Non-matrix varieties and nil-generated algebras whose units satisfy a group identity

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Abstract

Let $R^\times$ denote the group of units of an associative algebra R over an infinite field F. We prove that if R is unitarily generated by its nilpotent elements, then $R^\times$ satisfies a group identity precisely when R satisfies a non-matrix polynomial identity. As an application, we examine the group algebra FG of a torsion group G and the restricted enveloping algebra $u(L)$ of a p-nil restricted Lie algebra L. Giambruno, Sehgal and Valenti recently proved that if the group of units $(FG)^\times$ satisfies a group identity then FG satisfies a polynomial identity, thus confirming a conjecture of Brian Hartley. We show that, in fact, $(FG)^\times$ satisfies a group identity if and only if FG satisfies a non-matrix polynomial identity. In the case of restricted enveloping algebras, we prove that $u(L)^\times$ satisfies a group identity if and only if $u(L)$ satisfies the Engel condition.

1 Introduction

Let R be an associative unitary algebra over a field F of characteristic $p \geq 0$. Recall that R is said to satisfy a polynomial identity whenever there exists a nontrivial element $f(x_1, \ldots, x_m)$ of the free F-algebra generated by $\{x_1, x_2, \ldots\}$ such that $f(r_1, \ldots, r_m) = 0$ for all $r_i \in R$; whereas, the group of units $R^\times$ of R is said to satisfy a group identity if there exists a nontrivial word $w(y_1, \ldots, y_m)$ in the free group generated by $\{y_1, y_2, \ldots\}$ such that $w(u_1, \ldots, u_m) = 1$ for all $u_i \in R^\times$. There is ample evidence in the literature to suggest that there may be some general underlying relationship between group identities and polynomial identities. For example, Gupta and Levin proved that $R^\times$ is nilpotent whenever R is Lie nilpotent; Smirnov and Zalesskii established the fact that $R^\times$ is soluble whenever R is Lie soluble (if $p \neq 2$); and Shalev proved that $R^\times$ satisfies the Engel condition whenever R does ([GL],[ZSm],[Sm],[Sh1]). In the other direction, it follows from a result of Valitskas ([V]) that a radical algebra over an infinite field is a PI-algebra whenever its adjoint group satisfies a group identity. In radical algebras the adjoint group R° of R with the group operation given by $x \circ y = x + y + xy$ plays the role of the unit group in a unitary algebra. Also along this vein, Giambruno, Sehgal and Valenti ([GSV]) recently confirmed a conjecture of Brian Hartley by proving that the group algebra FG of a torsion group G over an infinite field F satisfies a polynomial identity whenever $(FG)^\times$ satisfies a group identity. Subsequently, Passman ([Pa2]) gave necessary and sufficient conditions for $(FG)^\times$ to satisfy an identity (*cf.* Theorem 4.1).

Two questions thus seem immediately relevant:

1. *Does $R^\times$ satisfy a group identity whenever R satisfies a PI?*

2. *Does R satisfy a PI whenever $R^\times$ satisfies a group identity?*

However, it quickly becomes clear that these questions are posed too generally. Indeed, free algebras have only the trivial units corresponding to $F^\times$; and $M_2(F)$, the algebra of 2×2 matrices over F, satisfies the standard polynomial identity of degree 4 even though $\mathrm{GL}_2(F)$ contains a non-abelian free group whenever F contains a transcendental element. Thus it is natural to concentrate on algebras containing many units, and on polynomial identities not satisfied by matrices. A polynomial identity not satisfied by $M_2(F)$ is called a non-matrix identity. It follows from well-known results that whenever R satisfies a non-matrix identity, then $R^\times$ satisfies a group identity. Namely, we have the following (we use brackets to denote Lie commutators and parentheses to denote group commutators):

Proposition 1.1 *Let R be a unitary algebra over a field of characteristic $p \geq 0$. Suppose that R satisfies a non-matrix identity. Then*

1. *$R^\times$ is soluble if $p = 0$; and,*

2. *$R^\times$ satisfies an identity of the form $(y_1, y_2)^{p^t} = 1$ if $p > 0$.*

In any case, $R^\times$ satisfies a group identity.

Proof. Suppose first that $p = 0$ and R satisfies a non-matrix identity. Then by a theorem of Kemer, [Ke], the ideal in R generated by the Lie commutators of the form $[[a, b], [c, d], e]$ is nilpotent. So, in particular, R is Lie soluble, and hence by Smirnov and Zalesskii's theorem mentioned above we find that $R^\times$ is soluble.

Now suppose that $p > 0$ and R satisfies a non-matrix identity f, say. Let A be the relatively-free algebra of rank 3 in the variety satisfying f. Then A is a finitely generated PI-algebra, so that by a theorem of Razmyslov and Braun, [Br], the Jacobson radical $J(A)$ of A is nilpotent. Also, $A/J(A)$ is a semiprimitive PI-algebra satisfying a polynomial identity not satisfied by $M_2(F)$; consequently, $A/J(A)$ is commutative. Therefore, A satisfies a polynomial identity of the form $([x_1, x_2]x_3)^{p^t} = 0$ for a suitable t. It follows that R also satisfies $([x_1, x_2]x_3)^{p^t} = 0$, and so $R^\times$ satisfies

$$(y_1, y_2)^{p^t} - 1 = ((y_1, y_2) - 1)^{p^t} = (y_1 y_2 y_1^{-1} y_2^{-1} - 1)^{p^t} = ([y_1, y_2] y_1^{-1} y_2^{-1})^{p^t} = 0.$$

□

The primary goal of this paper is to demonstrate that the converse to Proposition 1.1 holds for the class of nil-generated algebras over an infinite field. Let $\mathcal{N}(R)$ denote the set of nilpotent elements in R. We say that a unitary algebra R is nil-generated if it is generated by $\{1\} \cup \mathcal{N}(R)$. For example, $M_n(F)$ is nil-generated, as is the group algebra of a group generated by p-elements.

Theorem 1.2 *Let R be a nil-generated unitary algebra over an infinite field of characteristic $p \geq 0$. If $R^\times$ satisfies a group identity, then $\mathcal{N}(R)$ forms a locally nilpotent ideal and R satisfies a non-matrix identity.*

As a consequence of Proposition 1.1 and Theorem 1.2, we obtain the following characterisations:

Theorem 1.3 *Let R be a nil-generated unitary algebra over a field of characteristic 0. Then the following statements are equivalent:*

1. *$R^\times$ satisfies a group identity;*
2. *R satisfies a non-matrix identity;*

3. *R is Lie soluble; and,*
4. *$R^{\times}$ is a soluble group.*

Theorem 1.4 *Let R be a nil-generated unitary algebra over an infinite field of characteristic $p > 0$. Then the following statements are equivalent:*

1. *$R^{\times}$ satisfies a group identity;*
2. *R satisfies a non-matrix identity;*
3. *R satisfies the polynomial identity $([x_1, x_2]x_3)^{p^t} = 0$ for some t; and,*
4. *$R^{\times}$ satisfies the group identity $(y_1, y_2)^{p^t} = 1$ for some t.*

The following corollary is an analogue of the classical theorem of Kaplansky ([Ka]), which states that every nil-algebra satisfying a PI is locally nilpotent.

Corollary 1.5 *Let R be a nil-algebra over an infinite field. The adjoint group R° of R satisfies a group identity if and only if R satisfies a non-matrix identity. In this case R is locally nilpotent.*

Proof. It is well-known that R can be embedded into a unitary F-algebra R_1 in such a way that $\{1\} \cup R$ generates R_1 and $R_1^{\times} \cong F^{\times} \times R^{\circ}$. Theorems 1.2-1.4 now imply the result. □

As a further application of Theorem 1.4, we are able to study group algebras FG where G is any torsion group, and restricted enveloping algebras $u(L)$ where L is any p-nil restricted Lie algebra. In particular, we deduce that $(FG)^{\times}$ satisfies a group identity if and only if FG satisfies a non-matrix identity; whereas $u(L)^{\times}$ satisfies a group identity if and only if $u(L)$ satisfies the Engel condition. These results are detailed in Sections 4 and 5.

2 Existence of a polynomial identity

The following result, Corollary 2.2 of [GSV] (*cf.* Proposition 1 of [GJV]), plays a crucial role in the proof of our Theorem 1.2.

Lemma 2.1 *Let R be a semiprime algebra over an infinite commutative domain, such that its group of units $R^\times$ satisfies a group identity. Then for every nilpotent element $a \in R$, $bc = 0 \Rightarrow bac = 0$.*

Let $\mathcal{L}(R)$ denote the Levitzki radical of the algebra R; that is, the unique maximal locally nilpotent ideal in R. Then $R/\mathcal{L}(R)$ is semiprime (see Section 10 of [L], for example) and $\mathcal{L}(R)$ is contained in $\mathcal{N}(R)$, the set of all nilpotent elements in R.

Lemma 2.2 *Let R be a nil-generated unitary algebra over an infinite field. If $R^\times$ satisfies a group identity, then $\mathcal{L}(R) = \mathcal{N}(R)$. Consequently, $R = F \cdot 1 + \mathcal{N}(R)$ and every finite subset of $\mathcal{N}(R)$ generates nilpotent subalgebra in R.*

Proof. Let $\bar{R} = R/\mathcal{L}(R)$. Because $\mathcal{L}(R)$ is contained in the Jacobson radical $J(R)$ of R, $\bar{R}^\times$ is a homomorphic image of $R^\times$ and, hence, satisfies the same group identity. We claim that $\mathcal{N}(\bar{R}) = 0$.

Let $\bar{b}, \bar{c} \in \bar{R}$ be such that $\bar{b}\bar{c} = 0$. By induction, it follows that whenever $\bar{r}_1, \ldots, \bar{r}_k \in \bar{R}$ then $\bar{b}\bar{r}_1 \ldots \bar{r}_k\bar{c} = 0$. Indeed, we know that $(\bar{b}\bar{r}_1 \ldots \bar{r}_{k-1})\bar{c} = 0$ and hence Lemma 2.1 yields the result. Since $\bar{R}$ is nil-generated, it follows that $\bar{b}\bar{R}\bar{c} = 0$.

To show $\bar{R}$ has no nontrivial nilpotent elements, it suffices to show that the only square-zero element in $\bar{R}$ is 0. Suppose then there exists $\bar{x} \in \bar{R}$ with $\bar{x}^2 = 0$. The above argument yields $\bar{x}\bar{R}\bar{x} = 0$, so that $\bar{R}\bar{x}\bar{R}$ is a nilpotent ideal. Hence $\bar{x} = 0$ and it follows that $\mathcal{N}(\bar{R}) = 0$, as claimed. This implies $\mathcal{L}(R) = \mathcal{N}(R)$. $\square$

Proposition 2.3 *Let R be a nil-generated algebra over an infinite field. If $R^\times$ satisfies a group identity, then R satisfies polynomial identity.*

Proof. Let $w(y_1, \ldots, y_n) = 1$ be a group identity for $R^\times$. Put $k = F[t]$ and let A denote the completion of the free associative algebra $k\{x_1, \ldots, x_n\}$. The free group $F_n = \langle y_1, \ldots, y_n \rangle$ embeds into A by the well-known Magnus argument via the map φ induced by $\varphi(y_i) = 1 + tx_i$ (*cf.* [MKS], Section 5.5). It follows easily that

$$1 \neq \varphi(w(y_1, \ldots, y_n)) = 1 + \sum_{m=1}^{\infty} t^m p_m(x_1, \ldots, x_n),$$

where each $p_m(x_1, \ldots, x_n)$ is a homogeneous element of degree m in the free algebra $F\{x_1, \ldots, x_n\}$. Not all these elements can be trivial, so assume $p_{m_0}(x_1, \ldots, x_n) \neq 0$ in $F\{x_1, \ldots, x_n\}$.

Now consider $S = \langle r_1, r_2, \ldots, r_n \rangle$, the subalgebra in R generated by arbitrary elements $r_1, \ldots, r_n$ in $\mathcal{N}(R)$. Then S is nilpotent by Lemma 2.2, and so for each $\lambda \in F$ the map $t \to \lambda$, $x_i \to r_i$ induces a well-defined epimorphism ψ from the augmentation ideal of A to S. Since $w(y_1, \ldots, y_n) = 1$ is a group identity for $R^\times$, upon application of ψ we obtain

$$1 = w(1 + \lambda r_1, 1 + \lambda r_2, \ldots, 1 + \lambda r_n) = 1 + \sum_{m=1}^{\infty} \lambda^m p_m(r_1, \ldots, r_n)$$

for each $\lambda \in F$. (The sum is finite.) Now, using the fact that F is infinite, a routine Vandermonde matrix argument implies that each $p_m(r_1, \ldots, r_n) = 0$. Thus $p_{m_0}(x_1, \ldots, x_n) = 0$ is a polynomial identity for S, and hence for all of $\mathcal{N}(R)$. But $R = F \cdot 1 + \mathcal{N}(R)$, so that $p_{m_0}([x_1, x_2], \ldots, [x_{2n-1}, x_{2n}])$ is a nontrivial polynomial identity for R. $\square$

3 Existence of a non-matrix identity

To complete the proof of Theorem 1.2, it remains to prove the following:

Proposition 3.1 *Let R be a nil-generated algebra over an infinite field F. If $R^\times$ satisfies a group identity $\omega(y_1, \ldots, y_n) = 1$, then R satisfies a non-matrix identity.*

Proof. Recall that R is a PI-algebra by Proposition 2.3. To establish the statement about the existence of a non-matrix identity is more involved and we shall require some reductions (*cf.* proof of Proposition 1 in [GJV]).

First let us point out that it is enough to show that $\mathcal{N}(R)$ satisfies a non-matrix identity as R is a commutative extension of $\mathcal{N}(R)$ by Lemma 2.2.

Next, using that the fact that the derived subgroup of a free group of rank 2 is free of countably-infinite rank, we may also assume that ω is a word in two variables only. Furthermore, the substitution $y_1 = y_1y_2$ and $y_2 = y_2y_1$ allows us to assume that $R^\times$ satisfies a group identity of the form

$$\omega(y_1, y_2) = (y_1y_2)^{\alpha_1}(y_2y_1)^{\beta_1} \cdots (y_1y_2)^{\alpha_j}(y_2y_1)^{\beta_j}(y_1y_2)^{\alpha_{j+1}} = 1,$$

where $j \geq 1$ and the integers α_i and β_i are nonzero with the possible exception of α_{j+1}. The Magnus representation of ω now becomes

$$\omega(1 + x_1, 1 + x_2) = 1 + \sum_{m=1}^{\infty} p_m(x_1, x_2).$$

As is in the proof of Proposition 2.5, it follows that $\mathcal{N}(R)$ satisfies each of the polynomial identities $p_m(x_1, x_2) = 0$ (some of which may be trivial). In order to prove the proposition, it suffices for us to show that at least one of the p_m is not also satisfied by $M_2(F)$. Let us suppose then to the contrary. It follows from the Magnus representation of ω that $\omega(1 + a, 1 + b) = 1$ for each choice of nilpotent a, b in $M_2(F)$. Notice as well that $a^2 = 0$ implies that $(1 + a)^n = 1 + na$, for each integer n. It is easy to see that the reduced form of ω is of the type

$$\omega(y_1, y_2) = y_1^{\gamma_1} y_2^{\delta_1} \cdots y_1^{\gamma_k} y_2^{\delta_k} y_1^{\gamma_{k+1}},$$

where $k \geq 1$ and the integers γ_i and δ_i are one of $1, -1, 2$ or -2 with the possible exception of γ_{k+1}, which is one of $1, -1$ or 0. Now fix two square-zero elements $a, b \in M_2(F)$. Then for every $\lambda \in F$ we have

$$\omega(1 + \lambda a, 1 + \lambda b) = (1 + \gamma_1 \lambda a)(1 + \delta_1 \lambda b) \cdots (1 + \gamma_k \lambda a)(1 + \delta_k \lambda b)(1 + \gamma_{k+1} \lambda a).$$

But we also have

$$\omega(1 + \lambda a, 1 + \lambda b) = 1 + \sum_{m=1}^{l} \lambda^m p_m(a, b),$$

where $l = 2k+1$ unless $\gamma_{k+1} = 0$, in which case $l = 2k$. Comparing coefficients of λ^l we find that

$$(\gamma_1\gamma_2 \cdots \gamma_k)(\delta_1\delta_2 \cdots \delta_k)(ab)^{k+1} = 0.$$

In the case of characteristic $p \neq 2$, this yields $(ab)^{k+1} = 0$, which leads to the desired contradiction; for example, set $a = e_{12}$ and $b = e_{21}$.

It remains to consider the case of characteristic $p = 2$. Making the substitution $y_1 = y_1y_2$ and $y_2 = y_1y_3$ into the reduced form of ω above allows us to assume that $R^\times$ satisfies the following word:

$$\omega_2(y_1, y_2, y_3) = (y_1y_2)^{\gamma_1}(y_1y_3)^{\delta_1} \cdots (y_1y_2)^{\gamma_k}(y_1y_3)^{\delta_k}(y_1y_2)^{\gamma_{k+1}}.$$

Let us represent ω_2 by

$$\omega_2(1 + x_1, 1 + x_2, 1 + x_3) = 1 + \sum_{m=1}^{\infty} q_m(x_1, x_2, x_3).$$

As argued above, it suffices to show that $M_2(F)$ does not satisfy some q_m. Therefore, let us suppose otherwise and fix two square-zero elements $a, b \in M_2(F)$. Using the fact that the characteristic is 2, it is easy to check that $a + ba + ab + bab$ also has square zero. Evaluating the Magnus representation tells us

$$\omega_2(1 + \lambda a, 1 + \lambda b, 1 + \lambda(a + ba + ab + bab)) = 1.$$

Notice that in the reduced form of $\omega_2(y_1, y_2, y_3)$ the variables appear with exponents 1 or -1 only. Then because $(1 + a)^{-1} = 1 - a = 1 + a$, *etc.*, it follows that $\omega_2(1 + \lambda a, 1 + \lambda b, 1 + \lambda(a + ba + ab + bab))$ is merely an ordered product of the terms $1 + \lambda a, 1 + \lambda b$, and $1 + \lambda(a + ba + ab + bab)$ in which no two consecutive terms are equal. The triviality of the coefficient of the highest power of λ appearing in the resulting expansion leads to the fact that $(ab)^t = 0$ for some suitable $t > 0$. This in turn gives the desired contradiction. $\square$

4 Group algebras

Let us now consider group algebras FG of a torsion group over an infinite field F of prime characteristic. In [GSV] it was shown that whenever $(FG)^\times$ satisfies a group identity, then FG satisfies a polynomial identity. Passman ([Pa2]) subsequently characterised all torsion groups G such that $(FG)^\times$ satisfies a group identity. We are able to extend these results as follows:

Theorem 4.1 *Let FG be a group algebra of a torsion group over an infinite field F of characteristic $p > 0$. Then the following are equivalent:*

1. *$(FG)^\times$ satisfies a group identity;*

2. *FG satisfies a non-matrix identity;*

3. *G contains a normal subgroup A such that G/A and (A, A) are finite, and (G, G) is a p-group of finite exponent;*

4. *$[FG, FG]FG$ is nil of bounded index; and,*

5. *$((FG)^\times, (FG)^\times)$ is a p-group of finite exponent.*

Proof. Assume that (1) holds; we shall deduce (2). Let X be the set of p-elements in G and write P for the subgroup of G generated by X. Then the group algebra FP is nil-generated and $(FP)^\times$ satisfies a group identity. From Lemma 2.2 it follows that $\mathcal{N}(FP)$ is a locally nilpotent maximal ideal in FP. Therefore $\mathcal{N}(FP)$ coincides with the augmentation ideal of FP and P is a locally finite p-group. Using the normality of P in G, it follows that the ideal $\mathcal{N}(FP)FG$ in FG is also locally nilpotent. Now, according to Corollary 1.5, $\mathcal{N}(FP)FG$ satisfies a non-matrix identity as the adjoint group of $\mathcal{N}(FP)FG$ satisfies the identities of $(FG)^\times$. Also, because the kernel of the canonical projection $FG \to F(G/P)$ is $\mathcal{N}(FP)FG$, it follows that $(F(G/P))^\times$ satisfies a group identity. Since G/P is a p'-group, it follows that G/P is abelian as is shown in the semiprime case of [GSV]. Now FG satisfies some non-matrix identity by the fact that it is a commutative extension of an algebra satisfying a non-matrix identity.

The equivalence (1) $\Leftrightarrow$ (2) now follows from Proposition 1.1. (1) $\Rightarrow$ (3) is the statement of Lemma 2.4 in [Pa2]. The implication (3) $\Rightarrow$ (4) follows as in the proof Lemma 3.3 in [Pa2]. The remaining implications, (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1), are clear. $\square$

5 Restricted enveloping algebras

Let $u(L)$ be the restricted enveloping algebra of a restricted Lie algebra L over a field F of characteristic $p > 0$. Restricted enveloping algebras satisfying a polynomial identity were characterised independently by Passman and Petrogradski in [Pa1] and [Pe]: see Lemma 5.3 below. A restricted Lie algebra L is said to be p-nil if for every $x \in L$ there exists a natural number n such that $x^{p^n} = 0$. We are interested here in characterising p-nil restricted Lie algebras L for which $u(L)^\times$ satisfies a group identity. It follows from Jacobson's restricted analogue of the Poincaré-Birkhoff-Witt Theorem (see [J]) that for such an L, $u(L)$ is nil-generated. Therefore, according to Theorem 1.4, $u(L)$ satisfies a non-matrix identity precisely when $u(L)^\times$ satisfies a group identity. More specifically, we have:

Theorem 5.1 *If L is a p-nil restricted Lie algebra over an infinite field of characteristic $p > 0$, then the following statements are equivalent:*

1. *$u(L)^\times$ satisfies a group identity;*
2. *$u(L)$ satisfies a non-matrix identity;*
3. *$[L, L]$ is bounded p-nil and L contains a restricted ideal A of such that L/A and $[A, A]$ are finite-dimensional;*
4. *$u(L)$ satisfies the Engel condition; and,*
5. *$u(L)^\times$ satisfies an identity of the form $(y_1^{p^t}, y_2) = 1$ for some t.*

In fact, in Theorem 5.1 we need only assume that L can be generated by p-nil elements.

Corollary 5.2 *Let L be a virtually-(p-nil) restricted Lie algebra over an infinite field. If $u(L)^\times$ satisfies a group identity, then $u(L)$ satisfies a polynomial identity.*

Observe that some precondition on L is required in Corollary 5.2; indeed, the restricted enveloping algebra of a free restricted Lie algebra is a free associative algebra, and so has only the trivial unit group $F^\times$.

To prove Theorem 5.1, we shall make use of the result of Passman and Petrogradski mentioned above:

Lemma 5.3 *Let L be a restricted Lie algebra. Then its restricted enveloping algebra $u(L)$ satisfies a polynomial identity if and only if L possesses a restricted ideal (or subalgebra) A such that*

1. *A has finite codimension in L, and*

2. $[A, A]$ *is finite dimensional and* p*-nil.*

We shall also require Theorem 1.2 of [RS].

Lemma 5.4 *Let* L *be a restricted Lie algebra. Then its restricted enveloping algebra* $u(L)$ *satisfies the Engel condition if and only if*

1. L *is nilpotent,*

2. $[L, L]$ *is bounded* p*-nil, and*

3. L *possesses a restricted ideal* A *such that* L/A *and* $[A, A]$ *are finite dimensional.*

Only implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) in Theorem 5.1 do not follow directly from Theorem 1.4. To prove (3) $\Rightarrow$ (4), assume that (3) holds. Observe that the centraliser C of $[A, A]$ in A is of finite codimension in A, and in hence of finite codimension in L. Thus we may replace A by C, to assume that A is nilpotent of class 2. Now L is nilpotent-by-(finite-dimensional and p-nil). It follows from a result of Shalev ([Sh2], Proposition 5.1) that L is nilpotent. Now Lemma 5.4 yields the fact that $u(L)$ satisfies the Engel condition. It remains then to prove the following lemma.

Lemma 5.5 *If* L *is a* p*-nil restricted Lie algebra such that* $u(L)$ *satisfies a non-matrix identity, then* $[L, L]$ *is bounded* p*-nil and* L *contains a restricted ideal* A *such that* L/A *and* $[A, A]$ *are finite dimensional.*

Proof. The existence of A follows immediately from Lemma 5.3. From Theorem 1.4, there exists some t such that $u(L)$ satisfies an identity of the form $([x_1, x_2]x_3)^{p^t} = 0$. As argued above, L must be nilpotent. It remains to prove that any linear combination of commutators in L is p-nil of bounded index.

Claim 5.6 *For a sufficiently large integer* k*,* L *satisfies the identity*

$$(x + y)^{p^k} = x^{p^k} + y^{p^k}.$$

Proof. Let c be the nilpotency class of L, and choose k large enough that $p^k \geq p^t c$. Consider $\lambda \in F$ and expand $(x+\lambda y)^{p^k}$ to get:

$$(x+\lambda y)^{p^k} - x^{p^k} - \lambda^{p^k} y^{p^k} = \sum_{i \geq 1} \lambda^i h_i(x,y),$$

where each $h_i(x,y)$ in L is homogeneous in x, y of total degree p^k. Each h_i is a sum of elements of the form

$$[r_1^{p^{\alpha_1}}, \ldots, r_l^{p^{\alpha_l}}]^{p^\alpha},$$

where $p^\alpha(\sum_j p^{\alpha_j}) = p^k$ and $r_j \in \{x, y\}$. If $\alpha \geq t$ then this restricted Lie monomial is zero. On the other hand, if $\alpha < t$, then

$$\sum_j p^{\alpha_j} = p^{k-\alpha} \geq p^{k-t+1} \geq pc \geq c+1.$$

Therefore

$$\begin{aligned}
[r_1^{p^{\alpha_1}}, \ldots, r_l^{p^{\alpha_l}}] &= [r_1^{p^{\alpha_1}}, \underbrace{r_2, \ldots, r_2}_{p^{\alpha_2}}, \ldots, \underbrace{r_l, \ldots, r_l}_{p^{\alpha_l}}] \\
&= [-r_2, \underbrace{r_1, \ldots, r_1}_{p^{\alpha_1}}, \underbrace{r_2, \ldots, r_2}_{p^{\alpha_2}-1}, \ldots, \underbrace{r_l, \ldots, r_l}_{p^{\alpha_l}}] \\
&= 0,
\end{aligned}$$

being a commutator of length greater than c. $\square$

To finish the proof of Lemma 5.5, let r_i, s_i be arbitrary elements in L. Taking $p^k \geq p^t c$ as in the claim we have

$$(\sum_i \beta_i [r_i, s_i])^{p^k} = \sum_i \beta_i^{p^k} [r_i, s_i]^{p^k} = 0,$$

as required. $\square$

Corollary 5.2 follows by combining Theorem 5.1 with Lemma 5.3. $\square$

6 Concluding remark

Let us close by observing that not every nil PI-algebra satisfies a non-matrix identity. In light of our results, this is equivalent to the fact that the adjoint group of a nil PI-algebra need not satisfy a group identity.

Proposition 6.1 *Let F be an infinite field of characteristic $p \geq 0$. Then there exists a locally nilpotent associative algebra R over F such that R satisfies a polynomial identity, and yet its adjoint group R° does not satisfy any group identity.*

Proof. For the case of $p > 0$, consider the restricted Lie algebra L generated by the set $\{x, y_1, y_2, \ldots, z_1, z_2, \ldots\}$, subject to the relations: $[x, y_i] = z_i$ is central, $[y_i, y_j] = 0$, and $x^p = y_i^p = z_i^{p^i} = 0$, for all $i, j \geq 1$. Then the ideal of L generated by $\{y_1, y_2, \ldots\}$ is abelian and of codimension 1 in L. Hence, $R = L(u(L))$ satisfies a polynomial identity by Lemma 5.3. Furthermore, L is locally-(finite-dimensional and p-nil), so that R is locally nilpotent (see Lemma 2.4 of [RS], for example). However, R° does not satisfy any group identity for otherwise, by Theorem 5.1, $[L, L]$ would be bounded p-nil.

Now suppose that $p = 0$. Consider the exterior algebra E of an infinite dimensional F-space. Then E satisfies a PI: it is Lie nilpotent of class 2. Therefore, $R = E \otimes_F E \otimes_F E$ also satisfies a PI by a theorem of Regev, [Re]. Moreover, R is locally nilpotent since E is locally nilpotent. It was shown in [R], however, that R satisfies no non-matrix identity and so, by Theorem 1.3, R° cannot satisfy a group identity. $\square$

Notice that by Kaplansky's theorem any nil example to this effect cannot be finitely generated.

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