Irreducible Representations for Toroidal Lie Algebras

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Abstract: In this work a large number of irreducible representations with finite dimensional weight spaces are constructed for some toroidal Lie algebras. To accomplish this we develop a general theory of $\mathbb{Z}^n$-graded Lie algebras with polynomial multiplication. We construct modules by the standard inducing procedure and study their irreducible quotients using the vertex operator technics.

0. Introduction.

The purpose of this work is to construct a large class of irreducible representations, with finite dimensional weight spaces, for some toroidal Lie algebras. Let $\mathfrak{T}$ be a toroidal Lie algebra. The representations of this type which were investigated up till now appear in the context of studying some particular representations by vertex operators for $\mathfrak{T}$. Indeed, the works [F], [MRY], [EM] were the first to study vertex representations for toroidal Lie algebras and these use the homogeneous Heisenberg subalgebra while in [B1], [B2], and [T] a representation is studied using the principal Heisenberg subalgebra of $\mathfrak{T}$. This principal representation leads to an irreducible module with finite dimensional weight spaces. (Here, and throughout the paper, we use the term "weight space" to refer to the various homogeneous spaces in our algebras or modules. Most of the time this notation

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is transparent but when necessary, we will specify the grading in question. Often these spaces are, in fact, weight spaces relative to a Cartan subalgebra of the algebra in question, but there is no need to stress this.) Moreover, this principal vertex operator representation has the property that the image of the center of the core of $\mathfrak{T}$ (the core of $\mathfrak{T}$ is just the subalgebra generated by the non-isotropic root spaces) under the representation is infinite dimensional. We let $\mathcal{K}$ denote the center of the core of $\mathfrak{T}$. Besides these vertex operator representations some work has been done from the more abstract, Verma module, point of view. In [BC] (see also [CF]) Verma type modules, obtained as induced modules, were investigated and conditions were given to determine when such modules are irreducible. However, these Verma modules do not have finite dimensional weight spaces and their irreducible quotients are not integrable and seem very hard to work with. Moreover, for these modules, $\mathcal{K}$ is represented by a single scalar c (the central charge) so the image of $\mathcal{K}$ under this representation is one dimensional if $c \neq 0$. In the present paper we combine these two approaches using ideas from the known vertex operator representations as well as the basic Verma module approach of obtaining modules by first inducing and then factoring. In the end we obtain modules with finite dimensional weight spaces and where most of $\mathcal{K}$ acts non-trivially.

Before we go on to describe this work in more detail we recall some of the relevant results from the work [B1]. Thus we let

$$\mathfrak{T} = \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0^{\pm 1}, \ldots, t_n^{\pm 1}] \oplus \mathcal{K} \oplus \mathcal{D}^*$$

be a toroidal Lie algebra where $\dot{\mathfrak{g}}$ is any finite dimensional simple Lie algebra over the complex field $\mathbb{C}$ and $\mathbb{C}[t_0^{\pm 1}, \ldots, t_n^{\pm 1}]$ is the algebra of Laurent polynomials in $n+1$ variables. Also assume $n \geq 1$. Here $\mathcal{K}$ is the Kähler differentials of $\mathbb{C}[t_0^{\pm 1}, \ldots, t_n^{\pm 1}]$ modulo the exact forms and $\mathcal{D}^*$ the Lie subalgebra of the full derivation algebra of $\mathbb{C}[t_0^{\pm 1}, \ldots, t_n^{\pm 1}]$ given by $\mathcal{D}^* = \mathcal{D}^{**} \oplus \mathbb{C}d_0$ where

$$\mathcal{D}^{**} = \left\{ \sum_{p=1}^{n} f_p(t_0, \ldots, t_n) d_p \Big| f_1, \ldots, f_n \in \mathbb{C}[t_0^{\pm 1}, \ldots, t_n^{\pm 1}] \right\}$$

and d_i is the degree derivation associated to the variable t_i for $0 \leq i \leq n$. The algebra $\mathcal{D}^*$ is denoted by $\mathcal{D}^+$ in [B1] but here it is more convenient to use $\mathcal{D}^*$. Also, in [B1] there is a cocycle which comes into play and so the usual multiplication in $\mathcal{D}^*$ is adjusted by the cocycle

$$\tau(t_0^{r_0}\mathbf{t}^{\mathbf{r}} d_a, t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_b) = -m_a r_b \{ \sum_{p=0}^{n} r_p t_0^{r_0+m_0}\mathbf{t}^{\mathbf{r}+\mathbf{m}} k_p \},$$

which maps from $\mathcal{D}^* \times \mathcal{D}^*$ to $\mathcal{K}$. Here we are using the usual notation fk_i for the element of $\mathcal{K}$ corresponding to the differential $ft_i^{-1}dt_i$ for $f \in \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \dots, t_n^{\pm}]$. This is the very same toroidal Lie algebra which arose in [MRY] and [EM].

Let $\mathfrak{v} = \mathcal{K} \oplus \mathcal{D}^*$. The algebra $\mathfrak{v}$ has a (degenerate) Heisenberg subalgebra $\mathfrak{s}$ with basis $\left\{t_0^i d_p, t_0^i k_p, k_0\right\}$, $i \in \mathbb{Z}$, $p = 1, \dots, n$ Indeed, k_0 is its central element and the multiplication in $\mathfrak{s}$ is given by

$$[t_0^i d_a, t_0^j k_b] = i\delta_{ab}\delta_{i,-j}k_0,$$

$$[t_0^i d_a, t_0^j d_b] = 0, \qquad [t_0^i k_a, t_0^j k_b] = 0,$$

where $i, j \in \mathbb{Z}, a, b, p = 1, \dots, n$. This subalgebra is degenerate as a Heisenberg algebra since $[d_p, k_p] = 0$.

The Heisenberg algebra $\mathfrak{s}$ can be represented on the space

$$F = \mathbb{C}[q_p^{\pm 1}, u_{pi}, v_{pi}]_{i \in \mathbb{N}}^{p=1,\dots,n}$$

by differentiation and multiplication operators as follows.

$$\varphi(t_0^i d_p) = \frac{\partial}{\partial u_{pi}}, \qquad \varphi(t_0^{-i} d_p) = iv_{pi},$$

$$\varphi(t_0^i k_p) = \frac{\partial}{\partial v_{pi}}, \qquad \varphi(t_0^{-i} k_p) = iu_{pi},$$

$$\varphi(d_p) = q_p \frac{\partial}{\partial q_p}, \qquad \varphi(k_p) = 0,$$

$$\varphi(k_0) = \mathrm{Id},$$

where $i \geq 1$ and $p = 1, \dots, n$.

We give the module F a $\mathbb{Z}$-grading by assigning degrees of the variables in the following way,

$$\mathrm{deg} u_{pi} = \mathrm{deg} v_{pi} = -i, \qquad \mathrm{deg} q_p = 0.$$

One extends this representation from $\mathfrak{s}$ to $\mathfrak{v}$ using vertex operators. Indeed,using the usual notation from [FLM] consider the following elements of $\mathrm{End}_{gr}(F)[[z, z^{-1}]]$:

$$k_p(z) = \sum_{i \geq 1} iu_{pi} z^i + \sum_{i \geq 1} \frac{\partial}{\partial v_{pi}} z^{-i},$$

$$d_p(z) = \sum_{i \geq 1} iv_{pi} z^i + q_p \frac{\partial}{\partial q_p} + \sum_{i \geq 1} \frac{\partial}{\partial u_{pi}} z^{-i},$$

$$k(z,\mathbf{r}) = \mathbf{q}^{\mathbf{r}} \exp\left(\sum_{p=1}^{n} r_p \sum_{j\geq 1} z^j u_{pj}\right) \exp\left(-\sum_{p=1}^{n} r_p \sum_{j\geq 1} \frac{z^{-j}}{j} \frac{\partial}{\partial v_{pj}}\right).$$

Here $\mathbf{q}^{\mathbf{r}} = q_1^{r_1} \dots q_n^{r_n}$. Then from [B1] we have the following result.

Proposition. The mapping $\varphi : \mathfrak{v} \to \mathrm{End}(F)$ given by

$$\sum_{j\in\mathbb{Z}} \varphi(t_0^j \mathbf{t}^{\mathbf{r}} k_0) z^{-j} = k(z,\mathbf{r}),$$

$$\sum_{j\in\mathbb{Z}} \varphi(t_0^j \mathbf{t}^{\mathbf{r}} k_p) z^{-j} = k_p(z) k(z,\mathbf{r}),$$

$$\sum_{j\in\mathbb{Z}} \varphi(t_0^j \mathbf{t}^{\mathbf{r}} d_p) z^{-j} = :d_p(z) k(z,\mathbf{r}):,$$

$$\varphi(d_0) = -\sum_{p=1}^{n} \sum_{i=1}^{\infty} i \left(u_{pi} \frac{\partial}{\partial u_{pi}} + v_{pi} \frac{\partial}{\partial v_{pi}}\right),$$

defines a representation of $\mathfrak{v}$.

This proposition is part of the statement of Theorem 5 from [B1]. The following factorization property also holds and is very important for our purposes.

$$k_p(z,\mathbf{r}+\mathbf{m}) = k_p(z,\mathbf{r}) k(z,\mathbf{m}), \qquad p = 0,\dots,n. \tag{0.1}$$

The module F served as a model for all of the modules which we construct. Moreover, it is the factorization property which plays a crucial role in our construction. That this representation is irreducible is easy to see. Indeed, from the action of $d_1,\dots,d_n$ we see that every submodule is homogeneous with respect to $q_1,\dots,q_n$. Then, the action of the Heisenberg subalgebra on $\mathbb{C}[u_{pi}, v_{pi}]$ is irreducible, thus every non-zero submodule contains a vector $\mathbf{q}^{\mathbf{m}}$ for some $\mathbf{m} \in \mathbb{Z}^n$. The action of $\mathbb{C}[t_1^{\pm},\dots,t_n^{\pm}]k_0$ on such a vector generates the space $\mathbb{C}[q_1^{\pm},\dots,q_n^{\pm}]$ and finally considering again the action of the Heisenberg subalgebra we recover the whole module. Clearly this module has finite dimensional homogeneous spaces in the obvious $\mathbb{Z}^{n+1}$-grading. Thus, in the language introduced above, we simply say it has finite dimensional weight spaces.

Note that in this construction the variable t_0 plays a special role. However the appearance of a "distinguished direction" is inevitable. Indeed, in an irreducible module with weight decomposition every central element of weight zero must act as a scalar operator. Thus a subspace of codimension 1 in the span of $k_0, k_1, \dots, k_n$ must be represented by zero operators. This makes one direction special. All of the modules we study in this paper also have this property. They are defined with a specified "distinguished direction" singled out. Later, in the body of the paper, we will give a general argument showing that, in fact, the intersection of the kernel of the above representation with $\mathcal{K}$ is just of dimension n.

Our construction will work for algebras, $\mathfrak{T}$, slightly more general than the one described above. We will work with more general cocycles, but still with values in $\mathcal{K}$, and we can use the algebra $\mathcal{D}^*$, as above, or also (for most of the paper) the full derivation algebra of $\mathbb{C}[t_0^{\pm 1}, \dots, t_n^{\pm 1}]$. However, at one point, we do need to restrict ourselves to the case of $\mathcal{D}^*$.

Essentially our construction begins with a finite dimensional module, W, for the Lie algebra $gl_n(\mathbb{C})$. We then construct a module, $T(W)$, from this which is a module for the Lie algebra Der $\left(\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]\right)$. As a vector space $T(W) = \mathbb{C}[q_1^{\pm 1}, \dots, q_n^{\pm 1}] \otimes W$, where $\mathbb{C}[q_1^{\pm 1}, \dots, q_n^{\pm 1}]$ is the algebra of Laurent polynomials in the variables $q_1, \dots, q_n$. We then notice this is in fact a module for the zero component, $\mathfrak{v}_0$, in the $\mathbb{Z}$-grading, of the algebra $\mathfrak{v}$. Here, the $\mathbb{Z}$-grading corresponds to the grading in the variable t_0 so by eigenspaces for the degree derivation d_0. Thus, in our work it is the "zero direction" which is chosen as special. We call $T(W)$ a tensor module and note here it has been investigated in [L1] and [R] and is related to the modules studied in [Rud]. Noting that we have the decomposition $\mathfrak{v} = \mathfrak{v}_+ \oplus \mathfrak{v}_0 \oplus \mathfrak{v}_-$ we then extend the above action to $\mathfrak{v}_+ \oplus \mathfrak{v}_0$ by letting $\mathfrak{v}_+$ act as zero and form the induced module

$$M := \mathrm{Ind}_{\mathfrak{v}_0 \oplus \mathfrak{v}_+}^{\mathfrak{v}}(T).$$

As usual, it turns out that among the submodules of M, intersecting the top $T(W)$ trivially, there exists a unique maximal one which we denote by M^{rad}. Moreover, if W is irreducible as a gl_n-module then the factor module $L = M/M^{rad}$ is an irreducible $\mathfrak{v}$-module. Also, we are able to show that, in the obvious $\mathbb{Z}$-grading of L, the homogeneous spaces are finite dimensional. This fact is quite non-trivial and follows from a more general result (of independent interest) that we establish in Section 1.

The next step in our construction is to take an irreducible highest weight module for the affine Kac-Moody Lie algebra

$$\dot{\mathfrak{g}} \otimes \mathbb{C}[t_0, t_0^{-1}] \oplus \mathbb{C}k_0 \oplus \mathbb{C}d_0$$

and tensor this with the module L above and show the result can be made into a module for our toroidal algebra $\mathfrak{T}$. Here, we need to assume that the central charge is the same for both modules. It is in doing this that we need to assume we are working with $\mathcal{D}^*$. The factorization property, mentioned above, also comes into play here and we need to show the module L has this property.

The structure of this paper is as follows. In Section 1 we begin by presenting the basic results on $\mathbb{Z}$-graded Lie algebras which we will need later. In particular, we define a certain completion for the universal enveloping algebra of a $\mathbb{Z}$-graded Lie algebra and show how it is possible to exponentiate certain elements when the Lie algebra is abelian so the universal enveloping algebra is just the symmetric algebra. We then go on to define the notions of a Lie algebra with polynomial multiplication and of a module, for such an algebra, to have polynomial action. Working in great generality, we show, in this setting, how certain quotients of induced modules will have finite dimensional homogeneous spaces. This result is, no doubt, of independent interest and can be used for algebras other than toroidal Lie algebras. It is for this reason that we have presented these results in the first section.

In Section 2 we introduce the toroidal Lie algebras which will occupy us for the rest of the paper. As mentioned above we work with a general 2-cocycle from a 2-dimensional cohomology group and allow, until Section 4, either derivation algebra $\mathcal{D}^*$ or $\mathcal{D}$. We also introduce the tensor modules for our algebra $\mathfrak{v}$ and show how the results of Section 1 apply to these algebras and modules. In particular, we show that we have a large supply of $\mathfrak{v}$-modules, L, with finite dimensional homogeneous spaces. At the end of this section we make some comments on this construction when an algebra with a smaller $\mathcal{K}$ is used. Finally, we close the section by showing that if the central charge is non-zero then the kernel of the action of our $\mathcal{K}$ on the above module is spanned by the elements $k_1, \ldots, k_n$.

Section 3 deals with the factorization property (0.1). The goal here is to show the above modules have this property. This factorization property suggests that there is an exponential at play in our construction and indeed, to prove what we need, we must introduce a certain exponential of a generating series. Our motivation here came from [B1]. To do this in a mathematically sound way we need to work in a completion of a twisted version (twisted by automorphisms) of the universal enveloping algebra of $\mathfrak{v}$ tensored with Laurent polynomials in n variables. The factorization property follows by showing that the moments of certain of our generating series, which now make sense thanks to the above mentioned completion, when acting on the module L, act as zero. Here, since $L = M/M^{rad}$, it is enough to show the moments in question take M to M^{rad}. It should be noted here that the completed algebra which we have to introduce seems to be of independent interest.

In the final section, Section 4, we just need to put our previous results together. To do this we need to assume the derivation algebra is $\mathcal{D}^*$. Using this, as well as the factorization property, we show how to get a module for our toroidal Lie algebra $\mathfrak{T}$ from a highest weight irreducible affine Lie algebra module V, and one of the $\mathfrak{v}$-modules L above. Finally we see that this module is irreducible and has finite dimensional weight spaces.

After completing this work we learned of the preprints of Larsson, [L2], and Iohara et al., [ISW] where related problems are addressed. Larsson considers an algebra $\mathfrak{w} = \mathcal{D} \oplus \mathcal{M}$ where $\mathcal{M}$ is an abelian ideal of $\mathfrak{w}$ (containing $\mathcal{K}$), and $\mathfrak{w}/\mathcal{M} = \mathcal{D}$ is the algebra of vector fields on an $n+1$-dimensional torus. He constructs explicit realizations for a class of representations of $\mathfrak{w}$ and has an analog of Theorem 4.2 for the algebra $\mathfrak{w}$. It is clear that these modules can be also presented via an induction procedure similar to ours here. In [ISW] an analog of Theorem 4.2 is presented for 2-toroidal Lie algebras. Besides this, there is a lot of recent work dealing with different aspects of toroidal Lie algebras, see [IKU],[L3],[STU],[VV], and the references therein.

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1. Lie Algebras with polynomial multiplication.

In this section we work in a fairly general setting and establish some basic results about $\mathbb{Z}$-graded Lie algebras which will be used later. We then go on to introduce Lie algebras and modules which have multiplication (or action) tied to polynomials. The major result of this section says that a certain irreducible quotient of an induced module has finite dimensional homogeneous components. All of our vector spaces and algebras will be over the complex field $\mathbb{C}$.

We will use the notation in [FLM] throughout. In particular, if V is a vector space we let $V[[z, z^{-1}]]$ denote the formal Laurent series with coefficients in V so that

$$V[[z, z^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} x_n z^{-n} \middle| x_n \in V \right\}.$$

If $x(z) = \sum_{n\in\mathbb{Z}} x_n z^{-n}$ we say that x_{-n} is the n-th moment of the series $x(z)$.

If $V = \bigoplus_{n\in\mathbb{Z}} V_n$ is a $\mathbb{Z}$-graded vector space we let $V_{gr}[[z, z^{-1}]]$ be defined by

$$V_{gr}[[z, z^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} x_n z^{-n} \middle| x_n \in V_n \right\}.$$

We will say a Laurent series $\sum_{n\in\mathbb{Z}} x_n z^{-n}$ is *restricted* if and only if there is some N such that $j \geq N$ implies that $x_j = 0$. Also, if $G = \bigoplus_{n\in\mathbb{Z}} G_n$ is a $\mathbb{Z}$-graded algebra (either Lie or associative) and if V is a G-module then for $g(z) = \sum\limits_{n\in\mathbb{Z}} g_n z^{-n} \in G[[z, z^{-1}]]$ and $x \in V$ we let $g(z)x$ be defined by

$$g(z)x = \sum_{n\in\mathbb{Z}} (g_n x) z^{-n}.$$

We say that the G-module V is *restricted* if

$$\forall \quad v \in V \quad \exists \quad p \in \mathbb{Z} \quad \text{such} \quad \text{that} \quad \forall \quad i > p, \qquad G_i v = 0.$$

It is clear that a G-module is restricted if and only if for any $g(z) \in G_{gr}[[z, z^{-1}]]$ and any $v \in V$ the series $g(z)v \in V[[z, z^{-1}]]$ is restricted. For later use we record the following simple lemma.

Lemma 1.1. Let V (respectively W) be a restricted module for the $\mathbb{Z}$-graded algebra G(respectively H). Then for all $g(z) \in G_{gr}[[z, z^{-1}]]$ and $h(z) \in H_{gr}[[z, z^{-1}]]$ there is a well-defined mapping

$$g(z) \otimes h(z) : V \otimes W \to (V \otimes W)[[z, z^{-1}]]$$

given by

$$g(z) \otimes h(z)(v \otimes w) = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} (g_i v \otimes h_j w)\, z^{-i-j}.$$

Proof. The Lemma follows from the fact that the product of two restricted Laurent series is well-defined and restricted. **Q.E.D.**

A standard way of constructing restricted modules is via the inducing procedure. Let G be a $\mathbb{Z}$-graded Lie algebra and let T be a G_0-module. Let $G_+ = \bigoplus\limits_{i>0} G_i$ and $G_- = \bigoplus\limits_{i<0} G_i$ so that G_+, G_- and G_0 are subalgebras of G. We extend the action of G_0 on T to an action of $G_0 \oplus G_+$ by letting G_+ act trivially and then consider the induced module

$$M = \mathrm{Ind}_{G_0\oplus G_+}^{G}(T) = U(G) \otimes_{U(G_0\oplus G_+)} T \cong U(G_-) \otimes_{\mathbb{C}} T.$$

The module M is a restricted $U(G)$-module and is $\mathbb{Z}$-graded with $M_0 = T$ and $M_s = (0)$ for $s > 0$. We often call T the top of M.

We define the *radical* of M, denoted M^{rad}, to be the maximal $\mathbb{Z}$-graded submodule of M that intersects with T trivially. It is a standard result that M^{rad} is unique and its $\mathbb{Z}$-graded components are given by the following condition.

$$M_{-k}^{rad} = \left\{ v \in M_{-k} \middle| uv = 0 \quad \text{for} \quad \text{all} \quad u \in U(G_+)_k \right\}. \tag{1.2}$$

Due to the Poincare-Birkhoff-Witt theorem, the components of the $\mathbb{Z}$-grading of the universal enveloping algebra $U(G)$ can be written as

$$U(G)_k = \bigoplus_{\substack{j\geq 0, i\leq 0\\ i+j=k}} U(G_-)_i \otimes U(G_0) \otimes U(G_+)_j.$$

For later use we define a certain completion of $U(G)$ which we denote by $\overline{U}(G)$. Thus let

$$\overline{U}(G)_k = \prod_{\substack{j\geq 0, i\leq 0,\\ i+j=k}} U(G_-)_i \otimes U(G_0) \otimes U(G_+)_j, \tag{1.3}$$

and then let $\overline{U}(G) = \bigoplus_{k\in\mathbb{Z}} \overline{U}(G)_k$. It follows from the Poincare-Birkhoff-Witt theorem that $\overline{U}(G)$ becomes an algebra in the natural way and moreover this algebra acts naturally on any restricted $U(G)$-module. In particular it acts on M above.

We now assume that G is an abelian Lie algebra so that $U(G)$ is just the symmetric algebra $S(G)$. Note if we have elements $g(z) = \sum_{n\geq 0} g_n z^{-n}, h(z) = \sum_{n\geq 0} h_n z^{-n} \in S(G_+)_{gr}[[z^{-1}]]$ then their product is a well-defined element in $S(G_+)_{gr}[[z^{-1}]]$. In fact, if $g_0 = 0$ even $\exp(g(z)) = \sum_{m\geq 0} \frac{1}{m!} g(z)^m$ is a well-defined element in $S(G_+)_{gr}[[z^{-1}]]$. Similar remarks hold for series in $S(G_-)_{gr}[[z]]$.

For $g(z) = \sum_{n\geq 0} g_n z^{-n} \in S(G_+)_{gr}[[z^{-1}]], h(z) = \sum_{n\leq 0} h_n z^{-n} \in S(G_-)_{gr}[[z]]$ the product $g(z)h(z)$ is defined to be the element $\sum_{m\in\mathbb{Z}} a_m z^{-m} \in \overline{S}(G)_{gr}[[z, z^{-1}]]$ where for each $m \in \mathbb{Z}$ a_m is the element of $\prod_{j\leq 0, m-j\geq 0} S(G_-)_j S(G_+)_{m-j} \subset \overline{S}(G)_m$ specified by saying $a_m = \sum_{j\leq 0, m-j\geq 0} h_j g_{m-j}$. Thus $g(z)h(z) \in \overline{S}(G)_{gr}[[z, z^{-1}]]$.

Now let $d(z) = \sum_{n\in\mathbb{Z}} d_n z^{-n} \in G_{gr}[[z, z^{-1}]] \subset S(G)_{gr}[[z, z^{-1}]]$ and assume $d_0 = 0$. Let $d^+(z) = \sum_{n>0} d_n z^{-n}, d^-(z) = \sum_{n<0} d_n z^{-n}$. Then letting $e^+(z) = \exp(d^+(z)), e^-(z) = \exp(d^-(z))$, we find that the product $e^-(z)e^+(z)$ is defined, from what we said above, and is in $\overline{S}(G)_{gr}[[z, z^{-1}]]$. We then define $\exp(d(z))$ by the formula

$$\exp(d(z)) = e^-(z)e^+(z) = \exp\left(d^-(z)\right) \exp\left(d^+(z)\right). \tag{1.4}$$

Later we will need to multiply certain elements in $\overline{S}(G)_{gr}[[z, z^{-1}]]$. The following result allows us to do this.

Proposition 1.5. Under the natural product $\overline{S}(G)_{gr}[[z, z^{-1}]]$ is an associative algebra with identity.In fact,we can multiply elements of the form $z^i f(z), z^j g(z), i, j \in \mathbb{Z}$ where both $f(z), g(z)$ belong to $\overline{S}(G)_{gr}[[z, z^{-1}]]$.

Proof. We represent an element in $\overline{S}(G)_n = \prod_{k\geq 0, k\geq n} S(G_-)_{n-k}S(G_0)S(G_+)_k$ as a function

$$f_n : \mathbb{Z} \to \bigcup_{k\geq 0, k\geq n} S(G_-)_{n-k}S(G_0)S(G_+)_k.$$

where $f_n(k)$ is in $S(G_-)_{n-k}S(G_0)S(G_+)_k$ for all $k \in \mathbb{Z}$ and where we use the convention that $f_n(k) = 0$ if either $k < 0$ or if $k < n$.

Let $f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n}, g(z) = \sum_{n\in\mathbb{Z}} g_n z^{-n}$ be two elements in $\overline{S}(G)_{gr}[[z, z^{-1}]]$ where $f_n, g_n \in \overline{S}(G)_n$. We would like to define $f(z)g(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n}$ where for each $n \in \mathbb{Z}$ the element h_n is defined by

$$h_n = \sum_{i\in\mathbb{Z}} f_{n-i}g_i.$$

To do this we must see that for fixed $k, n \in \mathbb{Z}$ the sum

$$\sum_{i\in\mathbb{Z}} (f_{n-i}g_i)(k) = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} f_{n-i}(k-j)g_i(j)$$

has only a finite number of non-zero terms.

Clearly if $k < 0$ all terms are zero so suppose $k \geq 0$. We have that $f_{n-i}(k-j)g_i(j) = 0$ unless both $j \geq 0$ and $j \leq k$. Thus, there are only a finite number of indices j where there is some i for which $f_{n-i}(k-j)g_i(j)$ is non-zero. For each such j we have $f_{n-i}(k-j)g_i(j)$ can be non-zero only when $j \geq i$ and $k - j \geq n - i$. But these conditions imply that $j \geq i$ and $i \geq n+j-k$ so there are only a finite number of indices $i \in \mathbb{Z}$ with $f_{n-i}(k-j)g_i(j) \neq 0$ for some j. All in all it follows the product on $\overline{S}(G)_{gr}[[z, z^{-1}]]$ is well-defined and makes $\overline{S}(G)_{gr}[[z, z^{-1}]]$ into an associative algebra with identity. The statement about shifts in powers of z is clear. **Q.E.D.**

We now begin to describe algebras with polynomial multiplication. For this purpose we let G be a $\mathbb{Z}^n$-graded Lie algebra.

$$G = \bigoplus_{\mathbf{m}\in\mathbb{Z}^n} G_{\mathbf{m}}.$$

Definition 1.6. The $\mathbb{Z}^n$-graded Lie algebra G is said to be an algebra with *polynomial multiplication* if G has a homogeneous spanning set $\{g_i(\mathbf{m})\}_{i\in I, \mathbf{m}\in\mathbb{Z}^n}$ such that $g_i(\mathbf{m}) \in G_{\mathbf{m}}$, and there exists a family of polynomials in $2n$ variables $\{p_{ij}^s\}$ where $i, j, s \in I$ and where for each $i, j \in I$ the set $\{s | p_{ij}^s \neq 0\}$ is finite, and satisfies

$$[g_i(\mathbf{k}), g_j(\mathbf{m})] = \sum_s p_{ij}^s(\mathbf{k}, \mathbf{m}) g_s(\mathbf{k} + \mathbf{m}) \quad \text{for} \quad \text{all} \quad \mathbf{k}, \mathbf{m} \in \mathbb{Z}^n. \tag{1.7}$$

If this is the case we say the spanning set $\{g_i(\mathbf{m})\}_{i\in I,\mathbf{m}\in\mathbb{Z}^n}$ is *distinguished.*

Notice that in the above definition we used a spanning set and not a basis and that when $n = 0$ the above definition says nothing, as then the polynomials are just constants and the definition expresses that we have a spanning set. Thus, we are usually interested only in the case that $n \geq 1$.

Definition 1.8. Let G be a Lie algebra with polynomial multiplication and with the notation as above. We will say a $\mathbb{Z}^n$-graded G-module V is a module with *polynomial action* if V has a basis $\{v_j(\mathbf{m})\}_{j\in J}$ such that $v_j(\mathbf{m}) \in V_{\mathbf{m}}$ for $j \in J$, and there exists a family of polynomials in $2n$ variables $\{f_{ij}^s\}$ where $i \in I, j, s \in J$ and for each $i \in I, j \in J$ the set $\{s | f_{ij}^s \neq 0\}$ is finite and satisfies

$$g_i(\mathbf{k})v_j(\mathbf{m}) = \sum_s f_{ij}^s(\mathbf{k},\mathbf{m})v_s(\mathbf{k}+\mathbf{m}) \quad \text{for} \quad \mathbf{k},\mathbf{m} \in \mathbb{Z}^n.$$

Notice that in this definition we used a basis instead of a spanning set. Again we say the basis used is distinguished. Most of the modules we will consider will be induced, and hence have a specified special variable (we will later denote this t_0) and so in the next definition we need to consider both a $\mathbb{Z}^n$ and a $\mathbb{Z}$-grading of our Lie algebra G. We then have

$$G = \bigoplus_{\mathbf{m}\in\mathbb{Z}^n} G_{\mathbf{m}} \quad \text{and} \quad G = \bigoplus_{i\in\mathbb{Z}} G_i.$$

We will say these gradings are *compatible* if each space $G_{\mathbf{m}}$ is homogeneous in the $\mathbb{Z}$-grading and each space G_i is homogeneous in the $\mathbb{Z}^n$-grading. In this case we write $G_{i,\mathbf{m}}$ for the intersection $G_i \cap G_{\mathbf{m}}$. Thus G is $\mathbb{Z}^{n+1}$-graded so $G = \bigoplus_{(i,\mathbf{m})\in\mathbb{Z}^{n+1}} G_{i,\mathbf{m}}$.

Definition 1.9. An algebra G with polynomial multiplication is called *extragraded* if it has an additional $\mathbb{Z}$-grading compatible with the $\mathbb{Z}^n$-grading such that the distinguished spanning set is homogeneous in the $\mathbb{Z}^{n+1}$-grading and for fixed $i \in I$, $g_i(\mathbf{m})$ has the same degree in the $\mathbb{Z}$-grading for any $\mathbf{m} \in \mathbb{Z}^n$. We write this degree as $\deg(g_i)$ and call it the degree of g_i. Thus,

$$g_i(\mathbf{m}) \in G_{\deg g_i,\mathbf{m}} \quad \text{for} \quad \mathbf{m} \in \mathbb{Z}^n.$$

For G to be extragraded we also require that the number of $i \in I$, with g_i of a fixed degree, is finite.

We now assume that G is a Lie algebra which is extragraded and use the notation as above so that the decomposition $G = G_+ \oplus G_0 \oplus G_-$ refers to the $\mathbb{Z}$-grading. It is clear from the definition that the subalgebra $G_0 = \bigoplus_{\mathbf{m}\in\mathbb{Z}^n} G_{0,\mathbf{m}}$ is a Lie algebra with polynomial multiplication whose distinguished spanning set is $\{g_i(\mathbf{m}) | \deg g_i = 0, \mathbf{m} \in \mathbb{Z}^n\}$.

Let T be a G_0-module with polynomial multiplication and let the distinguished basis be $\{v_j(\mathbf{m}) | j \in J, \mathbf{m} \in \mathbb{Z}^n\}$. We can define an action of G_+ on T by $G_+ T = (0)$ and then consider the induced module

$$M = \mathrm{Ind}_{G_0 \oplus G_+}^{G} T \cong U(G_-) \otimes T.$$

Proposition 1.10. For a sequence $\gamma = (i_1, \ldots, i_\ell)$ of indicies from I such that $\sum_{b=1}^{\ell} \deg g_{i_b} = 0$, there exists a family of polynomials $p^s_{\gamma j}$, with $s, j \in J$, such that

$$g_{i_1}(\mathbf{m}_1) \ldots g_{i_\ell}(\mathbf{m}_\ell) v_j(\mathbf{m}_0) = \sum_s p^s_{\gamma j}(\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_\ell) v_s(\mathbf{m}_0 + \mathbf{m}_1 + \ldots + \mathbf{m}_\ell), \quad (1.11)$$

where the set $\{s | p^s_{\gamma j} \neq 0\}$ is finite.

Proof. We prove this by induction on ℓ. If $\ell = 1$ then $\deg g_{i_1} = 0$ and then

$$g_{i_1}(\mathbf{m}_1) v_j(\mathbf{m}_0) = \sum_s f^s_{i_1 j}(\mathbf{m}_1, \mathbf{m}_0) v_s(\mathbf{m}_0 + \mathbf{m}_1)$$

because T is a G_0-module with polynomial action. Here we have used the notation from Definition 1.6.

Next consider the case of $\ell \geq 2$. If $\deg g_{i_1} = \ldots = \deg g_{i_\ell} = 0$ then, by iteration of the previous case, we see that our hypothesis immediately implies the result. Otherwise, by the usual type of Poincare-Birkhoff-Witt argument, any expression of the form

$$g_{i_1}(\mathbf{m}_1) \ldots g_{i_\ell}(\mathbf{m}_\ell)$$

can be rearranged so that the terms of positive degree appear to the right, and thus applied to $v_j(\mathbf{m}_0)$ gives 0. However in doing this rearranging process, we must also add terms involving commutators. Notice that the terms involving commutators can be replaced using (1.7) by expressions of the form (1.11) of length shorter than ℓ with coefficients given by polynomials. Thus, by induction, the result follows. **Q.E.D.**

The main result about extragraded Lie algebras is the following Theorem. Notice that we need to use a basis for our module here, but that the distinguished spanning set for the algebra need not be a basis. Also, towards the end of the proof of the theorem we use a Vandermonde type argument. Here is what we mean by that. If a polynomial $a_n(y)x^n + a_{n-1}(y)x^{n-1} + \ldots + a_0(y)$ assumes zero values for all x then $a_n(y) = a_{n-1}(y) = \ldots = a_0(y) = 0$. One way of proving this is to evaluate this polynomial at $n+1$ distinct values of x, which gives a homogeneous system of linear equations on $a_n(y), a_{n-1}(y), \ldots, a_0(y)$

with a Vandermonde matrix. Then $a_n(y) = a_{n-1}(y) = \ldots = a_0(y) = 0$ follows from the fact that the Vandermonde determinant is non-zero.

Theorem 1.12. Let G be an extragraded Lie algebra with polynomial multiplication. Let T be an G_0-module with polynomial action and with distinguished basis $\{v_j(\mathbf{m})\}_{\mathbf{m}\in\mathbb{Z}^n}^{j\in J}$ in which the set J is finite. Then the homogeneous components of $L = M/M^{rad}$ are finite-dimensional. Furthermore if T is an irreducible G_0-module then L is an irreducible G-module.

Proof. Because M^{rad} is the sum of all graded submodules intersecting trivially with T then the statement about irreducibility is clear.

To prove that the homogeneous component $L_{s,\mathbf{m}}$, $s \leq 0$ is finite-dimensional, we will show that $M^{rad}_{s,\mathbf{m}}$ may be described inside of $M_{s,\mathbf{m}}$ by a system of finitely many linear equations.

Define the following finite sets, for $s \leq 0$ let

$$\Omega_s^+ = \left\{ \alpha = (i_1, \ldots, i_{l_\alpha}) \middle| i_p \in I, deg\left(g_{i_p}\right) > 0, \sum_{p=1}^{l_\alpha} deg\left(g_{i_p}\right) = -s \right\}$$

and

$$\Omega_s^- = \left\{ \beta = (i_1, \ldots, i_{l_\beta}) \middle| i_p \in I, deg\left(g_{i_p}\right) < 0, \sum_{p=1}^{l_\beta} deg\left(g_{i_p}\right) = s \right\}.$$

For $\gamma \in \Omega_s^\pm$ and a sequence $\overline{\mathbf{m}} = (\mathbf{m}_1, \ldots, \mathbf{m}_{l_\gamma}) \in \mathbb{Z}^{nl_\gamma}$ we define

$$u_\gamma(\overline{\mathbf{m}}) = g_{i_1}(\mathbf{m}_1) \ldots g_{i_{l_\gamma}}(\mathbf{m}_{l_\gamma}) \in U(G_\pm)_{\mp s}.$$

The set $\{u_\beta(\overline{\mathbf{m}})v_j(\mathbf{m}_0)\}$ with $\beta \in \Omega_s^-, j \in J, \overline{\mathbf{m}} \in \mathbb{Z}^{nl_\beta}$, $\sum_{p=0}^{l_\beta} \mathbf{m}_p = \mathbf{m}$, is a spanning set for $M_{s,\mathbf{m}}$. Thus a vector $v \in M_{s,\mathbf{m}}$ can be written as

$$v = \sum_{j\in J} \sum_{\beta\in\Omega_s^-} \sum_{\substack{(\overline{\mathbf{m}},\mathbf{m}_0)\in\mathbb{Z}^{n(l_\beta+1)} \\ \mathbf{m}_0+\mathbf{m}_1+\ldots+\mathbf{m}_{l_\beta}=\mathbf{m}}} c(j, \beta, \mathbf{m}_0, \overline{\mathbf{m}}) u_\beta(\overline{\mathbf{m}}) v_j(\mathbf{m}_0).$$

A vector $v \in M_{s,\mathbf{m}}$ belongs to the radical M^{rad} if and only if for every $u \in U(G_+)_{-s}$ we have $uv = 0$. The space $U(G_+)_{-s}$ is spanned by the set $\{u_\alpha(\overline{\mathbf{r}})\}$ with $\alpha \in \Omega_s^+, \overline{\mathbf{r}} \in \mathbb{Z}^{nl_\alpha}$. Thus $v \in M^{rad}$ if and only if for all $\alpha \in \Omega_s^+$ and all $\overline{\mathbf{r}} \in \mathbb{Z}^{nl_\alpha}$ we have

$$\sum_{j\in J} \sum_{\beta\in\Omega_s^-} \sum_{\substack{(\overline{\mathbf{m}},\mathbf{m}_0)\in\mathbb{Z}^{n(l_\beta+1)} \\ \mathbf{m}_0+\mathbf{m}_1+\ldots+\mathbf{m}_{l_\beta}=\mathbf{m}}} c(j, \beta, \mathbf{m}_0, \overline{\mathbf{m}}) u_\alpha(\overline{\mathbf{r}}) u_\beta(\overline{\mathbf{m}}) v_j(\mathbf{m}_0) = 0. \tag{1.13}$$

By Proposition 1.10 there exists a finite family of polynomials $\{p^k_{\alpha\beta j}\}$ such that

$$u_\alpha(\overline{\mathbf{r}})u_\beta(\overline{\mathbf{m}})v_j(\mathbf{m}_0) = \sum_{k\in J} p^k_{\alpha\beta j}(\mathbf{m}_0, \overline{\mathbf{m}}, \overline{\mathbf{r}})v_k(\mathbf{m}_0 + \mathbf{m}_1 + \ldots + \mathbf{m}_{l_\beta} + \mathbf{r}_1 + \ldots + \mathbf{r}_{l_\alpha}).$$

Since the set $\{v_k(\mathbf{m})\}$ forms a basis of T then we can rewrite (1.13) as follows:

$$\sum_{j\in J}\sum_{\beta\in\Omega_s^-}\sum_{\substack{(\overline{\mathbf{m}},\mathbf{m}_0)\in\mathbb{Z}^{n(l_\beta+1)}\\ \mathbf{m}_0+\mathbf{m}_1+\ldots+\mathbf{m}_{l_\beta}=\mathbf{m}}} c(j,\beta,\mathbf{m}_0,\overline{\mathbf{m}})p^k_{\alpha\beta j}(\mathbf{m}_0,\overline{\mathbf{m}},\overline{\mathbf{r}}) = 0 \tag{1.14}$$

for all $\alpha \in \Omega_s^+, \overline{\mathbf{r}} \in \mathbb{Z}^{nl_\alpha}$ and $k \in J$.

For a fixed s, the above expression involves a finite number of polynomials

$$p^k_{\alpha\beta j}(\mathbf{m}_0,\overline{\mathbf{m}},\overline{\mathbf{r}}) = \sum_{\overline{\mathbf{a}}\in\mathbb{Z}_+^{nl_\alpha}} p^k_{\alpha\beta j\overline{\mathbf{a}}}(\mathbf{m}_0,\overline{\mathbf{m}})\overline{\mathbf{r}}^{\overline{\mathbf{a}}},$$

where the summation on the right hand side has finite range.

Since (1.14) holds for an arbitrary $\overline{\mathbf{r}}$, we can apply a Vandermonde type argument to see that it is equivalent to the system of linear equations:

$$\sum_{j\in J}\sum_{\beta\in\Omega_s^-}\sum_{\substack{(\overline{\mathbf{m}},\mathbf{m}_0)\in\mathbb{Z}^{n(l_\beta+1)}\\ \mathbf{m}_0+\mathbf{m}_1+\ldots+\mathbf{m}_{l_\beta}=\mathbf{m}}} c(j,\beta,\mathbf{m}_0,\overline{\mathbf{m}})p^k_{\alpha\beta j\overline{\mathbf{a}}}(\mathbf{m}_0,\overline{\mathbf{m}}) = 0.$$

The number of these linear equations in $\{c(j,\beta,\mathbf{m}_0,\overline{\mathbf{m}})\}$ is determined by the ranges of k, α and $\overline{\mathbf{a}}$ and is finite.

Hence, $M^{rad}_{s,\mathbf{m}}$ is determined inside $M_{s,\mathbf{m}}$ by a system of finitely many linear equations, thus the dimension of $L_{s,\mathbf{m}} = M_{s,\mathbf{m}}/M^{rad}_{s,\mathbf{m}}$ is finite. **Q.E.D.**

2. Toroidal Lie Algebras.

In this section we introduce the toroidal Lie algebras which will concern us for the rest of the paper. We also introduce their associated $\mathfrak{v}$ algebras as well as some modules for these algebras. The results from [B1], recalled in the Introduction, serve as a model for our further constructions. The reader will find here numerous examples of algebras to which the results of Section 1 apply.

Let $\dot{\mathfrak{g}}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$. The algebra of Fourier polynomials on a torus of rank $n+1$ is isomorphic to the algebra of Laurent polynomials $\mathbb{C}[t_0^\pm, t_1^\pm, \ldots, t_n^\pm]$. Thus the tensor product $\tilde{\mathfrak{g}} = \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0^\pm, t_1^\pm, \ldots, t_n^\pm]$ can be interpreted as the algebra of $\dot{\mathfrak{g}}$-valued polynomial functions on a torus. Since the algebras $\tilde{\mathfrak{g}}$ are involved in all the algebras we construct the name toroidal Lie Algebras is used. The algebra $\tilde{\mathfrak{g}}$ itself may be called the loop toroidal algebra associated to $\dot{\mathfrak{g}}$ since if $n = 0$ then $\tilde{\mathfrak{g}}$ is just the untwisted loop affine Kac-Moody Lie algebra associated to $\dot{\mathfrak{g}}$. The algebra $\tilde{\mathfrak{g}}$ is perfect so has a universal central extension. When $n = 0$ the universal central extension is well-known to have the loop algebra of co-dimension 1 but in the general case the co-dimension is infinite. We give its description following [Kas] and [MRY].

Let $\tilde{\mathcal{K}}$ be the free $\mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}]$ module of rank $n+1$ with basis $\{k_0, k_1, \ldots, k_n\}$ and let $d\tilde{\mathcal{K}}$ be the subspace (not submodule) spanned by all elements of the form $r_0 t_0^{r_0} \mathbf{t}^{\mathbf{r}} k_0 + r_1 t_0^{r_0} \mathbf{t}^{\mathbf{r}} k_1 + \ldots + r_n t_0^{r_0} \mathbf{t}^{\mathbf{r}} k_n$ for $(r_0, \mathbf{r}) \in \mathbb{Z}^{n+1}$. We let

$$\mathcal{K} = \tilde{\mathcal{K}}/d\tilde{\mathcal{K}},$$

and we again denote the image of of fk_i in $\mathcal{K}$ by this same symbol. Thus $\mathcal{K}$ is spanned by the elements $t_0^{r_0} \mathbf{t}^{\mathbf{r}} k_p$, where $r_0 \in \mathbb{Z}, \mathbf{r} = (r_1, \ldots r_n) \in \mathbb{Z}^n$, and $\mathbf{t}^{\mathbf{r}} = t_1^{r_1} \ldots t_n^{r_n}, 1 \le p \le n$ subject to the the defining relations

$$r_0 t_0^{r_0} \mathbf{t}^{\mathbf{r}} k_0 + r_1 t_0^{r_0} \mathbf{t}^{\mathbf{r}} k_1 + \ldots + r_n t_0^{r_0} \mathbf{t}^{\mathbf{r}} k_n = 0. \tag{2.1}$$

We let

$$\hat{\mathfrak{g}} = \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}] \oplus \mathcal{K} \tag{2.2}$$

with the bracket

$$[g_1 \otimes f_1(t_0 \ldots t_n), g_2 \otimes f_2(t_0 \ldots t_n)] = [g_1, g_2] \otimes (f_1 f_2) + (g_1|g_2) \sum_{p=0}^{n} (d_p(f_1) f_2)\, k_p \tag{2.3}$$

and

$$[\hat{\mathfrak{g}}, \mathcal{K}] = 0. \tag{2.4}$$

where $(\quad | \quad)$ is a symmetric invariant bilinear form on $\dot{\mathfrak{g}}$ and d_p is the degree derivation of $\mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}]$ corresponding to the index p so that

$$d_p = t_p \frac{d}{dt_p}, \quad p = 0, \ldots, n. \tag{2.5}$$

Now we turn to the description of the $\mathfrak{v}$ algebras of rank $n+1$ which should be thought of as generalizations of the Virasoro algebra. One of these, using the algebra $\mathcal{D}^*$ below, arose naturally in the study of vertex operator representations in [EM]. Let $\mathcal{D}$ be the Lie algebra of derivations of $\mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}]$, so that

$$\mathcal{D} = \left\{ \sum_{p=0}^{n} f_p(t_0, \ldots, t_n) d_p \big| f_0, \ldots, f_n \in \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}] \right\}.$$

It is a general fact that a derivation acting on a perfect Lie algebra can be lifted, in a unique way, to a derivation of the universal central extension of this Lie algebra [BM] . Here, the natural action of $\mathcal{D}$ on $\tilde{\mathfrak{g}}$ is

$$f_1 d_p\,(g \otimes f_2) = g \otimes f_1 d_p(f_2) \qquad (2.6).$$

This has a unique extension to $\hat{\mathfrak{g}}$. Its action on the subspace $\tilde{\mathfrak{g}}$ is unchanged, while the action on $\mathcal{K}$ is given by the formula (see [EM]),

$$f_1 d_a(f_2 k_b) = f_1 d_a(f_2) k_b + \delta_{ab} \sum_{p=0}^{n} f_2 d_p(f_1) k_p. \qquad (2.7)$$

The space $\mathcal{D} \oplus \mathcal{K}$ can be made into a Lie algebra in several ways which we now describe. These will use (2.7) above to obtain the product of an element from $\mathcal{D}$ with an element from $\mathcal{K}$ so that,

$$[f_1 d_a, f_2 k_b] = f_1 d_a(f_2 k_b). \qquad (2.8)$$

$\mathcal{K}$ is to be an abelian ideal in $\mathcal{D} \oplus \mathcal{K}$ while the product of two elements from $\mathcal{D}$ can be either the usual product of two derivations

$$[t_0^{r_0}\mathbf{t}^{\mathbf{r}} d_a, t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_b] = m_a t_0^{r_0+m_0}\mathbf{t}^{\mathbf{r}+\mathbf{m}} d_b - r_b t_0^{r_0+m_0}\mathbf{t}^{\mathbf{r}+\mathbf{m}} d_a, \qquad (2.9)$$

or can be the usual product adjusted by a 2-cocycle $\tau: \quad \mathcal{D} \times \mathcal{D} \to \mathcal{K}$:

$$[t_0^{r_0}\mathbf{t}^{\mathbf{r}} d_a, t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_b] = m_a t_0^{r_0+m_0}\mathbf{t}^{\mathbf{r}+\mathbf{m}} d_b - r_b t_0^{r_0+m_0}\mathbf{t}^{\mathbf{r}+\mathbf{m}} d_a + \tau\left(t_0^{r_0}\mathbf{t}^{\mathbf{r}} d_a, t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_b\right). \quad (2.10)$$

The algebra $\mathcal{D}$ admits two non-trivial 2-cocycles with values in $\mathcal{K}$:

$$\tau_1(t_0^{r_0}\mathbf{t}^{\mathbf{r}} d_a, t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_b) = -m_a r_b \{\sum_{p=0}^{n} r_p t_0^{r_0+m_0}\mathbf{t}^{\mathbf{r}+\mathbf{m}} k_p\}, \qquad (2.11)$$

and

$$\tau_2(t_0^{r_0}\mathbf{t}^{\mathbf{r}} d_a, t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_b) = r_a m_b \{\sum_{p=0}^{n} r_p t_0^{r_0+m_0}\mathbf{t}^{\mathbf{r}+\mathbf{m}} k_p\}, \qquad (2.12)$$

for all $r_0, m_0 \in \mathbb{Z}, \mathbf{r}, \mathbf{m} \in \mathbb{Z}^n, 0 \leq a, b \leq n.$

The reader should see [BGK],[Dz],[EM],or [L2] for more on these cocycles. In fact, A. Dzhumadil'daev has informed us, in a private communication, that any cocycle with values in $\mathcal{K}$ is a linear combination of τ_1 and τ_2. This can be derived from the results in [Dz]. It is straightforward to see that in each case we obtain a Lie algebra structure on $\mathcal{D} \oplus \mathcal{K}$. To give some notation for this we fix $\tau : \mathcal{D} \times \mathcal{D} \to \mathcal{K}$ to be an arbitrary linear combination of τ_1 and τ_2. We then let $\mathfrak{v}(\mathcal{D},\tau)$ denote the resulting Lie algebra so that $\mathfrak{v}(\mathcal{D},\tau)=\mathcal{D}\oplus\mathcal{K}$ with multiplication given as above using the cocycle τ.

Next we note that $\mathcal{D}$ has an interesting subalgebra which will be denoted by $\mathcal{D}^*$. This is defined as follows.

$$\mathcal{D}^{**}=\left\{\sum_{p=1}^{n} f_p(t_0,\dots,t_n)d_p \big| f_1,\dots,f_n\in\mathbb{C}[t_0^{\pm},t_1^{\pm},\dots,t_n^{\pm}]\right\} \tag{2.13}$$

$$\mathcal{D}^{*}=\mathcal{D}^{**}\oplus\mathbb{C}d_0. \tag{2.14}$$

We note that this subalgebra arose naturally in the study of toroidal Lie algebras. Indeed $\mathcal{D}^*$ appeared in [EM] in connection with an untwisted vertex representation of $\hat{\mathfrak{g}}$ and in [B1] in connection with the principal vertex representation. In both of these works the authors denoted this algebra by $\mathcal{D}_+$ but since we need to make use of various positive (and negative) subalgebras we have chosen the notation $\mathcal{D}^*$. Any linear combinations of cocycles (2.11) and (2.12) can be used with this algebra of derivations so this now gives us two families of algebras $\mathfrak{v}(\mathfrak{D},\tau)$ where $\mathfrak{D}$ can be either $\mathcal{D}$ or $\mathcal{D}^*$ and τ is a linear combination (perhaps trivial) of τ_1 and τ_2. We call all of these $\mathfrak{v}$-algebras and will work with all of them simultaneously when possible. We usually just denote any of these algebras by $\mathfrak{v}$. Notice that for each of these algebras $\mathfrak{v}$ we get an associated toroidal Lie algebra $\dot{\mathfrak{g}}\otimes\mathbb{C}[t_0^{\pm},t_1^{\pm},\dots,t_n^{\pm}]\oplus\mathfrak{v}$ with multiplication given as in (2.3),(2.4),(2.6),(2.8),(2.10),(2.11) and (2.12). For notation we let $\mathfrak{T}(\mathfrak{v})$ denote this toroidal Lie algebra so that

$$\mathfrak{T}(\mathfrak{v})=\dot{\mathfrak{g}}\otimes\mathbb{C}[t_0^{\pm},t_1^{\pm},\dots,t_n^{\pm}]\oplus\mathfrak{v}. \tag{2.16}$$

Again we just use $\mathfrak{T}$ to refer to any of the algebras in this family. Note that the algebra $\hat{\mathfrak{g}}$ of (2.2) is a subalgebra of $\mathfrak{T}$ and that all of the degree derivations $d_0,d_1,\dots,d_n$ are also in $\mathfrak{T}$. With the obvious abuse of notation we will write $\mathfrak{T}(\mathfrak{D},\tau)$ for the algebra $\mathfrak{T}(\mathfrak{v}(\mathfrak{D},\tau))$ where $\mathfrak{D}\in\{\mathcal{D},\mathcal{D}^*\}$. We will be most concerned with the algebra $\mathfrak{T}(\mathcal{D}^*,\tau)$ where τ is one of the cocycles. It seems that these algebras are the most natural ones for a satisfying representation theory. Besides [EM] one should see [B1] and [B2] for more on this algebra when (2.11) is used. This particular algebra was discussed in the Introduction. We note the following result.

Proposition 2.17. Assume $n\geq 1$. All the algebras $\mathfrak{T}(\mathfrak{D},\tau)$ as well as the algebras $\mathfrak{v}(\mathfrak{D},\tau)$, where $\mathfrak{D}\in\{\mathcal{D},\mathcal{D}^*\}$ and τ is a linear combination of 2-cocycles (2.11) and (2.12), are algebras with polynomial multiplication which are extragraded.

Proof. This is easy to see by looking at (2.3),(2.4),(2.6),(2.8),(2.9),(2.10),(2.11) and (2.12). Indeed the t_0-direction gives the $\mathbb{Z}$-grading while the $\mathbb{Z}^n$-grading comes from the variables $t_1, \ldots, t_n$. For example, for the algebra $\mathfrak{v} = \mathcal{D} \oplus \mathcal{K}$ with any of our cocycles one may use the elements

$$d_{ai}(\mathbf{m}) = t_0^i \mathbf{t}^{\mathbf{m}} d_a, \qquad a = 0, \ldots, n,$$

$$k_{pi}(\mathbf{m}) = t_0^i \mathbf{t}^{\mathbf{m}} k_p, \qquad p = 0, \ldots, n,$$

where deg d_{ai} = deg $k_{pi} = i$ as our distinguished spanning set. The rest is straightforward. **Q.E.D.**

Remark. It is because of the central relations (2.1) that we have chosen to use a spanning set, rather than a basis, in Definition 1.9. This helps to make the previous result quite transparent.

Our next goal is to construct irreducible restricted $\mathfrak{v}$-modules with finite-dimensional weight spaces (recall from the Introduction that by weight spaces we just mean the homogeneous spaces relative to the specified grading) and non-trivial action of $\mathcal{K}$. As a starting point for our construction we take the tensor modules [L1] which are realized as the tensor product of a module for the Lie algebra $gl_n(\mathbb{C})$ with the Laurent polynomial algebra $\mathbb{C}[q_1^{\pm 1}, \ldots, q_n^{\pm 1}]$. We will eventually see that these modules have a factorization property similar to that discussed in the Introduction. We begin by recalling some results from [L1]. The reader should also see [R] for this.

Let $\mathcal{D}_n$ be the Lie algebra of derivations of the Laurent polynomial ring $\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ so

$$\mathcal{D}_n = \mathop{\oplus}_{p=1}^{n} \mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}] d_p.$$

An obvious family of modules for $\mathcal{D}_n$ consists of the the loop modules

$$\mathbb{C}[q_1^{\pm 1}, \ldots, q_n^{\pm 1}] \otimes W,$$

where W is an arbitrary vector space and $\mathcal{D}_n$ acts only by derivations on the first factor, the Laurent polynomials in $q_1, \ldots, q_n$.

This family admits an interesting non-trivial generalization (see [L1], [R], [Rud]). Let (W, ψ) be a finite dimensional $gl_n(\mathbb{C})$-module. We define an action of the algebra $\mathcal{D}_n$ on the space $\mathbb{C}[q_1^{\pm}, \ldots, q_n^{\pm}] \otimes W$ by

$$\varphi(\mathbf{t}^{\mathbf{r}} d_p) \mathbf{q}^{\mathbf{m}} w = m_p \mathbf{q}^{\mathbf{r}+\mathbf{m}} w + \sum_{i=1}^{n} r_i \mathbf{q}^{\mathbf{r}+\mathbf{m}} \psi(E_{ip}) w. \qquad (2.18)$$

Here, as usual, E_{ij} is the matrix with (i, j) entry 1 and zeroes everywhere else. Notice that the loop modules correspond to the trivial gl_n-action on W. The following result is known.

Lemma 2.19. **([L1])** The action (2.18) defines a representation of the algebra $\mathcal{D}_n$ on the space $\mathbb{C}[q_1^{\pm}, \ldots, q_n^{\pm}] \otimes W$.

For notation we let $T(W) = \mathbb{C}[q_1^{\pm}, \ldots, q_n^{\pm}] \otimes W$ denote the above module with action given by (2.18) associated to the module W and call this the *tensor module* associated to W. When no mention of W is necessary we will just let T denote a tensor module.

Remark. One can give a further generalization by considering shifts of these modules:

$$\mathbb{C}[q_1^{\pm}, \ldots, q_n^{\pm}]\mathbf{q}^{\alpha} \otimes W$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, and where (2.18) is replaced with

$$\varphi(\mathbf{t}^{\mathbf{r}} d_p)\mathbf{q}^{\mathbf{m}+\alpha} w = (m_p + \alpha_p)\mathbf{q}^{\mathbf{r}+\mathbf{m}+\alpha} w + \sum_{i=1}^{n} r_i \mathbf{q}^{\mathbf{r}+\mathbf{m}+\alpha} \psi(E_{ip}) w.$$

All further constructions in the present paper work for these modules, but for the sake of simplicity we set $\alpha = \mathbf{0}$.

Rao [R] classified the irreducible tensor modules (see also [Rud]). Obviously, a necessary condition for the irreducibility of a tensor module, $T(W)$, is the irreducibility of W. Recall that an irreducible $gl_n(\mathbb{C})$-module W is determined by a pair (λ, b), where λ is the highest weight with respect to the $sl_n(\mathbb{C})$-action and where the identity matrix acts as multiplication by b. Rao proved the following result.

Theorem 2.20.([R]) Let W be the finite dimensional irreducible $gl_n(\mathbb{C})$-module associated to the pair (λ, b). The tensor module $T(W)$ is irreducible if and only if $(\lambda, b) \neq (0,0), (0,n), (\omega_k, k), k = 1, \ldots, n-1$, where $\omega_1, \ldots, \omega_{n-1}$ are the fundamental weights of $sl_n(\mathbb{C})$.

Rao also describes the composition factors for the tensor modules in the above result which are not irreducible. Note that none of the proper submodules of these exceptional reducible $\mathcal{D}_n$-modules is $\mathbb{C}[q_1^{\pm}, \ldots, q_n^{\pm}]$-invariant.

We now consider one of the algebras $\mathfrak{v} = \mathfrak{v}(\mathfrak{D}, \tau)$ where τ is any of our cocycles and $\mathfrak{D} \in \{\mathcal{D}, \mathcal{D}^*\}$. Then $\mathfrak{v}$ is extragraded so we have the decomposition

$$\mathfrak{v} = \mathfrak{v}_- \oplus \mathfrak{v}_0 \oplus \mathfrak{v}_+.$$

This was done according to the degree in t_0 so that

$$\mathfrak{v}_+ = \mathfrak{D}_+ \oplus \sum_{p=0}^{n} t_0 \mathbb{C}[t_0, t_1^{\pm}, \ldots, t_n^{\pm}] k_p,$$

$$\mathfrak{v}_- = \mathfrak{D}_- \oplus \sum_{p=0}^{n} t_0^{-1}\mathbb{C}[t_0^{-1}, t_1^{\pm}, \ldots, t_n^{\pm}]k_p,$$

$$\mathfrak{v}_0 = \mathfrak{D}_0 \oplus \sum_{p=0}^{n} \mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}]k_p.$$

Here if $\mathfrak{D} = \mathcal{D}^*$ then we have

$$\mathfrak{D}_+ = \sum_{a=1}^{n} t_0\mathbb{C}[t_0, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]d_a,$$

$$\mathfrak{D}_- = \sum_{a=1}^{n} t_0^{-1}\mathbb{C}[t_0^{-1}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]d_a,$$

$$\mathfrak{D}_0 = \sum_{a=1}^{n} \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]d_a \oplus \mathbb{C}d_0,$$

while if $\mathfrak{D} = \mathcal{D}$ then we have

$$\mathfrak{D}_+ = \sum_{a=0}^{n} t_0\mathbb{C}[t_0, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]d_a,$$

$$\mathfrak{D}_- = \sum_{a=0}^{n} t_0^{-1}\mathbb{C}[t_0^{-1}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]d_a,$$

$$\mathfrak{D}_0 = \sum_{a=0}^{n} \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]d_a.$$

In all cases, $\mathfrak{v}_0$ contains the algebra $\mathcal{D}_n$. Let W be a finite-dimensional irreducible gl_n-module and let $T = \mathbb{C}[q_1^{\pm}, \ldots q_n^{\pm}] \otimes W$ be the corresponding tensor module for $\mathcal{D}_n$. We write aw for the element $a \otimes w$ of T. For arbitrary constants $c, d \in \mathbb{C}$, we can extend this action to all of $\mathfrak{v}_0$ as follows:

$$\varphi(\mathbf{t}^{\mathbf{r}}k_0)\mathbf{q}^{\mathbf{m}}w = c\mathbf{q}^{\mathbf{r}+\mathbf{m}}w, \tag{2.21}$$

$$\varphi(\mathbf{t}^{\mathbf{r}}k_p)\mathbf{q}^{\mathbf{m}}w = 0, \tag{2.22}$$

$$\varphi(d_0) = d \text{ Id}, \quad \text{and} \tag{2.23}$$

$$\mathbf{t}^{\mathbf{m}}d_0(\mathbf{q}^{\mathbf{r}}w) = d\mathbf{q}^{\mathbf{r}+\mathbf{m}}w \quad \text{for} \quad \mathbf{r}, \mathbf{m} \in \mathbb{Z}^n, 1 \le p \le n. \tag{2.24}$$

We call the scalar c the central charge of T.

It is straight-forward to see that this produces a well-defined $\mathfrak{v}_0$-module structure on T. Indeed, from (2.22) we see that the particular cocycle which we use makes no difference here. Also, $\mathfrak{v}_0$ has polynomial multiplication since the algebra $\mathfrak{v}$ is extragraded and this $\mathfrak{v}_0$-module has polynomial multiplication as the formulas (2.18),(2.21),(2.22),(2.23) and (2.24) show. As the dimension of W is finite we see all of the hypothesis of Theorem 1.12 hold. We state this as follows.

Proposition 2.25. Let $\mathfrak{v}$ be any of the Lie algebras $\mathfrak{v}(\mathfrak{D},\tau)$ considered above and let W be any finite dimensional irreducible gl_n-module. Let $T(W)$ be the tensor module for $\mathcal{D}_n$ constructed using (2.18). Then there is a unique $\mathfrak{v}_0$-module structure on $T(W)$ satisfying (2.21), (2.22),(2.23) and (2.24). Moreover, the $\mathfrak{v}_0$-module $T(W)$ has polynomial multiplication. If the central charge c is non-zero this module is irreducible as a $\mathfrak{v}_0$-module.

Proof. We only need to see the module T is irreducible as a $\mathfrak{v}_0$-module when $c \neq 0$ (even though it may not be irreducible as a $\mathcal{D}_n$-module (see Theorem 2.20)). Indeed (2.21) shows that when $c \neq 0$ the algebra $\mathfrak{v}_0$ will generate all operators of multiplication by $q_1^{\pm 1}, \ldots, q_n^{\pm 1}$. Thus, as W is irreducible so is T. **Q.E.D.**

We let $\mathfrak{v}_+$ act on T by 0 and so we have the induced module for $\mathfrak{v}$:

$$M := \mathrm{Ind}_{\mathfrak{v}_0 \oplus \mathfrak{v}_+}^{\mathfrak{v}}(T).$$

It is easy to see that the module M has a $\mathbb{Z}^{n+1}$-gradation. Note that most of its homogeneous spaces are infinite-dimensional. The only non-trivial finite-dimensional homogeneous spaces correspond to the tensor module $T = \mathbb{C}[q_1^{\pm 1}, \ldots q_n^{\pm 1}] \otimes W$, which is the *top* of M. Since the $\mathbb{Z}$-grading of M, by degrees in t_0, contains only non-positive components, the module M is restricted.

We now apply Theorem 1.12 to our situation. This gives us the following result.

Theorem 2.26. Let M be the above $\mathfrak{v}$-module. There is a vector space isomorphism $M \cong U(\mathfrak{v}_-) \otimes \mathbb{C}[q_1^{\pm 1}, \ldots q_n^{\pm 1}] \otimes W$.

(i) Among the submodules of M intersecting the top trivially there exists a unique maximal one which we denote M^{rad}.

(ii) If W is an irreducible as a gl_n-module then the factor-module $L = M/M^{rad}$ is an irreducible $\mathfrak{v}$-module.

(iii) The $\mathfrak{v}$-module $L = M/M^{rad}$ has finite-dimensional weight spaces.

Remark 2.27. The basic $\mathfrak{v}$-module F described in the Introduction is obtained as the irreducible factor of the module constructed above from the $\mathcal{D}_n$-module $\mathbb{C}[q_1^{\pm 1}, \ldots q_n^{\pm 1}]$ with trivial gl_n-action and with choice of constants $c = 1, d = 0$.

The construction of Theorem 2.26 can also be done for the algebra $\mathfrak{g} = \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0^{\pm}, \ldots, t_n^{\pm}]$ $\oplus \mathfrak{v}$. Again for the $\mathbb{Z}$-grading we use powers of t_0. One can see that $\mathfrak{g}_0 = \dot{\mathfrak{g}} \otimes \mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}] \oplus$

$\mathfrak{v}_0$. To construct a $\mathfrak{g}_0$-module, we take a $\mathfrak{v}_0$-module T as above and a finite-dimensional $\dot{\mathfrak{g}}$-module P. Then $T \otimes P$ admits the structure of a $\mathfrak{g}_0$-module by using (2.18), (2.21)-(2.24) together with

$$\varphi(\mathbf{t}^{\mathbf{r}} g)\mathbf{q}^{\mathbf{m}} w \otimes p = \mathbf{q}^{\mathbf{r}+\mathbf{m}} w \otimes (gp). \tag{2.28}$$

It is easy to check that this defines a $\mathfrak{g}_0$-action. Since $\mathfrak{g}$ is an extragraded algebra and the action of $\mathfrak{g}_0$ on $T \otimes P$ is polynomial then by Theorem 1.12 the irreducible factor of the $\mathfrak{g}$-module induced from $T \otimes P$ has finite-dimensional homogeneous spaces. We will recover this result later in Theorem 4.9 by constructing a realization for this irreducible $\mathfrak{g}$-module.

It is interesting to compare the above case with the case of a toroidal algebra with a smaller center. As in [BC], one can also consider a finite-dimensional central extension of $\tilde{\mathfrak{g}} = \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0^{\pm}, \ldots, t_n^{\pm}]$ with center $\dot{\mathcal{K}}$ having basis $\{k_0, \ldots, k_n\}$ of degree zero and with multiplication given by

$$[t_0^{r_0}\mathbf{t}^{\mathbf{r}} g_1, t_0^{m_0}\mathbf{t}^{\mathbf{m}} g_2] = t_0^{r_0+m_0}\mathbf{t}^{\mathbf{r}+\mathbf{m}}[g_1, g_2] + (g_1|g_2)\delta_{r_0,-m_0}\delta_{\mathbf{r},-\mathbf{m}}\sum_{p=0}^{n} r_p k_p.$$

We construct a module for $\tilde{\mathfrak{g}} \oplus \dot{\mathcal{K}}$ in the following way, using the induction procedure. The space $\mathbb{C}[q_1^{\pm}, \ldots, q_n^{\pm}]$ has a $\tilde{\mathfrak{g}}_0 \oplus \dot{\mathcal{K}}$-module structure defined by

$$\rho\,(\mathbf{t}^{\mathbf{r}} g)\,\mathbf{q}^{\mathbf{m}} = 0,$$

$$\rho\,(k_p)\,\mathbf{q}^{\mathbf{m}} = \delta_{p,0}\mathbf{q}^{\mathbf{m}}.$$

We let $\tilde{\mathfrak{g}}_+$ act on $\mathbb{C}[q_1^{\pm}, \ldots, q_n^{\pm}]$ trivially and consider the $\tilde{\mathfrak{g}} \oplus \dot{\mathcal{K}}$-module M induced from $\mathbb{C}[q_1^{\pm}, \ldots, q_n^{\pm}]$.

We are going to show now that in contrast to Theorem 1.12, some homogeneous spaces of the irreducible factor of M are infinite-dimensional. Indeed, let h be a non-zero element of a Cartan subalgebra $\dot{\mathfrak{h}}$ of $\dot{\mathfrak{g}}$. Then the vectors $\left\{\rho(t_0^{-1}\mathbf{t}^{-\mathbf{r}}h)\mathbf{q}^{\mathbf{r}}\right\}_{\mathbf{r}\in\mathbb{Z}^n}$ belong to the same homogeneous space of M. We claim that the images of these vectors in the irreducible factor are linearly independent. To prove this, we need to show that only the trivial linear combination of these elements belongs to the radical of M. Consider such a linear combination:

$$\sum_{\mathbf{r}\in\mathbb{Z}^n} a_{\mathbf{r}}\rho(t_0^{-1}\mathbf{t}^{-\mathbf{r}}h)\mathbf{q}^{\mathbf{r}} \in M^{rad}.$$

Let $h' \in \dot{\mathfrak{h}}$ satisfy $(h'|h) \neq 0$. Then for an arbitrary $\mathbf{m} \in \mathbb{Z}^n$

$$\rho(t_0\mathbf{t}^{\mathbf{m}}h')\sum_{\mathbf{r}\in\mathbb{Z}^n} a_{\mathbf{r}}\rho(t_0^{-1}\mathbf{t}^{-\mathbf{r}}h)\mathbf{q}^{\mathbf{r}} \in M^{rad}.$$

However

$$\rho(t_0\mathbf{t}^{\mathbf{m}}h')\sum_{\mathbf{r}\in\mathbb{Z}^n} a_{\mathbf{r}}\rho(t_0^{-1}\mathbf{t}^{-\mathbf{r}}h)\mathbf{q}^{\mathbf{r}}$$

$$=\sum_{\mathbf{r}\in\mathbb{Z}^n} a_{\mathbf{r}}\rho\left([t_0\mathbf{t}^{\mathbf{m}}h', t_0^{-1}\mathbf{t}^{-\mathbf{r}}h]\right)\mathbf{q}^{\mathbf{r}}$$

$$+\sum_{\mathbf{r}\in\mathbb{Z}^n} a_{\mathbf{r}}\rho(t_0^{-1}\mathbf{t}^{-\mathbf{r}}h)\rho(t_0\mathbf{t}^{\mathbf{m}}h')\mathbf{q}^{\mathbf{r}}.$$

The second term is zero since $\tilde{\mathfrak{g}}_+$ acts trivially on $\mathbb{C}[q_1^{\pm},\ldots,q_n^{\pm}]$, while the first equals

$$\sum_{\mathbf{r}\in\mathbb{Z}^n} a_{\mathbf{r}}\delta_{\mathbf{m},\mathbf{r}}(h'|h)\rho(k_0+m_1k_1+\ldots+m_nk_n)\mathbf{q}^{\mathbf{r}}$$

$$=a_{\mathbf{m}}\mathbf{q}^{\mathbf{m}}.$$

Since M^{rad} intersects $\mathbb{C}[q_1^{\pm},\ldots,q_n^{\pm}]$ trivially, we conclude that $a_{\mathbf{m}}=0$ for all $\mathbf{m}\in\mathbb{Z}^n$. Thus the irreducible module M/M^{rad} does have infinite-dimensional homogeneous spaces.

However notice the following. Since $\tilde{\mathfrak{g}}_0\oplus\dot{\mathcal{K}}$ is the zero component of the extra-graded Lie algebra $\dot{\mathfrak{g}}\oplus\dot{\mathcal{K}}$ then if we knew that the $\tilde{\mathfrak{g}}_0\oplus\dot{\mathcal{K}}$-module $\mathbb{C}[q_1^{\pm},\ldots,q_n^{\pm}]$ were a module with polynomial action then by Theorem 1.12 M/M^{rad} would have finite-dimensional homogeneous spaces. Thus, $\mathbb{C}[q_1^{\pm},\ldots,q_n^{\pm}]$ is not a $\tilde{\mathfrak{g}}_0\oplus\dot{\mathcal{K}}$-module with polynomial action. Intuitively, k_0 should be part of a family $k_0(\mathbf{m})$ where $k_0(\mathbf{m})=0$ for $\mathbf{m}\neq\mathbf{0}$. However, then $k_0(\mathbf{m})\mathbf{q}^{\mathbf{r}}=\delta_{\mathbf{m},\mathbf{0}}\mathbf{q}^{\mathbf{r}}$ and $\delta_{\mathbf{m},\mathbf{0}}$ is *not* polynomial in $\mathbf{m}$.

We finish this section with a result which shows that in the modules we constructed the space $\mathcal{K}$ acts "almost faithfully" when the central charge c is non-zero. (If $c=0$ then $\mathcal{K}$ does act trivially on L, see Remark 3.27.)

Proposition 2.29. If the central charge c is non-zero then the kernel N of the action of $\mathcal{K}$ on L is spanned by $k_1,\ldots,k_n$.

Proof. The idea of the proof is to show that if the kernel N of the action of $\mathcal{K}$ is larger than the span of $k_1,\ldots,k_n$ then N should contain k_0 which acts on L as cId. Since the module L is $\mathbb{Z}^{n+1}$-graded then so is the kernel N. Clearly, $N_{(0,\mathbf{0})}=\mathrm{Span}(k_1,\ldots,k_n)$ when $c\neq 0$.

We are going to show that $N_{(r_0,\mathbf{r})}=(0)$ for $(r_0,\mathbf{r})\neq(0,\mathbf{0})$ using the fact that N is $\mathcal{D}^*$-invariant. Indeed, let $\sum\limits_{p=0}^{n}\alpha_p t_0^{r_0}\mathbf{t}^{\mathbf{r}}k_p$ be a non-zero element of $N_{(r_0,\mathbf{r})}$. Let us consider first the case when $r_0\neq 0$ or $\alpha_0\neq 0$. Then the matrix

$$\begin{pmatrix} r_0 & r_1 & \ldots & r_n \\ \alpha_0 & \alpha_1 & \ldots & \alpha_n \end{pmatrix} \tag{2.30}$$

has rank 2 and its first column is non-zero.

Note that

$$\left[\varphi\left(t_0^{-r_0}\mathbf{t}^{-\mathbf{r}}d_a\right),\varphi\left(\sum_{p=0}^{n}\alpha_p t_0^{r_0}\mathbf{t}^{\mathbf{r}}k_p\right)\right] = r_a\sum_{p=0}^{n}\alpha_p\varphi(k_p) - \alpha_a\sum_{p=0}^{n} r_p\varphi(k_p)$$
$$= (r_a\alpha_0 - r_0\alpha_a)\varphi(k_0). \tag{2.31}$$

Since not every column in matrix (2.30) is a multiple of the first column then there exists $1 \le a \le n$ such that $r_a\alpha_0 - r_0\alpha_a \ne 0$. Thus $k_0 \in N$ which is a contradiction.

We reduce the case $r_0 = \alpha_0 = 0$ to the above case by a similar argument. Let $\sum\limits_{p=1}^{n}\alpha_p\mathbf{t}^{\mathbf{r}}k_p$ be a non-zero element of $N_{(0,\mathbf{r})}$. Then the matrix

$$\begin{pmatrix} r_1 & \dots & r_n \\ \alpha_1 & \dots & \alpha_n \end{pmatrix} \tag{2.32}$$

has rank 2. Choose $1 \le a \le n$ such that $r_a \ne 0$. Then

$$\left[\varphi\left(t_0 d_a\right),\varphi\left(\sum_{p=1}^{n}\alpha_p\mathbf{t}^{\mathbf{r}}k_p\right)\right] = r_a\sum_{p=1}^{n}\alpha_p\varphi(t_0\mathbf{t}^{\mathbf{r}}k_p) + \alpha_a\varphi(t_0\mathbf{t}^{\mathbf{r}}k_0). \tag{2.33}$$

Deleting the first column of the matrix

$$\begin{pmatrix} 1 & r_1 & \dots & r_n \\ \alpha_a & r_a\alpha_1 & \dots & r_a\alpha_n \end{pmatrix} \tag{2.34}$$

and comparing with (2.32) we see that the rank of (2.34) is also 2. Thus $\alpha_a t_0\mathbf{t}^{\mathbf{r}}k_0 + r_a\sum\limits_{p=1}^{n}\alpha_p t_0\mathbf{t}^{\mathbf{r}}k_p$ is a non-zero element of $N_{(1,\mathbf{r})}$ which is not possible from the first case.
Q.E.D.

Remark 2.35. In fact it is not hard to show, by an argument similar to the one above, that the span of $k_1, \dots, k_n$ is the kernel of the action of the entire algebra $\mathfrak{v}$ on L.

3. Factorization property for the modules $L(T)$.

In this section we show that the factorization property (0.1) holds for the modules $L(T)$ constructed from the tensor modules T. This is important for our purposes since in the next section we use this property to show that the tensor product of the module $L(T)$ with a module for affine Lie algebra $\dot{\mathfrak{g}}\otimes\mathbb{C}[t_0,t_0^{-1}]\oplus\mathbb{C}k_0\oplus\mathbb{C}d_0$ is a module for the toroidal algebra under consideration.

In order to prove our factorization property we need to show certain formal fields behave essentially as if they were exponentials. One problem here is to just write down this exponential in a well-defined manner and it is for this reason that we use a certain completion of a twisting, by Laurent polynomials, of the universal enveloping algebra $U(\mathfrak{v})$. We call the twisted algebra we need $U_q(\mathfrak{v})$ and let $\overline{U}_q(\mathfrak{v})$ denote the completion we work with. We begin by defining these algebras.

As before we let $\mathfrak{v} = \mathfrak{v}(\mathfrak{D}, \tau)$ where $\mathfrak{D} \in \{\mathcal{D}^*, \mathcal{D}\}$ and where τ is a linear combination of the cocycles from (2.11) and (2.12). Then $\mathfrak{v} = \mathfrak{D} \oplus \mathcal{K}$ where $\mathcal{K}$ is an abelian ideal of $\mathfrak{v}$ so that in the universal enveloping algebra of $\mathfrak{v}$ we have that $U(\mathfrak{v}) = U(\mathfrak{D})U(\mathcal{K})$. Throughout this section we fix a non-zero constant $c \in \mathbb{C}$. We are going to make the vector space $U(\mathfrak{v}) \otimes \mathbb{C}[q_1^{\pm}, \ldots, q_n^{\pm}]$ into an associative algebra via a twisting process similar to what is found in [Lam]. For this we need a supply of automorphisms of $U(\mathfrak{v})$ which we get by exponentiating some derivations. We do this on the level of $\mathfrak{v}$ and then extend to all of $U(\mathfrak{v})$.

Proposition 3.1. (a) For each $a = 1, \ldots, n$, there is a derivation Δ_a of $\mathfrak{v}$ which satisfies (for $m_0 \in \mathbb{Z}, \mathbf{m} \in \mathbb{Z}^n$)

$$\Delta_a(t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_b) = -\delta_{ab}t_0^{m_0}\mathbf{t}^{\mathbf{m}}k_0, \qquad \text{for} \quad 1 \leq b \leq n,$$

$$\Delta_a(t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_0) = t_0^{m_0}\mathbf{t}^{\mathbf{m}}k_a,$$

$$\Delta_a(\mathcal{K}) = (0).$$

(b) For $a, b = 1, \ldots, n$, we have $\Delta_a(\mathfrak{D}) \subset \mathcal{K}$ and $\Delta_a\Delta_b = 0$.

(c) For each $\mathbf{r} \in \mathbb{Z}^n$ there is an automorphism $\sigma^{\mathbf{r}}$ of $\mathfrak{v}$ (and hence of $U(\mathfrak{v})$) which preserves the $\mathbb{Z}^{n+1}$-grading and satisfies $\sigma^{\mathbf{r}} = 1 + \sum_{i=1}^{n} \frac{r_i}{c}\Delta_i$ for $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{Z}^n$.

(d) We have $\sigma^{\mathbf{r}}\sigma^{\mathbf{m}} = \sigma^{\mathbf{r}+\mathbf{m}}$ for $\mathbf{r}, \mathbf{m} \in \mathbb{Z}^n$.

Proof. (a) To prove this one can directly verify that the given formulas lead to a derivation of $\mathfrak{v}$. Another, somewhat enlightning argument goes as follows. We can enlarge the algebra $\mathfrak{v}$ by allowing arbitrary real powers of the variables $t_0, \ldots, t_n$ instead of just integers. This corresponds to replacing the algebra of Laurent polynomials $\mathbb{C}[t_0^{\pm}, \ldots, t_n^{\pm}]$ with the group algebra $\mathbb{C}[\mathbb{R}^{n+1}]$. (This was actually necessary to do in [B2] to get continuous families of soliton solutions to partial differential equations). Thus, in such an extension of $\mathfrak{v}$ we can consider the inner derivation $\Delta_a^\epsilon = -\mathrm{ad}(t_0^\epsilon k_a)$, for $\epsilon \in \mathbb{R}, 1 \leq a \leq n$. Then for $1 \leq b \leq n, m_0 \in \mathbb{Z}, \mathbf{m} \in \mathbb{Z}^n$, we obtain that (using (2.1))

$$\Delta_a^\epsilon(t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_b) = \delta_{ab}\sum_{p=0}^{n} m_p t_0^{m_0+\epsilon}\mathbf{t}^{\mathbf{m}}k_p = -\epsilon\delta_{ab}t_0^{m_0+\epsilon}\mathbf{t}^{\mathbf{m}}k_0,$$

$$\Delta_a^\epsilon(t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_0) = \epsilon t_0^{m_0+\epsilon}\mathbf{t}^{\mathbf{m}}k_a.$$

From these formulas we see that the derivations Δ_a, determined by saying

$$\Delta_a := \frac{d}{d\epsilon}\Delta_a^\epsilon\bigg|_{\epsilon=0}$$

are well-defined on $\mathfrak{v}$ and satisfy the formulas in (a).

(b) This is clear from the formulas given in (a).

(c) For $1 \le a \le n$ we have $\Delta_a^2 = 0$ so let $\sigma_a = \exp\left(\frac{1}{c}\Delta_a\right) = 1 + \frac{1}{c}\Delta_a$. Then $\sigma_a^{r_a} = 1 + \frac{r_a}{c}\Delta_a$ and so we let, for $\mathbf{r} \in \mathbb{Z}^n, \sigma^{\mathbf{r}}$ be defined by $\sigma^{\mathbf{r}} = \sigma_1^{r_1}\dots\sigma_n^{r_n}$. Clearly $\sigma^{\mathbf{r}} = 1 + \sum_{a=1}^{n}\frac{r_a}{c}\Delta_a$ and $\sigma^{\mathbf{r}}\sigma^{\mathbf{m}} = \sigma^{\mathbf{r}+\mathbf{m}}$ for $\mathbf{r},\mathbf{m} \in \mathbb{Z}^n$. **Q.E.D.**

We now let $U_q(\mathfrak{v}) = U(\mathfrak{v}) \otimes \mathbb{C}[q_1^{\pm},\dots,q_n^{\pm}]$ where $\mathbb{C}[q_1^{\pm},\dots,q_n^{\pm}]$ is just our algebra of Laurent polynomials. We write $u\mathbf{q}^{\mathbf{r}}$ for $u \otimes \mathbf{q}^{\mathbf{r}}$ when $u \in U(\mathfrak{v}), \mathbf{q}^{\mathbf{r}} \in \mathbb{C}[q_1^{\pm},\dots,q_n^{\pm}]$ and define a multiplication, extending the usual multiplications on both $U(\mathfrak{v})$ and $\mathbb{C}[q_1^{\pm},\dots,q_n^{\pm}]$ by defining

$$\mathbf{q}^{\mathbf{r}}u = \sigma^{\mathbf{r}}(u)\mathbf{q}^{\mathbf{r}} \quad \text{for any} \quad u \in U(\mathfrak{v}), \mathbf{r} \in \mathbb{Z}^n. \tag{3.2}$$

Here $\mathbf{q}^{\mathbf{r}}$ represents $q_1^{r_1}\dots q_n^{r_n}$ and we extend this definition linearly to all of $U_q(\mathfrak{v})$. This is similar to the twisting process found in [Lam] and it is easy to see that this process makes $U_q(\mathfrak{v})$ into an associative algebra with identity. Clearly this algebra is $\mathbb{Z}$-graded with $\mathbb{C}[q_1^{\pm},\dots,q_n^{\pm}]$ being of zero degree and where $U_q(\mathfrak{v})_i$ is just $U(\mathfrak{v})_i \otimes \mathbb{C}[q_1^{\pm},\dots,q_n^{\pm}]$. Moreover the following formulas, which are straightforward to verify, hold.

$$[t_0^i\mathbf{t}^{\mathbf{m}}d_a, \mathbf{q}^{\mathbf{r}}] = \frac{r_a}{c}(t_0^i\mathbf{t}^{\mathbf{m}}k_0)\mathbf{q}^{\mathbf{r}}, \quad a = 1,\dots,n, \tag{3.3}$$

$$[t_0^i\mathbf{t}^{\mathbf{m}}d_0, \mathbf{q}^{\mathbf{r}}] = -\frac{1}{c}\sum_{b=1}^{n} r_b(t_0^i\mathbf{t}^{\mathbf{m}}k_b)\mathbf{q}^{\mathbf{r}}, \tag{3.4}$$

$$[t_0^i\mathbf{t}^{\mathbf{m}}k_p, \mathbf{q}^{\mathbf{r}}] = 0, \quad p = 0,\dots,n. \tag{3.5}$$

Here $i \in \mathbb{Z}, \mathbf{r},\mathbf{m} \in \mathbb{Z}^n$. Notice that $U(\mathcal{K}) \otimes \mathbb{C}[q_1^{\pm},\dots,q_n^{\pm}]$ is a commutative algebra.

Next we let $T = T(W) = \mathbb{C}[q_1^{\pm},\dots,q_n^{\pm}]\otimes W$ be one of the tensor modules constructed in the previous section from a finite-dimensional gl_n-module with central charge non-zero and equal to the constant c we fixed at the beginning of this section. Clearly $\mathbb{C}[q_1^{\pm},\dots,q_n^{\pm}]$ acts on T by the natural action of multiplication on the left factor. We recall that $M = M(T(W)) = M(T) = \mathrm{Ind}_{\mathfrak{v}_0\oplus\mathfrak{v}_+}^{\mathfrak{v}}(T) = U(\mathfrak{v})\otimes_{U(\mathfrak{v}_0\oplus\mathfrak{v}_+)}T$ and $L = M/M^{rad}$. These modules have $\mathbb{Z}$ and $\mathbb{Z}^n$-gradings.

Proposition 3.6. (a) The action of $U(\mathfrak{v})$ on M can be extended to an action of $U_q(\mathfrak{v})$ on M which satisfies

$$(u_1\mathbf{q}^{\mathbf{r}})(u_2 \otimes \mathbf{q}^{\mathbf{m}} w) = u_1 \sigma^{\mathbf{r}}(u_2) \otimes \mathbf{q}^{\mathbf{r}+\mathbf{m}} w \quad \text{for} \quad u_1, u_2 \in U(\mathfrak{v}), \mathbf{r}, \mathbf{m} \in \mathbb{Z}^n.$$

Thus, M is a $U_q(\mathfrak{v})$-module.

(b) The $U(\mathfrak{v})$-submodule M^{rad} of M is a $U_q(\mathfrak{v})$-submodule.

(c) L is a $U_q(\mathfrak{v})$-module and $\dim L_{(s,\mathbf{r})} = \dim L_{(s,\mathbf{0})}$ for all $\mathbf{r} \in \mathbb{Z}^n$, $s \in \mathbb{Z}$.

Proof. (a) In order to show that the above action is well-defined it is enough to check that

$$\sigma^{\mathbf{r}}(x)\mathbf{q}^{\mathbf{r}+\mathbf{s}} w = \mathbf{q}^{\mathbf{r}}(x\mathbf{q}^{\mathbf{s}} w) \qquad \text{for} \quad x \in \mathfrak{v}_0 \oplus \mathfrak{v}_+, \quad \mathbf{r}, \mathbf{s} \in \mathbb{Z}^n.$$

If $x \in \mathfrak{v}_+$ then both sides of the equality are zero since $\sigma^{\mathbf{r}}$ preserves the grading and $\mathfrak{v}_+$ acts on T trivially.

Next we check the above equality for the basis elements $\mathbf{t}^{\mathbf{m}} d_p, \mathbf{t}^{\mathbf{m}} k_p$ of $\mathfrak{v}_0$:

$$\sigma^{\mathbf{r}}(\mathbf{t}^{\mathbf{m}} d_a)\mathbf{q}^{\mathbf{r}+\mathbf{s}} w = (\mathbf{t}^{\mathbf{m}} d_a - \frac{r_a}{c}\mathbf{t}^{\mathbf{m}} k_0)\mathbf{q}^{\mathbf{r}+\mathbf{s}} w =$$

$$= (r_a + s_a)\mathbf{q}^{\mathbf{r}+\mathbf{s}+\mathbf{m}} w + \sum_{i=1}^{n} m_i \mathbf{q}^{\mathbf{r}+\mathbf{s}+\mathbf{m}} \psi(E_{ia}) w - r_a \mathbf{q}^{\mathbf{r}+\mathbf{s}+\mathbf{m}} w =$$

$$= s_a \mathbf{q}^{\mathbf{r}+\mathbf{s}+\mathbf{m}} w + \sum_{i=1}^{n} m_i \mathbf{q}^{\mathbf{r}+\mathbf{s}+\mathbf{m}} \psi(E_{ia}) w =$$

$$= \mathbf{q}^{\mathbf{r}}(\mathbf{t}^{\mathbf{m}} d_a(\mathbf{q}^{\mathbf{s}} w)), \qquad a = 1, \ldots, n,$$

$$\sigma^{\mathbf{r}}(\mathbf{t}^{\mathbf{m}} d_0)\mathbf{q}^{\mathbf{r}+\mathbf{s}} w = (\mathbf{t}^{\mathbf{m}} d_0 + \frac{1}{c}\sum_{a=1}^{n} r_a \mathbf{t}^{\mathbf{m}} k_a)\mathbf{q}^{\mathbf{r}+\mathbf{s}} w =$$

$$= d\mathbf{q}^{\mathbf{r}+\mathbf{s}+\mathbf{m}} w = \mathbf{q}^{\mathbf{r}}(\mathbf{t}^{\mathbf{m}} d_0(\mathbf{q}^{\mathbf{s}} w)),$$

$$\sigma^{\mathbf{r}}(\mathbf{t}^{\mathbf{m}} k_p)\mathbf{q}^{\mathbf{r}+\mathbf{s}} w = \mathbf{t}^{\mathbf{m}} k_p(\mathbf{q}^{\mathbf{r}+\mathbf{s}} w) = \mathbf{q}^{\mathbf{r}}(\mathbf{t}^{\mathbf{m}} k_p(\mathbf{q}^{\mathbf{s}} w)), \quad p = 0, \ldots, n.$$

Finally, we verify that the action of $U_q(\mathfrak{v})$ on M agrees with the associative product (3.2) in $U_q(\mathfrak{v})$:

$$\mu(u_1\mathbf{q}^{\mathbf{r}} u_2 \mathbf{q}^{\mathbf{m}})(u_3 \otimes \mathbf{q}^{\mathbf{s}} w) = \mu(u_1 \mathbf{q}^{\mathbf{r}})\mu(u_2 \mathbf{q}^{\mathbf{m}})(u_3 \otimes \mathbf{q}^{\mathbf{s}} w).$$

However, both sides are equal to

$$u_1 \sigma^{\mathbf{r}}(u_2)\sigma^{\mathbf{r}+\mathbf{m}}(u_3) \otimes \mathbf{q}^{\mathbf{r}+\mathbf{m}+\mathbf{s}} w.$$

(b) A vector $v \in M_{-s}, s \geq 0$ belongs to M^{rad} if and only if $U(\mathfrak{v}_+)_s v = 0$. If $v \in M^{rad}_{-s}$ then for $u \in U(\mathfrak{v}_+)_s$ we have

$$u\mathbf{q}^{\mathbf{m}}v = \mathbf{q}^{\mathbf{m}}\sigma^{-\mathbf{m}}(u)v = 0$$

since $\sigma^{-\mathbf{m}}(U(\mathfrak{v}_+)_s) = U(\mathfrak{v}_+)_s$. Thus $\mathbf{q}^{\mathbf{m}}v \in M^{rad}_{-s}$ which shows that M^{rad} is a $U_q(\mathfrak{v})$-submodule.

Finally notice that (c) is clear since $\mathbf{q}^{\mathbf{r}}L_{(s,\mathbf{0})} \subseteq L_{(s,\mathbf{r})}$ for all $\mathbf{r} \in \mathbb{Z}^n$ and $\mathbf{q}^{\mathbf{r}}$ is invertible in $U_q(\mathfrak{v})$. **Q.E.D.**

Next we recall that we have the completion $\overline{U}(\mathcal{K}) = \overline{S}(\mathcal{K}) = \underset{k\in\mathbb{Z}}{\oplus} \overline{U}(\mathcal{K})_k$ of the abelian Lie algebra $\mathcal{K}$ which was given by (see (1.3))

$$\overline{U}(\mathcal{K})_k = \prod_{\substack{i\leq 0, j\geq 0\\ i+j=k}} U(\mathcal{K}_-)_i U(\mathcal{K}_0) U(\mathcal{K}_+)_j.$$

We also have the completion, $\overline{U}_q(\mathfrak{v})$, of our $\mathbb{Z}$-graded algebra $U_q(\mathfrak{v})$ and clearly $\overline{U}(\mathcal{K})$ is a subalgebra of $\overline{U}_q(\mathfrak{v})$. This lets us define the subalgebra

$$\widehat{U}_q(\mathfrak{v}) = U(\mathfrak{D})\overline{U}(\mathcal{K})\mathbb{C}[q_1^{\pm}, \ldots, q_n^{\pm}]. \tag{3.7}$$

Recalling that $\mathcal{K}$ is an ideal in $\mathfrak{v}$ and also taking into account (3.3)-(3.5), we conclude that the subalgebra $\overline{U}_q(\mathcal{K}) = \overline{U}(\mathcal{K})\mathbb{C}[q_1^{\pm}, \ldots, q_n^{\pm}]$ in $\widehat{U}_q(\mathfrak{v})$ is $\mathrm{ad}(\mathfrak{v})$-invariant.

We know $\overline{U}_q(\mathfrak{v})$ acts on M and hence so does $\widehat{U}_q(\mathfrak{v})$.

We now define some elements of $\widehat{U}_q(\mathfrak{v})[[z, z^{-1}]]$ we need to deal with. These are as follows,

$$k_0(z, \mathbf{r}) = \sum_{i\in\mathbb{Z}} (t_0^i \mathbf{t}^{\mathbf{r}} k_0) z^{-i}, \qquad \mathbf{r} \in \mathbb{Z}^n, \tag{3.8}$$

$$k_0(z) = k_0(z, \mathbf{0}), \tag{3.9}$$

$$k_p(z, \mathbf{r}) = \sum_{i\in\mathbb{Z}} (t_0^i \mathbf{t}^{\mathbf{r}} k_p) z^{-i}, \qquad \mathbf{r} \in \mathbb{Z}^n, \quad 1 \leq p \leq n, \tag{3.10}$$

$$k_p(z) = \sum_{\substack{i\in\mathbb{Z}\\ i\neq 0}} (t_0^i k_p) z^{-i}, \qquad 1 \leq p \leq n, \tag{3.11}$$

Thus $k_p(z, \mathbf{0}) = k_p + k_p(z)$ for $1 \leq p \leq n$. Notice that we have

$$k_p(z, \mathbf{r}) \in \mathcal{K}_{gr}[[z, z^{-1}]], \quad 0 \leq p \leq n, \quad \mathbf{r} \in \mathbb{Z}^n \tag{3.12}$$

and that since $t_0^i k_0 = 0$ in $\mathcal{K}$ (by (2.1)) if $i \neq 0$ then

$$k_0(z) = k_0(z, \mathbf{0}) = k_0. \tag{3.13}$$

We will also use the element

$$-\sum_{\substack{i \in \mathbb{Z} \\ i \neq 0}} (\frac{1}{i} t_0^i k_p) z^{-i} \in \mathcal{K}_{gr}[[z, z^{-1}]], \quad 1 \leq p \leq n,$$

and for notation we let (as usual) for $1 \leq p \leq n$,

$$\int \frac{k_p(z)}{z} dz \quad \text{denote} \quad -\sum_{\substack{i \in \mathbb{Z} \\ i \neq 0}} (\frac{1}{i} t_0^i k_p) z^{-i}. \tag{3.14}$$

Because $\int \frac{k_p(z)}{z} dz$ is in $\mathcal{K}_{gr}[[z, z^{-1}]]$ and has its zero moment equal to zero then this is also true for the series

$$\frac{1}{c} \sum_{p=1}^{n} r_p \int \frac{k_p(z)}{z} dz.$$

It follows from (1.4) that then we may exponentiate this to get the element, for $\mathbf{r} \in \mathbb{Z}^n$,

$$\exp\left(\frac{1}{c} \sum_{p=1}^{n} r_p \int \frac{k_p(z)}{z} dz \right) \in \overline{U}(\mathcal{K})_{gr}[[z, z^{-1}]] \subset \widehat{U}_q(\mathfrak{v})_{gr}[[z, z^{-1}]].$$

Thus, we have the element $k(z, \mathbf{r}) \in \widehat{U}_q(\mathfrak{v})_{gr}[[z, z^{-1}]]$ defined by

$$k(z, \mathbf{r}) := \mathbf{q}^{\mathbf{r}} \exp\left(\frac{1}{c} \sum_{p=1}^{n} r_p \int \frac{k_p(z)}{z} dz \right). \tag{3.15}$$

The reader should also note that by (1.5)

$$k_p(z) k(z, \mathbf{r}) \in \widehat{U}_q(\mathfrak{v})_{gr}[[z, z^{-1}]], \quad 0 \leq p \leq n. \tag{3.16}$$

We know that the moments of these series act on the restricted modules M, M^{rad} and L and we let φ denote the representation of $\mathfrak{v}$ on L. If $g(z) \in \widehat{U}_q(\mathfrak{v})[[z, z^{-1}]]$ is $g(z) = \sum_{i \in \mathbb{Z}} g_i z^{-i}$ we then let

$$\overline{g}(z) = \sum_{i \in \mathbb{Z}} \varphi(g_i) z^{-i} \in \mathrm{End}(L)[[z, z^{-1}]]. \tag{3.17}$$

Our goal is to show that for $1 \leq p \leq n, \mathbf{r} \in \mathbb{Z}^n$

$$\overline{k}_0(z, \mathbf{r}) = c\overline{k}(z, \mathbf{r}), \tag{3.18}$$

$$\overline{k}_p(z,\mathbf{r}) = \overline{k}_p(z)\overline{k}(z,\mathbf{r}). \tag{3.19}$$

To do this we consider the differences $k_0(z,\mathbf{r}) - ck(z,\mathbf{r})$ and $k_p(z,\mathbf{r}) - k_p(z)k(z,\mathbf{r})$ in $\widehat{U}_q(\mathfrak{v})[[z,z^{-1}]]$. Now let $\overline{U}(\mathcal{K})\mathbb{C}[q_1^{\pm},\ldots,q_n^{\pm}]$ be denoted by $\overline{U}_q(\mathcal{K}) \subset \widehat{U}_q(\mathfrak{v})$. $\overline{U}_q(\mathcal{K})$ is a commutative subalgebra of $\widehat{U}_q(\mathfrak{v})$. Recalling Proposition1.5 we see that $\overline{U}_q(\mathcal{K})_{gr}[[z,z^{-1}]]$ is an algebra since $\mathbb{C}[q_1^{\pm 1},\ldots,q_n^{\pm 1}]$ is in the degree zero space of $\overline{U}_q(\mathcal{K})$. Also, we will need the derivation D_z which acts by $D_z(\sum\limits_{i\in\mathbb{Z}} f_n z^{-n}) = -\sum\limits_{i\in\mathbb{Z}} nf_n z^{-n}$.

Definition 3.20. For $\mathbf{r} \in \mathbb{Z}^n$ let $b_p(z,\mathbf{r}) = k_p(z,\mathbf{r}) - k_p(z)k(z,\mathbf{r})$ for $1 \le p \le n$ and let $b_0(z,\mathbf{r}) = k_0(z,\mathbf{r}) - ck(z,\mathbf{r})$. Let I be the ideal of $\overline{U}_q(\mathcal{K})_{gr}[[z,z^{-1}]]$ generated by all the series $D_z^i(b_p(z,\mathbf{r}))$ for $i \in \mathbb{Z}, i \ge 0, 0 \le p \le n, \mathbf{r} \in \mathbb{Z}^n$. Also, let R be the subspace of $\overline{U}_q(\mathcal{K})$ spanned by all moments of the series in I.

Note that from the definition we have that R is a homogeneous ideal of $\overline{U}_q(\mathcal{K})$ in the $\mathbb{Z}^{n+1}$-grading. In particular, as R is homogeneous in the $\mathbb{Z}$-grading we write $R = \bigoplus\limits_{i\in\mathbb{Z}} R_i$. Also note that we have, by (3.13), that $b_0(z,\mathbf{0}) = k_0 - c$ and that $D_z^i(b_0(z,\mathbf{0})) = 0$ for $i \ge 1$. Thus, we have $k_0 f(z) \equiv cf(z) \mod I$, for any $f(z) \in \overline{U}_q(\mathcal{K})_{gr}[[z,z^{-1}]]$. Moreover, the moments of $D_z^i(b_p(z,\mathbf{r}))$, for $i \ge 1$, are just multiples of the corresponding ones of $b_p(z,\mathbf{r})$. Notice that we have that I is invariant under D_z. We are going to show that $RM \subset M^{rad}$. From this it will follow that (3.18) and (3.19) hold. The factorization property (0.1) will then follow easily from properties of the exponential map. To accomplish this we need several Lemmas.

Lemma 3.21. The zero component of R, R_0, acts trivially on the tensor module T. That is, $R_0T = (0)$.

Proof. Since $\overline{U}_q(\mathcal{K})$ is commutative and $\mathcal{K}_+T = (0)$ then to show that R_0 acts on T trivially, it is sufficient to check that the 0-moments of $k_0(z,\mathbf{r}) - ck(z,\mathbf{r})$ and $k_p(z,\mathbf{r}) - k_p(z)k(z,\mathbf{r})$ annihilate T. Indeed, if we let $b(z) = \sum\limits_{i\in\mathbb{Z}} b_i z^{-i}$ be one of the series $D_z^i(b_p(z,\mathbf{r}))$ from Definition 3.20 and let $g(z) = \sum\limits_{i\in\mathbb{Z}} g_i z^{-i}$ be an arbitrary series in $\overline{U}_q(\mathcal{K})_{gr}[[z,z^{-1}]]$ then the zero component in the product can be written as the function $\sum\limits_{k\in\mathbb{Z}} g_{-k}b_k$ in $\prod\limits_{k\ge 0} U_q(\mathcal{K}_-)_{-k}U_q(\mathcal{K}_0)U_q(\mathcal{K}_+)_k$. If $k > 0$,then b_k acts as zero while if $k < 0, g_{-k}b_k = b_kg_{-k}$ and g_{-k} acts as zero.

The 0-moment of $k_0(z,\mathbf{r})$ is $\mathbf{t}^{\mathbf{r}}k_0$, which acts on T as $c\mathbf{q}^{\mathbf{r}}$. Since $t_0^s k_p$ act trivially on T for $s \ge 0$ then the 0-moment of $k(z,\mathbf{r})$ acts on T as $\mathbf{q}^{\mathbf{r}}$, hence the 0-moment of $k_0(z,\mathbf{r}) - ck(z,\mathbf{r})$ annihilates T. Since $\mathbf{t}^{\mathbf{r}}k_p,\ p = 1,\ldots,n$ act on T trivially then the 0-moments of both $k_p(z,\mathbf{r})$ and $k_p(z)k(z,\mathbf{r})$ act trivially on T. **Q.E.D.**

Lemma 3.22. For any $m_0 \in \mathbb{Z}$, $\mathbf{m} \in \mathbb{Z}^n$ and $a, p = 0, 1, \ldots, n$ we have $[t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_a z^{-m_0}, b_p(z, \mathbf{r})] \in I$.

Proof. In the following proof we work with shifts in z of series from $\overline{U}_q(\mathcal{K})_{gr}[[z, z^{-1}]]$ so will work with $[t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_a, b_p(z, \mathbf{r})]$ rather than with $[t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_a z^{-m_0}, b_p(z, \mathbf{r})]$. Also, we will replace $b_0(z, \mathbf{r})$ by $k_0(z, \mathbf{r}) - k_0 k(z, \mathbf{r}) = k_0(z, \mathbf{r}) - k_0(z) k(z, \mathbf{r})$ to achieve uniformity of notation and not affect the conclusion. Indeed, if we know that $[t_0 t^{\mathbf{m}} d_a z^{-m_0}, k_0(z, \mathbf{r}) - k_0 k(z, \mathbf{r})] \in I$ then we get that $[t_0 t^{\mathbf{m}} d_a z^{-m_0}, k_0(z, \mathbf{r}) - c k(z, \mathbf{r})] \in I$ because $[t_0 t^{\mathbf{m}} d_a, k_0] = [t_0 t^{\mathbf{m}} d_a, c] = 0$ and so $k_0 [t_0 t^{\mathbf{m}} d_a, k(z, \mathbf{r})] \equiv c [t_0 t^{\mathbf{m}} d_a, k(z, \mathbf{r})] \mod I$. Furthermore, we note that we will freely use the central relations (2.1). For $a = 1, \ldots, n$ we have

$$[t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_a, k_p(z, \mathbf{r})] = \sum_j \left[t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_a, t_0^j \mathbf{t}^{\mathbf{r}} k_p\right] z^{-j}$$

$$= r_a \sum_j t_0^{j+m_0}\mathbf{t}^{\mathbf{r}+\mathbf{m}} k_p z^{-j} + \delta_{ap} \sum_{b=0}^{n} \sum_j m_b t_0^{j+m_0} \mathbf{t}^{\mathbf{r}+\mathbf{m}} k_b z^{-j}$$

$$= r_a z^{m_0} k_p(z, \mathbf{r}+\mathbf{m}) + \delta_{ap} z^{m_0} \sum_{b=0}^{n} m_b k_b(z, \mathbf{r}+\mathbf{m}).$$

To compute the commutator with $k_p(z)k(z, \mathbf{r})$, we use the fact that $[X, \exp(Y)] = [X, Y]\exp(Y)$, provided that $[[X, Y], Y] = 0$:

$$[t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_a, k_p(z) k(z, \mathbf{r})] =$$

$$= [t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_a, k_p(z)]\, k(z, \mathbf{r}) + k_p(z)\, [t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_a, k(z, \mathbf{r})] =$$

$$= \delta_{ap} z^{m_0} \sum_{b=0}^{n} m_b k_b(z, \mathbf{m}) k(z, \mathbf{r}) + k_p(z)\, [t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_a, \mathbf{q}^{\mathbf{r}}] \exp\left(\frac{1}{c}\sum_{b=1}^{n} r_b \int \frac{k_b(z)}{z} dz\right)$$

$$+ k_p(z)\mathbf{q}^{\mathbf{r}} \left[t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_a, \exp\left(\frac{1}{c}\sum_{b=1}^{n} r_b \int \frac{k_b(z)}{z} dz\right)\right] =$$

$$= \delta_{ap} z^{m_0} \sum_{b=0}^{n} m_b k_b(z, \mathbf{m}) k(z, \mathbf{r}) + k_p(z) \frac{r_a}{c} \left(t_0^{m_0}\mathbf{t}^{\mathbf{m}} k_0\right) k(z, \mathbf{r})$$

$$+ k_p(z) \left[t_0^{m_0}\mathbf{t}^{\mathbf{m}} d_a, \frac{r_a}{c} \sum_{j \neq 0} \frac{1}{(-j)} t_0^j k_a z^{-j}\right] k(z, \mathbf{r}) =$$

$$= \delta_{ap} z^{m_0} \sum_{b=0}^{n} m_b k_b(z, \mathbf{m}) k(z, \mathbf{r}) + k_p(z) \frac{r_a}{c} \left(t_0^{m_0}\mathbf{t}^{\mathbf{m}} k_0\right) k(z, \mathbf{r})$$

$$+k_p(z)\frac{r_a}{c}\sum_{j\neq 0}\frac{1}{(-j)}(-j)t_0^{j+m_0}\mathbf{t}^{\mathbf{m}}k_0 z^{-j}k(z,\mathbf{r}) =$$

$$= \delta_{ap} z^{m_0}\sum_{b=0}^{n} m_b k_b(z,\mathbf{m})k(z,\mathbf{r}) + r_a z^{m_0}k_p(z)\frac{1}{c}k_0(z,\mathbf{m})k(z,\mathbf{r}).$$

Thus

$$[t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_a, k_p(z,\mathbf{r}) - k_p(z)k(z,\mathbf{r})] =$$

$$r_a z^{m_0}\left(k_p(z,\mathbf{r}+\mathbf{m}) - \frac{1}{c}k_p(z)k_0(z,\mathbf{m})k(z,\mathbf{r})\right) +$$

$$\delta_{ap} z^{m_0}\sum_{b=0}^{n} m_b\left(k_b(z,\mathbf{r}+\mathbf{m}) - k_b(z,\mathbf{m})k(z,\mathbf{r})\right).$$

But

$$k_p(z,\mathbf{r}+\mathbf{m}) - \frac{1}{c}k_p(z)k_0(z,\mathbf{m})k(z,\mathbf{r}) =$$

$$= k_p(z,\mathbf{r}+\mathbf{m}) - k_p(z)k(z,\mathbf{r}+\mathbf{m}) + k_p(z)k(z,\mathbf{r})\left(k(z,\mathbf{m}) - \frac{1}{c}k_0(z,\mathbf{m})\right),$$

and

$$k_b(z,\mathbf{r}+\mathbf{m}) - k_b(z,\mathbf{m})k(z,\mathbf{r}) =$$

$$= k_b(z,\mathbf{r}+\mathbf{m}) - k_b(z)k(z,\mathbf{r}+\mathbf{m}) + k(z,\mathbf{r})\left(k_b(z)k(z,\mathbf{m}) - k_b(z,\mathbf{m})\right).$$

Hence the commutator $[t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_a, k_p(z,\mathbf{r}) - k_p(z)k(z,\mathbf{r})]$ can be written in the required form. That is, we have that $[t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_a z^{-m_0}, k_p(z,\mathbf{r}) - k_p(z)k(z,\mathbf{r})] \in I$.

The computation for $[t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_0, k_p(z,\mathbf{r}) - k_p(z)k(z,\mathbf{r})]$ is analogous but we need to make use of D_z.

$$[t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_0, k_p(z,\mathbf{r})] = \sum_j \left[t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_0, t_0^j\mathbf{t}^{\mathbf{r}}k_p\right] z^{-j} =$$

$$= \sum_j j t_0^{j+m_0}\mathbf{t}^{\mathbf{r}+\mathbf{m}}k_p z^{-j} + \delta_{p0}\sum_{b=0}^{n}\sum_j m_b t_0^{j+m_0}\mathbf{t}^{\mathbf{r}+\mathbf{m}}k_b z^{-j} =$$

$$= -D_z\left(z^{m_0}k_p(z,\mathbf{r}+\mathbf{m})\right) + \delta_{p0}z^{m_0}\sum_{b=0}^{n} m_b k_b(z,\mathbf{r}+\mathbf{m}).$$

$$[t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_0, k_p(z)k(z,\mathbf{r})] =$$

$$[t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_0, k_p(z)]\,k(z,\mathbf{r}) + k_p(z)\,[t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_0, \mathbf{q}^{\mathbf{r}}]\exp\left(\frac{1}{c}\sum_{b=1}^{n} r_b\int\frac{k_b(z)}{z}dz\right)$$

$$+k_p(z)\mathbf{q}^{\mathbf{r}}\left[t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_0, \exp\left(\frac{1}{c}\sum_{b=1}^{n} r_b\int\frac{k_b(z)}{z}dz\right)\right]=$$

$$=-D_z\left(z^{m_0}k_p(z,\mathbf{m})\right)k(z,\mathbf{r})+\delta_{p0}z^{m_0}\sum_{b=0}^{n} m_b k_b(z,\mathbf{m})k(z,\mathbf{r})$$

$$-k_p(z)\frac{1}{c}\left(\sum_{b=1}^{n} r_b t_0^{m_0}\mathbf{t}^{\mathbf{m}}k_b\right)k(z,\mathbf{r})-k_p(z)\frac{1}{c}\left(\sum_{b=1}^{n}\sum_{j\neq 0} r_b t_0^{j+m_0}\mathbf{t}^{\mathbf{m}}k_b z^{-j}\right)k(z,\mathbf{r})=$$

$$=-D_z\left(z^{m_0}k_p(z,\mathbf{m})\right)k(z,\mathbf{r})$$

$$+\delta_{p0}z^{m_0}\sum_{b=0}^{n} m_b k_b(z,\mathbf{m})k(z,\mathbf{r})-z^{m_0}k_p(z)\frac{1}{c}\sum_{b=1}^{n} r_b k_b(z,\mathbf{m})k(z,\mathbf{r}).$$

Finally,

$$[t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_0, k_p(z,\mathbf{r})-k_p(z)k(z,\mathbf{r})]=$$

$$-D_z\left(z^{m_0}k_p(z,\mathbf{r}+\mathbf{m})\right)+D_z\left(z^{m_0}k_p(z,\mathbf{m})\right)k(z,\mathbf{r})+z^{m_0}k_p(z)\frac{1}{c}\sum_{b=1}^{n} r_b k_b(z,\mathbf{m})k(z,\mathbf{r})+$$

$$+\delta_{p0}z^{m_0}\sum_{b=0}^{n} m_b\left(k_b(z,\mathbf{r}+\mathbf{m})-k_b(z,\mathbf{m})k(z,\mathbf{r})\right)=$$

$$-D_z\left(z^{m_0}(k_p(z,\mathbf{r}+\mathbf{m})-k_p(z)k(z,\mathbf{r}+\mathbf{m})\right)+D_z\left(z^{m_0}(k_p(z,\mathbf{m})-k_p(z)k(z,\mathbf{m})\right)k(z,\mathbf{r})$$

$$-z^{m_0}k_p(z)k(z,\mathbf{m})D_z k(z,\mathbf{r})+z^{m_0}k_p(z)\frac{1}{c}\sum_{b=1}^{n} r_b k_b(z)k(z,\mathbf{r})k(z,\mathbf{m})$$

$$+z^{m_0}k_p(z)\frac{1}{c}\sum_{b=1}^{n} r_b\left\{k_b(z,\mathbf{m})-k_b(z)k(z,\mathbf{m})\right\}k(z,\mathbf{r})$$

$$+\delta_{p0}z^{m_0}\sum_{b=0}^{n} m_b\left(k_b(z,\mathbf{r}+\mathbf{m})-k_b(z,\mathbf{m})k(z,\mathbf{r})\right).$$

Adding and subtracting the same term and using that $k(z,\mathbf{r}+\mathbf{m})=k(z,\mathbf{r})k(z,\mathbf{m})$ we find that the above becomes

$$-D_z\left(z^{m_0}(k_p(z,\mathbf{r}+\mathbf{m})-k_p(z)k(z,\mathbf{r}+\mathbf{m})\right)+D_z\left(z^{m_0}(k_p(z,\mathbf{m})-k_p(z)k(z,\mathbf{m})\right)k(z,\mathbf{r})$$

$$+z^{m_0}k_p(z)\frac{1}{c}\sum_{b=1}^{n} r_b\left\{k_b(z,\mathbf{m})-k_b(z)k(z,\mathbf{m})\right\}k(z,\mathbf{r})$$

$$-z^{m_0}k_p(z)k(z,\mathbf{m})D_z k(z,\mathbf{r})+z^{m_0}k_p(z)\frac{1}{c}\sum_{b=1}^{n} r_b k_b(z)k(z,\mathbf{r})k(z,\mathbf{m})$$

$$+\delta_{p0}z^{m_0}\sum_{b=0}^{n}m_b\left(k_b(z,\mathbf{r}+\mathbf{m})-k_b(z)k(z,\mathbf{r}+\mathbf{m})\right)$$

$$-\delta_{p0}z^{m_0}k(z,\mathbf{r})\sum_{b=0}^{n}m_b\left(k_b(z,\mathbf{m})-k_b(z)\right)k(z,\mathbf{m}).$$

Here we note that since

$$k(z,\mathbf{r})=\mathbf{q}^{\mathbf{r}}\exp\left(\frac{1}{c}\sum_{b=1}^{n}\sum_{i\neq 0}\frac{r_b t_0^i k_b}{(-i)}z^{-i}\right)$$

then we obtain

$$D_z k(z,\mathbf{r})=\mathbf{k}(z,\mathbf{r})\frac{1}{c}\left(\sum_{b=1}^{n}\sum_{i\neq 0}r_b t_0^i k_b z^{-i}\right)=\mathbf{k}(z,\mathbf{r})\frac{1}{c}\sum_{b=1}^{n}r_b k_b(z).$$

Canceling terms we get the above equals

$$=-D_z\left(z^{m_0}(k_p(z,\mathbf{r}+\mathbf{m})-k_p(z)k(z,\mathbf{r}+\mathbf{m}))\right)+D_z\left(z^{m_0}(k_p(z,\mathbf{m})-k_p(z)k(z,\mathbf{m}))\right)k(z,\mathbf{r})$$

$$+z^{m_0}k_p(z)k(z,\mathbf{r})\frac{1}{c}\sum_{b=1}^{n}r_b\left\{k_b(z,\mathbf{m})-k_b(z)k(z,\mathbf{m})\right\}+$$

$$+\delta_{p0}z^{m_0}\sum_{b=0}^{n}m_b\left(k_b(z,\mathbf{r}+\mathbf{m})-k_b(z)k(z,\mathbf{r}+\mathbf{m})\right)$$

$$-\delta_{p0}z^{m_0}k(z,\mathbf{r})\sum_{b=0}^{n}m_b\left(k_b(z,\mathbf{m})-k_b(z)k(z,\mathbf{m})\right).$$

Clearly it follows from this that

$$\left[t_0^{m_0}\mathbf{t}^{\mathbf{m}}d_0z^{-m_0},k_p(z,\mathbf{r})-k_p(z)k(z,\mathbf{r})\right]\in I,$$

and this is what we want. **Q.E.D.**

Corollary 3.23 The space R, in $\widehat{U}_q(\mathfrak{v})$, is $\mathrm{ad}(\mathfrak{v})$-invariant.

Proof. Since $\overline{U}_q(\mathcal{K})$ is $\mathrm{ad}(\mathfrak{v})$-invariant then

$$z^{-s}\mathrm{ad}(\mathfrak{v}_s)\overline{U}_q(\mathcal{K})_{gr}[[z,z^{-1}]]\subset\overline{U}_q(\mathcal{K})_{gr}[[z,z^{-1}]].$$

Also, the previous Lemma implies that for any $s\in\mathbb{Z}, 0\leq p,a\leq n,\mathbf{m},\mathbf{r}\in\mathbb{Z}^n$, we have

$$[t_0^s t^{\mathbf{m}}d_a z^{-s},b_p(z,\mathbf{r})]\in I.$$

We show, by induction on $i \in \mathbb{Z}, i \geq 0$, that $[t_0^s t^{\mathbf{m}} d_a z^{-s}, D_z^i b_p(z, \mathbf{r})] \in I$. We have

$$[t_0^s t^{\mathbf{m}} d_a z^{-s}, D_z^{i+1} b_p(z, \mathbf{r})] =$$

$$(t_0^s t^{\mathbf{m}} d_a z^{-s}) D_z^{i+1} b_p(z, \mathbf{r}) - D_z^{i+1} b_p(z, \mathbf{r})(t_0^s t^{\mathbf{m}} d_a z^{-s}) =$$

$$D_z((t_0^s t^{\mathbf{m}} d_a z^{-s}) D_z^i b_p(z, \mathbf{r})) - (D_z(t_0^s t^{\mathbf{m}} d_a z^{-s})) D_z^i b_p(z, \mathbf{r})$$

$$-D_z((D_z^i b_p(z, \mathbf{r})(t_0^s t^{\mathbf{m}} d_a z^{-s}) + (D_z^i b_p(z, \mathbf{r})(D_z(t_0^s t^{\mathbf{m}} d_a z^{-s}) =$$

$$D_z([t_0^s t^{\mathbf{m}} d_a z^{-s}, D_z^i b_p(z, \mathbf{r})]) + s[t_0^s t^{\mathbf{m}} d_a z^{-s}, D_z^i b_p(z, \mathbf{r})].$$

This is in I, by induction, since clearly, by its very definition, I is invariant under D_z.

Thus, it follows from what we said above, and the fact that $\mathrm{ad}(\mathfrak{v})$ acts as derivations, that

$$z^{-s}\mathrm{ad}(\mathfrak{v}_s) I \subset I. \tag{3.24}$$

We conclude from this that R is $\mathrm{ad}(\mathfrak{v})$-invariant. **Q.E.D.**

We can now prove our main result about R.

Lemma 3.25. $RM \subset M^{rad}$.

Proof. Recall from Theorem 2.26(c) that $M = U(\mathfrak{v}_-)T$ where T is our tensor module. Thus, a general element of M is a sum of terms of the form $x_1 \ldots x_m w$ where $w \in T, m \geq 0$ and $x_i \in \mathfrak{v}_-, 1 \leq i \leq m$.

Let f be a homogeneous element in R, say $f \in R_s$ and let $w \in T$. If $s > 0$ then clearly $fw = 0$ since M has trivial positive graded components. If $s = 0$ then Lemma 3.21 gives us that $fw = 0$. For $s < 0$ we need to show that $ufw = 0$ for all $u \in U(\mathfrak{v}_+)_{-s}$. It is enough to take u of the form $u = y_1 \ldots y_m$ with $y_j \in \mathfrak{v}_{k_j}, k_j > 0, \sum_{j=1}^{m} k_j = -s$. We have

$$y_1 \ldots y_m f w = y_1 \ldots y_{m-1}[y_m, f]w = [y_1, \ldots [y_{m-1}, [y_m f]] \ldots]w.$$

Using Corollary 3.23 and Lemma 3.21 we see this expression is zero since $[y_1, \ldots [y_{m-1}, [y_m f]] \ldots]w \in R_0 T = (0)$. Thus, $fw \in M^{rad}$.

We now show if $x_1, \ldots, x_m \in \mathfrak{v}_-$, $w \in T$ and $f \in R$ then $f x_1 \ldots x_m w \in M^{rad}$. We use induction on m where the case $m = 0$ has just been done. For $m > 0$ we have

$$f x_1 \ldots x_m w = x_1 f x_2 \ldots x_m w - [x_1, f] x_2 \ldots x_m w.$$

By induction the term $f x_2 \ldots x_m w \in M^{rad}$ as is the term $[x_1, f] x_2 \ldots x_m w$ because we know that $[x_1, f] \in R$. **Q.E.D.**

We now have the following result.

Theorem 3.26. Let $\mathfrak{v} = \mathfrak{v}(\mathfrak{D}, \tau)$ be as above and let $T = T(W)$ be a tensor module for $\mathfrak{v}_0$ with non-zero central charge c. Then on L we have

$$\overline{k}_p(z, \mathbf{r}) = \overline{k}_p(z)\overline{k}(z, \mathbf{r}) \quad \text{for} \quad 0 \leq p \leq n, \mathbf{r} \in \mathbb{Z}^n.$$

Moreover, $\overline{k}(z, \mathbf{r})\overline{k}(z, \mathbf{m}) = \overline{k}(z, \mathbf{r} + \mathbf{m})$ for $\mathbf{r}, \mathbf{m} \in \mathbb{Z}^n$ and hence we have

$$\overline{k}_p(z, \mathbf{r} + \mathbf{m}) = \overline{k}_p(z, \mathbf{r})\overline{k}(z, \mathbf{m}).$$

Remark 3.27. If $c = 0$ then, by arguments similar to the above, one can see all of $\mathcal{K}$ acts trivially on L.

4. Modules for Toroidal Algebras.

In this section we use our previous construction of modules L, for $\mathfrak{v} = \mathfrak{v}(\mathfrak{D}, \tau)$, to obtain modules for toroidal algebras. Our method is to take a tensor product of a module for the affine Lie algebra $\dot{\mathfrak{g}} \otimes \mathbb{C}[t_0, t_0^{-1}] \oplus \mathbb{C}k_0 \oplus \mathbb{C}d_0$ with one of our $\mathfrak{v}$-modules L. In showing that this is a module for the toroidal algebra we need to make use of the factorization property in Theorem 3.26. Moreover, it is here that we need to assume the derivation algebra $\mathfrak{D}$ equals $\mathcal{D}^*$.

We let V be a restricted module for the affine algebra $\dot{\mathfrak{g}} \otimes \mathbb{C}[t_0, t_0^{-1}] \oplus \mathbb{C}k_0 \oplus \mathbb{C}d_0$ and let ρ denote the corresponding representation. Moreover we assume the central element k_0 acts as the scalar c so the central charge is c. For $g \in \dot{\mathfrak{g}}$ we let $g(z)$ the formal series $\sum_{i \in \mathbb{Z}} \rho(t_0^i g) z^{-i}$. Note that $g(z) \in \mathrm{End}(V)[[z, z^{-1}]]$.

For a restricted representation (F, φ) of the algebra $\mathfrak{v} = \mathcal{D}^* \oplus \mathcal{K}$ we define $\overline{k}_p(z, \mathbf{r})$ and $\overline{d}_a(z, \mathbf{r})$ by

$$\overline{k}_p(z, \mathbf{r}) = \sum_{j \in \mathbb{Z}} \varphi(t_0^j \mathbf{t}^{\mathbf{r}} k_p) z^{-j}, p = 0, \ldots, n.$$

$$\overline{d}_a(z, \mathbf{r}) = \sum_{j \in \mathbb{Z}} \varphi(t_0^j \mathbf{t}^{\mathbf{r}} d_a) z^{-j}, a = 1, \ldots, n.$$

These series are in $\mathrm{End}(F)[[z, z^{-1}]]$. If the central charge c of F is non-zero then we set

$$\overline{k}(z, \mathbf{r}) = \frac{1}{c}\overline{k}_0(z, \mathbf{r}).$$

Note that (2.1) can be rewritten as

$$cD\overline{k}(z,\mathbf{r}) = \sum_{p=1}^{n} r_p \overline{k}_p(z,\mathbf{r}).$$

Also note that since F is restricted the seemingly infinite sum of endomorphisms

$$\sum_{j\in\mathbb{Z}} \varphi(t_0^j \mathbf{t}^{\mathbf{r}} k_p)\varphi(t_0^{i-j} t^{\mathbf{m}} k_0)$$

is a well defined element in $\mathrm{End}(F)$ for any $i \in \mathbb{Z}$ and hence the product $\overline{k}_p(z,\mathbf{r})\overline{k}(z,\mathbf{m})$ makes sense in $\mathrm{End}(F)[[z,z^{-1}]]$. Also since both modules V and F are restricted we may use Lemma 1.1 to get series of the form $g(z) \otimes h(z)$.

We need to make use of the formal analog of the delta-function so will use the series

$$\delta(z) = \sum_{j\in\mathbb{Z}} z^j.$$

For the differential operator $D = z\frac{d}{dz}$ consider also the series

$$D\delta(z) = \sum_{j\in\mathbb{Z}} jz^j.$$

The following result is well-known.

Lemma 4.1. (cf. Proposition 2.2.2. in [FLM]) For a formal Laurent series $X(z_1, z_2)$ the following equalities hold provided the products on the left hand sides exist.

$$(i) \qquad X(z_1,z_2)\delta\left(\frac{z_2}{z_1}\right) = X(z_2,z_2)\delta\left(\frac{z_2}{z_1}\right) = X(z_1,z_1)\delta\left(\frac{z_2}{z_1}\right),$$

$$(ii) \qquad X(z_1,z_2)D\delta\left(\frac{z_2}{z_1}\right) = X(z_2,z_2)D\delta\left(\frac{z_2}{z_1}\right) + \left(D_{z_1}X(z_1,z_2)\right)\Big|_{z_1=z_2}\delta\left(\frac{z_2}{z_1}\right),$$

Note that our products, in the proof of the following result, will exist since we are working with restricted modules.

Theorem 4.2. Let (V,ρ) be a restricted representation of the affine algebra

$$\dot{\mathfrak{g}} \otimes \mathbb{C}[t_0, t_0^{-1}] \oplus \mathbb{C}k_0 \oplus \mathbb{C}d_0$$

and let (F,φ) be a restricted representation of the algebra

$$\mathfrak{v} = \mathcal{D}^* \oplus \mathcal{K}.$$

Suppose that both modules V and F have the same central charge $c \neq 0$. Moreover we require that the module F satisfies the factorization condition

$$\overline{k}_p(z, \mathbf{r} + \mathbf{m}) = \overline{k}_p(z, \mathbf{r}) \overline{k}(z, \mathbf{m}) \tag{4.3}$$

for all $\mathbf{r}, \mathbf{m} \in \mathbb{Z}^n, p = 0, \ldots, n$.

Then the following defines a representation, μ, of the toroidal algebra $\mathfrak{T} = \tilde{\mathfrak{g}} \oplus \mathcal{K} \oplus \mathcal{D}^*$ on the space $V \otimes F$.

$$\sum_{j \in \mathbb{Z}} \mu(t_0^j \mathbf{t}^{\mathbf{r}} g) z^{-j} = g(z) \otimes \overline{k}(z, \mathbf{r}) \quad \text{for} \quad g \in \dot{\mathfrak{g}},$$

$$\mu\big|_{\mathcal{D}^* \oplus \mathcal{K}} = \mathrm{Id} \otimes \varphi\big|_{\mathcal{D}^* \oplus \mathcal{K}},$$

$$\mu(d_0) = \rho(d_0) \otimes \mathrm{Id} + \mathrm{Id} \otimes \varphi(d_0).$$

Proof. The relations that we must check can be written down as the following equalities for our generating series:

$$\left[\sum_{i \in \mathbb{Z}} \mu(t_0^i \mathbf{t}^{\mathbf{r}} g_1) z_1^{-i}, \sum_{j \in \mathbb{Z}} \mu(t_0^j \mathbf{t}^{\mathbf{m}} g_2) z_2^{-j} \right] = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \mu(\left[t_0^i \mathbf{t}^{\mathbf{r}} g_1, t_0^j \mathbf{t}^{\mathbf{m}} g_2 \right]) z_1^{-i} z_2^{-j}, \tag{4.4}$$

$$\left[\sum_{i \in \mathbb{Z}} \mu(t_0^i \mathbf{t}^{\mathbf{r}} g) z_1^{-i}, \sum_{j \in \mathbb{Z}} \mu(t_0^j \mathbf{t}^{\mathbf{m}} k_p) z_2^{-j} \right] = 0, \tag{4.5}$$

$$\left[\mu(t_0^i \mathbf{t}^{\mathbf{r}} d_a), \sum_{j \in \mathbb{Z}} \mu(t_0^j \mathbf{t}^{\mathbf{m}} g) z^{-j} \right] = m_a \sum_{j \in \mathbb{Z}} \mu(t_0^{i+j} \mathbf{t}^{\mathbf{r}+\mathbf{m}} g) z^{-j}, \tag{4.6}$$

$$[\mu(d_0), g(z, \mathbf{r})] = -D g(z, \mathbf{r}), \tag{4.7}$$

$$\left[\mu(d_0), \overline{d}_a(z, \mathbf{r})\right] = -D \overline{d}_a(z, \mathbf{r}), \quad \left[\mu(d_0), \overline{k}_p(z, \mathbf{r})\right] = -D \overline{k}_p(z, \mathbf{r}), \tag{4.8}$$

for all $g, g_1, g_2 \in \dot{\mathfrak{g}}$, $p = 0, \ldots, n$, $a = 1, \ldots, n$.

To get this we have just used (2.3),(2.4), (2.6), and (2.8) from the definition of the toroidal algebra $\mathfrak{T}$.

First we establish establish the relation (4.4). The right hand side of (4.4) can be transformed as follows:

$$\sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \mu(\left[t_0^i \mathbf{t}^{\mathbf{r}} g_1, t_0^j \mathbf{t}^{\mathbf{m}} g_2 \right]) z_1^{-i} z_2^{-j} =$$

$$= \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \mu(t_0^{i+j}\mathbf{t}^{\mathbf{r}+\mathbf{m}}[g_1,g_2]) z_1^{-i} z_2^{-j}$$

$$+(g_1|g_2) \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} i\mu(t_0^{i+j}\mathbf{t}^{\mathbf{r}+\mathbf{m}}k_0) z_1^{-i} z_2^{-j} + (g_1|g_2)\sum_{p=1}^{n} r_p \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \mu(t_0^{i+j}\mathbf{t}^{\mathbf{r}+\mathbf{m}}k_p) z_1^{-i} z_2^{-j}$$

$$= \sum_{i\in\mathbb{Z}} \sum_{s=i+j\in\mathbb{Z}} \mu(t_0^{s}\mathbf{t}^{\mathbf{r}+\mathbf{m}}[g_1,g_2]) z_1^{-i} z_2^{-s+i} + (g_1|g_2) \sum_{i\in\mathbb{Z}} \sum_{s=i+j\in\mathbb{Z}} i\mu(t_0^{s}\mathbf{t}^{\mathbf{r}+\mathbf{m}}k_0) z_1^{-i} z_2^{-s+i}$$

$$+(g_1|g_2)\sum_{p=1}^{n} r_p \sum_{i\in\mathbb{Z}} \sum_{s=i+j\in\mathbb{Z}} \mu(t_0^{s}\mathbf{t}^{\mathbf{r}+\mathbf{m}}k_p) z_1^{-i} z_2^{-s+i}$$

$$= \sum_{s\in\mathbb{Z}} \mu(t_0^{s}\mathbf{t}^{\mathbf{r}+\mathbf{m}}[g_1,g_2]) z_2^{-s} \delta\left(\frac{z_2}{z_1}\right)$$

$$+(g_1|g_2) \sum_{s\in\mathbb{Z}} \mu(t_0^{s}\mathbf{t}^{\mathbf{r}+\mathbf{m}}k_0) z_2^{-s} D\delta\left(\frac{z_2}{z_1}\right) + (g_1|g_2)\sum_{p=1}^{n} r_p \sum_{s\in\mathbb{Z}} \mu(t_0^{s}\mathbf{t}^{\mathbf{r}+\mathbf{m}}k_p) z_2^{-s} \delta\left(\frac{z_2}{z_1}\right)$$

$$= [g_1,g_2](z_2)\overline{k}(z_2,\mathbf{r}+\mathbf{m})\delta\left(\frac{z_2}{z_1}\right)$$

$$+(g_1|g_2)c\overline{k}(z_2,\mathbf{r}+\mathbf{m})D\delta\left(\frac{z_2}{z_1}\right) + (g_1|g_2)\sum_{p=1}^{n} r_p \overline{k}_p(z_2,\mathbf{r}+\mathbf{m})\delta\left(\frac{z_2}{z_1}\right).$$

Analogous calculation carried out for the representation (V,ρ) for the affine algebra gives (see e.g. [Kac])

$$[g_1(z_1), g_2(z_2)] = [g_1,g_2](z_2)\delta\left(\frac{z_2}{z_1}\right) + (g_1|g_2)cD\delta\left(\frac{z_2}{z_1}\right).$$

We use this together with Lemma 4.1 and assumption (4.3) as well as (2.1) to establish (4.4).

$$\left[\sum_{i\in\mathbb{Z}} \mu(t_0^{i}\mathbf{t}^{\mathbf{r}}g_1) z_1^{-i}, \sum_{j\in\mathbb{Z}} \mu(t_0^{j}\mathbf{t}^{\mathbf{m}}g_2) z_2^{-j}\right] =$$

$$= \left[g_1(z_1)\overline{k}(z_1,\mathbf{r}), g_2(z_2)\overline{k}(z_2,\mathbf{m})\right] = [g_1(z_1), g_2(z_2)]\,\overline{k}(z_1,\mathbf{r})\overline{k}(z_2,\mathbf{m}) =$$

$$= [g_1,g_2](z_2)\delta\left(\frac{z_2}{z_1}\right)\overline{k}(z_1,\mathbf{r})\overline{k}(z_2,\mathbf{m}) + (g_1|g_2)cD\delta\left(\frac{z_2}{z_1}\right)\overline{k}(z_1,\mathbf{r})\overline{k}(z_2,\mathbf{m}) =$$

$$= [g_1,g_2](z_2)\overline{k}(z_2,\mathbf{r})\overline{k}(z_2,\mathbf{m})\delta\left(\frac{z_2}{z_1}\right) + (g_1|g_2)c\overline{k}(z_2,\mathbf{r})\overline{k}(z_2,\mathbf{m})D\delta\left(\frac{z_2}{z_1}\right)$$

$$+(g_1|g_2)c\left(D_{z_1}\overline{k}(z_1,\mathbf{r})\right)\Big|_{z_1=z_2}\overline{k}(z_2,\mathbf{m})\delta\left(\frac{z_2}{z_1}\right) =$$

$$= [g_1, g_2](z_2, \mathbf{r}+\mathbf{m})\delta\left(\frac{z_2}{z_1}\right) + (g_1|g_2)c\overline{k}(z_2, \mathbf{r}+\mathbf{m})D\delta\left(\frac{z_2}{z_1}\right) + (g_1|g_2)\sum_{p=1}^{n} r_p\overline{k}_p(z_2, \mathbf{r}+\mathbf{m})\delta\left(\frac{z_2}{z_1}\right).$$

This establishes (4.4).

Relation (4.5) follows immediately from

$$[\overline{k}_a(z_1, \mathbf{r}), \overline{k}_b(z_2, \mathbf{m})] = 0 \quad \text{and} \quad [g(z), \overline{k}_a(z_1, \mathbf{r})] = 0 \quad \text{for} \quad \text{all} \quad a, b = 0, \ldots, n.$$

We now verify (4.6):

$$\left[\mu(t_0^i \mathbf{t}^{\mathbf{r}} d_a), \sum_{j\in\mathbb{Z}} \mu(t_0^j \mathbf{t}^{\mathbf{m}} g) z^{-j}\right] = g(z)\left[\varphi(t_0^i \mathbf{t}^{\mathbf{r}} d_a), \overline{k}(z, \mathbf{m})\right] =$$

$$= g(z) z^i m_a \overline{k}(z, \mathbf{r}+\mathbf{m}) = m_a z^i g(z, \mathbf{r}+\mathbf{m}).$$

From (2.6) and (2.7) we have

$$[\rho(d_0), g(z)] = -Dg(z), \quad [\varphi(d_0), \overline{k}(z, \mathbf{r})] = -D\overline{k}(z, \mathbf{r}).$$

Thus,

$$[\mu(d_0), g(z, \mathbf{r})] = -\left(Dg(z)\right)\overline{k}(z, \mathbf{r}) - g(z)\left(D\overline{k}(z, \mathbf{r})\right)$$

$$= -Dg(z, \mathbf{r}),$$

which establishes (4.7).

Finally, (4.8) follows immediately from (2.7) and the definition of our cocycle τ. **Q.E.D.**

Notice that the cocycle τ does not enter into the above computations. However, we are only working with $\mathcal{D}^*$ here so have only needed to check (4.6). If we tried to use $\mathcal{D}$ in place of $\mathcal{D}^*$ we run into difficulties (as the reader can easily verify) in trying to see if

$$[\mu(t_0^i \mathbf{t}^{\mathbf{r}} d_0), \sum_{j\in\mathbb{Z}} \mu(t_0^j \mathbf{t}^{\mathbf{m}} g)] = \mu([t_0^i \mathbf{t}^{\mathbf{r}} d_0, \sum_{j\in\mathbb{Z}} t_0^j \mathbf{t}^{\mathbf{m}} g]).$$

It seems that to solve this problem one may need to enlarge $\mathfrak{v}$ by replacing $\mathcal{K}$ by a bigger algebra. See [L2] for some particular incidences of this.

The following result summarizes the main results of our work.

Theorem 4.9. Let (V, ρ) be an irreducible highest weight module with non-zero central charge c for the affine algebra $\dot{\mathfrak{g}} \otimes \mathbb{C}[t_0, t_0^{-1}] \oplus \mathbb{C}k_0 \oplus \mathbb{C}d_0$ and let (L, φ) be one of the restricted irreducible representations with the same central charge of the algebra $\mathfrak{v} = \mathcal{D}^* \oplus \mathcal{K}$ constructed in Theorem 2.26. Then the module $V \otimes L$, for the toroidal algebra

$\mathfrak{T} = \tilde{\mathfrak{g}} \oplus \mathcal{K} \oplus \mathcal{D}^*$, constructed in Theorem 4.2, is irreducible and has finite dimensional homogeneous spaces in the $\mathbb{Z}^{n+1}$-grading.

Proof. We need to show that the submodule generated by an arbitrary nonzero vector is the whole space $V \otimes L$. Indeed, acting on a nonzero vector by operators from the positive part of affine algebra $\dot{\mathfrak{g}} \otimes \mathbb{C}[t_0, t_0^{-1}] \oplus \mathbb{C}k_0$, we can obtain a vector of the form $v_0 \otimes w$, where v_0 is the highest weight vector in V and w is some nonzero vector in L. Since L is irreducible and the $\mathfrak{v}$-subalgebra $\mathcal{D}^{**} \oplus \mathcal{K}$ acts only on the second factor in the tensor product, (recall that $\mathcal{D}^* = \mathcal{D}^{**} \oplus \mathbb{C}d_0$ and d_0 acts as a scalar on v_0) then the $\mathfrak{v}$-submodule generated by $v_0 \otimes w$ is $v_0 \otimes L$. Finally, applying the action of the affine subalgebra again, we generate the space $V \otimes L$.

Q.E.D.

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