the former group into the latter. We will recover the Virasoro-Bott cocycle by evaluating the Heisenberg cocycle on the jacobians of the diffeomorphisms.

Let us recall the construction of the infinite dimensional Heisenberg group. We will in fact give two versions of the Heisenberg group, one algebraic, and another analytic.

In our first construction, we will directly exponentiate the infinite dimensional Heisenberg Lie algebra. This Lie algebra has the basis $\{H_j, c | j \in \mathbb{Z}\}$ and the Lie bracket

$$[H_n, H_m] = n\delta_{n,-m}c,$$

and c is central.

By the direct exponentiation we will get the Heisenberg group:

$$\left\{ \exp(\alpha c)\exp(\sum_{j\le 0} a_j H_j)\exp(\sum_{j>0} a_j H_j) \right\}, \tag{2.2}$$

where $a_k \in \mathbb{R}$ with only finitely many non-zero a_k. Using the formula

$$\exp(A)\exp(B) = \exp([A,B])\exp(B)\exp(A),$$

which holds when $[A,B]$ commutes with A and B, we get the multiplication law in the Heisenberg group:

$$\exp(\alpha c)\exp(\sum_{j\leq 0} a_j H_j)\exp(\sum_{j>0} a_j H_j) \times \exp(\beta c)\exp(\sum_{j\leq 0} b_j H_j)\exp(\sum_{j>0} b_j H_j)$$

$$= \exp((\alpha+\beta+\sum_{j>0} j a_j b_{-j})c)\exp(\sum_{j\leq 0}(\alpha_j+\beta_j)H_j)\exp(\sum_{j>0}(\alpha_j+\beta_j)H_j).$$

This version is an algebraic version of the Heisenberg group since only finite products are allowed in (2.2). The group of diffeomorphisms on the other hand is of analytic nature, so we would need to consider a certain completion of the algebraic version of the Heisenberg group. The construction of the analytic version of the Heisenberg group will follow the general pattern of the affine extensions of the loop groups [PS].

Consider the group $Map(S^1, \mathbb{R}^*)$ of $\mathcal{C}^\infty$ loops in $\mathbb{R}^* = GL_1(\mathbb{R})$, and its connected component of identity $Map_0(S^1, \mathbb{R}^*) = Map(S^1, \mathbb{R}^*_+)$. The group structure in a loop group is given by pointwise multiplication. In order to define a central extension of $Map(S^1, \mathbb{R}^*_+)$, we need to consider contractions of elements $f \in Map(S^1, \mathbb{R}^*_+)$ to identity. The general scheme of affine extensions simplifies in our case since there exists a canonical contraction for each f. This is due to the fact that the exponential map from $\mathbb{R}$ (the Lie algebra of $\mathbb{R}^*_+$) to $\mathbb{R}^*_+$ is bijective.

Indeed let $x = \ln(f)$, $x \in Map(S^1, \mathbb{R})$. Then $\tilde{f}(\tau) = \exp(\tau x), \tau \in [0,1]$ is a homotopy between identity and f. This canonical contraction respects multiplication: $\widetilde{fg} = \widetilde{f}\widetilde{g}$ because $\mathbb{R}^*$ is commutative, and the usual properties of the logarithm and the exponential functions hold. In the general scheme of affine extensions one has to consider all possible contractions of elements of a loop group to identity. In our case this is not necessary because of the existence of a canonical contraction. The affine cocycle is given by the formula [PS]:

$$C(f,g) = \int_{S^1}\int_{[0,1]} \widetilde{f}^{-1} d\widetilde{f} \wedge d\widetilde{g}\, \widetilde{g}^{-1}.$$

Here $\widetilde{f}, \widetilde{g}$ are viewed as functions of two variables: t – the parameter along S^1 and τ – the parameter of the homotopy. Let $x(t) = \ln(f(t))$, $y(t) = \ln(g(t))$. Then we can calculate $C(f,g)$ explicitly in the following way:

$$C(f,g) = \int_{S^1}\int_{[0,1]} \frac{1}{\widetilde{f}\widetilde{g}} \left(\frac{\partial \widetilde{f}}{\partial \tau}\frac{\partial \widetilde{g}}{\partial t} - \frac{\partial \widetilde{f}}{\partial t}\frac{\partial \widetilde{g}}{\partial \tau} \right) d\tau \wedge dt$$

$$= \int_{S^1}\int_{[0,1]} \tau \left(x(t)\frac{dy}{dt} - \frac{dx}{dt}y(t) \right) d\tau \wedge dt.$$

Integrating in τ, and then integrating the second term by parts with respect to t we get

$$C(f,g) = \int_{S^1} x(t)dy(t).$$

We see that the formula for the Heisenberg cocycle becomes very similar to the formula for the Virasoro-Bott cocycle:

$$C(f,g) = \int_{S^1} \ln(f) d\ln(g). \tag{2.3}$$

Now $x(t)$ and $y(t)$ are periodic $\mathcal{C}^\infty$ functions. These can be represented by Fourier series:

$$x(t) = \frac{a_0}{2} + \sum_{j>0} a_j \cos(2\pi jt) + \sum_{j>0} a_{-j} \sin(2\pi jt),$$

$$y(t) = \frac{b_0}{2} + \sum_{j>0} b_j \cos(2\pi jt) + \sum_{j>0} b_{-j} \sin(2\pi jt).$$

The functions $x(t)$, $y(t)$ are of class $\mathcal{C}^\infty$ if and only if the Fourier coefficients satisfy the condition

$$\lim_{j\to\pm\infty} j^k a_j = \lim_{j\to\pm\infty} j^k b_j = 0 \text{ for all } \ k \in \mathbb{N}. \tag{2.4}$$

Now by direct integration of (2.3) we can get the formula for the cocycle in terms of the Fourier coefficients:

$$C(f,g) = \pi \sum_{j\in\mathbb{Z}} j a_j b_{-j}.$$

The analytic version of the Heisenberg group is the set of pairs (f,α), where $f \in Map(S^1, \mathbb{R}^*_+)$, $\alpha \in \mathbb{R}$, and the group law is

$$(f,\alpha)(g,\beta) = (fg, \alpha + \beta + C(f,g)).$$

It is easy to see that this cocycle is equivalent to the cocycle from the first version of the Heisenberg group. Indeed, the isomorphism between the latter analytic version and the completion of the algebraic version can be given as follows:

$$\varphi\left(\exp\left(\frac{a_0}{2} + \sum_{j>0} a_j \cos(2\pi jt) + \sum_{j>0} a_{-j} \sin(2\pi jt)\right), \alpha\right) =$$

$$\exp((\frac{\alpha}{2\pi} + \frac{1}{2}\sum_{j>0} j a_j a_{-j})c) \exp(\sum_{j\leq 0} a_j H_j) \exp(\sum_{j>0} a_j H_j).$$

The condition on coefficients $\{a_j\}$ for the completion of the first version of the Heisenberg group should be taken the same as the $\mathcal{C}^\infty$ condition (2.4).

Remark 5. We can lift the cocycle (2.3) to the group $Map(S^1, \mathbb{R}^*) = Map(S^1, \mathbb{R}^*_+) \times \mathbb{Z}_2$ by taking absolute values of the loops:

$$C(f,g) = \int_{S^1} \ln|f| d \ln|g|. \tag{2.5}$$

Now we can give interpretation of the Virasoro-Bott cocycle by means of the cocycle of the Heisenberg group.

Theorem 2.1. Let J be the jacobian map $J : Diff(S^1) \to Map(S^1, \mathbb{R}^*)$. The Virasoro-Bott cocycle on $Diff(S^1)$ can be obtained by evaluating the Heisenberg cocycle (2.5) on $F^J(G)$ and G^J:

$$B(F,G) = C(F^J(G), G^J).$$

Proof. This follows immediately from comparing the formulas (2.1) and (2.5).

Remark 6. Using the jacobian map we also get a similar relation between the Virasoro and Heisenberg cocycles at the level of Lie algebras. The Heisenberg cocycle on the abelian Lie algebra of functions on a circle is

$$h(u_1(x), u_2(x)) = \int_{S^1} u_1(x) du_2(x).$$

The Virasoro cocycle on W_1 is obtained by evaluating the Heisenberg cocycle on the jacobians (i.e. first derivatives) of vector fields:

$$\left(v(x)\frac{\partial}{\partial x}, w(x)\frac{\partial}{\partial x} \right) = h(\mathbf{v}^J, \mathbf{w}^J) = \int_{S^1} v'(x) dw'(x).$$

Remark 7. Integration along S^1 in the formulas (2.3) and (2.5) realizes the isomorphism between $\Omega^1/d\Omega^0$ and $\mathbb{R}$. We thus can also write the Heisenberg cocycle as a $\Omega^1/d\Omega^0$ valued cocycle

$$C(f,g) = \ln|f| d\ln|g|. \tag{2.6}$$

Proposition 2.2. Let X be a $\mathcal{C}^\infty$ real manifold. Then the formula (2.6) gives an $\Omega^1(X)/d\Omega^0(X)$ valued central 2-cocycle on the group $Map(X, \mathbb{R}^*)$.

Proof. We first verify the claim for the subgroup $Map(X, \mathbb{R}^*_+)$. Applying the logarithm function, we get that the multiplicative group $Map(X, \mathbb{R}^*_+)$ is isomorphic to the additive group of $\mathcal{C}^\infty$ functions on X. Under this isomorphism (2.5) transforms into

$$C_1(u,v) = u(x) dv(x), \tag{2.7}$$

where $u = \ln(f), v = \ln(g)$. To see that (2.7) satisfies the cocycle condition, we need to check that

$$C_1(u,v) + C_1(u+v,w) = C_1(v,w) + C_1(u, v+w),$$

which follows immediately from the linearity of (2.7).

Finally, the group $Map(X, \mathbb{R}^*)$ is a direct product $Map(X, \mathbb{R}^*_+) \times \mathbb{Z}_2^n$, where n is the number of connected components of X, thus a central cocycle on the subgroup $Map(X, \mathbb{R}^*_+)$ lifts to a cocycle on $Map(X, \mathbb{R}^*)$.

Even though the above theorems are just very simple observations, they allow us to construct interesting generalizations of the Virasoro-Bott cocycle that will be explored in the next two sections.

3. Abelian extension of the group of diffeomorphisms of a torus.

The Virasoro cocycle can not be generalized as a central cocycle to the case of a Lie algebra of vector fields in dimension greater than one. However in the case of a torus, it is known that there are abelian generalizations of the Virasoro cocycle. We thus may expect that we can also construct abelian extensions of the group of diffeomorphisms of a torus.

In this section we will construct an abelian extension that corresponds to Lie algebra cocycle $d\tau_1$. The space of the extension will be $\Omega^2(\mathbb{T}^N)$. This cocycle will be non-vanishing on the subgroup of volume-preserving diffeomorphisms.

As we have seen in Section 1, the two abelian cocycles on W_N, τ_1 and τ_2, correspond to the decomposition of gl_N into $sl_N \oplus \mathbb{R}I$. The cocycle that will be constructed in this section corresponds to the sl_N component. In Section 4 we will construct a cocycle that corresponds to the component of scalar matrices.

Motivated by Theorem 2.1, our idea is to use the crossed homomorphism (1.2) and pull back a central 2-cocycle on $Map(\mathbb{T}^N, GL_N)$ to an abelian 2-cocycle on $Diff(\mathbb{T}^N)$.

We will first describe our construction in a very general set-up.

Let a group D acts on a group M on the right by automorphisms:

$$m \mapsto m(d), \text{ for } m \in M, d \in D.$$

Definition. A crossed homomorphism is a map

$$j : D \to M,$$

such that

$$j(d_1 d_2) = j(d_1)(d_2) \cdot j(d_2) \tag{3.1}$$

for all $d_1, d_2 \in D$.

Suppose we indeed have a crossed homomorphism $j : D \to M$.

Let K be an abelian (additive) group and let

$$c : M \times M \to K$$

be a central 2-cocycle on M. Suppose that D also acts on K on the right by automorphisms.

Lemma 3.1. If the K valued 2-cocycle c on M is D invariant, i.e.,

$$c(m_1, m_2)(d) = c(m_1(d), m_2(d)), \text{ for all } m_1, m_2 \in M, d \in D,$$

then

$$b(d_1, d_2) = c(j(d_1)(d_2), j(d_2)) \tag{3.2}$$

defines an abelian K valued 2-cocycle on D.

Proof. Using the fact that c satisfies the central cocycle condition

$$c(m_1, m_2) + c(m_1 m_2, m_3) = c(m_1, m_2 m_3) + c(m_2, m_3), \tag{3.3}$$

we need to show that b is an abelian cocycle:

$$b(d_1, d_2)(d_3) + b(d_1 d_2, d_3) = b(d_1, d_2 d_3) + b(d_2, d_3). \tag{3.4}$$

From the definition (3.2), we get that the left hand side is

$$c(j(d_1)(d_2), j(d_2))(d_3) + c(j(d_1 d_2)(d_3), j(d_3)),$$

which using the invariance of cocycle c and (3.1), becomes

$$c(j(d_1)(d_2 d_3), j(d_2)(d_3)) + c(j(d_1)(d_2 d_3) j(d_2)(d_3), j(d_3)).$$

By (3.3) this equals

$$c(j(d_1)(d_2 d_3), j(d_2)(d_3) j(d_3)) + c(j(d_2)(d_3), j(d_3))$$

$$= c(j(d_1)(d_2 d_3), j(d_2 d_3)) + c(j(d_2)(d_3), j(d_3))$$
$$= b(d_1, d_2 d_3) + b(d_2, d_3),$$

which is the left hand side of (3.4). The Lemma is proved.

We will apply this Lemma to the groups $D = \mathit{Diff}(\mathbb{T}^N)$ and $M = \mathit{Map}(\mathbb{T}^N, GL_N)$. The extension group K will be $\Omega^2(\mathbb{T}^N)$ and the crossed homomorphism between $\mathit{Diff}(\mathbb{T}^N)$ and $\mathit{Map}(\mathbb{T}^N, GL_N)$ is the jacobian map (1.2). The action of $\mathit{Diff}(\mathbb{T}^N)$ on $\mathit{Map}(\mathbb{T}^N, GL_N)$ and Ω^2 is given by (1.4) and (1.1).

The central 2-cocycle on $\mathit{Map}(\mathbb{T}^N, GL_N)$ that we will use is (cf. [PS], [LNMS]):

$$C(f(\mathbf{x}), g(\mathbf{x})) = \mathit{Tr}\left(f^{-1} df \wedge dg\, g^{-1}\right). \tag{3.5}$$

Let us verify the cocycle condition

$$C(f, g) + C(fg, h) = C(f, gh) + C(g, h). \tag{3.6}$$

The left hand side here is

$$\mathit{Tr}\left(f^{-1} df \wedge dg\, g^{-1}\right) + \mathit{Tr}\left((fg)^{-1} d(fg) \wedge dh\, h^{-1}\right)$$

$$= \mathit{Tr}\left(f^{-1} df \wedge dg\, g^{-1}\right) + \mathit{Tr}\left(g^{-1} f^{-1} d(f) g \wedge dh\, h^{-1}\right) + \mathit{Tr}\left(g^{-1} dg \wedge dh\, h^{-1}\right).$$

Similarly, the left hand side of (3.6) becomes

$$Tr\left(f^{-1}df \wedge d(gh)(gh)^{-1}\right) + Tr\left(g^{-1}dg \wedge dh\, h^{-1}\right)$$

$$= Tr\left(f^{-1}df \wedge dg\, g^{-1}\right) + Tr\left(f^{-1}d(f) \wedge gdh\, h^{-1}g^{-1}\right) + Tr\left(g^{-1}dg \wedge dh\, h^{-1}\right).$$

Both sides are equal because

$$Tr\left(g^{-1}f^{-1}d(f)g \wedge dh\, h^{-1}\right) = Tr\left(f^{-1}d(f) \wedge g\, dh\, h^{-1}g^{-1}\right)$$

by the commutativity of the trace: $Tr(AB) = Tr(BA)$.

Finally we note that the cocycle (3.5) on $Map(\mathbb{T}^N, GL_N)$ is $Diff(\mathbb{T}^N)$ invariant, since the action of diffeomorphisms on differential forms (1.1) agrees with the action on functions (1.4):

$$Tr\left(f^{-1}df \wedge dgg^{-1}\right)(H) = Tr\left(f^{-1}(H)df(H) \wedge dg(H)g^{-1}(H)\right).$$

We thus obtain the following result:

Theorem 3.2. The set $Diff(\mathbb{T}^N) \times \Omega^2(\mathbb{T}^N)$ becomes an extension of the group of diffeomorphisms with the abelian group $\Omega^2(\mathbb{T}^N)$ with multiplication defined as follows:

$$(F,\alpha)(G,\beta) = \left(FG, \alpha(G) + \beta + Tr\left(f^{-1}df \wedge dgg^{-1}\right)\right),$$

where $f = F^J(G), g = G^J$.

Remark 8. It is easy to compute that the corresponding Lie algebra extension of W_N is given by cocycle $d\tau_1$.

Problem. Construct an abelian extension of $Diff(\mathbb{T}^N)$ with abelian subgroup $\Omega^1/d\Omega^0$ which corresponds to the Lie algebra 2-cocycle τ_1.

4. Abelian extension of the group of diffeomorphisms of a manifold with a volume form.

In this section we introduce an abelian extension of the group of diffeomorphisms of a manifold with a volume form that generalizes the Virasoro-Bott central extension.

Let X be a real $\mathcal{C}^\infty$ manifold with the volume form ω. We note that for an arbitrary diffeomorphism $F \in Diff(X)$, the quotient $\frac{\omega(F)}{\omega}$ is a well-defined non-vanishing function. Moreover, the map

$$\delta : Diff(X) \to Map(X, \mathbb{R}^*),$$

given by

$$\delta(F) = \frac{\omega(F)}{\omega}$$

is a crossed homomorphism. Indeed,

$$\delta(FG) = \frac{\omega(FG)}{\omega} = \frac{\omega(FG)}{\omega(G)}\frac{\omega(G)}{\omega} = \delta(F)(G)\delta(G).$$

As an immediate consequence of this observation, Lemma 3.1 and Proposition 2.2, we get the following:

Theorem 4.1. Let X be a $\mathcal{C}^\infty$ real manifold with a volume form ω and let $C(f,g)$ be the Heisenberg cocycle (2.6) on $Map(X, \mathbb{R}^*)$. Then

$$B(F,G) = C\left(\frac{\omega(FG)}{\omega(G)}, \frac{\omega(G)}{\omega}\right)$$

$$= \ln\left|\frac{\omega(FG)}{\omega(G)}\right| d\ln\left|\frac{\omega(G)}{\omega}\right| \tag{4.1}$$

is an abelian $\Omega^1(X)/d\Omega^0(X)$ valued 2-cocycle on the group of diffeomorphisms $Diff(X)$.

Corollary 4.2. The 2-cocycle on the Lie algebra of vector fields on a manifold with a volume form ω that corresponds to the group cocycle (4.1) is

$$c(\mathbf{v}, \mathbf{w}) = div(\mathbf{v})\, d\, div(\mathbf{w}),$$

where the divergence of a vector field $\mathbf{v}$ is given by $div(\mathbf{v}) = \frac{\mathbf{v}\cdot\omega}{\omega}$.

Example 4.3. If X is a torus $\mathbb{T}^N$ with the volume form $\omega = dx_1 \wedge \ldots \wedge dx_N$, then $\frac{\omega(F)}{\omega} = det(F^J)$, and the cocycle (3.1) can be written as

$$B(F,G) = \ln|det(F^J(G))|d\ln|det(G^J)|.$$

One can easily calculate that the corresponding Lie algebra cocycle on W_N is τ_2.

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