Conjugacy theorem for the strictly hyperbolic Kac-Moody algebras.

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ABSTRACT. In this article we prove the conjugacy theorem for the (non-symmetrizable) Kac-Moody algebras corresponding to strictly hyperbolic generalized Cartan matricies, i.e., without submatricies of finite type.

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Introduction.

In [PK] Peterson and Kac proved the conjugacy theorem for the Cartan subalgebras of a symmetrizable Kac-Moody algebra. This theorem allows in particular to describe the group of the automorphisms of a Kac-Moody algebra.

The Peterson-Kac proof involves an interesting analysis of the action of a Kac-Moody group on the Kostant cone. To any element of the Kostant cone one associates a polyhedron which is a convex closure of the support of this element in the weight lattice. Peterson and Kac prove that

(i) every vertex of this polyhedron belongs to the W-orbit of the highest weight

(ii) every edge is parallel to a real root.

This allows us to apply the procedure of "stripping off edges" which is the crucial part of the proof.

The detailed version of the Peterson-Kac proof is presented in the book [MP].

In order to deal with the sophisticated construction of the Kostant cone one needs to use strong technical machinery, so the restriction to the symmetrizable Kac-Moody algebras seems to be unavoidable.

In this article we prove the conjugacy theorem for the strictly hyperbolic (non - symmetrizable) Kac-Moody algebras, i.e., corresponding to the generalized Cartan matricies with no rank 2 submatricies of finite type.

Our idea is to work directly with the convex closure of the support of an ad-locally finite element in the root space. We prove that

(i') every vertex of this polyhedron is a real root

(ii') every edge is parallel to some (not necessarily real) root.

This allows us to apply the same procedure of "stripping off the edges". Unfortunately, in the finite root system, edges may be not parallel to any root, so this proof doesn't work in the finite type case.

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Definitions and notations.

Let A be a generalized Cartan matrix, i.e., $a_{ij} \in \mathbb{Z}$; $a_{ii} = 2$; $a_{ij} \leq 0$ if $i \neq j$; $a_{ij} = 0 \iff a_{ji} = 0$; $i, j = 1, \ldots l$.

Fix a base field, $\mathbb{F}$, of characteristic zero. A realization of A is a triple $< \mathfrak{h}_0, \Pi, \Pi^\vee >$, where $\mathfrak{h}_0$ is a vector space over $\mathbb{F}$, dim $\mathfrak{h}_0 = l$+corank(A), $\Pi = \{\alpha_1, \ldots, \alpha_l\} \subset \mathfrak{h}_0^*$ and $\Pi^\vee = \{h_1, \ldots, h_l\} \subset \mathfrak{h}_0$, satisfying $\alpha_j(h_i) = a_{ij}$.

Introduce first the Lie algebra $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(A)$ over $\mathbb{F}$ with the generators e_i, f_i and $\mathfrak{h}_0$, $i = 1, \ldots, l$ and the following defining relations:

$$[h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i,$$

$$[e_i, f_j] = \delta_{ij}h_i, \quad [\mathfrak{h}_0, \mathfrak{h}_0] = 0.$$

There exists the unique maximal ideal (radical) I in $\tilde{\mathfrak{g}}$ that intersects with $\mathfrak{h}_0$ trivially. The factor-algebra $\mathfrak{g}(A) = \tilde{\mathfrak{g}}/I$ is called the Kac-Moody algebra.

The following relations hold in $\mathfrak{g}(A)$:

$$(\mathrm{ad} e_i)^{-a_{ij}+1}e_j = (\mathrm{ad} f_i)^{-a_{ij}+1}f_j = 0, \quad i \neq j.$$

Denote by $\mathfrak{g}_+$ (resp. $\mathfrak{g}_-$) the subalgebra in $\mathfrak{g}$, generated by $e_1, \ldots, e_l$ (resp. $f_1, \ldots, f_l$). Then we have a triangular decomposition for $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h}_0 \oplus \mathfrak{g}_+.$$

The Borel subalgebra $\mathfrak{b}_\pm$ is defined as $\mathfrak{h}_0 \oplus \mathfrak{g}_\pm$.

We also have a root decomposition for $\mathfrak{g}$:

$$\mathfrak{g} = \underset{\alpha\in\mathfrak{h}_0^*}{\oplus} \mathfrak{g}_\alpha, \quad \text{where}$$

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | \text{ for all } h \in \mathfrak{h}_0 : [h,x] = \alpha(h)x\}.$$

The root system $\Delta = \Delta(A)$ is the set of all $\alpha \in \mathfrak{h}_0^*$ such that $\mathfrak{g}_\alpha \neq 0$. Obviously, Δ is a subset of the lattice $\mathbb{Z}^l$ generated by the fundamental roots $\{\pm\alpha_i\}$, $i = 1, \ldots, l$. It follows from the triangular decomposition for $\mathfrak{g}$ that $\Delta = \Delta_- \cup \{0\} \cup \Delta_+$, where $\Delta_+ = \Delta \cap \mathbb{N}^l$ and $\Delta_- = -\Delta_+$.

Let (V, ρ) be an infinite-dimensional $\mathfrak{g}$ -module. An element $x \in \mathfrak{g}$ is called ρ-locally finite if for any $v \in V$ the linear span of $\{\rho(x)^n v\}, n \in \mathbb{N}$, is finite-dimensional.

An element $x \in \mathfrak{g}$ is called ρ-locally nilpotent if for any $v \in V$ there exists $n \in \mathbb{N}$ such that $\rho(x)^n v = 0$.

We shall call $x \in \mathfrak{g}$ locally finite (resp. locally nilpotent) if it is ad-locally finite (resp. ad-locally nilpotent).

Let W denote the subgroup of $\mathrm{Aut}_{\mathbb{F}}(\mathfrak{h}_0^*)$ generated by the "simple reflections" $\beta \mapsto \beta - \langle\beta, \alpha^\vee\rangle\alpha$ for $\alpha \in \Pi$. The set $\{\, w\alpha \mid w \in W,\ \alpha \in \Pi \,\}$ of real roots will be denoted Δ^{re}. Define positive and negative real roots as usual by $\Delta^{re}_\pm = \Delta^{re} \cap \Delta_\pm$.

A $\mathfrak{g}$-module V, or (V, ρ) where $\rho \colon \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(V)$, is said to be integrable if $\mathfrak{g}_\alpha$ is ρ-locally nilpotent for all $\alpha \in \Delta^{re}$. One associates a Kac-Moody group $G(A)$ to $\mathfrak{g}(A)$ as in [PK]; $G(A)$ is the quotient $G = G^*/N$ of the free product G^* of the root subgroups $\mathfrak{g}_\alpha$ (for real roots α) by the largest normal subgroup N of G^* acting trivially on all integrable $\mathfrak{g}$-modules V (where $\mathfrak{g}_\alpha \subset G^*$ acts on each such (V, ρ) by $(e, v) \mapsto \exp(\rho(e))v$). For real roots α, one has the canonical injection (also denoted exp) $\mathfrak{g}_\alpha \to G^* \to G$ with image the one-parameter subgroup.

Highest weight representations for sl_2 and for the Heisenberg algebra $\mathfrak{n}$.

Consider Lie algebra $sl_2$with the basis $\{e, h, f\}$ and relations $[h, e] = 2e$, $[h, f] = 2f$, $[e, f] = h$.

Let (V, ρ) be an sl_2-module generated by a highest weight vector v : $\rho(e)v = 0$, $\rho(h)v = cv$.

The space V is a linear span of $\{\rho(f)^k v\}, k \geq 0$. We can compute the action of $\rho(e)$ on these vecors:

$$\rho(e)\rho(f)^k v = \sum_{i=0}^{k-1} \rho(f)^{k-1-i}\rho(h)\rho(f)^i v$$

$$= \big(\sum_{i=0}^{k-1}(c-2i)\big)\rho(f)^{k-1}v$$

$$= k(c-(k-1))\rho(f)^{k-1}v,$$

$$\rho(e)^k\rho(f)^k v = k!\prod_{i=0}^{k-1}(c-i)v.$$

This computation shows that if c is not a non-negative integer then $\rho(e)^k v \neq 0$ for all $k \in \mathbb{N}$ and V is an infinite-dimensional simple module.

If V is a finite-dimensional module then $c \in \mathbb{N} \cup \{0\}$ and $\rho(f)^{c+1}v = 0$, so $\{\rho(f)^k v\}, k = 0, \ldots, c$, is a basis for V.

We can rewrite the last formula as follows:

$$\rho(e)^k\rho(f)^k v = \frac{k!c!}{(c-k)!}v.$$

Denote $v_k = \frac{1}{k!}\rho(f)^k v$. Then

$$\rho(f)v_k = (k+1)v_{k+1}, \quad k < c,$$

$$\rho(e)v_k = (c-(k-1))v_{k+1}, \quad k>0,$$

$$\rho(f)v_c = \rho(e)v_0 = 0.$$

As the action of $\rho(e)$ and $\rho(f)$ on V is nilpotent, we can consider exponentials of these operators:

$$E(\lambda) = \exp(\lambda\rho(e)) = \sum_{k=0}^{c} \frac{1}{k!}\rho(e)^k,$$

$$F(\lambda) = \exp(\lambda\rho(f)) = \sum_{k=0}^{c} \frac{1}{k!}\rho(f)^k.$$

The families of operators $\{E(\lambda)\}$ and $\{F(\lambda)\}, \lambda \in \mathbb{F}$, form abelian (additive) groups and together generate a group $SL_2(\mathbb{F})$ acting on V (if $\dim V > 1$).

Consider now a 3-dimensional nilpotent Heisenberg algebra $\mathfrak{n}$ with the basis $\{x,y,z\}$, where z is a central element of $\mathfrak{n}$ and $[x,y]=z$.

Let (V,ρ) be a highest weight $\mathfrak{n}$ -module generated by vector v : $\rho(x)v = 0$, $\rho(z)v = cv$. As $\rho(z)$ commutes with $\rho(x)$ and $\rho(y)$ we have $\rho(z) = c\ \mathrm{Id}_V$. The space V is a linear span of $\{\rho(y)^k v\}, k \geq 0$. Also,

$$\rho(x)\rho(y)^k v = \sum_{i=0}^{k-1} \rho(y)^{k-1-i}\rho(z)\rho(y)^i v =$$

$$= kc\rho(y)^{k-1}v, \quad \text{and}$$

$$\rho(x)^k\rho(y)^k v = k!c^k v,$$

so , if $c \neq 0$, then $\rho(y)^k v \neq 0$ and $\{\rho(y)^k v\}$ are linearly independent. Hence V is an infinite-dimensional simple $\mathfrak{n}$-module.

Conjugacy Theorem.

Theorem 1. *Let A be a generalized Cartan matrix such that $a_{ij}a_{ji} \geq 4$ for all i,j and let $\mathfrak{g}(A)$ be the corresponding Kac-Moody algebra over the field*

$\mathbb{F}$ *of characteristic 0. Then all Cartan subalgebras in $\mathfrak{g}(A)$ are conjugated under the action of a Kac-Moody group $G(A)$.*

For an element $x \in \mathfrak{g}$ consider a decomposition in homogeneous components:

$$x = \sum_{\alpha \in \Delta} x_\alpha.$$

For a subset P of Δ denote

$$x_P = \sum_{\alpha \in P} x_\alpha.$$

Define the support of x in Δ as follows:

$$\text{supp}(x) = \{\alpha \in \Delta | x_\alpha \neq 0\},$$

and if $\mathfrak{h}$ is a subalgebra in $\mathfrak{g}$ then define

$$\text{supp}(\mathfrak{h}) = \underset{x \in \mathfrak{h}}{\cup} \text{supp}(x).$$

Let S(x) be a convex closure of supp(x) in $\mathbb{Z}^l$. As supp(x) is finite it follows that $S(x)$ is a polyhedron in $\mathbb{Z}^l$. Note that every vertex of $S(x)$ belongs to supp(x).

Proposition 1. *Let x be a locally finite element in $\mathfrak{g}$ and let Γ be a generalized face of $S(x)$.*
(i) If Γ contains 0 then x_Γ is locally finite.
(ii) If Γ doesn't contain 0 then x_Γ is locally nilpotent.

Proof. Clearly, in order to prove local finitness (nilpotency) of an element, it is sufficient to consider its action on homogeneous elements.
(i). Let π be a linear subspace spanned by Γ and let $u \in \mathfrak{g}_\alpha$. Then

$$((\text{ad}x)^n u)_{\alpha+\pi} = (\text{ad}x_\Gamma)^n u.$$

As the linear span of $\{(\mathrm{ad}x)^n u\}, n \in \mathbb{N}$, is finite-dimensional then its projection on $\alpha + \pi$ is also finite-dimensional, hence x_Γ is locally finite.

(ii). In the same way

$$((\mathrm{ad}x)^n u)_{\alpha+n\Gamma} = (\mathrm{ad}x_\Gamma)^n u.$$

As Γ doesn't contain 0 there exists $n \in \mathbb{N}$ such that $\alpha+n\Gamma = \{\alpha+\gamma_1+\ldots+\gamma_n | \gamma_i \in \Gamma\}$ doesn't intersect with the support of the span of $\{(\mathrm{ad}x)^n u\}$. Consequently, $(\mathrm{ad}x_\Gamma)^n u = 0$.

Two following lemmas supply us with our main technical tools.

Lemma 1. *Let* $V = \bigoplus_{k\in\mathbb{Z}} V_k$ *be a graded* sl_2*-module for which* $\rho(h)v_k = 2kv_k$ *for all* $v_k \in V_k$*. If* $\mathrm{dim}V_0 < \infty$ *and* $\mathrm{dim}V_k > 0$ *for all* $k > 0$ *then* $\rho(e)$ *is not locally nilpotent.*

Proof. Suppose, by contradiction, that $\rho(e)$ is locally nilpotent. As $\mathrm{dim}V_0 < \infty$ then there exists $m \in \mathbb{N}$ such that $\rho(e)^m V_0 = 0$. Consider a non-zero element $v_n \in V_n$ for some $n > m$. Again using local nilpotency of $\rho(e)$ we can find $s \geq 0$ such that $\rho(e)^{s+1}v_n = 0$ and $u = \rho(e)^s v_n \neq 0$. Note that $\rho(f)^{n+s}u \in V_0$, hence $\rho(e)^{n+s}\rho(f)^{n+s}u = 0$, but $\rho(e)^{n+s}\rho(f)^{n+s}u = (2(n+s))!u \neq 0$. This is a contradiction.

Lemma 2. *Let* $V = \bigoplus_{k\in\mathbb{Z}} V_k$ *be a graded* $\mathfrak{n}$*-module for which* $\rho(x)V_k \subset V_{k+1}, \rho(y)V_k \subset V_{k-1}$ *and for all* $v \in V$, $\rho(z)v = cv, c \neq 0$*. If* $\mathrm{dim}V_0 < \infty$ *and* $\mathrm{dim}V_k > 0$ *for all* $k > 0$ *then* $\rho(x)$ *is not locally nilpotent.*

Proof. The proof is essentially the same as for the previous lemma. Assume, by contradiction, that $\rho(x)$ is locally nilpotent, then for some $m > 0$ $\rho(e)^m V_0 = 0$. Let u be a non-zero element in V_k for some $k > m$ such that $\rho(x)u = 0$. Then $\rho(x)^k\rho(y)^k u = 0$. On the other hand, $\rho(x)^k\rho(y)^k u = k!c^k u \neq 0$, a contradiction.

Lemma 3. *Let* $\alpha, \beta \in \Delta^{\mathrm{re}}, \alpha \neq \pm\beta$ *and let* h_α, h_β *be generators of* $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ *and* $[\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}]$ *, respectively. If some non-zero linear combination* $c_1 h_\alpha + c_2 h_\beta$ *belongs to the center of* $\mathfrak{g}(A)$ *then* $\mathfrak{g}(A)$ *is an affine Lie algebra.*

Proof. Considering the action of W on $\mathfrak{g}$ we may assume that α is a fundamental root: $\alpha = \alpha_i \in \Pi, h_\alpha = h_i$ and $\beta \in \Delta^{\mathrm{re}}_+$. From the action of W on the real co-roots we can deduce that in the decomposition $c_2 h_\beta = \sum_{j=1}^{l} b_j h_j$ all coefficients b_j have the same sign, so we may assume that they are all non-negative. The element $c_1 h_i + \sum_{j=1}^{l} b_j h_j$ belongs to $Z(\mathfrak{g})$ if and only if the corresponding linear combination of rows of the generalized Cartan matrix A is zero.

If $c_1 + b_i < 0$ then for the i-th column of A we have $\sum_{\substack{j=1 \\ j\neq i}}^{l} b_j a_{ji} + 2(c_1 + b_i) < 0$, so this linear combination of rows of A is not a zero row.

If $c_1 + b_i \geq 0$ then this linear combination has all coefficients non-negative, so A is of affine type [K].

Lemma 4. *Let* $\mathfrak{g}$ *be a non-affine Kac-Moody algebra and let* $\beta \in \Delta^{\mathrm{re}}_+$, $\alpha_i \in \Pi$, $\beta - \alpha_i \notin \Delta$, $\alpha = \beta + n\alpha_i \in \Delta$. *If* $x \in \mathfrak{g}_\alpha, x \neq 0$, *then there exists* $y \in \mathfrak{g}_{-\alpha}$ *such that* $[x, y] \notin Z(\mathfrak{g})$.

Proof. Consider the finite-dimensional $< e_i, h_i, f_i > \cong sl_2$-module $V = \bigoplus_{k\geq 0} \mathfrak{g}_{\beta + k\alpha_i}$. Without loss of generality we may assume that $\alpha(h_i) = -m \leq 0$. As V is completely reducible, x can be written as the following sum:

$$x = \sum_{k=0}^{n} (\mathrm{ad} e_i)^{n-k} v_k,$$

where $v_k \in \mathfrak{g}_{\beta + k\alpha_i}, [f_i, v_k] = 0$.

Let u_0 be a non-zero element of a real root space $\mathfrak{g}_{-\beta}$ and let $y = (\mathrm{ad} f_i)^n u_0$. We shall prove that $[x, y] \notin Z(\mathfrak{g})$.

We have

$$[(\mathrm{ad}e_i)^{n-k}v_k, (\mathrm{ad}f_i)^n u_0]$$

$$= \sum_{j=0}^{n-k} (-1)^j C_{n-k}^j (\mathrm{ad}e_i)^{n-k-j}(\mathrm{ad}v_k)(\mathrm{ad}e_i)^j(\mathrm{ad}f_i)^n u_0$$

$$= \sum_{j=0}^{n-k} (-1)^j C_{n-k}^j \frac{n!(m-j)!}{(n-j)!(m-n)!}(\mathrm{ad}e_i)^{n-k-j}(\mathrm{ad}v_k)(\mathrm{ad}f_i)^{n-j}u_0 =$$

$$= \sum_{j=0}^{n-k} (-1)^j C_{n-k}^j \frac{n!(m-j)!}{(n-j)!(m-n)!}(\mathrm{ad}e_i)^{n-k-j}(\mathrm{ad}f_i)^{n-j}[v_k, u_0].$$

As $[v_k, u_0] \in \mathfrak{g}_{k\alpha_i}$, it follows $[v_k, u_0] = 0$ when $k \neq 0, 1$.

If $k = 0$ then $[v_0, u_0] \in \mathfrak{h}_0$, so $(\mathrm{ad}f_i)^{n-j}[v_0, u_0] = 0$ when $n - j > 1$.

If $k = 1$ then $[v_1, u_0] \in \mathfrak{g}_{\alpha_i}$, so $(\mathrm{ad}f_i)^{n-j}[v_1, u_0] = 0$ when $n - j > 2$.

Hence,

$$[x, y] = (-1)^n(\mathrm{ad}v_0)(\mathrm{ad}e_i)^n(\mathrm{ad}f_i)^n u_0 +$$

$$+(-1)^{n-1}n(\mathrm{ad}e_i)(\mathrm{ad}v_0)(\mathrm{ad}e_i)^{n-1}(\mathrm{ad}f_i)^n u_0 +$$

$$+(-1)^{n-1}(\mathrm{ad}v_1)(\mathrm{ad}e_i)^{n-1}(\mathrm{ad}f_i)^n u_0$$

$$= (-1)^n \frac{n!m!}{(m-n)!}[v_0, u_0] +$$

$$+(-1)^{n-1}n\frac{n!(m-1)!}{(m-n)!}(\mathrm{ad}e_i)(\mathrm{ad}f_i)[v_0, u_0] +$$

$$+(-1)^{n-1}\frac{n!(m-1)!}{(m-n)!}(\mathrm{ad}f_i)[v_1, u_0].$$

Note that $[v_0, u_0]$ is a non-zero element of $[\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}]$. As $[v_0, u_0] \in \mathfrak{h}_0$ and $[v_1, u_0] \in \mathfrak{g}_{\alpha_i}$, we have $[e_i[f_i[v_0, u_0]]], [f_i[v_1, u_0]] \in \mathbb{F}h_i$.

As $\mathfrak{g}$ is non-affine , applying Lemma 3 we conclude that $[x, y] \notin Z(\mathfrak{g})$.

Proposition 2. *If x_α is a non-zero homogeneous locally-nilpotent element of $\mathfrak{g}(A)$ and $\alpha \in \Delta_+$ then α is a real root.*

Proof. This statement is obvious for affine Kac-Moody algebras, so we may assume that $\mathfrak{g}(A)$ is not affine. Suppose, by contradiction, that there exist imaginary roots in Δ_+ with non-zero locally nilpotent elements in the corresponding root spaces. Let α be minimal with this property and let $x_\alpha \in \mathfrak{g}_\alpha$ be locally nilpotent. Considering the action of the Weyl group we get that $\alpha \in -C^\vee$. Since $\mathfrak{g}(A)$ is radical-free, $[f_i, x_\alpha] \neq 0$ for some $i = 1, \dots, l$.

Consider an automorphism $F_i(1)$ of $\mathfrak{g}(A)$:

$$F_i(1)x_\alpha = \sum_{k=0}^{\infty} \frac{(\mathrm{ad} f_i)^k}{k!} x_\alpha.$$

Let n be the maximal integer for which $(\mathrm{ad} f_i)^n x_\alpha \neq 0, n \geq 1$. According to Proposition 1(ii) $(\mathrm{ad} f_i)^n x_a$ is locally nilpotent. As α is the minimal positive imaginary root with non-zero locally nilpotent elements, $\alpha - n\alpha_i$ should be a real root. Now, by Lemma 4 , there exists $y_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $h_\alpha = [x_\alpha, y_{-\alpha}] \notin Z(\mathfrak{g})$.

Consider a 3-dimensional subalgebra $< x_\alpha, h_\alpha, y_{-\alpha} >$. If this subalgebra is isomorphic to sl_2then taking a graded module $\bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{k\alpha}$ and applying Lemma 1 we get a contradiction to local nilpotency of x_α.

If $< x_\alpha, h_\alpha, y_{-\alpha} >$ is isomorphic to the Heisenberg algebra then choose $\delta \in \Delta_+^{\mathrm{im}} \cap -C^\vee$ such that δ has the full support (i.e. all coefficients in the decomposition of δ as a linear combination of $\{\alpha_j\}$ are non-zero) and for all $j = 1, \dots, l$ $\delta + \alpha_j \in \Delta_+^{\mathrm{im}} \cup -C^\vee$ [K]. Since $h_\alpha \notin Z(\mathfrak{g})$ then for at least one of these roots $\mu = \delta + \alpha_i$ we have $\mu(h_\alpha) \neq 0$.

As $\mu, \alpha \in \Delta_+^{\mathrm{im}} \cup -C^\vee$ and μ has the full support it follows that $\mu + k\alpha \in \Delta_+^{\mathrm{im}}$ for all $k \in \mathbb{N}$.

Now we can get a contradiction to the local nilpotency of x_α if we apply Lemma 2 to $< x_\alpha, h_\alpha, y_{-\alpha} >$ -module $\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\mu+k\alpha}$. This completes the proof of Proposition 2.

Proposition 3. *Let generalized Cartan matrix A satisfy $a_{ij}a_{ji} \geq 4$ for all i, j. If x is a locally finite element of $\mathfrak{g}(A)$ and Γ is an edge of $S(x)$ then Γ is parallel to some root in $\Delta(A)$.*

Proof. If Γ contains 0 then the proposition is obvious. If Γ doesn't contain 0 then according to Proposition 1(ii) x_Γ is locally nilpotent. It follows from Proposition 2 that the vertices $\{\beta, \gamma\}$ of edge Γ are real roots.

Consider now non-zero elements $y_{-\beta} \in \mathfrak{g}_{-\beta}$ and $y_{-\gamma} \in \mathfrak{g}_{-\gamma}$. Let $y = c_1 y_{-\beta} + c_2 y_{-\gamma}$.

Suppose, by contradiction, that Γ is not parallel to any root. Then $(\Gamma + (-\Gamma)) \cap \Delta = \{0\}$, hence

$$[x_\Gamma, y] = c_1[x_\beta, y_{-\beta}] + c_2[x_\gamma, y_{-\gamma}] = c_1 h_\beta + c_2 h_\gamma.$$

As Γ doesn't contain 0 then β and γ (resp. h_β and h_γ) are linearly independent. It is possible to see that a subalgebra $\mathfrak{g}_1$ in $\mathfrak{g}(A)$, generated by $< x_\beta, x_\gamma, y_{-\beta}, y_{-\gamma} >$, is isomorphic to a rank 2 Kac-Moody algebra, corresponding to a rank 2 root subsystem in $\Delta(A)$.

Since A doesn't contain submatrices of rank 2 of finite type, $\Delta(A)$ doesn't contain rank 2 root subsystems of finite type [HMR] . Hence $\mathfrak{g}_1$ is infinite-dimensional.

If $\mathfrak{g}_1$ is a Kac-Moody algebra of indefinite type then

$$\det \begin{vmatrix} \beta(h_\beta) & \beta(h_\gamma) \\ \gamma(h_\beta) & \gamma(h_\gamma) \end{vmatrix} \neq 0,$$

and there exist constants c_1, c_2 such that $\beta(c_1 h_\beta + c_2 h_\gamma) = \gamma(c_1 h_\beta + c_2 h_\gamma) = 2$.

The subalgebra $< x_\Gamma, c_1 h_\beta + c_2 h_\gamma, y >$ is isomorphic to $sl_2$and the module $\bigoplus_{k\in\mathbb{Z}} (\bigoplus_{t\in\mathbb{Q}} \mathfrak{g}_{k\pi})$, where π is a line through Γ , satisfies the conditions of Lemma 1 which gives a contradiction.

If $\mathfrak{g}_1$ is affine then

$$\det \begin{vmatrix} \beta(h_\beta) & \beta(h_\gamma) \\ \gamma(h_\beta) & \gamma(h_\gamma) \end{vmatrix} = 0,$$

hence there exist constants c_1, c_2 such that $\beta(c_1 h_\beta + c_2 h_\gamma) = \gamma(c_1 h_\beta + c_2 h_\gamma) = 0$.

If algebra $\mathfrak{g}(A)$ itself is non-affine then by Lemma 3 $c_1 h_\beta + c_2 h_\gamma \notin Z(\mathfrak{g})$ and again there exists $\mu \in \Delta_+^{\mathrm{im}} \cap -C^\vee$ such that μ has the full support amd $\mu(c_1 h_\beta + c_2 h_\gamma) \neq 0$. If δ is a null-root of $\mathfrak{g}_1$ then $\mu + k\delta \in \Delta_+^{\mathrm{im}}$ for all $k \geq 0$. Now the algebra $< x_\Gamma, c_1 h_\beta + c_2 h_\gamma, y >$ is isomorphic to the Heisenberg algebra and the module $\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\mu+k\pi}$ satisfies the conditions of Lemma 2, which leads again to a contradiction.

Finally, suppose that both $\mathfrak{g}_1$ and $\mathfrak{g}(A)$ are affine. As $a_{ij}a_{ji} \geq 4$ for all i, j we have that $\mathfrak{g}(A)$ should have rank 2. In this case the required statement may be obtained by direct computation.

Lemma 5. *Let α_i be a fundamental root and let the root string through a real root β_1 along α_i contains four real roots $\beta_1 < \beta_2 < \beta_3 < \beta_4$; $\beta_2 - \beta_1 = \beta_4 - \beta_3 = \alpha_i; \beta_3 - \beta_2 = n\alpha_i$. If $x = \sum_{j=0}^{n} x_{\beta_2 + j\alpha_i} \neq 0$ then x is not locally nilpotent.*

Proof. Roots $\pm\alpha_i, \pm\beta_1$ generate a rank 2 subsystem Δ_1 in $\Delta(A)$.

Consider $y = c_1 y_{-\beta_1} + c_2 y_{-\beta_2} + c_3 y_{-\beta_3} + c_4 y_{-\beta_4}$. Then

$$[x, y] = c_1[x_{\beta_2}, y_{-\beta_1}] + c_2[x_{\beta_2}, y_{-\beta_2}] + c_2[x_{\beta_2+\alpha_i}, y_{-\beta_2}]+$$

$$+c_3[x_{\beta_3-\alpha_i}, y_{-\beta_3}] + c_3[x_{\beta_3}, y_{-\beta_3}] + c_4[x_{\beta_3}, y_{-\beta_4}].$$

If Δ_1 is of indefinite type (resp. affine type) then there exist constants c_2 and c_3 such that $h = c_2[x_{\beta_2}, y_{-\beta_2}] + c_3[x_{\beta_3}, y_{-\beta_3}]$ satisfies $\beta_2(h) = \beta_3(h) = 2$ (resp. $\beta_2(h) = \beta_3(h) = 0$).

Choose now constants c_1 and c_4 satisfying

$$c_1[x_{\beta_2}, y_{-\beta_1}] + c_2[x_{\beta_2+\alpha_i}, y_{-\beta_2}] = 0,$$

$$c_3[x_{\beta_3-\alpha_i}, y_{-\beta_3}] + c_4[x_{\beta_3}, y_{-\beta_4}] = 0.$$

If Δ_1 is of indefinite type then $< x, y, h > \cong sl_2$ and applying Lemma 1 to the module $\bigoplus_{k\in\mathbb{Z}} (\bigoplus_{m\in\mathbb{Z}} \mathfrak{g}_{k\beta_1+m\alpha_i})$ we get that x is not locally nilpotent.

If Δ_1 is of affine type and $\Delta(A)$ is non-affine then again choose $\mu \in \Delta^{\mathrm{im}} \cap -C^\vee$ with full support and satisfying $\mu(h) \neq 0$. Now applying Lemma 2 to the module $\bigoplus_{k\in\mathbb{Z}} (\bigoplus_{m\in\mathbb{Z}} \mathfrak{g}_{\mu+k\beta_1+m\alpha_i)}$ we get the required statement.

If, finally, both Δ_1 and $\Delta(A)$ are affine then Δ_1 is of type $A_2^{(2)}$ and $\Delta(A)$ is of type $A_{2k}^{(2)}$. In this case the statement of the lemma may be obtained by direct computation.

Now we have all requisites to prove the theorem if the field $\mathbb{F}$ is uncountable and algebraically closed. The idea could be the following: over an uncountable field every finite-dimensional abelian Cartan subalgebra $\mathfrak{h}$ contains a regular element x, such that $C_{\mathfrak{g}}(x) = \mathfrak{h}$, then applying the procedure of " stripping off edges " [PK] to $S(x)$ we construct an automorphism $g \in G(A)$ such that $g(x) \in \mathfrak{h}_0$, hence $g(\mathfrak{h}) = \mathfrak{h}_0$. However, to be able to prove the theorem over an arbitrary field of characteristic 0 a little more work needs to be done.

Let V be a finite-dimensional sl_2-module with the set of weights P and the highest weight μ. Let $v \in V, v = \sum_{\gamma\in P} v_\gamma \neq 0$. Consider a convex closure $S(v)$ of the support of v in P, so $S(v)$ is a segment or a point. Let $\bar{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$ and let $\bar{V} = V \otimes_{\mathbb{F}} \bar{\mathbb{F}}$. Consider non-zero vectors in V with the following property:

$$\text{For every } g \in SL_2(\bar{\mathbb{F}}) \text{ vertices of } S(gv) \text{ belong to the set } \{\pm\mu\}. \qquad (*)$$

Lemma 6. *Let V be a finite-dimensional $< e_\alpha, h_\alpha, f_\alpha >\cong sl_2$-module over the field $\mathbb{F}$ with the highest weight $\mu, 2\mu = n\alpha, n \geq 2, \dim V_\mu = 1$. Let v be a non-zero vector in V , satisfying the property (*). Then*
(i). There exists $g_0 \in SL_2(\mathbb{F})$ such that $g_0 v \in V_{-\mu}$.
(ii). If for vector $w \in V$ $w + cv$ satisfies (*) for all $c \in \mathbb{F}$ then $g_0 w \in V_{-\mu}$.

Proof. *(i)* If μ is not a vertex of $S(v)$ then $-\mu$ is the only vertex, hence $v \in V_{-\mu}$ and there is nothing to prove.

If μ is a vertex of $S(v)$, take a highest weight vector u_0 and construct a basis for the irreducible submodule U generated by V_μ:

$$u_k = \frac{(\mathrm{ad} f_\alpha)^k}{k!} u_0.$$

Since V is completely reducible, there exists a canonical projection $\tau : V \mapsto U$. Let $\tau(v) = \sum_{k=0}^{n} a_k u_k, a_0 \neq 0$. Note that for every $w \in V$, $\tau(w)_{\pm\mu} = w_{\pm\mu}$.

Consider an automorphism $F(\lambda)$ applied to $\tau(v)$:

$$(F(\lambda)\tau(v))_{-\mu} = \sum_{k=0}^{n} C_n^k a_k \lambda^{n-k}.$$

As $a_0 \neq 0$, there exists $\lambda_0 \in \bar{\mathbb{F}}$ such that $(F(\lambda_0)\tau(v))_{-\mu} = 0$. Consequently, $(F(\lambda_0)v) \in \bar{V_\mu}$ and $\tau(v) = v$. Note that $(F(\lambda_0)v)_{\mu-\alpha} = a_1 + \lambda_0 a_0 = 0$, hence λ_0 in fact belongs to $\mathbb{F}$.

Finally, $\sigma_\alpha F(\lambda_0)v \in V_{-\mu}$ as was to be proven.

(ii). Now we may assume that $v \in V_{-\mu}$. As in (i) we can prove that $w \in U, w = \sum_{k=0}^{n} b_k u_k$. As the vertices of $S(cv+w)$ belong to $\{\pm\mu\}$ then $w = b_0 u_0 + b_n u_n$. However, $F(1)v = v$ and $F(1)w = b_0 \sum_{k=0}^{n} u_k + b_n u_n$, hence $b_0 = 0$ and $w \in V_{-\mu}$.

Consider now non-zero vectors in sl_2-module V with the following property:

$$\text{For every } g \in SL_2(\bar{\mathbb{F}}) \text{ vertices of } S(gv) \text{ belong to the set}$$

$$\{\pm\mu, \pm(\mu-\alpha)\}, \text{ but } S(gv) \text{ is not a segment } [\alpha-\mu, \mu-\alpha]. \qquad (**)$$

The following lemma is an analogue of Lemma 6 adjusted for the situation of four real roots in a root string.

Lemma 7. *Let V be a finite-dimensional $< e_\alpha, h_\alpha, f_\alpha > \cong sl_2$-module over the field $\mathbb{F}$ with the highest weight $\mu, 2\mu = n\alpha, n \geq 3, \dim V_\mu = \dim V_{\mu-\alpha} = 1$. Let v be a non-zero vector in V , satisfying property (**). Then*
(i). There exists $g_0 \in SL_2(\mathbb{F})$ such that $g_0 v \in V_{-\mu} \oplus V_{-\mu+\alpha}$.
(ii). If for vector $w \in V$ $w + cv$ satisfies property (**) for all $c \in \mathbb{F}$ then $g_0 w \in V_{-\mu} \oplus V_{-\mu+\alpha}$.

Proof. *(i)* Let $\tau(v) = \sum_{k=0}^{n} a_k u_k$. If $a_0 = a_1 = 0$ then $v \in V_{-\mu} \oplus V_{-\mu+\alpha}$.

If $a_0 = 0, a_1 \neq 0$ then as in Lemma 6 there exists $\lambda_0 \in \mathbb{F}$ such that $(F(\lambda_0)v)_{-\mu} = 0$ and hence $F(\lambda_0)v \in V_{\mu-\alpha}$, so $\sigma_\alpha F(\lambda_0)v \in V_{\alpha-\mu}$.

If $a_0 \neq 0$ then there exists $\lambda_0 \in \bar{\mathbb{F}}$ such that $(F(\lambda_0)v)_{-\mu} = 0$. If $(F(\lambda_0)v)_{-\mu+\alpha} = 0$ then $F(\lambda_0)v \in \bar{V}_\mu \oplus \bar{V}_{\mu-\alpha}$ and $\sigma_\alpha F(\lambda_0)v \in \bar{V}_{-\mu} \oplus \bar{V}_{\alpha-\mu}$.

If $(F(\lambda_0)v)_{-\mu+\alpha} \neq 0$ then there exists $\epsilon_0 \in \bar{\mathbb{F}}$ such that $(E(\epsilon_0)F(\lambda_0)v)_\mu = 0$. Hence $E(\epsilon_0)F(\lambda_0)v \in \bar{V}_{\alpha-\mu}$.

We proved that there exists $g_1 \in SL_2(\bar{\mathbb{F}})$ such that $g_1 v \in \bar{V}_{-\mu} \oplus \bar{V}_{\alpha-\mu}$, let us prove now that the required element can be found in $SL_2(\mathbb{F})$.

Using the Bruhat decomposition for $SL_2(\bar{\mathbb{F}})$ we get that g_1^{-1} may be written in the form $\sigma_\alpha F(\kappa)H(\eta)$ or in the form $E(\lambda)F(\kappa)H(\eta)$ for some $\lambda,\kappa,\eta\in\bar{\mathbb{F}}$.

Note that $F(\kappa)H(\eta)g_1(v)\in\bar{V}_{-\mu}\oplus\bar{V}_{\alpha-\mu}$.

If $g_1^{-1}=\sigma_\alpha F(\kappa)H(\eta)$ then $\sigma_\alpha^{-1}v\in\bar{V}_{-\mu}\oplus\bar{V}_{\alpha-\mu}$.

If $g_1^{-1}=E(\lambda)F(\kappa)H(\eta)$ then $v=g_1^{-1}g_1v=E(\lambda)(au_{n-1}+bu_n)$, where $au_{n-1}+bu_n=F(\kappa)H(\eta)g_1(v),\ a,b\in\bar{\mathbb{F}}$.

To prove that $\lambda,a,b\in\mathbb{F}$ let us compute several components of v : $v_{-\mu}=(E(\lambda)(au_{n-1}+bu_n))_{-\mu}=bu_n$, hence $b\in\mathbb{F}$ as $v\in V$. Also,

$$v_{-\mu+\alpha}=(a+\lambda b)u_{n-1},$$

$$v_{-\mu+2\alpha}=(2\lambda a+\lambda^2 b)u_{n-2},$$

$$v_{-\mu+3\alpha}=(3\lambda^2 a+\lambda^3 b)u_{n-3}.$$

Consequently, $c_1=a+\lambda b; c_2=\lambda(2a+\lambda b); c_3=\lambda^2(3a+\lambda b)\in\mathbb{F}$.

Clearly, we may assume that $b\neq 0$.

We get:

$$\lambda(2c_1-\lambda b)=c_2,\quad \lambda^2(3c_1-2\lambda b)=c_3$$

and

$$\lambda^2 c_1=2\lambda c_2-c_3.$$

If $c_1=0$ then $\lambda=\frac{c_3}{2c_2}\in\mathbb{F}$, or $a=0,\lambda b=0$. If $c_1\neq 0$ then $\lambda^2=\frac{2c_2}{c_1}\lambda-\frac{c_3}{c_1}=-\frac{2a}{b}\lambda-\frac{c_2}{b}$, hence $a,\lambda\in\mathbb{F}$. Consequently, $E(-\lambda)v\in V_{-\mu}\oplus V_{\alpha-\mu},\lambda\in\mathbb{F}$, as was to be proven.

The proof of part (ii) is analogous to the proof of (ii) of the previous lemma.

Proof of the Theorem 1. It is sufficient to prove that every finite-dimensional abelian subalgebra $\mathfrak{h}$ acting on $\mathfrak{g}(A)$ semisimply can be conjugated into $\mathfrak{h}_0$. This will imply that every Cartan subalgebra in $\mathfrak{g}(A)$ is finite-dimensional and hence is conjugated with $\mathfrak{h}_0$.

The arguments from [PK] show that it is enough to prove that $\mathfrak{h}$ can be conjugated into $\mathfrak{b}_-$.

Since $\mathfrak{h}$ is finite-dimensional, supp($\mathfrak{h}$) is finite and there exists $x \in \mathfrak{h}$ such that supp$(x) =$ supp($\mathfrak{h}$). As x is locally finite then by Proposition 2 all non-zero vertices of $S(x)$ are real roots.

Consider a natural partial ordering on $\mathbb{Z}^l \supset \Delta$. Let us prove that $S(x)$ has a unique maximal vertex with respect to this partial ordering.

Indeed, let α be a maximal vertex of $S(x)$. By Proposition 3 all edges going out of α are parallel to some roots, but $\Delta = \Delta_+ \cup \{0\} \cup \Delta_-$, $\Delta_\pm \subset \mathbb{Z}^l_\pm$. As α is maximal then no edges from α go in a positive direction, hence all edges from α go in a negative direction. This implies that α is the unique maximal vertex as $S(x)$ is convex.

Applying the procedure of "stripping off edges" we shall show that we can always lower the maximal positive vertex α of supp($\mathfrak{h}$) .

Let σ_i be a fundamental reflection such that $\sigma_i(\alpha) < \alpha$. The root string Γ through α along α_i in Δ contains either 2 or 4 real roots. Due to Proposition 2 and Lemma 5, x_Γ satisfies the conditions of Lemma 6 or Lemma 7. By these lemmas there exists $g_0 \in SL_2(\mathbb{F})_i$ such that supp$(g_0(x))$ has a maximal vertex in Γ which is lower than α. Moreower , by Lemmas 6,7 (ii) the maximal vertex of supp$(g_0(\mathfrak{h}))$ is strictly less than α.

Continuing the process of lowering the maximal vertex of supp($\mathfrak{h}$) we shall eventually construct $g \in G(A)$ such that $g(\mathfrak{h}) \subset \mathfrak{b}_-$, as was to be proven.

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