The ABC's of Calculus, Volume 1

2026 Solutions Manual

Angelo B. Mingarelli

Source edition
Oct 14, 2026

Contents

1 Solutions Manual **1**

1.1 1

1.2 Exercise Set 1 (page 13) 1

1.3 Exercise Set 2 (page 23) 2

1.4 Exercise Set 3 (page 24) 2

1.5 Chapter Exercises (page 26) 5

2 Solutions **7**

2.1 Exercise Set 4 (page 38) 7

2.2 Exercise Set 5 (page 48) 7

2.2 Exercise Set 6 (page 51) 8

2.2 Exercise Set 7 (page 57) 8

2.2 Exercise Set 8 (page 58) 9

2.3 Exercise Set 9 (page 63) 10

2.4 Exercise Set 10 (page 78) 10

2.5 Chapter Exercises (page 79) 12

3 Solutions **15**

3.1 Exercise Set 11 (page 93) 15

3.2 Exercise Set 12 (page 99) 15

3.3 Exercise Set 13 (page 112) 16

3.4 Exercise Set 14 (page 119) 18

3.5 Exercise Set 15 (page 128) 18

3.6 Exercise Set 16 (page 136) 19

3.7 Exercise Set 17 (page 144) 20

3.8 Exercise Set 18 (page 149) 21

3.9 Exercise Set 19 (page 161) 22

3.10 Chapter Exercises (page 162) . . . 22

4 Solutions **25**

4.1 Exercise Set 20 (page 173) . . . 25
4.2 Exercise Set 21 (page 179) . . . 26
4.3 . . . 26
4.4 Exercise Set 22 (page 189) . . . 26
4.5 Exercise Set 23 (page 192) . . . 26
4.6 Exercise Set 24 (page 198) . . . 28
4.7 Chapter Exercises (page 199) . . . 29

5 Solutions **31**

5.1 Exercise Set 25 (page 206) . . . 31
5.2 Exercise Set 26 (page 214) . . . 31
5.3 Exercise Set 27 (page 220) . . . 33
5.4 Chapter Exercises: Use plotting software (page 250) . . . 37

6 Solutions **40**

6.1 Exercise Set 28 (page 268) . . . 40
6.2 Exercise Set 29 (page 279) . . . 41
6.3 Exercise Set 30 (page 285) . . . 42
6.4 . . . 44
6.5 Chapter Exercises (page 300) . . . 44

7 Solutions **50**

7.1 . . . 50
7.2 Exercise Set 31 (page 319) . . . 50
7.3 Exercise Set 32 (page 342) . . . 52
7.4 Exercise Set 33 (page 346) . . . 53
7.4.1 Exercise Set 34 (page 364) . . . 54
7.5 Exercise Set 35 (page 373) . . . 57
7.5.1 Exercise Set 36 (page 382) . . . 59
7.6 Exercise Set 37 (page 387) . . . 59
7.6.1 Exercise Set 38 (page 396) . . . 61
7.7 Exercise Set 39 (page 411) . . . 64
7.8 Exercise Set 40 (page 420) . . . 67

7.8.1 Exercise Set 41 (page 430) 68
7.9 Exercise Set 42 (page 446) 69
7.10 Exercise Set 43 (page 457) 71
7.11 Exercise Set 44 (page 470) 73
7.12 Chapter Exercises (page 472) 75
7.12.1 Chapter Exercises, continued (page 472) 76
7.12.2 Review Exercises on Integration (page 473) 76

8 APPENDIX A **99**
8.1 Exercise Set 45 (page 487) 99

9 APPENDIX B **101**
9.1 Exercise Set 46 (page 495) 101

10 APPENDIX C **102**
10.1 Exercise Set 47 (page 509) 102

11 APPENDIX D **105**
11.1 Exercise Set 48 (page 515) 105
11.2 Exercise Set 49 (page 523) 105

12 APPENDIX E **109**
12.1 Special Exercise Set (page 530) 109
12.2 Exercise Set 50 (page 538) 110

Chapter 1

Solutions Manual

1.1

1.2 Exercise Set 1 (page 13)

1. $-2, -1, 2, \frac{1}{4}$.
2. $(x+1)^3 \sin(x+1)$. Use the Box method.
3. $z - 2 + 2\sin(z-2) - \cos z$.
4. $-2\cos(x+ct)$.
5. $f(\pi/2) = \sin(\cos(\pi/2)) = \sin 0 = 0$.
6. $2x + h$. You get this by dividing by h since $h \neq 0$.
7. $\sin(t+3)\dfrac{\cos h - 1}{h} + \cos(t+3)\dfrac{\sin h}{h}$.
8. (a) $-\pi^2$, (b) $4\pi^2$.
9. (a) $f(0) = 1$, (b) $f(0.142857) = 0.857143$, (c) Since $0 < x < 1$ we see that $2 < 3x + 2 < 5$. So, $f(3x+2) = (3x+2)^2 = 9x^2 + 12x + 4$.
10. $f(F(x)) = x$, $F(f(x)) = |x|$.
11. The Box method gives that $g(x+1) = (x+1)^2 - 2(x+1) + 1 = x^2$.
12. Again we use the Box method with the quantity $(x-1)/(2-x)$ *inside* the Box. Since $h(\Box) = (2\Box+1)/(1+\Box)$, we use some simple algebra to see that the right-hand side becomes just x.
13. 8. Observe that $f(x+h) - 2f(x) + f(x-h) = 8h^2$, so that, for $h \neq 0$ the cancellation of the h^2-terms gives the stated result.
14. The definition of the funcion tells us that (using the Box method), $f(x+1) = (x+1) - 1 = x$ whenever $0 \leq x + 1 \leq 2$, which is equivalent to saying that $f(x+1) = x$ whenever $-1 \leq x \leq 1$. We use the same idea for the other interval. Thus, $f(x+1) = 2(x+1) = 2x + 2$ whenever $2 < x + 1 \leq 4$, equivalently, $f(x+1) = 2x + 2$ whenever $1 < x \leq 3$. Since the interval $\{1 < x \leq 2\}$ is contained inside the interval $\{1 < x \leq 3\}$ it follows that $f(x+1) = 2x + 2$ for such x.

1.3 Exercise Set 2 (page 23)

1. $$f(x) = \begin{cases} x^2 - 1, & x \geq 1 \text{ or } x \leq -1, \\ 1 - x^2, & -1 < x < 1. \end{cases}$$

2. $$f(x) = \begin{cases} 3x + 4, & \text{if } x \geq -4/3, \\ -3x - 4, & \text{otherwise.} \end{cases}$$

3. $$h(x) = \begin{cases} x^2, & \text{if } x \geq 0, \\ -x^2, & \text{otherwise.} \end{cases}$$

4. $$f(x) = \begin{cases} 1 - t, & \text{if } t \geq 0, \\ 1 + t, & \text{if } t < 0. \end{cases}$$

5. $$g(w) = \begin{cases} \sin w & \text{for } w \text{ in any interval of the form } [2\pi n, 2\pi n + \pi], \\ -\sin w & \text{otherwise,} \end{cases}$$

   where n is an integer.

6. $$f(x) = \begin{cases} \frac{1}{x\sqrt{x^2-1}}, & \text{if } x > 1, \\ -\frac{1}{x\sqrt{x^2-1}}, & \text{if } x < -1. \end{cases}$$

7. $$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases}$$

8. $$f(x) = \begin{cases} 2x, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

9. $$f(x) = \begin{cases} 0, & \text{if } x \geq 0, \\ 2x, & \text{if } x < 0. \end{cases}$$

1.4 Exercise Set 3 (page 24)

;

1. Correction: If $A < 0$, then $-A < B$ implies $-1/A > 1/B$.
2. This is false. To see this, let $A = 1$ and $B = 0$.
3. Correction: $0 \leq A < B$ implies $A^2 < B^2$.
4. Correction: $A > B > 0$ implies $1/A < 1/B$.
5. Correction: $A < B$ implies $-A > -B$.
6. Correction: If $A^2 < B^2$ and $B > 0$, then $A < B$.
7. This statement is correct. There is nothing wrong!
8. $(0, \pi)$. (Note: To complete our argument we need $\sin x > 0$, which is guaranteed by $0 < x < \pi$.)
9. It's values are less than or equal to 6. Actually, its largest value occurs when $x = 2$ in which case $f(2) \approx 5.8186$.

10. g is unbounded: This means that it can be greater than (resp. less than) any given number. The problem occurs at $x = 0$.

11. From $x > 1$ we see that both x and $x-1$ are positive. Hence we can square both sides of the inequality $x > x-1$ to arrive at $x^2 > (x-1)^2$. (Alternately, since both x and $x-1$ are positive, $x^2 > x^2 - x - (x-1) = x^2 - 2x + 1 = (x-1)^2$.)

12. From $p \leq 1$ we see that $1 - p \geq 0$. Since $x \geq 1$ (certainly this implies the positivity of x), we have $x^{1-p} \geq 1^{1-p}$, or $x^{1-p} \geq 1$. Now $x^{1-p} = x^{-(p-1)} = \frac{1}{x^{p-1}}$. So the last inequality can be rewritten as $\frac{1}{x^{p-1}} \geq 1$. We can multiply both sides of this inequality by $\sin x$ because $1 \leq x \leq \pi$ guarantees that $\sin x$ is positive.

13. Since both x and x^2 are ≥ 0, we can apply the AG-inequality to get $\frac{x+x^2}{2} \geq \sqrt{x \cdot x^2} = \sqrt{x^3}$. Since $x + x^2 \geq 0$, we have $x + x^2 \geq \frac{x+x^2}{2}$. So $x + x^2 \geq \sqrt{x^3}$. Yes, we can square both sides since $x \geq 0$, and so both terms in the inequality are greater than or equal to 0.

14. Yes. Under no further conditions on the symbol, since it is true that $(\square - 1)^2 \geq 0$ for any symbol, $\square$. Expanding the square and separating terms we get that $\square^2 \geq 2\square - 1$.

15. Since $1 - p \geq 0$ and $|x| \geq 1$, we have $|x|^{1-p} \geq 1^{1-p} = 1$, or $|x|\ |x|^{-p} \geq 1$, which gives $|x| \geq |x|^p$. Taking reciprocals, we get $\frac{1}{|x|} \leq \frac{1}{|x|^p}$. (The last step is legitimate because both $|x|^p$ and $|x|$ are positive.)

16. $|v| < c$. This is because we need $1 - v^2/c^2 > 0$. Now solve this inequality for v.

17. If $n = 2$, the result is clear, because $2 < (1.5)^2 < 3$. So let's assume that $n > 2$, now. We use (11.9) with the quantity "$1/n$" inside the box symbol (or replacing the box by $1/n$, if you like). We'll see that

$$\begin{aligned}\left(1 + \boxed{\tfrac{1}{n}}\right)^n &= 1 + n\boxed{\tfrac{1}{n}} + \tfrac{n(n-1)}{2!}\boxed{\tfrac{1}{n}}^2 + \cdots + \tfrac{n(n-1)\cdots(2)(1)}{n!}\boxed{\tfrac{1}{n}}^n \\ &= 1 + n(\tfrac{1}{n}) + \tfrac{n(n-1)}{2!}(\tfrac{1}{n})^2 + \cdots + \tfrac{n(n-1)(n-2)\cdots(2)(1)}{n!}(\tfrac{1}{n})^n. \\ &= 1 + n(\tfrac{1}{n}) + \tfrac{n(n-1)}{n^2}(\tfrac{1}{2!}) + \cdots + \tfrac{n(n-1)(n-2)\cdots(2)(1)}{n^n}(\tfrac{1}{n!}).\end{aligned}$$

Now, we regroup all the terms in the above display in the following way .... Note that the following term is not apparent in the display above, but it IS there! See Equation (11.9).

$$\begin{aligned}\tfrac{n(n-1)(n-2)}{n^3} &= \left(\tfrac{n}{n}\right)\left(\tfrac{n-1}{n}\right)\left(\tfrac{n-2}{n}\right) \\ &= (1)\left(\tfrac{n-1}{n}\right)\left(\tfrac{n-2}{n}\right) \\ &= \left(\tfrac{n-1}{n}\right)\left(\tfrac{n-2}{n}\right), \\ &= \left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right).\end{aligned}$$

A similar idea is used for the other terms. Okay, so using this rearrangement of terms we can rewrite $(1 + \frac{1}{n})^n$ as

$$\begin{aligned}\left(1 + \tfrac{1}{n}\right)^n &= \ldots \\ &= 1 + 1 + \left(1 - \tfrac{1}{n}\right)\tfrac{1}{2!} + \left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right)\tfrac{1}{3!} + \ldots \\ &+ \left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right)\left(1 - \tfrac{3}{n}\right)\cdots\left(1 - \tfrac{n-1}{n}\right)\tfrac{1}{n!}.\end{aligned}$$

(where there are $(n + 1)$ terms in the right hand side). Now, notice that for every integer $n > 2$, each term of the form "$1 - (something)/n$" is less than 1 and bigger than zero, because we're subtracting something positive from 1. So,

$$\left(1 - \frac{1}{n}\right) \quad < \quad 1$$

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \;<\; (1)\left(1-\frac{2}{n}\right) < 1, \ldots$$

where we have used Figure 242 with $A = 1-2/n$, $\square = 1-1/n$ (or with the symbols "$1-1/n$" inside the box), and $\triangle = 1$ (or with "1" inside the triangle). Using these estimates we can see that we can replace every term inside the "large brackets" by 1 so that

$$\begin{aligned}\left(1+\tfrac{1}{n}\right)^n &= \ldots \\ &= 1+1+\left(1-\tfrac{1}{n}\right)\tfrac{1}{2!}+\cdots+\left(1-\tfrac{1}{n}\right)\left(1-\tfrac{2}{n}\right)\left(1-\tfrac{3}{n}\right)\cdots\left(1-\tfrac{n-1}{n}\right)\tfrac{1}{n!} \\ &< 1+1+\tfrac{1}{2!}+\tfrac{1}{3!}+\cdots+\tfrac{1}{n!}\end{aligned} \tag{1.1}$$

We're almost done! Now we use the following inequalities ...

$$\begin{aligned} 3! = 3\times 2\times 1 &> 2\times 2\times 1 = 2^2 \\ 4! = 4\times 3\times 2\times 1 &> 2\times 2\times 2\times 1 = 2^3 \\ 5! = 5\times 4\times 3\times 2\times 1 &> 2\times 2\times 2\times 2\times 1 = 2^4 \\ &\cdots \\ n! &> 2^{n-1}\end{aligned}$$

Now since we must "reverse the inequality when we take reciprocals of positive numbers" (Table 1.1, Table 1.2) we get that for every integer $n > 2$,

$$n! > 2^{n-1} \qquad \text{implies} \qquad \frac{1}{n!} < \frac{1}{2^{n-1}}$$

Combining this estimate with Equation (11.1) we get a new estimate, namely,

$$\begin{aligned}\left(1+\tfrac{1}{n}\right)^n &< 1+1+\tfrac{1}{2!}+\tfrac{1}{3!}+\cdots+\tfrac{1}{n!} \\ &< 1+1+\tfrac{1}{2}+\tfrac{1}{2^2}+\cdots+\tfrac{1}{2^{n-1}}.\end{aligned} \tag{1.2}$$

Now, the sum on the right above is a **finite geometric series** and we know that, if $n > 2$,

$$1+\tfrac{1}{2^1}+\tfrac{1}{2^2}+\cdots+\tfrac{1}{2^n} = \tfrac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}} < \tfrac{1}{1-\frac{1}{2}} = 2.$$

Now you can see that, when we combine this latest estimate with (11.2) we find

$$\begin{aligned}\left(1+\tfrac{1}{n}\right)^n &< 1+1+\tfrac{1}{2}+\tfrac{1}{2^2}+\cdots+\tfrac{1}{2^{n-1}} \\ &< 1+2=3\end{aligned}$$

which is what we wanted to show. Okay, this looks a bit long, but we did include all the details, right? Eventually, you'll be able to skip many of the details and do them in your head, so to speak, and the whole thing will get shorter and faster, you'll see.

NOTE: It looks tough, but we'll be using this 200 yr old inequality later on, in Chapter 4, when we define Euler's number, 2.7182818284590.

1.5 Chapter Exercises (page 26)

1. $6,\ 1,\ 2,\ 2\frac{3}{4} = \frac{11}{4}$.
2. $(x^2+1)^3\cos(x^2+1)$.
3. $z + 3 + 2\sin(z+3) - \cos(z+5)$.
4. $-\dfrac{\sin h}{h}\sin x + \dfrac{\cos h - 1}{h}\cos x$.
5. From $\frac{3}{x} > 6$ we see that x must be positive: $x > 0$. So we can rewrite it as $3 > 6x$, which gives $x < \frac{1}{2}$. Thus the solution is $0 < x < \frac{1}{2}$.
6. $x \geq -\frac{4}{3}$, since we an subtract 4 from both sides ...
7. $x < \frac{1}{2}$. Note that $2x - 1 < 0$ and so $2x < 1$.
8. $|x| > \sqrt{5}$. In other words, either $x > \sqrt{5}$ or $x < -\sqrt{5}$.
9. $|t| < \sqrt[4]{5}$. That is, $-\sqrt[4]{5} < t < \sqrt[4]{5}$.
10. $-\infty < x < +\infty$. That is, x can be any real number. This is because the stated inequality implies that $\sin x \leq 1$ and this is *always* true!
11. $z \geq 2^{1/p}$. (Note: For general p, z^p is defined only for $z > 0$.)
12. $|x| \leq 3$. Or $-3 \leq x \leq 3$.
13. $$f(x) = \begin{cases} x+3, & \text{for } x \geq -3, \\ -x-3, & \text{for } x < -3. \end{cases}$$
14. $$g(x) = \begin{cases} t - 0.5, & \text{if } t \geq 0.5, \\ -t + 0.5, & \text{otherwise.} \end{cases}$$
15. $$g(t) = \begin{cases} 1 - t, & \text{if } t \leq 1, \\ t - 1, & \text{otherwise.} \end{cases}$$
16. $$f(x) = \begin{cases} 2x - 1, & x \geq \frac{1}{2} \\ 1 - 2x, & x < \frac{1}{2} \end{cases}$$
17. $$f(x) = \begin{cases} 1 - 6x, & \text{if } x \leq 1/6, \\ 6x - 1, & \text{otherwise.} \end{cases}$$
18. $$f(x) = \begin{cases} x^2 - 4, & \text{if either } x \geq 2 \text{ or } x \leq -2, \\ 4 - x^2, & \text{if } -2 < x < 2. \end{cases}$$
19. $$f(x) = \begin{cases} 3 - x^3, & \text{if } x \leq \sqrt[3]{3}, \\ x^3 - 3, & \text{if } x > \sqrt[3]{3}. \end{cases}$$
20. $f(x) = |(x-1)^2| = (x-1)^2 = x^2 - 2x + 1$ for all x. (Note that $(x-1)^2$ is always ≥ 0 for any value of x.)
21. $$f(x) = |x(2-x)| = \begin{cases} x(2-x), & \text{if } 0 \leq x \leq 2, \\ x(x-2), & \text{otherwise.} \end{cases}$$
22. $f(x) = |x^2 + 2| = x^2 + 2$ for all x, because $f(x) = x^2 + 2 \geq 2 > 0$ to begin with.

23. From $p \leq 1$ we have $1 - p \geq 0$. So $x \geq 1 > 0$ gives $x^{1-p} \geq 1^{1-p} = 1$. Now $x^{1-p} = x^{-(p-1)} = \frac{1}{x^{p-1}}$. Thus $\frac{1}{x^{p-1}} \geq 1$. On the other hand, from $0 \leq x \leq \pi/2$ we have $\cos x \geq 0$. So we can multiply $\frac{1}{x^{p-1}} \geq 1$ throughout by $\cos x$ to arrive at $\frac{\cos x}{x^{p-1}} \geq \cos x$.

24. 2, 2.25, 2.370370, 2.44141, 2.48832, 2.52163, 2.54650, 2.56578, 2.58117, 2.59374. Actually, these numbers approach the value $2.71828\ldots$.

25. From $0 \leq x \leq \frac{\pi}{2}$ we have $\sin x \geq 0$ and $\cos x \geq 0$. Thus we may apply the AG-inequality to get $\frac{\sin x + \cos x}{2} \geq \sqrt{\sin x \cos x}$. Since $\sin 2x = 2 \sin x \cos x$, we see that $\sqrt{\sin x \cos x} = \sqrt{\frac{\sin 2x}{2}}$ and so $\frac{\sin x + \cos x}{2} \geq \sqrt{\frac{\sin 2x}{2}}$. Multiplying both sides by $\sqrt{2}$ we get the desired inequality.

Chapter 2

Solutions

2.1 Exercise Set 4 (page 38)

1. 4
2. 1
3. 0
4. $+\infty$, since $t > 2$ and $t \to 2$.
5. 0
6. -1, since $|x| = -x$ for $x < 0$.
7. 0
8. $-\frac{1}{\pi}$
9. 0
10. $+\infty$, since $|x-1| = 1 - x > 0$ for $x < 1$.
11. 0
12. $\dfrac{1}{6}$
13. i) 0, ii) 1. Since the limits are different the graph must have a break at $x = 1$.
14. i) 1, ii) 1, iii) 0, iv) 1; since the one-sided limits are equal at $x = 0$ and $g(0) = 1$, the graph has no break at $x = 0$. But since these limits are different at $x = 1$, it must have a break at $x = 1$.
15. i) 1, ii) 2, iii) 1, iv) 2.

2.2 Exercise Set 5 (page 48)

1. No, because the left and right-hand limits at $x = 0$ are different, $(2 \neq 0)$.
2. Yes, the value is 4, because the two one-sided limits are equal (to 4).
3. Yes, the value is 0, because the two one-sided limits are equal (to 0).
4. Yes, the value is 0, because the two one-sided limits are equal (to 0).
5. Yes, the value is 0, because the two one-sided limits are equal; remove the absolute value, first, and note that $\sin 0 = 0$.

6. No, because the left-hand limit at $x = 0$ is $-\infty$ while the right-hand limit there is $+\infty$.
7. No, because the left-hand limit at $x = 0$ is $-\infty$ and the right-hand limit there is $+\infty$.
8. Yes, the answer is $1/2$ because the two-one sided limits are equal (to $\frac{1}{2}$).
9. Yes, because the two-one sided limits are equal (to 2).
10. No, because the left-hand limit at $x = 0$ is $+3$ and the right-hand limit there is $+2$ ($3 \neq 2$).
11. a) Yes, the left and right-hand limits are equal (to 0) and $f(0) = 0$;
    b) Yes, because g is a polynomial;
    c) Yes, because the left and right-limits are equal to 3 and $h(0) = 3$;
    d) Yes, since by Table 2.4d, the left and right-limits exist and are equal and $f(0) = 2$;
    e) Yes, because f is the quotient of two continuous functions with a non-zero denominator at $x = 0$. Use Table 2.4d again.
12. Follow the hints.

2.2 Exercise Set 6 (page 51)

1. $x = 0$ only; this is because the right limit is 2 but the left-limit is 0. So, f cannot be continuous at $x = 0$.
2. $x = 0$ only; this is because the right limit is 1 but the left-limit is 0. So, f cannot be continuous at $x = 0$.
3. $x = \pm 1$ because these are the roots of the denominator, so the function is infinite there, and so it cannot be continuous there.
4. $x = 0$ only. In this case the right limit is the same as the left-limit, 1, but the value of $f(0) = 2$ is not equal to this common value, so it cannot be continuous there.
5. $x = 0$ only. This is because the right-limit at $x = 0$ is $+\infty$, so even though $f(0)$ is finite, it doesn't matter, since one of the limits is infinite. So, f cannot be continuous at $x = 0$.
6. $x = 0$ only, because the left-limit there is 1.62 while its right-limit there is 0. There are no other points of discontinuity.

2.2 Exercise Set 7 (page 57)

1. -1. Use the trigonometric identity, $\sin(\square + \pi) = -\sin\square$.
2. -1. Use the hint.
3. 2. Multiply the expression by $1 = \frac{2}{2}$ and rearrange terms.
4. 0. Let $\square = 3x$, rearrange terms and simplify.
5. 2. Multiply the whole expression by "1" or $\frac{2x}{4x} \cdot \frac{4x}{2x}$.
6. 1. Let $\square = \sqrt{x-1}$. As $x \to 1$ we have $\square \to 0$ and $\frac{\sin\square}{\square} \to 1$.

2.2 Exercise Set 8 (page 58)

1. 0. Continuity of the quotient at $x = 2$.
2. 0. Note that $\cos 0 = 1$.
3. $\frac{1}{6}$. Factor the denominator.
4. -1. Rewrite the secant function as the reciprocal of the cosine function and use the trig. identity $\cos\Box = -\sin(\Box - \frac{\pi}{2})$.
5. -2. Factor out the 2 from the numerator and then use the idea of Exercise 4, above.
6. 0. The function is continuous at $x = 2$, and $\sin 2\pi = 0$.
7. 3. Multiply and divide the expression by 3 and rewrite it in a more familiar form.
8. $-\infty$. Use your calculator for a test of this limit. The numerator approaches -1 and the denominator approaches 0 through positive values. So the quotient must approach the stated value.
9. $+\infty$. The denominator approaches 0 through negative values, while the numerator approaches -1. Thus, the quotient approaches the stated value.
10. 0. The function is continuous at $x = 0$.
11. $x = \pi$. The denominator is 0 and the numerator isn't.
12. $x = 0$. Since $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1 \neq f(0)$, we know that f cannot be continuous there, by definition.
13. None. This is because f is a polynomial and so it is continuous everywhere.
14. $x = \pm 1$, the roots of the denominator.
15. $x = \pm 2$. For $x = 2$ the numerator is of the form $0/0$ and $f(2)$ is not defined at all, so the function is not continuous here (by definition). Next, the denominator is zero for $x = -2$, but the numerator isn't zero here. So the function is of the form $-4/0 = -\infty$ and so once again, f is not continuous here because its value here is $-\infty$.
16. $\frac{3}{2}$. Use the Hint. We know from the Hint (with $A = x, B = 2x$) that $\cos x - \cos 2x = -2\sin(3x/2)\sin(-x/2)$. Then

$$\begin{aligned}\frac{\cos x - \cos 2x}{x^2} &= -\frac{2\sin(3x/2)}{x}\frac{\sin(-x/2)}{x},\\ &= -\frac{2\left(\frac{3}{2}\right)\sin(3x/2)}{\left(\frac{3x}{2}\right)}\frac{\left(\frac{-1}{2}\right)\sin(-x/2)}{\left(\frac{-x}{2}\right)},\\ &= -\left(-\frac{3}{2}\right)\frac{\sin(3x/2)}{\left(\frac{3x}{2}\right)}\frac{\sin(-x/2)}{\left(\frac{-x}{2}\right)}.\end{aligned}$$

   Now use the hint with $\Box = \frac{3x}{2}$ and $\Box = -\frac{x}{2}$, as $x \to 0$. Both limits approach 1 and so their product approaches $3/2$.
17. 0. Use the Hint. We can rewrite the expression as

$$\begin{aligned}\frac{\tan x - \sin x}{x^2} &= \frac{\tan x\,(1 - \cos x)}{x^2},\\ &= \frac{\tan x}{x}\left(\frac{1 - \cos x}{x}\right),\\ &= \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right)\left(\frac{1 - \cos x}{x}\right).\end{aligned}$$

As $x \to 0$, the first term approaches 1, the second term approaches 1, while the last term approaches 0, by Table 2.12. So, their product approaches 0.

18. $+\infty$. The limit exists and is equal to $+\infty$.

19. $a = -\frac{\pi}{2}$, $b = -\pi a = \frac{\pi^2}{2}$.

20. 1. Rationalize the denominator. Note that the function is continuous at $x = 0$.

2.3 Exercise Set 9 (page 63)

1. 0. This is a limit as $x \to \infty$, not as $x \to 0$.

2. 0. Divide the numerator and denominator by x and simplify.

3. 1. Divide the numerator and denominator by x and simplify.

4. $\frac{1}{2}$. Rationalize the numerator first, factor out $\sqrt{x}$ out of the quotient, simplify and then take the limit.

5. 0. Use the Sandwich Theorem.

6. The graph of the function $\sin x$ isn't going anywhere definite; it just keeps oscillating between 1 and -1 forever and so it cannot have a limit. This is characteristic of *periodic functions* in general.

2.4 Exercise Set 10 (page 78)

1. $-\infty$. Since $x \to 0^+$ it's necessary that $x > 0$. So, simplifying, we get $(x-1)/x = 1 - (1/x)$ and since $x > 0$ it must be that the limit of $1/x$ as $x \to 0^+$ exists and is equal to plus infinity. Hence, as $x \to 0^+$, $(x-1)/x \to 1 - (+\infty) = -\infty$.

2. $+\infty$. As before, as $x \to 0^+$, x must be positive and so, $(2+x)/x = (2/x) + 1$. Since $2/x \to +\infty$ as $x \to 0^+$ it follows that $(2+x)/x \to +\infty + 1 = +\infty$.

3. The limit does not exist. Note that this is a two-sided limit so we (usually) need to check each one of the one-sided limits at 0. Since $x \to 0$ we have $x \neq 0$, so $(3-x)/x = (3/x) - 1$. Now for the one-sided limits. As $x \to 0^+$, $x > 0$ so the right-hand limit is equal to $3/0 - 1 = +\infty - 1 = +\infty$. For $x \to 0^-$, $x < 0$ so the left-hand limit is equal to $-3/0 - 1 = -\infty - 1 = -\infty$. Since each of these one-sided limits are different, the required two-sided limit cannot exist.

4. $+\infty$. As $x \to 0^+$ we must have $x > 0$ so, $(2x+1)/x = 2 + (1/x)$. So, as $x \to 0^+$ this quotient tends to $2 + \infty = +\infty$.

5. $-\infty$. Now $x \to 0^-$ means that $x < 0$ and x approaches zero. Thus, $(x^2+1)/x = x + (1/x)$ and $1/x \to -\infty$ (since $x < 0$ in its approach to zero). It follows that the limit exists and is equal to $0 - \infty = -\infty$.

6. $+\infty$. This is a good question because it is a two-sided limit and it has an absolute value in it, so one doesn't normally expect such a limit to even exist. We remove the absolute value first. Simplifying the expression and then applying the definition of the absolute value we get, since $x \neq 0$, $(x+1)/|x| = (x/|x|) + (1/|x|)$ so that,

$$\frac{x+1}{|x|} = \begin{cases} 1 + 1/x, & \text{if} \quad x > 0, \\ -1 - 1/x, & \text{if} \quad x < 0. \end{cases}$$

Now letting $x \to 0^+$, we see that $x > 0$ so $(x+1)/|x| \to 1 + 1/0 = 1 + \infty = +\infty$. However, letting $x \to 0^-$ means that $x < 0$ so $(x+1)/|x| = -1 - 1/x$. But as $x \to 0^-$, $x < 0$ and the expression $-1/x \to -(-\infty) = +\infty$. So, $-1 - 1/x \to -1 - (-\infty) = -1 + \infty = +\infty$. See? Both one-sided limits are actually equal, so the limit exists and is equal to $+\infty$.

7. The limit does not exist. Again we have a two-sided limit and an absolute value inside the expression. Applying the definitions we get

$$\frac{2x^2 + x}{|x|} = \begin{cases} 2x + 1, & \text{if} \quad x > 0, \\ -2x - 1, & \text{if} \quad x < 0. \end{cases}$$

But as $x \to 0^+$ we have $2x + 1 \to 1$ and as $x \to 0^-$, $-2x - 1 \to -1$. Since both limits are different at 0 the two sided limit cannot exist.

8. The limit does not exist. This is because the two one sided limits are different. On the one hand, $x \to 1^+$ means that $x - 1 > 0$ which implies that $x/(x-1) \to 1/0 = +\infty$. On the other hand, $x \to 1^-$ means that $x - 1 < 0$ which implies that $x/(x-1) \to -\infty$, since the quotient is always negative and its denominator is approaching zero.

9. $+\infty$. First observe that the numerator is continuous at $x = 2$ (x is always in radians, remember?). Thus, $1 + \sin(x) \to 1 + \sin 2 \approx 1.909$. The limit's existence is now a matter for the denominator to decide. But since the denominator $x - 2 > 0$ as $x \to 2^+$ and it approaches zero, it follows that the quotient, $(1 + \sin x)/(x-2) \to (1.909)/0 = +\infty$.

10. $+\infty$. This is simple because $x \to -3^+$ is equivalent to saying that $x + 3 \to 0^+$ (from the right too, basically by adding 3 to both sides of $x \to -3^+$ and thinking for a minute). Hence the quotient $1/(x+3) \to 1/0 = +\infty$ in this case.

11. $-\infty$. The numerator tends to $1/2$ as $x \to 1/2^-$, while $x \to 1/2^-$ actually means that $2x - 1 \to 0^-$ (seen by multiplying both sides of $x \to 1/2^-$ by 2 and rearranging terms). So the denominator tends to zero from the left (that is, through negative values) and so the quotient must tend to $-\infty$.

12. The limit does not exist. The function defined by the numerator $\cos(x-2) \to cos0 = 1$ as $x \to 2$ by continuity. Being a two-sided limit we see that the sign of the approach of the denominator $x - 2$ to 0 will depend on whether the limit is from the right (in which case we get 0 through positive values) or from the left (in which case we get 0 through negative values). It follows the right hand limit is $+\infty$ while the left-hand limit is $-\infty$. So the limit cannot exist.

13. $+\infty$. (Similar to Exercises 6 and 7, above.) The point here is that the presence of the absolute value $|x-2|$ in the denominator ensures that the quotient always approaches 0 from the right regardless of how $x \to 2$. So now the limit does, in fact, exist and is equal to $+\infty$.

14. $+\infty$. (See Example 45 (c) in this section.)

15. The limit does not exist. (See Example 45 (a) in this section with the same identity.)

16. The limit does not exist. (See Example 45 (b) in this section.) The oscillations of $2^x \sin x$ get larger and larger as $x \to \infty$ so there can be no limit.

17. 10^{-11}. As $x \to +\infty$ the quotient, $1/(1+x^2) \to 0$ through positive values of x. It follows that the sum $1/(1+x^2) + 10^{-11} \to 0 + 10^{-11} = 10^{-11}$. Note that if you did this problem on a hand-held conventional calculator you might think the answer is 0 since the machine only gives 9 decimal places accuracy! But it isn't equal to zero, is it!?

18. -10^{-8}. This is similar to the preceding one except that now $x \to -\infty$ means that $x < 0$, i.e., $-1/x^3 > 0$ and so it tends to zero anyhow as $x \to -\infty$.

19. $+\infty$. The first term approaches 0 while the second term approaches $-(-\infty) = +\infty$. The result follows from this.

20. $+\infty$. Same idea as the previous one ... The first term approaches 0 while the second term approaches $+\infty$ as $x \to \infty$.

21. $-\infty$. The first term approaches zero as $x \to 0^+$, while the second term approaches $-\cos(0)/0 = -1/0 = -\infty$ through positive values of x. The result follows.

22. 0. Recall that $\sqrt{x^2} = |x|$ by definition, so since $x \to +\infty$ it follows that $x > 0$ for all large values of x. So, it must be the case that the denominator $2x - \sqrt{x^2} = 2x - x = x$ approaches plus infinity, through positive values of x. In other words, the quotient approaches $1/\infty = 0$.

23. 1. The reason for this is the classic trig identity, $\sin^2 x + \cos^2 x = 1$ valid for any real numbver x. So, we are basically taking the limit as $x \to \infty$ of the constant function 1 which, of course, gives 1. Although each term in this expression is oscillating and has no limit, their sum *does* have a limit. (It doesn't generally happen, but it does happen here!)

24. The limit does not exist. We know from trig that the tan function is infinite at $\pi/2$, because $\cos(\pi/2) = 0$ and the numerator is 1. But what is the approach like? Well, as $x \to \pi/2^+$, the expression $3\tan x \to +\infty$ while as as $x \to \pi/2^-$, the expression $3\tan x \to -\infty$ (check the graph of the tan function if you're not sure). It follows that the required two-sided limit cannot exist because both one-sided limits will give different "infinities".

25. $-\infty$. This is similar to the preceding one. We know from trig that the cot function is infinite at π, because $\sin \pi = 0$ and $\cos \pi = -1$. Again, what is the approach like? As $x \to \pi^-$ we know that for x close to π and just less than π, the expression $\cos x < 0$. On the other hand, for such x the expression $\sin x > 0$. Their quotient is therefore negative and so the quantity $3x + \cot x \to -\infty$ as $x \to \pi^-$. This means that $3x + \cot x \to -\infty$ as $x \to \pi^-$.

2.5 Chapter Exercises (page 79)

1. Since f is a polynomial, it is continuous everywhere and so also at $x = 1$.

2. g is the product of two continuous functions (continuous at 0) and so it is itself continuous at $t = 0$.

3. h is the sum of three continuous functions and so it is continuous at $z = 0$.

4. f is a constant multiple of a continuous function and so it is continuous too (at $x = \pi$).

5. The graph of f is 'V'-shaped at $x = -1$ but it is continuous there nevertheless.

6. The limit is $3 - 2 + 1 = 2$ since f is continuous at $x = 1$.

7. The limit is $0 \cdot 1 = 0$ since g is continuous at $t = 0$.

8. The limit is $0 + (2)(0) - \cos 2 = -\cos 2 \approx 0.416$ since h is continuous at $z = 0$.

9. The limit is $2 \cdot \cos \pi = (2)(-1) = -2$ since f is continuous at $x = \pi$.

10. The limit is $|-1 + 1| = |0| = 0$ since f is continuous at $x = -1$.

11. 0. The function is continuous at $t = 2$.

12. $\frac{1}{8}$. Factor the denominator first, then take the limit.

13. $+\infty$. Use extended real numbers.

14. 1. Remove the absolute value first.

15. $+\infty$.

16. i) 1; ii) 1; iii) 0; iv) 1; v) Since (i) and (ii) are equal we see that g is continuous at $x = 0$ as $g(0) = 1$, by definition. Since the left and right limits at $x = 1$ are different (by (iii) and (iv)), we see that g is not continuous at $x = 1$ and so the graph has a break there.

17. The limit from the left is 2 and the limit from the right is 1. So the limit cannot exist.

18. $|-2| = 2$. The absolute value function is continuous there.

19. $0/(-1) = 0$. The quotient is continuous at $x = -2$.

20. 0. The function is continuous at that point.

21. Does not exist. The left-hand limit as $x \to 1$ is 1, but the right-hand limit as $x \to 1$ is $|1-1| = 0$, so the limit cannot exist.

22. $x = 0$. This is because the left-and right-hand limits there are not equal. For example, the left limit is -2 while the right-limit is 0. Use the definition of the absolute value, OK?

23. $x = 0$. The left-hand limit is -1 while the right-hand limit is 1.

24. None. The denominator is $x^3 - 1 = (x-1)(x^2+x+1)$ with $x = 1$ as its only real root. Why? By "completing the square", we have $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} > 0$ and hence $x^2 + x + 1$ does not have real roots. The only possible point of discontinuity is $x = 1$. But both the left and right limits at $x = 1$ are $-1/3$, which is also the value of f at $x = 1$. Hence f is continuous at $x = 1$ and so everywhere.

25. $x = 0$. Even though the values of the left and right limits here are 'close' they are not equal, since $-0.99 \neq -1$.

26. $x = 0$. The left and right-hand limits there are both equal to $+\infty$, so f cannot be continuous there.

27. $\frac{a}{b}$. Multiply the expression by $1 = \frac{bx}{ax}\frac{ax}{bx}$, simplify. Then take the limit.

28. $+\infty$. This limit actually exists in the extended reals. Observe that the numerator approaches 1 regardless of the direction (left or right) because it is continuous there, while the denominator approaches 0 regardless of the direction, too, and for the same reason. The quotient must then approach $1/0 = +\infty$ in the extended reals.

29. 0. Break up the expression into three parts, one involving only the term x, another with the term $\sin x/x$ and the remaining one with the term $x/\sin 2x$. The first term approaches 0, the next term term approaches 1 while the last term approaches 1/2, by Exercise 27, with $a = 2, b = 1$ and Table 2.4, (d). So, the product of these three limits must be equal to zero.

30. 1. Let $\square = \sqrt{3-x}$. As $x \to 3^-$, we have $\square \to 0^+$ and so $\frac{\sin \square}{\square} \to 1$.

31. $\frac{b}{a}$. See Exercise 27 in this Section: Multiply the expression by ax/ax, re-arrange terms and evaluate.

32. 0. This limit actually exists. This is because the numerator oscillates between the values of ± 1 as $x \to \infty$, while the denominator approaches ∞. The quotient must then approach $(something)/\infty = 0$ in the extended reals.

33. Does not exist. There are many reasons that can be given for this answer. The easiest is found by studying its graph and seeing that it's not 'going anywhere'. You can also see that this function is equal to zero infinitely often as $x \to -\infty$ (at the zeros or roots of the sine function). But then it also becomes as large as you want it to when x is chosen to be anyone of the values which makes $\sin x = -1$. So, it oscillates like crazy as $x \to -\infty$, and so its limit doesn't exist.

34. 0. Hard to believe? Rationalize the numerator by multiplying and dividing by the expression $\sqrt{x^2+1} + x$. The numerator will look like $(x^2+1) - x^2 = 1$, while the denominator looks like $\sqrt{x^2+1} + x$. So, as $x \to +\infty$, the numerator stays at 1 while the denominator tends to ∞. In the end you should get something like $1/\infty = 0$ in the extended reals.

35. Set $a = -5, b = 1$ in Bolzano's Theorem and set your calculator to radians. Now, calculate the values of $f(-5), f(1)$. You should find something like $f(-5) = -4.511$ and $f(1) = 1.382$ so that their product $f(-5) \cdot f(1) < 0$. Since the function is a product of continuous functions, Bolzano's Theorem guarantees that $f(x) = 0$ somewhere inside the interval $[-5, 1]$. So, there is a root there.

36. Set $a = -3, b = 0$. Now, calculate the values of $f(-3), f(0)$.Then $f(-3) = -9$ and $f(0) = 2$ so that their product $f(-3) \cdot f(0) < 0$. Since the function is a polynomial, it is a continuous function, so Bolzano's Theorem guarantees that $f(x) = 0$ somewhere inside the interval $[-3, 0]$. So, there is a root there.

37. Let $f(x) = x^2 - \sin x$. Write $f(a) \cdot f(b)$. Now let a, b with $a < b$ be any two numbers whatsoever. Check that your calculator is in radian mode, and calculate the values $f(a) \cdot f(b)$ like crazy! As soon as you find values of a, b where $f(a) \cdot f(b) < 0$, then STOP. You have an interval $[a, b]$ where $f(x) = 0$ somewhere inside, by Bolzano's Theorem. For example, $f(-0.3) \cdot f(2.5) = 2.179$, $f(0.3) \cdot f(1.5) = -0.257 < 0$. STOP. So we know there is a root in the interval $[0.3, 1.5]$.

Chapter 3

Solutions

3.1 Exercise Set 11 (page 93)

1. 4. Use the binomial theorem to expand and simplify.
2. -1. Note that $f(x) = -x$ for $x < 0$ and so for $x = -1$, too.
3. $+\infty$. The quotient is equal to $1/h^2 \to +\infty$ as $h \to 0$.
4. a) $+\infty$, b) 1. Note that $f(1+h) = 1+h$ for $h < 0$ and $f(1+h) = 2+h$ for $h > 0$.
5. $\dfrac{1}{2\sqrt{2}} \approx 0.3536$.
6. 4. Use the binomial theorem to expand and simplify.
7. 3.
8. -4.
9. 6.
10. 1. Note that $f(x) = x$ near $x = 1$.
11. 0. Note that $f(x) = x^2$ for $x > 0$ and $f(x) = -x^2$ for $x < 0$.
12. 0 for all $x \neq 0$, and the slope does not exist when $x = 0$.
13. The derivative does not exist since f is not continuous there.
14. The derivative does not exist because $f(x)$ is undefined for any x slightly less than -1. However, its right-derivative at $x = -1$ is $+\infty$.
15. Yes. The absolute value can be removed so that $f(x) = x^2$. It turns out that $f'(0) = 0$.
16. $f'(1) = -\frac{1}{2}$.
17. $f'(1) = -2$.
18. a) $f'(1)$ does not exist since f is *not continuous* at $x = 1$. Alternately note that the left- and right-derivatives at $x = 1$ are unequal: $f'_+(1) = 1,\ f'_-(1) = \infty$.
    b) No. In this case f is continuous at $x = 2$ but the one-sided derivatives are unequal: $f'_+(2) = -4,\ f'_-(2) = 1$.
    c) Since $2 < \frac{5}{2} < 3$, we see that $f'(\frac{5}{2}) = -5$.

3.2 Exercise Set 12 (page 99)

1. $\dfrac{3}{2}x^{\frac{1}{2}} = \dfrac{3}{2}\sqrt{x}$.

2. $-2t^{-3} = -\dfrac{2}{t^3}$.
3. 0.
4. $\dfrac{2}{3}x^{-1/3} = \dfrac{2}{3\sqrt[3]{x}}$.
5. $\dfrac{t^{-4/5}}{5} = \dfrac{1}{5\sqrt[5]{t^4}}$.
6. 0.
7. $4t^3$.
8. $-3x^{-4} = -\dfrac{3}{x^4}$.
9. $-x^{-2} = -\dfrac{1}{x^2}$.
10. $\pi x^{\pi-1}$.
11. $2t$. Use the Power Difference Rules
12. $6x + 2$. Use the Power, Sum and Difference Rules
13. $1(t^2+4) + 2t(t-1)$. Use the Product Rule
14. $f(x) = 3x^{5/2} + x^{1/2}$ so $f'(x) = \frac{15}{2}x^{3/2} + \frac{1}{2\sqrt{x}}$, Use the Power Rule
15. $\dfrac{(2x+1)\ (0.5)x^{-0.5} - 2x^{0.5}}{(2x+1)^2}$. Use the Quotient Rule
16. $\dfrac{(x+1)-(x-1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$. Use the Quotient Rule
17. $\dfrac{(x^2+x-1)\ (3x^2) - (x^3-1)\ (2x+1)}{(x^2+x-1)^2}$. Use the Quotient Rule
18. $\dfrac{(\sqrt{x}+3x^{3/4})\ ((2/3)x^{-1/3}) - (x^{2/3})\ \left((1/2)x^{-1/2} + (9/4)x^{-1/4}\right)}{(\sqrt{x}+3x^{3/4})^2}$. Use the Quotient Rule

3.3 Exercise Set 13 (page 112)

1. 0.
2. 3.
3. $\frac{2}{3}$.
4. $\frac{3}{2}\sqrt{x-4}$.
5. $-\frac{5}{2}x^{-7/2}$.
6. $\frac{1}{3}(2t+1)(t^2+t-2)^{-2/3}$.
7. $\frac{d^2f}{dx^2} = 6$.
8. $4x(x+1)^3 + (x+1)^4 = (x+1)^3(5x+1)$.
9. $-\frac{1}{2}$.
10. $(t+2)^2 + 2(t-1)(t+2) = 3t^2 + 6t$.
11. $32(\frac{4}{3}x^2 - x)(x-1)^{-1/3} = \frac{32}{3}(4x^2-3x)(x-1)^{-1/3}$.
12. $210(2x+3)^{104}$.

13. $\frac{1}{2}\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$.

14. $3x^2 - 6x + 3$, or $3(x-1)^2$: Both are identical.

15. $-\dfrac{1}{x^2} + \dfrac{x}{\sqrt{x^2-1}}$.

16. $\dfrac{1}{4}\dfrac{1+3\sqrt{x}}{x\sqrt{x}(1+\sqrt{x})^3} = \dfrac{1+3\sqrt{x}}{4x^{3/2}(1+\sqrt{x})^3}$.

17. -10. Note that $f''(x) = 6x - 10$.

18. $3.077(x+0.5)^{-3.324}$.

19. Use the **Chain Rule;** For instance, let $\square = x^2$, from which we get $\frac{d}{dx}f(\square) = f'(\square)D\square$. Put x^2 in the Box, note that $D\square = 2x$ and simplify. You'll find $\frac{d}{dx}f(x^2) = 2x\ f'(x^2)$.

20. Use another form of the Chian Rule: Putting $u = g(x)$ and $w = \sqrt[3]{u} \equiv u^{1/3}$, we have $w = \sqrt[3]{g(x)}$ and

$$\frac{d}{dx}\sqrt[3]{g(x)} = \frac{dw}{dx} = \frac{dw}{du}\cdot\frac{du}{dx} = \frac{1}{3}u^{-2/3}\cdot g'(x) = \frac{g'(x)}{3\sqrt[3]{g(x)^2}}.$$

21. Let $y(x) = f(x^2)$. By the Chain Rule, we have $y'(x) = f'(x^2)\cdot 2x = 2xf'(x^2)$. Replacing x by x^2 in $f'(x) + f(x) = 0$, we have $f'(x^2) + f(x^2) = 0$, or $f'(x^2) = -f(x^2) = -y(x)$. So $y'(x) = 2xf(x^2)$ can be rewritten as $y'(x) = -2xy(x)$, that is, $y'(x) + 2xy(x) = 0$.

22. Use the Chain Rule once again on both sides of $f(F(x)) = x$. We find $f'(F(x))F'(x) = 1$, which gives $F'(x) = \frac{1}{f'(F(x))}$.

23. Use another form of the Chain Rule: $\frac{dy}{du} = \frac{dy}{dt}\cdot\frac{dt}{du} = 3t^2\cdot\frac{1}{2\sqrt{u}}$. At $u = 9$ we have $t = \sqrt{9} + 6 = 9$ and $\frac{dy}{du} = 3\cdot 9^2\cdot\frac{1}{2\sqrt{9}} = \frac{81}{2}$.

24. $y = 32(x-2) + 1$ (or $32x - y - 63 = 0$).

25. Just use the Chain Rule. You don't even have to know f, g explicitly, just their values: So, $y'(2) = f'(g(2))\cdot g'(2) = f'(0)\cdot 1 = 1$.

26. $\left(1 - \frac{2}{(3t-2\sqrt{t})^2}\right)\left(3 - \frac{1}{\sqrt{t}}\right)$. Use the Chain Rule in the form: $\frac{dy}{dt} = \frac{dy}{dr}\ \frac{dr}{dt}$. But $\frac{dy}{dr}\ \frac{dr}{dt} = \left(1 - 2r^{-2}\right)\left(3 - t^{-1/2}\right)$. Now set $r = 3t - 2\sqrt{t}$.

27. $f'(9) = \frac{7}{24\sqrt{3}}$, since $f'(x) = \frac{1}{2}\frac{1+\frac{1}{2\sqrt{x}}}{\sqrt{x+\sqrt{x}}}$. On the other hand, since $\sqrt{t^2} = |t|$, we see that $\frac{df}{dt} = \frac{2t+1}{2\sqrt{t^2+t}}$, if $t \geq 0$ and $\frac{df}{dt} = \frac{2t-1}{2\sqrt{t^2-t}}$ if $t < 0$.

28. Let $y = |x| = \sqrt{x^2}$. Now, set $g(u) = \sqrt{u}, \quad u = u(x) = x^2$. Then, $y = g(u(x))$. Using the Chain Rule we get $y'(x) = g'(u(x))\cdot u'(x) = \frac{1}{2\sqrt{u}}\cdot(2x) = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$, whenever $x \neq 0$.

29. By definition, $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$. Look at the limit

$$\lim_{h\to 0}[f(x_0+h) - f(x_0)] = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\cdot h = f'(x_0)\cdot 0 = 0.$$

We have shown that $\lim_{h\to 0} f(x_0+h) - f(x_0) = 0$, which forces

$$\lim_{h\to 0} f(x_0+h) = f(x_0).$$

This, however, is another way of writing

$$\lim_{x\to x_0} f(x) = f(x_0).$$

Hence f is continuous at x_0 (by an equivalent definition of continuity).

3.4 Exercise Set 14 (page 119)

1. -2. Implicit differentiation gives $(2x+y)+y'(x)(x+2y)=0$. Now set $x=1, \quad y=0$ and solve for $y'(1)$.
2. $\frac{dy}{dx}=\frac{3x^2-2y^2}{4xy-4y^3}$. $\frac{dx}{dy}=\frac{4xy-4y^3}{3x^2-2y^2}$.
3. $-\frac{1}{129}$. Implicit differentiation gives an expression of the form $\frac{1}{2}(x+y)^{-1/2}(1+y')+xy'+y=0$. Now solve for y' after setting $x=16$ and $y=0$.
4. $\frac{1}{2y}$. Implicit differentiation gives an expression of the form $1-2yy'(x)=0$. Now solve for y'.
5. 0. Implicit differentiation gives an expression of the form $2x+2yy'(x)=0$. Now set $x=0$, $y=3$. You see that $y'(0)=0$.
6. $y+1=\frac{1}{2}(x+1)$. Note that $y'(x)=\frac{x}{2y}$.
7. $y-1=\frac{1}{3}(x-1)$, or $x-3y+2=0$. Note that $y'(x)=\frac{2-y}{x+2y}$.
8. $y=\frac{5}{2}(x-4)$, or $5x-2y-20=0$. Note that $y'(x)=\frac{x+1}{2-y}$.
9. $y=-(x-1)-1$, or $x+y=0$. Note that $y'(x)=-\frac{y(2x+y)}{x(x+2y)}$.

3.5 Exercise Set 15 (page 128)

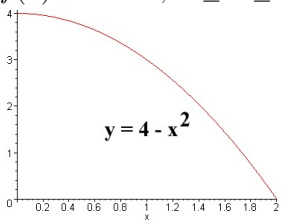
1. $\dfrac{\cos 1}{2}$. The derivative is given by $\frac{\cos\sqrt{x}}{2\sqrt{x}}$.
2. $2\sec(2x)\cdot\tan(2x)\cdot\sin x+\sec(2x)\cdot\cos x$.
3. 1. The derivative is given by $\cos^2 x-\sin^2 x$. Now evaluate this at $x=0$.
4. $\dfrac{1}{1-\sin x}$. The derivative is given by $\frac{1+\sin x}{\cos^2 x}$. Now use an identity in the denominator and factor.
5. $\dfrac{1}{2}$. Note that $y'(t)=\frac{\cos t}{2\sqrt{1+\sin t}}$. Now set $t=0$.
6. $-2x\sin(x^2)\cos(\cos(x^2))$.
7. $2x\cos 3x-3x^2\sin 3x$.
8. $\dfrac{2}{3}x^{-1/3}\tan(x^{1/3})+\dfrac{1}{3}\sec^2(x^{1/3})$.
9. $-(1+\cos x)\csc^2(2+x+\sin x)$. Don't forget the *minus* sign here!
10. $-3\cot 3x\csc 3x$. The original function is the same as $\csc 3x$.
11. 1. In this case, the derivative is given by $\dfrac{-\sin x+x\cos x+\cos x}{\cos^2 x-1}$. Remember that $\cos(\pi/2)=0, \quad \sin(\pi/2)=1$.
12. $4x\cos(2x^2)$.
13. 1. In this case, the derivative is given by $2\sin x\cos x$. When $x=\frac{\pi}{4}$ we know that $\cos\frac{\pi}{4}=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$.
14. $-3\csc^2(3x-2)$.
15. $2\csc x-(2x+3)\csc x\cot x$.
16. $-(\sin x+x\cos x)\sin(x\sin x)$.
17. $\dfrac{1}{2\sqrt{x}}\sec\sqrt{x}+\dfrac{1}{2}\sec\sqrt{x}\cdot\tan\sqrt{x}$.
18. 0, except when $x^2=2+2n\pi$, where $n\geq 0$ is an integer. This is because $\csc\square\cdot\sin\square=1$ for any symbol, $\square$, by definition, whenever the cosecant is defined.

19. $-\sin 2(x-6) - 2\csc 2x \cot 2x$. (Use the identity $2\sin u \cos u = \sin 2u$ to simplify.)
20. $4\sec^2 2x \tan 2x$. The given function is equal to $\sec^2(2x)$.
21. Notice that, for $x \neq 0$, $y(x) = \sin x / \tan x = \sin x \cdot \cot x = \cos x$. On the other hand, at $x = 0$, we have $y(0) = 1$, which coincides with the value of the cosine function at $x = 0$. Therefore, $y(x) = \cos x$ for all x. Now all three parts are clear.

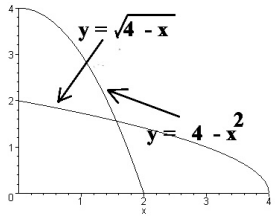
3.6 Exercise Set 16 (page 136)

You can use any plotting software to sketch the graphs.

1. $f(x) = 4 - x^2,\ 0 \leq x \leq 2.$ $\quad f^{-1}(x) = \sqrt{4-x},\ 0 \leq x \leq 4.$

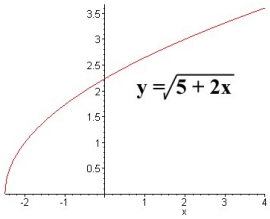


$f(x) = 4 - x^2, x\ in\ [0, 2].$

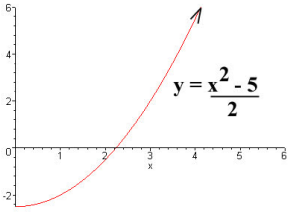


$f^{-1}(x) = \sqrt{4-x}, x\ in\ [0, 4].$

2. $g(x) = (x-1)^{-1},\ 1 < x < \infty.$ $\quad g^{-1}(x) = x^{-1} + 1,\ 0 < x < \infty.$
3. $f(x) = 2 - x^3,\ -\infty < x < \infty.$ $\quad f^{-1}(x) = \sqrt[3]{2-x},\ -\infty < x < \infty.$
4. $f(x) = \sqrt{5+2x}, -\frac{5}{2} \leq x < \infty.$ $\quad f^{-1}(x) = \frac{1}{2}(x^2-5),\ 0 \leq x < \infty.$



$f(x) = \sqrt{5+2x}$, $x\ in\ [-5/2, \infty)$.



$f^{-1}(x) = \frac{x^2-5}{2},\ x\ in\ [0, \infty).$

5. $f(y) = (2+y)^{1/3},\ -2 < y < \infty.$ $\quad f^{-1}(y) = y^3 - 2,\ 0 < y < \infty.$

6. (i) $F(0) = 2$, since $f(2) = 0$ forces $2 = F(f(2)) = F(0)$.
(ii) $f(-1) = 6$, since $F(6) = -1$ means that $6 = f(F(6)) = f(-1)$.
(iii) Indeed, if $f(x) = 0$ then $x = F(f(x)) = F(0) = 2$, and so this is the only possibility.
(iv) $y = 8$, because $f(-2) = 8$ means (by definition) that $F(8) = -2$ so $y = 8$ is a solution. No, there are no other solutions since if we set $F(y) = -2$ then $y = f(F(y)) = f(-2) = 8$, so that $y = 8$ is the only such solution.
(v) No. The reasoning is the same as the preceding exercise. Given that $f(-1) = 6$, the solution x of $f(x) = 6$ must satisfy $x = F(f(x)) = F(6) = -1$, by definition of the inverse function, F.
7. We know that $F'(-1) = \dfrac{1}{f'(F(-1))} = \dfrac{1}{f'(-2.1)} = \dfrac{1}{4}$.
8. $F(x) = x$, $Dom(f) = Ran(F) = (-\infty, +\infty) = \{x : -\infty < x < +\infty\}$, and $Dom(F) = Ran(f) = (-\infty, +\infty)$ too.
9. $F(x) = \dfrac{1}{x}$, $Dom(f) = Ran(F) = \{x : x \neq 0\}$, and $Dom(F) = Ran(f) = \{x : x \neq 0\}$.
10. $F(x) = \sqrt[3]{x}$, $Dom(f) = Ran(F) = \{x : -\infty < x < +\infty\} = Dom(F) = Ran(f)$.
11. $F(t) = \dfrac{t-4}{7}$, $Dom(f) = Ran(F) = \{x : 0 \leq t \leq 1\}$ while $Dom(F) = Ran(f) = \{x : 4 \leq t \leq 11\}$
12. $G(x) = \dfrac{x^2-1}{2}$, $Dom(g) = Ran(G) = \{x : -\dfrac{1}{2} \leq x < +\infty\}$ while $Dom(G) = Ran(g) = \{x : 0 \leq x < \infty\}$.
13. Note that g is one-to-one on this domain. Its inverse is given by $G(t)$ where $G(t) = \dfrac{\sqrt{1-t^2}}{2}$, $Dom(g) = Ran(G) = \{t : 0 \leq t \leq \dfrac{1}{2}\}$ while $Dom(G) = Ran(g) = \{t : 0 \leq t \leq 1\}$.
14. This f is also one-to-one on its domain. Its inverse is given by $F(x)$ where $F(x) = \dfrac{3x-2}{2x+3}$, $Dom(f) = Ran(F) = \{x : x \neq \dfrac{3}{2}\}$ while $Dom(F) = Ran(f) = \{x : x \neq -\dfrac{3}{2}\}$.
15. This g is one-to-one if $y \geq -\frac{1}{2}$ and so it has an inverse, G. Its form is $G(y)$ where $G(y) = \dfrac{-1+\sqrt{1+4y}}{2}$, $Dom(g) = Ran(G) = \{y : -\dfrac{1}{2} \leq y < +\infty\}$ while $Dom(G) = Ran(g) = \{y : -\dfrac{1}{4} \leq y < +\infty\}$.

3.7 Exercise Set 17 (page 144)

1. $\sin(\text{Arccos}(0.5)) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.
2. $\cos(\text{Arcsin}(0)) = \cos 0 = 1$.
3. $\sec(\sin^{-1}(\frac{1}{2})) = \sec(\frac{\pi}{6}) = \frac{2}{\sqrt{3}}$.
4. $-\sqrt{5}$. (This is hard!) Let $\tan^{-1}(-\frac{1}{2}) = \alpha$. Then $-\frac{\pi}{2} < \alpha < 0$; (see the graph of the Arctangent function in this Section.) Also, $\tan\alpha = -\frac{1}{2}$. Thus

$$\sec^2\alpha = 1 + tan^2\alpha = 1 + (-1/2)^2 = \frac{5}{4}.$$

But $-\frac{\pi}{2} < \alpha < 0$ implies that $\sec\alpha = 1/\cos\alpha > 0$. Therefore

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{\cos\alpha}{\sin\alpha}\cdot\frac{1}{\cos\alpha} = \frac{1}{\tan\alpha}\cdot\sec\alpha = (-2)\frac{\sqrt{5}}{2} = -\sqrt{5}.$$

5. $\sec(\sin^{-1}(\frac{\sqrt{3}}{2})) = \sec\frac{\pi}{3} = 2$.
6. $\text{Arcsin}(\tan(-\frac{\pi}{4})) = \text{Arcsin}(-1) = -\frac{\pi}{2}$.

7. $\pi/4$, as we are dealing with the principal branch here.
8. 1, since this is true regardless of the branch.
9. $\sqrt{2}/2$, since this is true regardless of the branch.
10. -1.
11. 0, since $\sin \pi = 0$ and $\text{Arcsin}(0) = 0$.
12. $\pi/2$, since we are dealing with the principal branch of arccos.
13. $-\pi/3$, since $\sin(-2\pi/3) = -\sqrt{3}/2$ and $\text{Arcsin}(-\sqrt{3}/2) = -\pi/3$.
14. $3\pi/4$, since $\cos(5\pi/4) = -\sqrt{2}/2$ and so $\text{Arccos}(-\sqrt{2}/2) = 3\pi/4$.
15. $-\pi/4$, since $\tan(3\pi/4) = -1$ and so $\text{Arctan}(-1) = -\pi/4$.
16. $-\pi/2$, since $\sin(-\pi/2) = -1$ and $\text{Arcsin}(-1) = -\pi/2$.
17. $-\pi/4$.
18. π.
19. $\pi/4$, since $\tan(\pi) = 0$, and $\text{Arctan}(1) = \pi/4$.
20. -0.8082, (use your calculator here).
21. $\pi/3$, since $\tan \pi = 0$ and $\text{Arctan}(\sqrt{3}) = \pi/3$.

3.8 Exercise Set 18 (page 149)

1. $\dfrac{d}{dx}\text{Arcsin}(x^2) = \dfrac{2x}{\sqrt{1-x^4}}$ which is 0 at $x = 0$.
2. $2x\text{Arccos } x - \dfrac{x^2}{\sqrt{1-x^2}}$.
3. $\dfrac{1}{2(1+x)\sqrt{x}}$.
4. $-\dfrac{\sin x}{|\sin x|}$. Remember the identity?
5. $\dfrac{\sin x - \sin^{-1} x \cdot \cos x\sqrt{1-x^2}}{\sin^2 x \cdot \sqrt{1-x^2}}$.
6. $\dfrac{1}{2|x|\sqrt{(x^2-1)\sec^{-1} x}}$.
7. 2, because $\cos(2\text{Arcsin } x) \cdot \dfrac{2}{\sqrt{1-x^2}}$ which is 2 at $x = 0$.
8. $-\dfrac{16x}{\sqrt{1-16x^2}}$.
9. $-\dfrac{1}{(\text{Arctan } x)^2(1+x^2)}$.
10. $3x^2\text{Arcsec } (x^3) - \dfrac{3|x|^3}{x\sqrt{x^6-1}}$.

3.9 Exercise Set 19 (page 161)

1. $-\frac{1}{2}$.
2. 1.
3. 0.
4. -2. Factor the numerator and simplify.
5. 2. Factor the numerator and simplify.
6. 2. Use L'Hospital's Rule.
7. 0. Use L'Hospital's Rule.
8. 0. Indeed, $\lim_{t\to 0}\dfrac{\sin^2 t-\sin(t^2)}{t^2}=\lim_{t\to 0}\left[\left(\dfrac{\sin t}{t}\right)^2-\dfrac{\sin(t^2)}{t^2}\right]=1-1=0.$
9. $\frac{1}{2}$. The quotient is continuous at $x=1$. Use of L'Hospital's Rule will give nonsense here.
10. This limit does not exist. In fact, applying L'Hospital's rule to the one-sided limits at $x=0$ shows that $\lim_{x\to 0^+}\dfrac{\text{Arctan } x}{x^2}=\lim_{x\to 0^+}\dfrac{(1+x^2)^{-1}}{2x}=+\infty$ and $\lim_{x\to 0^-}\dfrac{\text{Arctan } x}{x^2}=\lim_{x\to 0^-}\dfrac{(1+x^2)^{-1}}{2x}=-\infty$. So there is NO limit at $x=0$.
11. 1. By L'Hospital's rule, $\lim_{x\to 0}\dfrac{(1-x^2)^{-1/2}}{(1+x^2)^{-1}}=1$.
12. $\frac{6}{5}$. Multiply the expression by $1=\dfrac{5x}{3x}\cdot\dfrac{3x}{5x}$, re-arrange terms and take the limit.
13. $\frac{1}{3}$. Use L'Hospital's Rule.
14. $\frac{1}{36}$. In this exercise we must apply L'Hospital's rule "three" times before we can see the answer.
15. 2. Indeed,

$$\begin{aligned}\lim_{x\to 0}\frac{x\sin(\sin x)}{1-\cos(\sin x)} &= \lim_{x\to 0}\frac{\sin(\sin x)+x\cos(\sin x)\cdot\cos x}{\sin(\sin x)\cdot\cos x}\\ &= \lim_{x\to 0}\left(\frac{1}{\cos x}+\frac{x\cos(\sin x)}{\sin(\sin x)}\right)\\ &= 1+\lim_{x\to 0}\left(\frac{x}{\sin(\sin x)}\cdot\cos(\sin x)\right)\\ &= 1+\lim_{x\to 0}\left(\frac{x}{\sin x}\cdot\frac{\sin x}{\sin(\sin x)}\cdot\cos(\sin x)\right)=2.\end{aligned}$$

3.10 Chapter Exercises (page 162)

1. $27(x+1)^{26}$.
2. $-3\cos^2 x\sin x$.
3. $\csc 2x-2(x+1)\csc 2x\cot 2x$. Note that $csc(2x)=\frac{1}{\sin 2x}$.
4. $2(x+5)\cos((x+5)^2)$. You can easily do this one using the "Box" form of the Chain Rule!
5. $\dfrac{\sin x-\cos x}{(\sin x+\cos x)^2}$.

6. $\dfrac{1}{\sqrt{2x-5}}$. Use the Generalized Power Rule.
7. $2\cos 2x$.
8. $-4\cos 4x \cdot \sin(\sin 4x)$.
9. $6\tan 2x \cdot \sec^3 2x$. The two *minus* signs cancel out!
10. $2x\sec 2x + 2(x^2+1)\tan 2x \sec 2x$.
11. $-3\csc 3x \cdot \cot 3x$.
12. $\sec 2x + 2(x+2)\tan 2x \sec 2x$.
13. $\dfrac{2x^2+6x-2}{(2x+3)^2}$.
14. $3\cos 3x \cdot (x^{1/5}+1) + \frac{1}{5}x^{-4/5} \cdot \sin 3x$.
15. $(2x+6)\cos(x^2+6x-2)$.
16. 2.8.
17. $\dfrac{2}{3\sqrt[3]{2}} = \dfrac{\sqrt[3]{4}}{3}$.
18. $-\dfrac{\sqrt{3}}{3(2+\sqrt{3})}$. Be careful with the square root terms.
19. $210 \times 5^{104} = 42 \times 5^{105}$.
20. 0.
21. 4. The derivative is $4\cos(\sin(4x))\cos(4x)$.
22. 2.
23. 1. $f(x) = x+2$ for $x > -2$. In this case, $x = -1 > -2$ so this is our f.
24. $\dfrac{1}{2\sqrt{2}}$.
25. $87,318 \cdot (3x-2)^{97}$, $\quad 87,318$.
26. Putting $u = 3x^2$ and $y = f(u)$, we have

$$\frac{d}{dx}f(3x^2) = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = f'(u)\cdot 6x = 6x \cdot f'(3x^2).$$

27. $y - 1 = 24(x-2)$, or $24x - y - 47 = 0$.
28. $\dfrac{81}{2} - \dfrac{\sin 9}{6}$. When $u = 9$ we have $t = 9$ also. We know that $\frac{dy}{du} = \frac{dy}{dt}\cdot\frac{dt}{du}$ or, $(3t^2 - \sin t)\cdot\frac{1}{2\sqrt{u}} = \frac{1}{6}\left(3\cdot(81) - \sin 9\right)$.
29.

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dr}\cdot\frac{dr}{dt} = \left(\frac{1}{2}r^{-1/2} + 3r^{-2}\right)(3 - t^{-1/2}),\\ &= \left(\frac{1}{2}(3t - 2t^{1/2})^{-1/2} + 3(3t-2t^{1/2})^{-2}\right)(3 - t^{1/2}).\end{aligned}$$

30. Notice that, for $x > 0$ we have $y(x) = x^2$ and hence y is differentiable at x with $y'(x) = 2x$. Similarly, for $x < 0$ we have $y(x) = -x^2$ and hence $y'(x) = -2x$. Finally, for $x = 0$ we have

$$\frac{y(h) - y(0)}{h} = \frac{h|h|}{h} = |h| \to 0 \text{ as } h \to 0$$

and hence y is also differentiable at $x = 0$ with $y'(0) = 0$. From the above argument we see that $y'(x) = 2|x|$ for all x. It is well-known that the absolute value function $|x|$ is not differentiable at $x = 0$. Therefore the derivative of y' at 0 does not exist. In other words, $y''(0)$ does not exist.

31. $-\dfrac{3}{2}$. The derivative is $3x^2 + 2xy' + 2y + 2yy' = 0$. Set $x = 1$, $y = 0$ and solve for y'.

32. $\dfrac{dy}{dx} = \dfrac{3x^2 - 2y^2}{4xy - 4y^3}, \quad \dfrac{dx}{dy} = \dfrac{4xy - 4y^3}{3x^2 - 2y^2}.$

33. Implicit differentiation gives $\frac{1+y'}{2\sqrt{x+y}} + 2xy^2 + 2x^2yy' = 0$. So, at $(0, 16)$, we have $y' = -1$.

34. $\dfrac{dy}{dx} = \dfrac{3y^2 + y}{5y^4 - 6xy - x}.$

35. The tangent line to the curve at $(4, 0)$ is vertical. Here $2x + 2yy' = 0$ and we are dividing by 0 at $x = 4$.

36. $y + 1 = 2(x + 1)$, or $2x - y + 1 = 0$.

37. The vertical line through the origin: $x = 0$ (or the $y-$axis itself.) In this case, $(x + 2y)y' + (2 + y) = 0$. The derivative is undefined (or infinite) at $x = 0$.

38. $y = \frac{5}{2}(x - 4)$, or $5x - 2y - 20 = 0$.

39. $y = x$. At $(1, 1)$ we have $y' = 1$. So $y - 1 = 1(x - 1)$ and the result follows.

40. $y = x - \pi$. The derivative is $\cos x + y' \cos y - 6yy' = 0$. Set $x = \pi,\ y = 0$ and solve for y'.

41. $\dfrac{1}{2}$. Use L'Hospital's Rule.

42. 1

43. 0. Find a common denominator and use L'Hospital's Rule.

44. 0. Divide the numerator and denominator by x^2 and let $x \to -\infty$.

45. By L'Hospital's Rule, we have

$$\begin{aligned}
\lim_{x\to-\infty} x^2\left(\frac{\pi}{2} + \operatorname{Arctan} x\right) &= \lim_{x\to-\infty} \frac{\frac{\pi}{2} + \operatorname{Arctan} x}{x^{-2}} \\
&= \lim_{x\to-\infty} \frac{(1+x^2)^{-1}}{-2x^{-3}} \\
&= \lim_{x\to-\infty} -\frac{x^3}{2(1+x^2)} \\
&= +\infty.
\end{aligned}$$

Chapter 4

Solutions

4.1 Exercise Set 20 (page 173)

1. $x^2 + 1$
2. x
3. $\frac{x}{2} - 2x^2$. Note that $log_4(2^x) + log_4(16^{-x^2}) = x\ log_4(2) - x^2 log_4(16)$ $= \frac{x}{2} - 2x^2$.
4. $1 - \log_3(4)$
5. 0, since $2^x\ 2^{-x} = 1$ for any x.
6. The graph looks like Figure 55. Its values are:

-2	-1	0	1	2
$\frac{1}{16}$	$\frac{1}{4}$	1	4	16

7. The graph looks like Figure 56. Its values are:

-2	-1	0	1	2
16	4	1	$\frac{1}{4}$	$\frac{1}{16}$

8. The graph is similar to $y = \sqrt{2}^x$ in Figure 59. Its values are:

-2	-1	0	1	2
0.333	0.577	1	1.73	3

9. $\log_{\sqrt{2}}(1.6325) = \sqrt{2}$
10. $\log_2\left(\frac{1}{16}\right) = -4$
11. $\log_3\left(\frac{1}{9}\right) = -2$
12. $f(x) = 2^x$
13. $3^4 = 81$
14. $\left(\frac{1}{2}\right)^{-2} = 4$
15. $\left(\frac{1}{3}\right)^{-3} = 27$
16. $a^0 = 1$
17. $\sqrt{2}^{\sqrt{2}} = 1.6325$
18. $x = \frac{16}{3}$, since $log_2(3x) = 4$ means that $2^4 = 3x$.

19. $x = -\dfrac{3}{2}$, since $3 = \frac{x}{x+1}$ forces $3x + 3 = x$, etc.
20. $x = \pm\sqrt{2}$, since $\sqrt{2}^0 = x^2 - 1$, or $x^2 = 2$, etc.
21. $x = 2$, since $\frac{1}{2}^{-1} = x$ is equivalent to $x = 2$.
22. $y = x^2$, since $y = log_2(2^{x^2}) = x^2\ log_2(2) = x^2 \cdot 1 = 2$.

4.2 Exercise Set 21 (page 179)

1. $b_1 = 0$, $b_2 = 0.25$, $b_3 = 0.29630$, $b_4 = 0.31641$, $b_5 = 0.32768$, $b_6 = 0.33490$, $b_7 = 0.33992$, $b_8 = 0.34361$, $b_9 = 0.34644$, $b_{10} = 0.34868$,
2. $e^{-1} = \frac{1}{e} \approx 0.3679$. See the following exercise.
3. Let $x = \frac{a}{n}$, so that $n = \frac{a}{x}$ and as $n \to \infty$ we get $x \to 0$, then

$$\begin{aligned} \lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n &= \lim_{x\to 0} (1+x)^{\frac{a}{x}} \\ &= \left[\lim_{x\to 0} (1+x)^{\frac{1}{x}}\right]^a \\ &= e^a \text{ by Known Fact\#5} \end{aligned}$$

4.3

4.4 Exercise Set 22 (page 189)

1. $\dfrac{3}{4}$, since $y' = \frac{3x^2}{x^3+3}$. Evaluate this at $x = 1$.
2. $3 \cdot e^{3x} \log x + e^{3x} \cdot \dfrac{1}{x}$
3. $\dfrac{e^x(x \log x - 1)}{x(\log x)^2}$.
4. $\dfrac{13}{6}$, since $y = 2x + \ln(x+6)$ and so $y' = 2 + \dfrac{1}{x+6}$. Evaluate this at $x = 0$.
5. $\dfrac{1}{x + \sqrt{x^2+3}} \left(1 + \dfrac{x}{\sqrt{x^2+3}}\right)$.
6. $\dfrac{4}{x+2}$.
7. $\dfrac{x}{x^2+4}$, since $\ln(\sqrt{x^2+4}) = \frac{1}{2}\ln(x^2+4)$.

4.5 Exercise Set 23 (page 192)

1. $a_n = n+2$ and $a_{n+1} = (n+1)+2 = n+3$. Clearly $n+3 > n+2$ and so $\{a_n\}$ is increasing and $\lim\limits_{n\to\infty} a_n = \infty$.
2. $a_n = \dfrac{n-1}{n}$ and $a_{n+1} = \dfrac{(n+1)-1}{(n+1)} = \dfrac{n}{n+1}$. Consider $a_{n+1} - a_n$:

$$\frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2 - (n-1)(n+1)}{n(n+1)}$$

$$\begin{aligned} &= \frac{n^2-(n^2-1)}{n(n+1)} \\ &= \frac{1}{n(n+1)} > 0 \text{ for all } n \geq 1. \end{aligned}$$

Therefore $a_{n+1} > a_n$ and the series increases.
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1.$

3. $a_n = \frac{n(n-2)}{n^2} = \frac{n-2}{n}$ and $a_{n+1} = \frac{(n+1)((n+1)-2)}{(n+1)^2} = \frac{n-1}{n+1}$

$$\begin{aligned} a_{n+1} - a_n = \frac{n-1}{n+1} - \frac{(n-2)}{n} &= \frac{n(n-1)-(n+1)(n-2)}{n(n+1)} \\ &= \frac{n^2-n-(n^2-n-2)}{n(n+1)} \\ &= \frac{2}{n(n+1)} > 0 \text{ for all } n \geq 1. \end{aligned}$$

Therefore $a_{n+1} > a_n$ and the series increases. Furthermore,
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{2}{n}\right) = 1.$

4. $a_n = \frac{n}{n+3}$ and $a_{n+1} = \frac{(n+1)}{(n+1)+3} = \frac{n+1}{n+4}$.

$$\begin{aligned} a_{n+1} - a_n = \frac{n+1}{n+4} - \frac{n}{n+3} &= \frac{(n+1)(n+3)-n(n+4)}{(n+3)(n+4)} \\ &= \frac{n^2+4n+3-(n^2+4n)}{(n+3)(n+4)} \\ &= \frac{3}{(n+3)(n+4)} > 0 \text{ for all } n \geq 1 \end{aligned}$$

Thus $\{a_n\}$ is increasing and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{3}{n+3}\right) = 1.$

5. $a_n = \frac{(n-1)}{(n+1)}$ and $a_{n+1} = \frac{(n+1)-1}{(n+1)+1} = \frac{n}{n+2}$.

$$\begin{aligned} a_{n+1} - a_n = \frac{n}{n+2} - \frac{n-1}{n+1} &= \frac{n(n+1)-(n+2)(n-1)}{(n+2)(n+1)} \\ &= \frac{n^2+n-(n^2+n-2)}{(n+2)(n+1)} \\ &= \frac{2}{(n+2)(n+1)} > 0 \text{ for all } n \geq 1. \end{aligned}$$

So $\{a_n\}$ is increasing and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{2}{n+1}\right) = 1.$

6. For n from 1 to 15, a_n runs like 0, 0.70711, 0.81650, 0.86603, 0.89443, 0.91287, 0.92582, 0.93541, 0.94281, 0.94868, 0.95346, 0.95743, 0.96077, 0.96362, 0.96609. You can guess that the limit must be 1. For the graph, see Figure 60.

7. $\lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^n = e^{-2}$

8. Apply the *Box method* to Fact # 5:

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e \quad \Rightarrow \quad \lim_{\square\to 0} (1+\square)^{\frac{1}{\square}} = e.$$

Now let $\square = x^2$. Note that $x^2 \to 0$ as $x \to 0$ and we are done.

9. Use $n = 2000$ in the expression $\left(1 + \frac{1}{n}\right)^n \approx 2.7176$. You don't want to write down the rational number, though! The numerator alone has about $11,300$ digits!!

10. The graph of $y(x) = e^{2(x-1)}$ has the same shape as the graph of $y = e^x$ except for three minor differences: first, it is steeper, second, it 'shoots' through $(1, 1)$ instead of $(0, 1)$, and third, it is a translate of the graph $y = e^{2x}$ by one unit to the right.

11. a) 1, since $e^{3 \ln x} = x^3$.
    b) 0
    c) 5
    d) 1
    e) 0, since $sin^2 x + \cos^2 x = 1$.
    f) 0, since $\ln 1 = 0$.
    g) $2^{2x} = 4^x$
    h) $\ln(x - 1)$
    i) $x - 1$
    j) 0, since $\ln\left(e^{e^{x^2}}\right) = e^{x^2} \ln e = e^{x^2}$.

12. 0. Use L'Hospital's Rule twice.

13. a) $e^{0.38288} = 1.46650$.
    b) $e^{-1.38629} = 0.250000$.
    c) $e^{4.33217} = 76.1093$.
    d) $e^{-2.86738} = 0.05685$.
    e) $e^{2.42793} = 11.33543$.

14. $f(x) = e^{(\sin x) \ln x}$.

15. a) $4e^{2x}$.
    b) $-3.4e^2$.
    c) $3^{\cos x} \ln 3 \cdot (-\sin x) = -\ln 3 \cdot \sin x \cdot 3^{\cos x}$.
    d) $-\frac{6}{e^6}$. Be careful, $(e^{3x})^{-2} = e^{-6x}$!
    e) 1. The derivative is $e^{x^2} \cos x + 2xe^{x^2} \sin x$.
    f) $e^x(\cos x - \sin x)$.
    g) $(x^2 - 2x)e^{-x}$.
    h) $2x\ e^{2x}\ (1 + x)$.
    i) $-2(1 + x)x^{-3}e^{-2x}$.
    j) $(1.2)^x \ln(1.2)$.
    k) $x^{0.6}e^{-x}(1.6 - x)$.

4.6 Exercise Set 24 (page 198)

1. a) $\frac{1}{\ln a} \cdot \frac{3x^2 + 1}{x^3 + x + 1}$.
   b) $\log_3 x + \frac{1}{\ln 3} = \frac{\ln x}{\ln 3} + \frac{1}{\ln 3}$.
   c) $x^x(\ln x + 1)$, since $x^x = e^{x \ln x}$.

d) $\dfrac{1}{\ln 3}\cdot\dfrac{1}{4x-3}\cdot 4=\dfrac{4}{\ln 3\cdot(4x-3)}$.

e) $-\dfrac{4}{\ln 3}$.

f) $(3^x\ln 3)\log_2(x^2+1)+3^x\cdot\dfrac{1}{\ln 2}\cdot\dfrac{2x}{x^2+1}$.

g) $1+\ln x$.

h) $\dfrac{e^x}{\ln 2}(1+x)$, since $\ln_2(e^x)=\dfrac{\ln(e^x)}{\ln 2}=\dfrac{x}{\ln 2}$.

i) $\dfrac{1}{\ln 2}\cdot\dfrac{1}{3x+1}\cdot 3=\dfrac{3}{\ln 2\cdot(3x+1)}$.

j) $\dfrac{1}{2}\left(\dfrac{1}{\ln 2}\cdot\dfrac{1}{x+1}\right)=\dfrac{1}{2\ln 2\cdot(x+1)}$.

4.7 Chapter Exercises (page 199)

1. $a_n=n+3$ and $a_{n+1}=(n+1)+3=n+4$. Clearly $n+4>n+3$ and so $\{a_n\}$ is increasing and $\lim\limits_{n\to\infty}a_n=\infty$.

2. $a_n=\dfrac{n-3}{n}$ and $a_{n+1}=\dfrac{(n+1)-3}{(n+1)}=\dfrac{n-2}{n+1}$.

$$\begin{aligned}a_{n+1}-a_n=\frac{n-2}{n+1}-\frac{n-3}{n} &= \frac{n(n-2)-(n-3)(n+1)}{n(n+1)}\\ &= \frac{n^2-2n-(n^2-2n-3)}{n(n+1)}\\ &= \frac{3}{n(n+1)}>0 \text{ for all } n\geq 1.\end{aligned}$$

Therefore $a_{n+1}>a_n$ and the series increases. Furthermore,
$\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\left(1-\dfrac{3}{n}\right)=1$.

3. $a_n=\dfrac{n(n-1)}{n^2}=\dfrac{(n-1)}{n}$ and $a_{n+1}=\dfrac{(n+1)((n+1)-1)}{(n+1)^2}=\dfrac{n}{(n+1)}$

$$\begin{aligned}a_{n+1}-a_n=\frac{n}{n+1}-\frac{(n-1)}{n} &= \frac{n^2-(n+1)(n-1)}{n(n+1)}\\ &= \frac{n^2-(n^2-1)}{n(n+1)}\\ &= \frac{1}{n(n+1)}>0 \text{ for all } n\geq 1.\end{aligned}$$

Therefore $a_{n+1}>a_n$ and the series increases. Furthermore,
$\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)=1$.

4. $a_n=\dfrac{n}{n+4}$ and $a_{n+1}=\dfrac{(n+1)}{(n+1)+4}=\dfrac{n+1}{n+5}$.

$$\begin{aligned}a_{n+1}-a_n=\frac{n+1}{n+5}-\frac{n}{n+4} &= \frac{(n+1)(n+4)-n(n+5)}{(n+5)(n+4)}\\ &= \frac{n^2+5n+4-(n^2+5n)}{(n+5)(n+4)}\\ &= \frac{4}{(n+5)(n+4)}>0 \text{ for all } n\geq 1.\end{aligned}$$

Thus $\{a_n\}$ is increasing. Furthermore,
$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 - \frac{4}{n+4}\right) = 1.$

5. Sketch this as in Figure 60. Note that

$$\lim_{n\to\infty}\sqrt{\frac{n-1}{2n}} = \lim_{n\to\infty}\sqrt{\frac{n}{2n} - \frac{1}{2n}} = \lim_{n\to\infty}\sqrt{\frac{1}{2} - \frac{1}{2n}} = \frac{1}{\sqrt{2}}.$$

6. a) x^x
   b) $\sqrt{x}$

7. a) $e^{0.89032} = 2.43592$
   b) $e^{-1.5314} = 0.21623$
   c) $e^{0.17328} = 1.18920$

8. a) $15e^{5x}$
   b) $6e^2$
   c) $-\sin(xe^x)\cdot e^x(1+x)$
   d) $-8e^{-8}$, since $(e^{4x})^{-2} = e^{-8x}$.
   e) 0
   f) $e^x(\ln(\sin x) + \cot x)$
   g) 1
   h) $2x(1-x)e^{-2x}$
   i) $e^{-2x}\left(-2\operatorname{Arctan} x + \frac{1}{1+x^2}\right)$.
   j) 2. Note that $(x^2)^x = x^{2x} = e^{2x\ \ln x}$.
   k) $\frac{1}{2\sqrt{x}}(\ln\sqrt{x} + 1)$.
   l) $2^x\cdot\ln 2\cdot\log_{1.6}(x^3) + 2^x\cdot\frac{1}{\ln 1.6}\cdot\frac{3}{x}$
   m) $-3^{-x}\cdot\ln 3\cdot\log_{0.5}(\sec x) + 3^{-x}\cdot\frac{1}{\ln 0.5}\cdot\tan x$

9. Amount after t years is: $A(t) = Pe^{rt}$, where $P = \$500$ and $r = 0.10$, so $A(t) = 500e^{0.1t}$
   (a) Thus after 5 years the amount in the account will be $A(5) = 500e^{0.1\times 5} = \$824.36$
   (b) Want t such that $A = 3P = 1500 = 500e^{0.1t}$, so $3 = e^{0.1t}$ giving $ln(3) = 0.1t$, therefore, $t = 10.986 \approx 11$ years.

10. $A = Pe^{rt}$, where $A = \$2400$ if $r = 0.12$, $t = 8$. So $2400 = Pe^{0.12\times 8} = 2.6117P$. Thus $P = \frac{2400}{2.6117} = \$918.94$

11. Sales after t months: $y(t) = y(0)e^{kt} = 10,000e^{kt}$. At $t = 4$, $8,000 = 10,000e^{4k}$, so $0.8 = e^{4k}$, $ln(.8) = 4k$, and $k = -0.0558$. Thus, $y(t) = 10,000e^{-0.0558t}$, and when $t = 6$ (2 more months), sales $= 10,000e^{-0.0558\times 6} = \$7154.81$

12. (a) Revenue at time t is : $y(t) = y(0)e^{kt} = 486.8e^{kt}$, taking 1990 as $t = 0$. In 1999, $t = 9$, so $y(9) = 1005.8 = 486.8e^{9k}$. Thus $\frac{1005.8}{486.8} = e^{9k}$, so $ln\frac{1005.8}{486.8} = 9k$, giving $k = \frac{1}{9}ln(2.066) \approx 0.08$, so $y(t) = 486.8e^{0.08t}$. In 2001, $t = 11$, so revenue $= 486.8e^{0.08\times 11} = \$1173.62$ million.
    (b) Want t such that $1400 = 486.8e^{0.08t}$, so $ln\frac{1400}{486.8} = 0.08t$, and $t = 13.2$ years.

13. (a) $\lim_{t\to\infty} S = 30,000 = \lim_{t\to\infty} Ce^{k/t} = C$. Thus, $S = 30,000e^{k/t}$. When $t = 1$ $S = 5000$, therefore, $5000 = 30,000e^k$, so $\frac{1}{6} = e^k$, and $k = ln\frac{1}{6} = -1.79$. Thus, $S = 30,000e^{-1.79/t}$.
    (b) When $t = 5$, number of units sold is $S(5) = 30,000e^{-1.79/5} = 30,000\times 0.699 = 20,972.19 \approx 20,972$ units

Chapter 5

Solutions

Use your **Plotter Software** available on the author's web site to obtain the missing graphs of the functions in the Chapter Exercises, at the end.

5.1 Exercise Set 25 (page 206)

1. $(x-1)(x+1)$, all Type I.
2. $(x-1)(x^2+1)$. One Type I and one Type II factor.
3. $(x+3)(x-2)$, all Type I.
4. $(x-1)^2(x+1)$, all Type I.
5. $(x-2)(x+2)(x^2+4)$. Two Type I factors and one Type II factor.
6. $(2x-1)(x+1)$, all Type I.
7. $(x+1)^2(x-1)^2$, all Type I.
8. $(x+1)(x^2+1)$. One Type I and one Type II factor.

5.2 Exercise Set 26 (page 214)

1. a). $\pm\frac{1}{3}$, -1. b). ± 1, -3. c). -2, 1. d). 1. e). ± 1.
2. **a)**

	$(x-(1/3))$	$(x+(1/3))$	$(x+1)$	Sign of $p(x)$
$(-\infty,-1)$	$-$	$-$	$-$	$-$
$(-1,-1/3)$	$-$	$-$	$+$	$+$
$(-1/3,1/3)$	$-$	$+$	$+$	$-$
$(1/3,\infty)$	$+$	$+$	$+$	$+$

b) Note that $x^2+1>0$ so it need not be included in the SDT.

	$(x-1)$	$(x+1)$	$(x+3)$	Sign of $q(x)$
$(-\infty,-3)$	$-$	$-$	$-$	$-$
$(-3,-1)$	$-$	$-$	$+$	$+$
$(-1,1)$	$-$	$+$	$+$	$-$
$(1,\infty)$	$+$	$+$	$+$	$+$

c) Note that x^2+x+1 is a Type II factor. You may leave it out of the SDT if you want.

	$(x-1)$	$(x+2)$	(x^2+x+1)	Sign of $r(x)$
$(-\infty,-2)$	$-$	$-$	$+$	$+$
$(-2,1)$	$-$	$+$	$+$	$-$
$(1,\infty)$	$+$	$+$	$+$	$+$

d) Note that $t^3-1=(t-1)(t^2+t+1)$ and the quadratic is a Type II factor.

	$(t-1)$	(t^2+t+1)	Sign of $p(t)$
$(-\infty,1)$	$-$	$+$	$-$
$(1,\infty)$	$+$	$+$	$+$

e) Note that
$w^6-1=(w^3-1)(w^3+1)=(w-1)(w^2+w+1)(w+1)(w^2-w+1)$.

	$w-1$	$w+1$	w^2+w+1	w^2-w+1	Sign $q(w)$
$(-\infty,-1)$	$-$	$-$	$+$	$+$	$+$
$(-1,-1)$	$-$	$+$	$+$	$+$	$-$
$(1,\infty)$	$+$	$+$	$+$	$+$	$+$

3. $p(x)=(x-3)(x+3)(4-x)(x+4)$. Note the *minus* sign here, since $16-x^2=(4-x)(4+x)$!

	$x+4$	$x+3$	$x-3$	$4-x$	Sign of "$p(x)$"
$(-\infty,-4)$	$-$	$-$	$+$	$-$	$-$
$(-4,-3)$	$+$	$-$	$-$	$+$	$+$
$(-3,3)$	$+$	$+$	$-$	$+$	$-$
$(3,4)$	$+$	$+$	$+$	$-$	$-$
$(4,\infty)$	$+$	$+$	$+$	$-$	$-$

4. 2. This is because $2+\sin x>0$ so it doesn't contribute any break-points.

5. Note that $3+\cos x\geq 3-1=2>0$, since $\cos x\geq -1$ for any real x. So it doesn't contribute any break-points. On the other hand, $x^4-1=(x^2-1)(x^2+1)$ and so x^2+1 (being a Type II factor) doesn't have any break-points either. Thus the only break points are those of $x^2-1=(x-1)(x+1)$ and so the SDT is equivalent to the SDT of x^2-1 which is easy to build.

6. First, we find the SDT of this polynomial, $p(x)$. The only break-points are at $x=-1,0,1$ since the quadratic is a Type II factor. So the SDT looks like,

	$(x-1)$	$(x+1)$	x	Sign of $p(x)$
$(-\infty,-1)$	$-$	$-$	$-$	$-$
$(-1,0)$	$-$	$+$	$-$	$+$
$(0,1)$	$-$	$+$	$+$	$-$
$(1,\infty)$	$+$	$+$	$+$	$+$

We can now *read-off* the answer: $p(x)<0$ whenever $-\infty<x<-1$ or $0<x<1$.

7. $-3<x<-1$, or $x>1$. Add an extra row and column to the SDT of Table 5.1.

8. All factors are Type I, so the SDT looks like,

	$x+1$	$x-2$	$x-3$	$x+4$	Sign $p(x)$
$(-\infty,-4)$	$-$	$-$	$-$	$-$	$+$
$(-4,-1)$	$-$	$-$	$-$	$+$	$-$
$(-1,2)$	$+$	$-$	$-$	$+$	$+$
$(2,3)$	$+$	$+$	$-$	$+$	$-$
$(3,\infty)$	$+$	$+$	$+$	$+$	$+$

The solution of the inequality $p(x) \leq 0$ is given by: $-4 \leq x \leq -1$, or $2 \leq x \leq 3$.

9. Let $p(x) = (x-1)^3(4-x^2)(x^2+1)$. The SDT of $p(x)$ is the same as the SDT of the polynomial $r(x) = (x-1)^3(4-x^2)$. This factors as $(x-1)^3(2-x)(x+2)$. Its SDT is given by:

	$(x-1)^3$	$x+2$	$2-x$	Sign of $r(x)$
$(-\infty,-2)$	$-$	$-$	$+$	$+$
$(-2,1)$	$-$	$+$	$+$	$-$
$(1,2)$	$+$	$+$	$+$	$+$
$(2,\infty)$	$+$	$+$	$-$	$-$

It follows that the solution of the inequality $p(x) \geq 0$ is given by solving $r(x) \geq 0$ since the exra factor in $p(x)$ is positive. Thus, $p(x) \geq 0$ whenever $-\infty < x \leq 2$, or $1 \leq x \leq 2$.

10. $x \geq \frac{1}{3}$, or , $-1 \leq x \leq -\frac{1}{3}$. See Exercise 2 a), above.

5.3 Exercise Set 27 (page 220)

1. **a)**. 2. **b)**. $1, \frac{2}{3}$. **c)**. $1, -\frac{1+\sqrt{13}}{2}, -\frac{1-\sqrt{13}}{2}$. **d)**. ± 1. e). ± 1. f). $\pm 1, \pm 2$.
2. **a)** $t = 2$ is the only break-point. Its SDT looks like:

	$(t-2)$	(t^2+1)	Sign of $r(t)$
$(-\infty,2)$	$-$	$+$	$-$
$(2,\infty)$	$+$	$+$	$+$

b) $t = \frac{2}{3}$, $t = 1$ are the only break-points. Note that we factored out the 3 out of the numerator so as to make its leading coefficient equal to a 1. Its SDT now looks like:

	$(t-\frac{2}{3})$	$(t-1)$	Sign of $r(t)$
$(-\infty,\frac{2}{3})$	$-$	$-$	$+$
$(\frac{2}{3},1)$	$+$	$-$	$-$
$(1,\infty)$	$+$	$+$	$+$

c) Write this as a rational function, first. Taking a common denominator we get that

$$t + 2 - \frac{1}{t-1} = \frac{t^2+t-3}{t-1}.$$

Its break-points are given by $t = 1$ and, using the quadratic formula,

$$t = \frac{-1 \pm \sqrt{13}}{2}.$$

The SDT looks like:

	$(t-\left(\frac{-1-\sqrt{13}}{2}\right))$	$(t-\left(\frac{-1+\sqrt{13}}{2}\right))$	$(t-1)$	Sign $r(t)$
$(-\infty,-2.303)$	$-$	$-$	$-$	$-$
$(-2.303,1)$	$+$	$-$	$-$	$+$
$(1,1.303)$	$+$	$-$	$+$	$-$
$(1.303,\infty)$	$+$	$+$	$+$	$+$

where we have used the approximations: $\dfrac{-1-\sqrt{13}}{2}\approx -2.303$, and

$\dfrac{-1+\sqrt{13}}{2}\approx +1.303$.

d) Write the rational function as

$$r(t)=\frac{t^3+1}{t^3-1}.$$

The factors of the numerator and denominator in this quotient are given by: $t^3+1=(t+1)(t^2-t+1)$ and $t^3-1=(t-1)(t^2+t+1)$, where each quadratic is Type II, and so does not contribute any new sign to its SDT. The SDT looks like the SDT for a polynomial having only the factors $t-1$ and $t+1$, that is:

	$(t+1)$	$(t-1)$	Sign of $r(t)$
$(-\infty,-1)$	$-$	$-$	$+$
$(-1,1)$	$+$	$-$	$-$
$(1,\infty)$	$+$	$+$	$+$

e) This rational function may be rewritten as

$$\frac{t^2-2t+1}{t^2-1}=\frac{(t-1)^2}{(t-1)(t+1)}=\frac{t-1}{t+1}.$$

Its SDT is basically the same as the one for a polynomial having only the factors $t-1$ and $t+1$. See Exercise 2 d), in this Set.

f) The break-points are easily found to be: $-2,-1,1,2$. The corresponding SDT is then

	$t+2$	$t+1$	$t-1$	$t-2$	Sign $r(t)$
$(-\infty,-2)$	$-$	$-$	$-$	$-$	$+$
$(-2,-1)$	$+$	$-$	$-$	$-$	$-$
$(-1,1)$	$+$	$+$	$-$	$-$	$+$
$(1,2)$	$+$	$+$	$+$	$-$	$-$
$(2,\infty)$	$+$	$+$	$+$	$+$	$+$

3. Use the SDT's found in Exercise 2 in this Set. From these we see that

a) $\dfrac{1+t^2}{t-2}\leq 0$ only when $-\infty<t<2$.

b) $\dfrac{3t-2}{t^3-1}\geq 0$ only when $-\infty<t\leq\frac{2}{3}$, or $1\leq t<\infty$.

c) $t+2-\dfrac{1}{t+1}>0$ only when $\frac{-1-\sqrt{13}}{2}<t<1$, or $\frac{-1+\sqrt{13}}{2}<t<\infty$.

d) $\dfrac{t^3+1}{t^3-1}<0$ only when $-1<t<1$.

e) $\dfrac{t^2-2t+1}{t^2-1} \geq 0$ only when $-\infty < t < -1$, or $1 < t < \infty$. You may also allow $t = 1$ in the reduced form of $r(t)$.

f) $\dfrac{4-t^2}{1-t^2} < 0$ only when $-2 < t < -1$, or $1 < t < 2$.

4. **a)** Break-points: -4 only. This is because the numerator factors as $x^2 - 16 = (x-4)(x+4)$ and one of these cancels out the corresponding one in the denominator. So, its SDT looks like the SDT of the polynomial $x + 4$ only, and this is an easy one to describe.

	$x+4$	$\dfrac{x^2-16}{x-4} = x+4$
$(-\infty, -4)$	$-$	$-$
$(-4, +4)$	$+$	$+$
$(+4, +\infty)$	$+$	$+$

The solution of the inequality $\dfrac{x^2-16}{x-4} > 0$ is given by $x > -4$.

b) The only break-point is at $x = 0$, since the other term is a Type II factor. Its SDT looks like:

	x	$3x + \frac{5}{x}$
$(-\infty, 0)$	$-$	$-$
$(0, +\infty)$	$+$	$+$

so the solution of the inequality $3x + \dfrac{5}{x} < 0$ is given by $x < 0$.

c) The break-points are at $x = 5, \quad \pm\sqrt{5}$; Its SDT looks like:

	$x-5$	$x-\sqrt{5}$	$x+\sqrt{5}$	$\frac{x^2-5}{x-5}$
$(-\infty, -\sqrt{5})$	$-$	$-$	$-$	$-$
$(-\sqrt{5}, +\sqrt{5})$	$-$	$-$	$+$	$+$
$(+\sqrt{5}, 5)$	$-$	$+$	$+$	$-$
$(5, +\infty)$	$+$	$+$	$+$	$+$

So, the solution of the inequality $\dfrac{x^2-5}{x-5} > 0$ is given by $x > 5 \quad$ or $\quad -\sqrt{5} < x < \sqrt{5}$.

d) The break-points are at $-10, 2$, since the numerator is a Type II factor. Its SDT is the same as the one for

	$x+10$	$x-2$	$3x^2+4x+5$	$\frac{3x^2+4x+5}{x^2+8x-20}$
$(-\infty, -10)$	$-$	$-$	$+$	$+$
$(-10, 2)$	$+$	$-$	$+$	$-$
$(2, +\infty)$	$+$	$+$	$+$	$+$

So, the solution of the inequality $\dfrac{3x^2+4x+5}{x^2+8x-20} < 0$ is given by $-10 < x < 2$.

e) The break-points are at $x = 0, 1$ only, since

$$\frac{x^3+x^2}{x^4-1} = \frac{x^2(x+1)}{(x^2+1)(x+1)(x-1)} = \frac{x^2}{(x-1)(x^2+1)},$$

and the only non-Type II factor is $x^2 + 1$. Its SDT is basically the same as the one below:

	$x-1$	x^2	x^2+1	$\frac{x^2}{(x^2+1)(x-1)}$
$(-\infty, 0)$	$-$	$+$	$+$	$-$
$(0, 1)$	$-$	$+$	$+$	$-$
$(1, +\infty)$	$+$	$+$	$+$	$+$

So, the solution of the inequality $\dfrac{x^3+x^2}{x^4-1} \geq 0$ is given by
$x > 1$ along with the single point, $x = 0$.

f) The break-points are at $0, 2$.

interval	x^2	$x-2$	$\lvert\cos x\rvert$	$\frac{x^2\lvert\cos x\rvert}{x-2}$
$(-\infty, 0)$	$+$	$-$	$+$	$-$
$(0, 2)$	$+$	$-$	$+$	$-$
$(2, +\infty)$	$+$	$+$	$+$	$+$

The solution of the inequality $\frac{x^2\lvert\cos x\rvert}{x-2} < 0$ is given by $x < 2$.

5. **a)** The only break-points are at $x = -1, 1$ and so the SDT is basically like the one in Exercise 2 d), above. Since $x^2 + 4 > 0$ we see that the solution of the inequality $\dfrac{x^2-1}{x^2+4} > 0$, is given by the set $x < -1$ or $x > 1$. This can also be written as $|x| > 1$.

b) There are no break-points here since $x^4 + 1 > 0$ and $x^2 + 1 > 0$ as well, for any value of x. So, no SDT is needed. We see that the solution of the inequality $\dfrac{x^2+1}{x^4+1} > 0$, is given by the set of all real numbers, namely, $-\infty < x < \infty$.

c) The only break-points are at $x = -3, 3$ and so the SDT is basically like the one in Exercise 2 d), above, with 1's replaced by 3's. Since
$x^2 + x + 1 > 0$ we see that the solution of the inequality $\dfrac{x^2-9}{x^2+x+1} < 0$, is given by the set $-3 < x < 3$. This can also be written as $|x| < 3$.

d) There are 2 break-points here, namely, at $x = \frac{3}{2}, 4$. Since $(x-4)^2 \geq 0$ for any value of x, this term will not contribute anything to the signs in the SDT. So, the only contributions come from the term $2x - 3 = 2(x - \frac{3}{2})$. It's now a simple matter to see that the solution of the inequality $\frac{2x-3}{(x-4)^2} < 0$ is given by the set $x < \frac{3}{2}$.

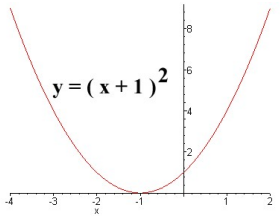
e) The break-points here are at $x = -3, -2, -1$, as this is easy to see. The SDT looks like:

	$(x+3)$	$(x+2)$	$(x+1)$	Sign of $\dfrac{x+1}{(x+2)(x+3)}$
$(-\infty, -3)$	$-$	$-$	$-$	$-$
$(-3, -2)$	$+$	$-$	$-$	$+$
$(-2, -1)$	$+$	$+$	$-$	$-$
$(-1, \infty)$	$+$	$+$	$+$	$+$

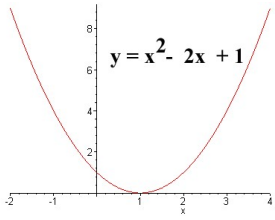
So, the solution of the inequality $\dfrac{x+1}{(x+2)(x+3)} > 0$ is given by
$-3 < x < -2$, or $-1 < x < \infty$.

f) The only break-points here are at $x = -1, 1$. This is because the rational function factors as $\dfrac{x^3-1}{x+1} = \dfrac{(x-1)(x^2+x+1)}{x+1}$ where the quadratic expression is Type II. So, the SDT looks like the one in Exercise 2 d), above. It follows that the solution of the inequality $\dfrac{x^3-1}{x+1} < 0$ is given by $-1 < x < 1$, or, written more compactly, as $|x| < 1$.

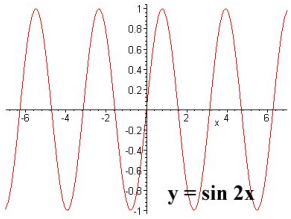
5.4 Chapter Exercises: Use plotting software (page 250)



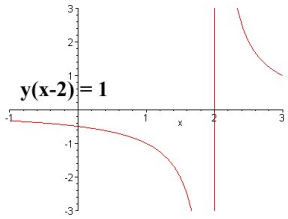
Exercise # 1



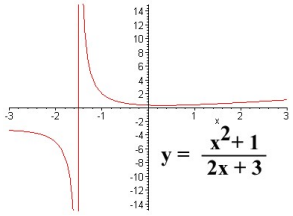
Exercise # 4



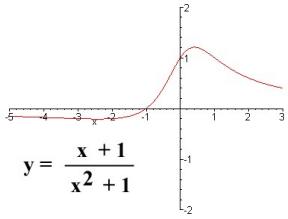
Exercise # 7



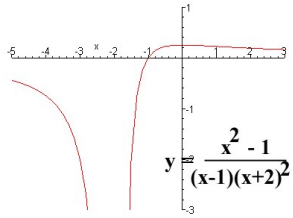
Exercise # 10



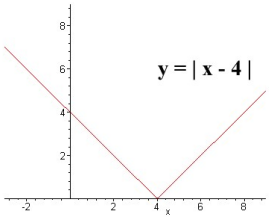
Exercise # 13



Exercise # 16

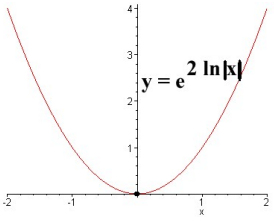


Exercise # 19

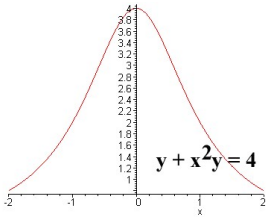


Exercise # 22

Chapter Exercises: (cont'd.)



Exercise # 25



Exercise # 28

16. $y = \frac{x+1}{x^2+1}$, $y' = -\frac{x^2+2x-1}{(x^2+1)^2}$, $y'' = \frac{2(x-1)^3}{(x^2+1)^3}$.

interval	$x+1$	$x+1-\sqrt{2}$	$x+1+\sqrt{2}$	$(x-1)^3$	y	y'	y''
$(-\infty, -1-\sqrt{2})$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$(-1-\sqrt{2}, -1)$	$-$	$-$	$+$	$-$	$-$	$+$	$-$
$(-1, -1+\sqrt{2})$	$+$	$-$	$+$	$+$	$+$	$+$	$+$
$(-1+\sqrt{2}, +\infty)$	$+$	$+$	$+$	$+$	$+$	$-$	$+$

19. $y = \frac{x^2-1}{(x-1)(x+2)^2}$, $y' = -\frac{x}{(x+2)^3}$, $y'' = \frac{2(x-1)}{(x+2)^4}$,

SDT:

interval	$x+1$	$(x+2)^3$	x	y	y'	y''
$(-\infty, -2)$	$-$	$-$	$-$	$-$	$-$	$-$
$(-2, -1)$	$-$	$+$	$-$	$-$	$+$	$-$
$(-1, 0)$	$+$	$+$	$-$	$+$	$+$	$-$
$(0, 1)$	$+$	$+$	$+$	$+$	$-$	$-$
$(1, +\infty)$	$+$	$+$	$+$	$+$	$-$	$+$

32. Profit $P(x) = R(x) - C(x) = 32x - (5 + 35x - 1.65x^2 + 0.1x^3) = -5 - 3x + 1.65x^2 - 0.1x^3$, $0 \le x \le 20$. For a local extremum, $\frac{dP}{dx} = -3 + 3.3x - 0.3x^2 = 0$, so $x^2 - 11x + 10 = 0$, and $(x-10)(x-1) = 0$. Thus $x = 1$ or $x = 10$. Now $\frac{d^2R}{dx^2} = 3.3 - 0.6x$. This is < 0 when $x = 10$ and > 0 when $x = 1$. So $x = 10$ gives a local maximum of $P(10) = -5 - 30 + 165 - 100 = 30$. Check end points: $P(0) = -5$, $P(20) = -5 - 60 + 660 - 800 < 0$. A production level of $x = 10$ stereos per day yields the maximum profit of \$30 per day.

33. (a) Revenue $R(x) = xp = 4x - 0.002x^2$. For a local maximum, $\frac{dR}{dx} = 4 - 0.004x = 0$, so $x = 1000$. Checks: $\frac{d^2R}{dx^2} = -0.004$, so $x = 1000$ is a local maximum. Endpoints: $R(0) = 0$, $R(1200) = \$1920$, $R(1000) = \$2000$. So a production of $x = 1000$, and hence price of $p = 4 - 0.002(1000) = \$2$ will maximize revenue.

(b) Profit $P = R - C = 2.5x - 0.002x^2 - 200$. $\frac{dP}{dx} = 2.5 - 0.004x = 0$ when $x = 625$. $(625, P(625))$, where $P(625) = 2.5(625) - .002(625)^2 - 200 = 581.25$, is a local maximum since $\frac{d^2P}{dx^2} < 0$. Now $P(0) = -200$, $P(1200) = 2.5(1200) - .002(1200)^2 - 200 = -80$, so a production level of $x = 625$ maximizes daily profit.

(c) \$581.25 from (b)

(d) marginal cost $MC = \frac{dC}{dx} = 1.5$. marginal revenue $MR = \frac{dR}{dx} = 4 - 0.004x$

(e) $4 - .004x = 1.5$, therefore $x = \frac{2.5}{0.004} = 625$ as in (c).

34. Average cost $AC = (800 + .04x + .0002x^2)/x = \frac{800}{x} + .04 + .0002x$, $x \ge 0$. For a local minimum, $\frac{d(AC)}{dx} = \frac{-800}{x^2} + .0002 = 0$, so $0.0002x^2 = 800$. Thus $x = 2000$ cabinets. (Check: $\frac{d^2(AC)}{dx^2} = \frac{1600}{x^3} > 0$ for $x > 0$, so $x = 2000$ gives a local minimum.)

35. (a) $45 = Ce^{1000k}$ and $40 = Ce^{1200k}$. Dividing gives $\frac{45}{40} = e^{-200k}$, so $ln\ \frac{9}{8} = -200k$, and $k = -.0005889 \approx -.0006$. Thus, $45 = Ce^{-.0006 \times 1000} = Ce^{-.6}$. Thus, $C = 45e^{.6} \approx 82$.

(b) Revenue, $R = xp$, where $p = 82e^{-.0006x}$ from (a). Thus, $R = 82xe^{-.0006x}$, and $\frac{dR}{dx} = 82e^{-.0006x} - .0006(82)xe^{-.0006x} = 82e^{-.0006x}[1 - .0006x]$. $\frac{dR}{dx} = 0$ for $1 - .0006x = 0$, so $x = 1666.7$.

	$e^{-.0006x}$	$1 - .0006x$	$R'(x)$
$(-\infty, 1666.7)$	+	+	+
$(1666.7, \infty)$	+	-	-

So maximum revenue occurs when $x \approx 1667$, and hence $p = 82e^{-.0006(1667)} = \$30.16$.

Chapter 6

Solutions

6.1 Exercise Set 28 (page 268)

1. $-5x + C$. Use Table 6.2 with $r = 1$, $\square = x$, $c = -5$.
2. $x + C$.
3. C. Use Table 6.2 with $r = 1$, $\square = 0$.
4. $\frac{1}{1.6}\ x^{1.6} + C$. Use Table 6.2 with $r = 0.6$, $\square = x$.
5. $\frac{3}{2}x^2 + C$.
6. $\frac{1}{2}x^2 - x + C$.
7. $\frac{1}{3}x^3 + x + C$. See Example 246.
8. $\frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$.
9. $\frac{3}{2}x^2 + C$. (Actually, this is the same as Exercise 5 above.)
10. $x^4 + x^2 - 1.314\ x + C$. See Example 246.
11. $\frac{1}{3}(2x - 2)^{3/2} + C$.
12. $\frac{2}{9}(3x + 4)^{3/2} + C$.
13. $-\frac{2}{3}(1 - x)^{3/2} + C$. See Example 247.
14. $\frac{1}{12}(4x^2 + 1)^{3/2} + C$.
15. $-\frac{1}{6}(1 - 2x^2)^{3/2} + C$.
16. $\frac{1}{3.5}(1 + x^2)^{1.75} + C$.
17. $\frac{1}{5}(2 + x^3)^{5/3} + C$.
18. $-\frac{1}{54}(4 + 9x^4)^{3/2} + C$. See Example 251.
19. $\frac{1}{3.6}(1 + x^{2.4})^{3/2} + C$.
20. $\mathcal{F}(x) = \frac{1}{4}\sin^4 x - 1$.
21. $\mathcal{F}(x) = \frac{1}{3}(1 - \cos^3 x)$.
22. $\mathcal{F}(x) = \frac{1}{2}(e^{-2} - e^{-2x})$.
23. $\dfrac{y^4(x)}{4} = \dfrac{x^3}{3} + \dfrac{1}{4}$. See Example 252.

24. $y = x^4 - 1$.

25. $y(x) = x^4 - 1$. See Example 255.

6.2 Exercise Set 29 (page 279)

1. $\frac{3}{2}$. $\displaystyle\int_0^1 3x\ dx = \frac{3x^2}{2}\Big|_0^1 = \frac{3(1^2)}{2} - 0 = \frac{3}{2}$.

2. $-\frac{1}{2}$. $\displaystyle\int_{-1}^0 x\ dx = \frac{x^2}{2}\Big|_{-1}^0 = -\frac{1}{2}$.

3. 0. $\displaystyle\int_{-1}^1 x^3\ dx = \frac{x^4}{4}\Big|_{-1}^0 = 0$; (note: x^3 is an odd function.)

4. $-\frac{4}{3}$. $\displaystyle\int_0^2 (x^2 - 2x)\ dx = \frac{x^3}{3} - x^2\Big|_0^2 = (\frac{2^3}{3} - 2^2) - 0 = -\frac{4}{3}$.

5. 16. $\displaystyle\int_{-2}^2 (4 - 4x^3)dx = 4x - x^4\Big|_{-2}^2 = 16$.

6. $\frac{1}{2}$. $\displaystyle\int_0^{\pi/2} \sin x\, \cos x\ dx = \int_0^{\pi/2} \sin x \left(\frac{d}{dx}\sin x\right)\ dx = \frac{\sin^2 x}{2}\Big|_0^{\pi/2} = \frac{1}{2}$.

7. $\frac{2}{3}$. Let $\square = \cos x$. Then $D\square = -\sin x$. So, $\mathcal{F}(x) = -\frac{1}{3}\cos^3 x + C$ and, by definition, $\displaystyle\int_0^{\pi} \cos^2 x \sin x\ dx = -\frac{\cos^3 x}{3}\Big|_0^{\pi} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$.

8. $\frac{1}{4}$. $\displaystyle\int_{-\pi}^{\pi/2} \sin^3 x\, \cos x\ dx = \frac{\sin^4 x}{4}\Big|_{-\pi}^{\pi/2} = \frac{1}{4}$.

9. -0.26. (Notice that the upper limit 1.2 of the integral is less than the lower one, namely 1.5; nevertheless we can proceed in the usual way.) $\displaystyle\int_{1.5}^{1.2} (2x - x^2)\ dx = x^2 - \frac{x^3}{3}\Big|_{1.5}^{1.2} = -0.26$.

10. $\frac{\pi}{2}$. $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\ dx = \text{Arcsin } x\Big|_0^1 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$.

11. $\frac{1}{2}(e-1)$. $\displaystyle\int_0^1 xe^{x^2} dx = \int_0^1 e^{x^2}\frac{d}{dx}\left(\frac{x^2}{2}\right)\ dx = \frac{1}{2}e^{x^2}\Big|_0^1 = \frac{1}{2}(e-1)$.

12. $2(1 - e^{-4})$.

$$\begin{aligned}\int_0^2 4xe^{-x^2} dx &= \int_0^4 2e^{-x^2}\left(\frac{d}{dx}x^2\right)\ dx \\ &= -2e^{-x^2}\Big|_0^2 = 2(1-e^{-4}).\end{aligned}$$

13. $\frac{2}{\ln 3}$. If we set $f(x) = 3^x$ then $f'(x) = 3^x\ \ln 3$. So $\displaystyle\int 3^x\ dx = \frac{3^x}{\ln 3} + C$. Thus $\displaystyle\int_0^1 3^x\ dx = \frac{3^x}{\ln 3}\Big|_0^1 = \frac{3}{\ln 3} - \frac{1}{\ln 3} = \frac{2}{\ln 3}$.

14. $\frac{1}{3}\left(e^{3\triangle} - e^{3\square}\right)$.

15. 0.1340. $\displaystyle\int_0^{0.5} \frac{x}{\sqrt{1-x^2}}\ dx = -(1-x^2)^{1/2}\Big|_0^{0.5} = 1 - \sqrt{0.75} \approx 0.1340$.

16. $\dfrac{1}{\ln 2}$. We know that $D(a^{\square}) = a^{\square} D(\square) \ln a$, where D as usual denotes the operator of taking derivative. It follows $\displaystyle\int a^{\square} \frac{d\square}{dx}\, dx = \frac{a^{\square}}{\ln a} + C$. Now, setting $a = 2$, $\square = x^2 + 1$, and $D\square = 2x$, we see that

$$\int_0^1 x\ 2^{x^2+1}\, dx = \frac{1}{2}\ \frac{2^{x^2+1}}{\ln 2}\bigg|_0^1 = \frac{2}{\ln 2} - \frac{1}{\ln 2} = \frac{1}{\ln 2}.$$

17. $\dfrac{\sqrt{2}-1}{2}$.

$$\begin{aligned} I &\equiv \int_0^{\sqrt{\pi}/2} x \sec(x^2)\tan(x^2)\, dx \\ &= \int_0^{\sqrt{\pi}/2} \frac{1}{2}\frac{d}{dx}\sec(x^2)\, dx \\ &= \frac{1}{2}\sec(x^2)\bigg|_0^{\sqrt{\pi}/2} = \frac{1}{2}\left(\sec\frac{\pi}{4} - \sec 0\right) = \frac{1}{2}(\sqrt{2}-1). \end{aligned}$$

18. 0. Let $\square = x^2$ So $D\square = 2x$ and the antiderivative looks like

$$\frac{1}{2}\int \frac{1}{1+\square^2}\ \frac{d\square}{dx}\, dx,$$

which reminds one of the derivative of the Arctangent function. In fact,

$$\int_{-1}^1 \frac{x}{1+x^4}\, dx = \frac{1}{2}\tan^{-1} x^2\ \bigg|_{-1}^1 = \frac{1}{2}(\tan^{-1} 1 - \tan^{-1} 1) = 0.$$

(Notice that 0 is the expected answer because the integrand is an odd function.)

19. Following the hint, we have $\displaystyle\frac{d}{dx}\int_0^{x^2} e^t\, dt = e^{x^2}\frac{d}{dx}x^2 = 2xe^{x^2}$.

20. These identities can be seen from the respective symmetry in the graph of f. Here is an analytic argument. Assume that f is even: $f(-x) = f(x)$. Let $\mathcal{F}(x) = \int_0^x f(t)dt$, $(-\infty < x < \infty)$. Then $\frac{d}{dx}\mathcal{F}(x) = f(x)$ and

$$\begin{aligned} \int_{-x}^x f(t)dt &= \int_{-x}^0 f(t)dt + \int_0^x f(t)dt \\ &= -\int_0^{-x} f(t)dt + \int_0^x f(t)dt = -\mathcal{F}(-x) + \mathcal{F}(x). \end{aligned}$$

Thus we will have $\int_{-x}^x f(t)dt = 2\int_0^x f(t)dt$ if we can show $-\mathcal{F}(-x) = \mathcal{F}(x)$. Let $\mathcal{G}(x) = -\mathcal{F}(-x)$. We are going to show $\mathcal{G} = \mathcal{F}$. Now

$$\begin{aligned} \frac{d}{dx}\mathcal{G}(x) &= \frac{d}{dx}(-\mathcal{F}(-x)) = -\left(\frac{d}{dx}\mathcal{F}(-x)\right) \\ &= -\left(\mathcal{F}'(-x)\cdot(-1)\right) = \mathcal{F}'(-x) = f(-x) = f(x). \end{aligned}$$

Thus, by the Fundamental Theorem of Calculus, $\mathcal{G}(x) = \int_0^x f(t)\ dt + C$ for some constant C, or $\mathcal{G}(x) = \mathcal{F}(x) + C$. Now $\mathcal{G}(0) = -\mathcal{F}(-0) = -\mathcal{F}(0) = -0 = 0$, which is the same as $\mathcal{F}(0)$ $(= 0)$. So C must be zero. Thus $\mathcal{G} = \mathcal{F}$. Done! (The second part of the exercise which involves an even function f can be dealt with in the same manner.)

6.3 Exercise Set 30 (page 285)

1. $\displaystyle\sum_{i=1}^{10} i$.

2. $\displaystyle\sum_{i=1}^{9}(-1)^{i-1}$, or $\displaystyle\sum_{i=1}^{9}(-1)^{i+1}$, or $\displaystyle\sum_{i=0}^{8}(-1)^{i}$.

3. $\displaystyle\sum_{i=1}^{5}\sin i\pi$.

4. $\displaystyle\sum_{i=1}^{n}\frac{i}{n}$

5. -0.83861

6. 0.19029.

7. 0. Note that $\sin n\pi = 0$ for any integer n.

8. $1+2+3+\cdots+50=\dfrac{50\times 51}{2}=1275$.

9. $1^2+2^2+\cdots+100^2=\dfrac{100\times 101\times 201}{6}=338350$.

10. $\displaystyle\sum_{i=1}^{n}\frac{i}{n}=\frac{1}{n}\sum_{i=1}^{n}i=\frac{1}{n}\cdot\frac{n(n+1)}{2}=\frac{n+1}{2}$.

11. $\displaystyle\sum_{i=1}^{n}6\left(\frac{i}{n}\right)^2=\frac{6}{n^2}\sum_{i=1}^{n}i^2=\frac{6}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}=\frac{(n+1)(2n+1)}{n}$.

12. This is a telescoping sum:

$$\sum_{i=1}^{6}(a_i-a_{i-1})=(a_1-a_0)+(a_2-a_1)+\cdots+(a_6-a_5)=a_6-a_0.$$

The final expression stands for what is left after many cancellations.

13. We prove this identity by induction. For $n=1$, we have

$$\text{LHS}=\sum_{i=1}^{1}(a_i-a_{i-1})=a_1-a_0=\text{RHS}.$$

Now we assume $\sum_{i=1}^{k}(a_i-a_{i-1})=a_k-a_0$, that is, the identity holds for $n=k$. Then, for $n=k+1$, we have

$$\begin{aligned}\sum_{i=1}^{k+1}(a_i-a_{i-1}) &= \sum_{i=1}^{k}(a_i-a_{i-1})+(a_{k+1}-a_k)\\ &= (a_k-a_0)-(a_{k+1}-a_k)=a_{k+1}-a_0.\end{aligned}$$

So the identity is also valid for $n=k+1$. Done.

14. Indeed,

$$\begin{aligned}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{i}{n}\right)^2 &= \left(\frac{1}{n}\right)^3\sum_{i=1}^{n}i^2=\left(\frac{1}{n}\right)^3\ \frac{n(n+1)(2n+1)}{6}\\ &= \frac{n}{n}\ \frac{(n+1)}{n}\ \frac{(2n+1)}{n}\ \frac{1}{6}=\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\ \frac{1}{6}.\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{i}{n}\right)^2=1\cdot 2\cdot\frac{1}{6}=\frac{1}{3}.$$

15. For convenience, we write

$$A_n = \sum_{i=1}^{n-1} \frac{n^3}{n^4 + in^3 + p_n}.$$

We have to show that $\lim_{n\to\infty} A_n = \ln 2$. We know that $\int_1^2 \frac{1}{x}\,dx = \ln 2$. Divide the interval $[1,2]$ into n subintervals of the same length $1/n$ by means of subdivision points $x_i = 1 + \frac{i}{n}$ $(i = 0, 1, 2, \ldots, n-1)$ and form the corresponding Riemann sum S_n for the function $f(x) = 1/x$:

$$S_n = \sum_{i=0}^{n-1} f(x_i)\cdot\Delta x_i = \sum_{i=0}^{n-1} \frac{1}{x_i}\cdot\Delta x_i = \sum_{i=0}^{n-1} \frac{n}{n+i}\cdot\frac{1}{n} = \sum_{i=0}^{n-1} \frac{1}{n+i}.$$

Since f is continuous on $[1,2]$, from the theory of Riemann integration we know that $\lim_{n\to\infty} S_n = \ln 2$. It suffices to show that $\lim_{n\to\infty}(S_n - A_n) = 0$. Now

$$\begin{aligned} S_n - A_n &= \sum_{i=0}^{n-1}\left(\frac{1}{n+i} - \frac{n^3}{n^4+in^3+p_n}\right) \\ &= \sum_{i=0}^{n-1} \frac{(n^4+in^3+p_n) - n^3(n+i)}{(n^4+in^3+p_n)(n+i)} = \sum_{i=0}^{n-1} \frac{p_n}{(n^4+in^3+p_n)(n+i)}. \end{aligned}$$

Thus

$$0 \le S_n - A_n \le \sum_{i=0}^{n-1} \frac{p_n}{n^4\cdot n} = p_n/n^4;$$

(dropping something positive from the denominator of a positive expression would diminish the denominator and hence would increase the size of this expression.) By the *Hint*, we have $p_n < 36n\ln n$. It is well-known that $\ln x \le x$ for all $x > 0$. So $p_n < 36n^2$ for all $n \ge 2$. Thus $0 < S_n - A_n \le p_n/n^4 < 36n^2/n^4 = 36/n^2$ for $n \ge 2$. Now it is clear that $S_n - A_n$ tends to 0 as $n \to \infty$, by the Sandwich Theorem of Chapter 2.

6.4

6.5 Chapter Exercises (page 300)

1. $\frac{1}{27}(x+1)^{27} + C$. Use Table 6.5, $\square = x+1, r = 26$.
2. $\frac{1}{2}\sin 2x + C$.
3. $\frac{1}{3}(2x+1)^{3/2} + C$.
4. $-\frac{1}{12}(1-4x^2)^{3/2} + C$. Use Table 6.5, $\square = 1-4x^2, r = 1/2$.
5. $-\cos x + \sin x + C$.
6. $-\frac{1}{3}(5-2x)^{3/2} + C$.
7. $-\frac{1}{2}\cos(2x) + C$. Use Table 6.6, $\square = 2x$.
8. $0.4\ x^{2.5} + 0.625\ \cos(1.6x) + C$.
9. $3\tan x + C$.
10. $\frac{1}{200}(x^2+1)^{100} + C$. Use Table 6.5, $\square = x^2+1, r = 99$.
11. $-\frac{1}{3}\csc 3x + C$.

12. $-\dfrac{1}{6}\,e^{-3\,x^2}+C$. Use Table 6.5, $\square=-3x^2$.

13. $-\dfrac{1}{k}\,e^{-kx}+C$.

14. $\dfrac{\sin k\,x}{k}+C$. Use Table 6.6, $\square=kx$.

15. $-\dfrac{\cos k\,x}{k}+C$.

16. 2. $\displaystyle\int_0^1 (2x+1)\,dx = x^2+x\Big|_0^1 = 2.$

17. 0. Note that $f(x)=x^3$ is an *odd function.*

18. 10. $\displaystyle I=\int_0^2 (3x^2+2x-1)\,dx = x^3+x^2-x\Big|_0^2 = 10.$

19. $\dfrac{1}{5}$. $\displaystyle\int_0^{\pi/2} \sin^4 x\,\cos x\,dx = \frac{\sin^5 x}{5}\Big|_0^{\pi/2} = \frac{1}{5}.$

20. $\dfrac{1}{\ln 3}$. $\displaystyle\int_0^1 x\cdot 3^{x^2}\,dx = \frac{1}{2}\frac{3^{x^2}}{\ln 3}\Big|_0^1 = \frac{1}{\ln 3}.$ Use Table 6.5, $\square=x^2, a=3$.

21. $\dfrac{1}{\ln 4}$. $\displaystyle\int_0^1 2^{-x}dx = -\frac{1}{\ln 2}\,2^{-x}\Big|_0^1 = -\frac{1}{\ln 2}\left(\frac{1}{2}-1\right) = \frac{1}{2\ln 2} = \frac{1}{\ln 4}.$

22. $\dfrac{2}{3}$. $\displaystyle\int_0^{\pi} \cos^2 x\cdot\sin x\,dx = -\,\frac{\cos^3 x}{3}\Big|_0^{\pi} = \left(-\frac{(-1)^3}{3}\right)-\left(-\frac{1}{3}\right) = \frac{2}{3}.$

23. $\dfrac{28}{3}$. Note that $f(x)=x^2+1$ is an *even function.*

24. $\dfrac{\pi}{6}$. $\displaystyle\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\,dx = \text{Arcsin}\ x\Big|_0^{0.5} = \frac{\pi}{6}.$

25. $\dfrac{1}{2}\,e^4-\dfrac{1}{2}$. Use Table 6.5, $\square=x^2$.

26. $\displaystyle\sum_{i=1}^{n} 12\left(\frac{i}{n}\right)^2 = \frac{12}{n^2}\sum_{i=1}^{n} i^2 = \frac{12}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{2(n+1)(2n+1)}{n}.$

27. Consider the partition

$$0=x_0<x_1<x_2<\cdots<x_n=1$$

with $x_i=\frac{i}{n}$, which divides $[0,1]$ into n subintervals $[\frac{i}{n},\ \frac{i+1}{n}]$ of the same length $1/n$, $(i=0,1,2,\ldots,n-1)$. In each subinterval $[\frac{i}{n},\ \frac{i+1}{n}]$ we take c_i to be the left end point $\frac{i}{n}$. Then the corresponding Riemann sum for the function $f(x)=e^x$ is

$$\sum_{i=0}^{n-1} f(c_i)(x_{i+1}-x_i) = \sum_{i=0}^{n-1} e^{i/n}\left(\frac{i+1}{n}-\frac{i}{n}\right) = \sum_{i=0}^{n-1} e^{i/n}\cdot\frac{1}{n}.$$

But we know from the definition of Riemann integration that

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)(x_k-x_{k-1}) = \int_0^1 f(x)dx = \int_0^1 e^x dx = e-1.$$

Now the assertion is clear.

28. Indeed,

$$\left(\frac{n}{n^2+0^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+...+\frac{n}{n^2+(n-1)^2}\right)=$$
$$\left(\frac{n}{n^2(1+\left(\frac{0}{n}\right)^2)}+\frac{n}{n^2(1+\left(\frac{1}{n}\right)^2)}+\frac{n}{n^2(1+\left(\frac{2}{n}\right)^2)}+...+\frac{n}{n^2(1+\left(\frac{n-1}{n}\right)^2)}\right)=$$
$$\left(\frac{1}{n\,(1+\left(\frac{0}{n}\right)^2)}+\frac{1}{n\,(1+\left(\frac{1}{n}\right)^2)}+\frac{1}{n\,(1+\left(\frac{2}{n}\right)^2)}+...+\frac{1}{n\,(1+\left(\frac{n-1}{n}\right)^2)}\right)=$$
$$\frac{1}{n}\left(\frac{1}{(1+\left(\frac{0}{n}\right)^2)}+\frac{1}{(1+\left(\frac{1}{n}\right)^2)}+\frac{1}{(1+\left(\frac{2}{n}\right)^2)}+...+\frac{1}{(1+\left(\frac{n-1}{n}\right)^2)}\right)=$$

$$\sum_{i=0}^{n-1}\frac{1}{(1+\left(\frac{i}{n}\right)^2)}\left(\frac{1}{n}\right) \;=\; \sum_{i=0}^{n-1} f(c_i)\,(\Delta x_i),$$

once we choose the c_i as $c_i = x_i = i/n$ and f as in the Hint. Next, we let $n \to \infty$ so that the norm of this subdivision approaches 0 and, by the results of this Chapter, the Riemann Sum approaches the definite integral

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=0}^{n-1} f(c_i)\Delta x_i &= \int_0^1 \frac{1}{1+x^2}\,dx,\\ &= \operatorname{Arctan}1 - \operatorname{Arctan}0 = \frac{\pi}{4}.\end{aligned}$$

29. **Method 1** First we interpret the integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ (whose value is $\frac{\pi}{2}$) as the limit of a sequence of Riemann sums S_n defined as follows. For fixed n, we divide $[0,1]$ into n subintervals of length $1/n$ by $x_i \equiv i/n$ $(0 \le i \le n)$ and we take c_i to be x_i. Then the corresponding Riemann sum for $f(x) \equiv \frac{1}{\sqrt{1-x^2}}$ is

$$S_n = \sum_{i=0}^{n-1} f(c_i)(x_{i+1}-x_i) = \sum_{i=0}^{n-1}\frac{1}{\sqrt{1-(i/n)^2}}\cdot\frac{1}{n} = \sum_{i=0}^{n-1}\frac{1}{\sqrt{n^2-i^2}}.$$

For convenience, let us put $A_{n,i} = n^8 - i^2n^6 + 2ip_n - p_n^2$. It is enough to show that $S_n - \sum_{i=0}^{n-1}\frac{n^3}{\sqrt{A_{n,i}}} \to 0$ as $n \to \infty$. Now, for each n and each i,

$$\frac{1}{\sqrt{n^2-i^2}} - \frac{n^3}{\sqrt{A_{n,i}}} = \frac{\sqrt{A_{n,i}} - n^3\sqrt{n^2-i^2}}{\sqrt{n^2-i^2}\sqrt{A_{n,i}}} = \frac{A_{n,i} - n^6(n^2-i^2)}{(n^2-i^2)\sqrt{A_{n,i}} + A_{n,i}\sqrt{n^2-i^2}}.$$

The denominator is too bulky here and we have to sacrifice some terms to tidy it up. But we have to wait until the numerator is simplified:

$$A_{n,i} - n^6(n^2-i^2) = (n^8 - i^2n^6 + 2ip_n - p_n^2) - (n^8 - i^2n^6) = 2ip_n - p_n^2.$$

Now we drop every thing save $A_{n,i}$ in the denominator. Then within

$$A_{n,i} \equiv n^8 - in^6 + 2ip_n - p_n^2 = n^6(n^2-i^2) + 2ip_n - p_n^2$$

we drop the positive term $2ip_n$ and the factor $n^2 - i^2$ which is ≥ 1. (We still have to keep the burdensome $-p_n^2$ because it is negative.) Ultimately, the denominator is replaced by a smaller expression, namely $n^6 - p_n^2$. Recall that $p_n < 36n^2$ for $n \ge 2$; (see Exercise 15 in the previous **Exercise Set.**) Using this we see that

$$n^6 - p_n^2 \ge n^6 - 36n^2 = n^2(n^4 - 36).$$

Thus, for $n \ge 2$, $n^2(n^4-36)$ is a lower bound of the denominator. Next we get an upper bound for the numerator:

$$|2ip_n - p_n^2| \le 2ip_n + p_n^2 \le 2np_n + p_n^2 \le 2n(36n^2) + (36n^2)^2 = 72n^3 + 1296n^4.$$

Now we can put all things together:

$$\left|S_n - \sum_{i=0}^{n-1} \frac{n^3}{\sqrt{A_{n,i}}}\right| \leq \sum_{i=1}^{n-1} \left|\frac{1}{\sqrt{n^2 - i^2}} - \frac{n^3}{\sqrt{A_{n,i}}}\right| \leq n \cdot \frac{72n^3 + 1296n^4}{n^2(n^4 - 36)}.$$

The last expression approaches to 0 as n tends to infinity. Done!

Method 2 Let $f(x) = \dfrac{1}{\sqrt{1-x^2}}$, on $[0, 1)$. Let $\mathcal{P}$ denote the partition with $x_0 = 0$, and $x_i = \frac{i}{n}$, $i = 1, 2, \ldots, n$. It is clear that, as $n \to \infty$, the norm of this partition approaches 0. Next, by Sierpinski's estimate we know that

$$p_n < 36n \ \ln n.$$

But by L'Hospital's Rule, $\lim\limits_{n\to\infty} \dfrac{\ln n}{n^3} = 0$. This means that

$$\lim_{n\to\infty} \frac{p_n}{n^4} \leq \lim_{n\to\infty} \frac{36 \ln n}{n^3} = 0.$$

So

$$\lim_{n\to\infty} \frac{p_n}{n^4} = 0$$

by the Sandwich Theorem of Chapter 2. Okay, now choose our interior points t_i in the interval $(x_i, \ x_{i+1})$, as follows: Let

$$t_i = \frac{i}{n} + \frac{p_n}{n^4}.$$

By what has been said, note that if n is sufficiently large, then t_i lies *indeed in this interval*. By definition of the Riemann integral it follows that this specific Riemann sum given by

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i) \ \Delta x_i$$

tends to, as $n \to \infty$, the Riemann integral of f over $[0, 1)$. But

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i) \ \Delta x_i &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{\sqrt{1 - {t_i}^2}} \cdot \frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{\sqrt{1 - \left(\frac{i}{n} + \frac{p_n}{n^4}\right)^2}} \cdot \frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{n^3}{\sqrt{n^8 - i^2 n^6 - 2ip_n - {p_n}^2}}.
\end{aligned}$$

The conclusion follows since the Riemann integral of this function f exists on $[0, 1)$ and

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} = \operatorname{Arcsin} 1 - \operatorname{Arcsin} 0 = \operatorname{Arcsin} 1 = \frac{\pi}{2}.$$

30. 2. $\displaystyle\int_0^{\pi} \sin x \ dx = -\cos x \Big|_0^{\pi} = 2.$

31. $2\sqrt{2} - 2$. Notice that when x runs from 0 to $\pi/2$, the cosine curve drops from 1 to 0 and the sine curve elevates from 0 to 1. Between 0 and $\pi/2$, the sine curve and the cosine curve meet at $x = \frac{\pi}{4}$. Hence

$$|\cos x - \sin x| = \begin{cases} \cos x - \sin x & \text{if } 0 \leq x \leq \pi/4, \\ \sin x - \cos x & \text{if } \pi/4 \leq x \leq \pi/2. \end{cases}$$

Thus the required integral is equal to

$$\begin{aligned}
&\int_0^{\pi/4} (\cos x - \sin x) \ dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \ dx \\
= \ &(\sin x + \cos x)|_0^{\pi/4} + (-\cos x - \sin x)|_{\pi/4}^{\pi/2} = 2\sqrt{2} - 2.
\end{aligned}$$

32. $1-\dfrac{\sqrt{2}}{2}$. $\displaystyle\int_{\pi/12}^{\pi/8}\frac{\cos 2x}{\sin^2 2x}\,dx=\int_{\pi/6}^{\pi/4}\csc 2x\cot 2x\,dx$
$\displaystyle=-\frac{1}{2}\csc 2x\Big|_{\pi/12}^{\pi/8}=1-\frac{\sqrt{2}}{2}$.

33. $\dfrac{4}{9}\sqrt{2}-\dfrac{2}{9}$. $\displaystyle\int_0^1 t^2\sqrt{1+t^3}\,dt=\frac{1}{3}\cdot\frac{(1+t^3)^{3/2}}{3/2}\Big|_0^1=\frac{4}{9}\sqrt{2}-\frac{2}{9}$. Use Table 6.5, $\square=1+t^3, r=1/2$.

34. $\displaystyle\int_0^1\frac{x}{1+x^4}\,dx=\frac{1}{2}\text{Arctan }x^2\Big|_0^1=\frac{1}{2}\left(\text{Arctan }1-\text{Arctan }0\right)=\frac{\pi}{8}$.

35. $\displaystyle\frac{d}{dx}\int_1^{x^2}\frac{\sin t}{t^{3/2}}dt=\frac{\sin(x^2)}{x^3}\cdot 2x=\frac{2\sin(x^2)}{x^2}\longrightarrow 2\quad\text{as } x\to 0+$.

36. As $x\to\infty$, we have

$$\frac{d}{dx}\int_{\sqrt{3}}^{\sqrt{x}}\frac{r}{(r+1)(r-1)}dr=\frac{x^{3/2}}{(x^{1/2}+1)(x^{1/2}-1)}\cdot\frac{1}{2\sqrt{x}}=\frac{x}{2(x-1)}\longrightarrow\frac{1}{2}.$$

37. $\displaystyle\frac{d}{dx}\int_x^{x^2}e^{-t^2}\,dt=2xe^{-x^4}-e^{-x^2}=2\frac{x}{e^{x^4}}-e^{-x^2}$ the first of which has the indefinite form $\frac{\infty}{\infty}$ when $x\to\infty$, while the second term tends to zero. By L'Hospital's rule and the fact that $e^x\to+\infty$ as $x\to+\infty$ we see that $2\frac{x}{e^{x^4}}\to 0$ as $x\to\infty$ as well.

38. $\displaystyle\frac{d}{dx}\int_1^{\sqrt{x}}\frac{\sin(y^2)}{2y}\,dy=\frac{\sin(\sqrt{x}^2)}{2\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}=\frac{\sin x}{4x}\longrightarrow\frac{1}{4}\quad\text{as } x\to 0$.

39.

$$\begin{aligned}
&\lim_{x\to 0^+}\frac{d}{dx}\int_1^{\sin x}\frac{\ln t}{\ln(\text{Arcsin }t)}\,dt\\
=&\lim_{x\to 0^+}\left(\frac{\ln(\sin x)}{\ln(\text{Arcsin }(\sin x))}\cdot\cos x-0\right),\\
=&\lim_{x\to 0^+}\left(\frac{\ln(\sin x)}{\ln x}\cdot\cos x\right)\\
=&\lim_{x\to 0^+}\left(\frac{\ln(\sin x)}{\ln x}\right)\cdot\cos 0\\
=&\lim_{x\to 0^+}\left(\frac{\cot x}{1/x}\right)\\
=&\lim_{x\to 0^+}x\,\cot x=\lim_{x\to 0^+}\frac{x}{\sin x}\cos x=(1)(1)=1.
\end{aligned}$$

40. Indeed, as $t\to 0$,

$$\frac{d}{dt}\int_{2\pi-ct}^{2\pi+ct}\frac{\sin x}{cx}\,dx=\frac{\sin(2\pi+ct)}{c(2\pi+ct)}\cdot c-\frac{\sin(2\pi-ct)}{c(2\pi-ct)}(-c)\longrightarrow\frac{\sin 2\pi}{\pi}=0.$$

41.

$$\begin{aligned}
&\lim_{h\to 0+}\frac{d}{dx}\left(\frac{1}{h}\int_{x-h}^{x+h}\sqrt{t}\,dt\right)=\lim_{h\to 0+}\frac{\sqrt{x+h}\,(1)-\sqrt{x-h}\,(1)}{h},\\
=&\lim_{h\to 0+}\frac{\sqrt{x+h}-\sqrt{x-h}}{h}=\lim_{h\to 0+}\frac{2h}{h(\sqrt{x+h}+\sqrt{x-h})},\\
=&\lim_{h\to 0+}\frac{2}{\sqrt{x+h}+\sqrt{x-h}}=\frac{1}{\sqrt{x}}.
\end{aligned}$$

42. $\lim_{x\to 0}\dfrac{1}{2x}\displaystyle\int_{-x}^{x}\cos t\ dt = \lim_{x\to 0}\dfrac{1}{2x}(\sin x - \sin(-x)) = \lim_{x\to 0}\dfrac{2\sin x}{2x} = 1$. [Remark: Actually, for every continuous function f defined on the real line, we have

$$\lim_{x\to 0+}\frac{1}{2x}\int_{-x}^{x} f(t)\ dt = f(0).$$

Do you know why?]

43. $\dfrac{y^5}{5} = \dfrac{x^4}{4} + \dfrac{1}{5}$.

44. $\sin(y(x)) + \cos x = C$ is the most general antiderivative. But $y = \pi/2$ when $x = 0$. This means that $\sin(\pi/2) + \cos 0 = C$, or $C = 2$. So, the solution in implicit form is given by $\sin(y(x)) + \cos x = 2$.

45. $y = \tan\left[\dfrac{1}{2}(e^{2x} - 1) + \dfrac{\pi}{4}\right]$.

46. $y = 2x^4 + \frac{4}{3}x^3 + x$.

47. $y(x) = C_1 + C_2\,x + C_3\,x^2 - x^4$ is the most general antiderivative. Now, the initial conditions $y(0) = 0, y'(0) = 0, y''(0) = -1$ imply that $C_1 = 0, C_2 = 0, C_3 = -1/2$. The required solution is given by

$$y(x) = -\frac{1}{2}\,x^2 - x^4.$$

48. $y = e^x - x - 1$.

49. $y = \dfrac{x^4}{12} + \dfrac{x^3}{3}$.

50. Since marginal cost $= \frac{dC}{dx} = 60 + \frac{40}{x+10}$,

    (a) total increase in cost as x goes from 20 to 40 is

$$\int_{20}^{40}\left[60 + \frac{40}{x+10}\right]\ dx = [60x + 40ln|x + 40|\ ]_{20}^{40}$$

$= 60 \times 40 + 40\ ln(50) - [60 \times 20 + 40ln(30)] = 1200 + 40(ln\ 50 - ln\ 30) = 1200 + 40ln(5/3) = \$1220.43$

    (b) Let $I(t)$ be value of investment at time t, t in years. $\frac{dI}{dt} = (500e^{\sqrt{t}})/\sqrt{t}$, thus

$$I(t) = \int \frac{500e^{\sqrt{t}}}{\sqrt{t}}dt = 500e^{\sqrt{t}} + C.$$

When $t = 0$, $I = 1000$, so $1000 = 500 + C$, and $C = 500$. Therefore, at $t = 4$, $I = 500e^2 + 500 = \$4194.53$.

Chapter 7

Solutions

7.1

7.2 Exercise Set 31 (page 319)

1. $\frac{1}{200}(2x-1)^{100}+C$.
2. $3\cdot\dfrac{(x+1)^{6.1}}{6.1}+C$.
3. $I=\displaystyle\int_0^1(3x+1)^{-5}dx=\frac{1}{3}\cdot\left.\frac{(3x+1)^{-4}}{-4}\right|_0^1\approx 0.0830$.
4. $I=\displaystyle\int(x-1)^{-2}dx=-(x-1)^{-1}+C=\frac{1}{1-x}+C$.
5. $-\dfrac{1}{202}(1-x^2)^{101}+C=\dfrac{1}{202}(x^2-1)^{101}+C$.
6. $\dfrac{1}{\ln 2}\,2^{x^2-1}+C$. Let $u=x^2,\ du=2x\ dx$, etc.
7. $\displaystyle\int_0^{\pi/4}\tan x\ dx=-\ln|\cos x|=\ln|\sec x|\Big|_0^{\pi/4}=\ln\sqrt{2}-\ln 1=\frac{\ln 2}{2}$.
8. $\dfrac{1}{3}e^{z^3}+C$.
9. $-\dfrac{3}{4}\,(2-x)^{4/3}+C$.
10. $\frac{1}{2}\sin 8\approx 0.49468$.
11. $I=\displaystyle\int\frac{1}{1+\sin t}\cdot\frac{d(1+\sin t)}{dt}\ dt=\ln|1+\sin t|+C$.
12. $-\sqrt{1-x^2}+C$.
13. $\frac{1}{2}\ln|y^2+2y|+C$. Let $u=y^2+2y,\ du=2(y+1)\ dy$, etc.
14. $I=\displaystyle\int\frac{\sec^2 x\ dx}{\sqrt{1+\tan x}}=\int\frac{\left(\frac{d}{dx}\tan x\right)\ dx}{\sqrt{1+\tan x}}=2\sqrt{1+\tan x}+C$.

15. $I = -\displaystyle\int_0^{\pi/4} \frac{1}{\cos^2 x} \cdot \frac{d\cos x}{dx}\, dx = \frac{1}{\cos x}\Big|_0^{\pi/4} = \sqrt{2} - 1$. Alternatively,

$$I = \int_0^{\pi/4} \tan x \sec x\, dx = \sec x \Big|_0^{\pi/4} = \sqrt{2} - 1.$$

16. **Hard! Very hard!** The function $\sec x + \tan x$ in the hint seems to be extremely tricky and unthinkable; see Example 364 in §8.5.2 for manipulating this integral according to the hint. Here is a slightly more natural way (although just as unthinkable): Try to put everything in terms of sines or cosines. Let's begin. Don't feel bad if you find this still too slick for you.

$$\begin{aligned}
\int \sec x\, dx &= \int \frac{1}{\cos x}\, dx = \int \frac{\cos x}{\cos^2 x}\, dx = \int \frac{1}{\cos^2 x} \cdot \frac{d\sin x}{dx}\, dx \\
&= \int \frac{1}{1-\sin^2 x} \cdot \frac{d\sin x}{dx}\, dx = \int \frac{1}{1-u^2} du \quad (u = \sin x) \\
&= \int \frac{1}{(1-u)(1+u)}\, du = \int \frac{1}{2}\left[\frac{1}{1-u} + \frac{1}{1+u}\right] du \\
&= \frac{1}{2}\left[-\ln|1-u| + \ln|1+u|\right] + C = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C \\
&= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C \\
&= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}\right| + C \\
&= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{\cos^2 x}\right| + C \\
&= \ln\left|\frac{1+\sin x}{\cos x}\right| + C = \ln|\sec x + \tan x| + C.
\end{aligned}$$

17. One way to do this is to multiply out everything and then integrate term by term. But this way is very messy! Observe that $4z^3 + 1$ is nothing but the derivative of $z^4 + z$. So we have an easy way out:

$$I = \int (z^4 + z)^4 \cdot \frac{d}{dz}(z^4 + z)\, dz = \frac{1}{5}(z^4 + z)^5 + C.$$

18. $-\text{Arctan}(\cos x) + C$. Let $u = \cos x$, $du = -\sin x\, dx$, etc.

19. $I = \dfrac{1}{2}\,\text{Arctan}(t^2)\Big|_0^1 = \dfrac{\pi}{8}$.

20. $\frac{1}{8}\sin^4(x^2+1) + C$. Let $u = x^2 + 1$ first, then $v = \sin u$ as the next substitution.

21. $\frac{3}{2}\ln(x^2+1) - \text{Arctan}\, x + C$. (Since $x^2 + 1$ is always positive, there is no need to put an absolute value sign around it.)

22. $I = \displaystyle\int_e^{e^2} \frac{1}{\ln x} \cdot \frac{d\ln x}{dx}\, dx = \ln(\ln x)\Big|_e^{e^2} = \ln 2 - \ln 1 = \ln 2$.

23. $\frac{1}{3}(\text{Arctan}\, x)^3 + C$.

24. $I = \int \cosh(e^t) \cdot e^t\, dt = \sinh(e^t) + C$. (Recall that $D\sinh\square = \cosh\square\ D\square$ and $D\cosh\square = \sinh\square\ D\square$.)

25. $\frac{1}{5}\text{Arcsin}\ 5s + C$.

26. $I = \displaystyle\int_{\pi^2}^{4\pi^2} \cos\sqrt{x} \cdot 2\frac{d\sqrt{x}}{dx}\, dx = 2\sin\sqrt{x}\Big|_{\pi^2}^{4\pi^2} = 2(\sin 2\pi - \sin\pi) = 0$.

27. $\dfrac{1}{2}e^{x^2}+C$.

28. $-\sqrt{1-y^2}+\text{Arcsin } y+C$. Split this integral up into two pieces and let $u=1-y^2$, etc.

29. $\sec(\ln x)+C$. Let $u=\ln x$, etc.

30. $I=\displaystyle\int \sin^{-2/3}x\cdot\frac{d}{dx}\sin x\ dx=3\sin^{1/3}x+C$.

31. $I=\displaystyle\int_0^1 e^{e^t}\cdot\frac{de^t}{dt}\ dt=e^{e^t}\Big|_0^1=e^e-e$.

32. $\dfrac{1}{2\ln(1.5)}\,1.5^{x^2+1}+C=1.23316\,1.5^{x^2+1}+C$.

7.3 Exercise Set 32 (page 342)

1. Using the normal method, we have:

$$I=\int x\frac{d}{dx}\sin x\ dx=x\sin x-\int\sin x\ dx=x\sin x+\cos x+C.$$

2. $-x\cos x+\sin x+C$.

3. $-1/2$.

4. Using the normal method, we have:

$$\begin{aligned}\int x^2\sin x\ dx&=\int x^2\frac{d}{dx}(-\cos x)dx\\ &=-x^2\cos x+\int 2x\cdot\cos x\ dx\\ &=-x^2\cos x+\int 2x\frac{d}{dx}\sin x\ dx\\ &=-x^2\cos x+2x\sin x-\int 2\sin x\ dx\\ &=-x^2\cos x+2x\sin x+2\cos x+C.\end{aligned}$$

   Now you can see the advantage of the Table method over the above normal method: you don't have to copy down some expressions several times and the minus signs are no longer a worry!

5. $x\tan x+\ln|\cos x|+C$.

6. $x\sec x-\ln|\sec x+\tan x|+C$. (Here you have to recall the answer to a very tricky integral: $\int\sec x\ dx=\ln|\sec x+\tan x|$. See Exercise Set 32, Number 16.)

7. $(x^2-2x+2)e^x+C$.

8. $I=-\dfrac{1}{3}x^2e^{-3x}+\dfrac{2}{9}xe^{-3x}-\dfrac{2}{27}e^{-3x}\Big|_0^\infty=\dfrac{2}{27}$. Notice that here we have used the fact $p(x)e^{-3x}\longrightarrow 0$ as $x\to+\infty$, where $p(x)$ is any polynomial, that is, the exponential growth is faster than the polynomial growth. Alternately, use L'Hospital's Rule for each limit except for the last one.

9. $\dfrac{1}{5}x^5\ln x-\dfrac{1}{25}x^5+C$.

10. $-\dfrac{1}{3}\left(x^3+x^2+\dfrac{2}{3}x+\dfrac{2}{9}\right)e^{-3x}+C$.

11. $x\sin^{-1}x+\sqrt{1-x^2}+C$.

12. $x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$.

13. Let $u = \ln x$. Then $x = e^u$ and $dx = e^u du$. Thus the integral can be converted to $\int u^5 e^{2u} e^u \, du = \int u^5 e^{3u} \, du$. Using the Table method to evaluate the last integral, we have

$$\int u^5 e^{3u} \, du = e^{3u}\left(\frac{1}{3}u^5 - \frac{5}{9}u^4 + \frac{20}{27}u^3 - \frac{20}{27}u^2 + \frac{40}{81}u - \frac{40}{243}\right) + C.$$

Substituting $u = \ln x$ back, we get the answer to the original integral $\int x^2(\ln x)^5 \, dx$:

$$x^3\left(\frac{1}{3}(\ln x)^5 - \frac{5}{9}(\ln x)^4 + \frac{20}{27}(\ln x)^3 - \frac{20}{27}(\ln x)^2 + \frac{40}{81}\ln x - \frac{40}{243}\right) + C.$$

14. $\frac{x^2}{2}\sec^{-1}x - \frac{1}{2}\sqrt{x^2-1} + C$, if $x > 0$.

15. Use the Table method for this problem.

$$\int (x-1)^2 \sin x \, dx = -(x-1)^2\cos x + 2(x-1)\sin x + 2\cos x + C.$$

16. $-\frac{1}{13}(2\sin 3x + 3\cos 3x)e^{-2x} + C$.

17. $\frac{1}{17}(\cos 4x + 4\sin 4x)e^x + C$.

18. $-\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C$, or $-\frac{1}{5}(2\sin 3x \sin 2x + 3\cos 3x\cos 2x) + C$.
Use the identity $\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$ with $A = 3x$ and $B = 2x$ and integrate. Alternately, this is also a **three-row problem**: This gives the second equivalent answer.

19. $-\frac{1}{12}\cos 6x + \frac{1}{4}\cos 2x + C$, or $-\frac{1}{3}\cos^3 2x + \frac{1}{2}\cos 2x + C$, or
$\frac{1}{12}(4\sin 2x\sin 4x + 2\cos 2x\cos 4x) + C$. This is a **three-row problem** as well. See the preceding exercise.

20. $\frac{1}{14}\sin 7x + \frac{1}{2}\sin x + C$, or $\frac{1}{7}(4\cos 3x\sin 4x - 3\sin 3x\cos 4x) + C$. Use the identity
$\cos A\cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$ with $A = 4x$ and $B = 3x$ and integrate. Alternately, this is also a **three-row problem**: This gives the second equivalent answer.

21. $e^{2x}\left(\frac{1}{2}x^5 - \frac{5}{4}x^4 + \frac{5}{2}x^3 - \frac{15}{4}x^2 + \frac{15}{4}x - \frac{15}{8}\right) + C$. For this exercise you really should use the Table method, otherwise you will find the amount of work overwhelming!

22. $\frac{x}{2}(\cos\ln x + \sin\ln x) + C$. See Example 314.

7.4 Exercise Set 33 (page 346)

1. $x - 3 + \frac{4}{x+1}$

2. $2 - \frac{3x^2+x+3}{x^3+2x+1}$

3. $\frac{1}{3}\left(x^2 - \frac{2}{3} + \frac{7/3}{3x^2-1}\right)$

4. $x^2 - 1 + \frac{2}{x^2+1}$

5. $x^4 + x^3 + 2x^2 + 2x + 2 + \dfrac{3}{x-1}$
6. $\dfrac{3}{2}\left(x - 1 + \dfrac{13x+15}{6x^2+6x+3}\right)$

7.4.1 Exercise Set 34 (page 364)

1. $\displaystyle\int \frac{x}{x-1}\,dx = \int \left(1 + \frac{1}{x-1}\right)\,dx = x + \ln|x-1| + C.$
2. $\displaystyle\int \frac{x+1}{x}\,dx = \int \left(1 + \frac{1}{x}\right)\,dx = x + \ln|x| + C.$
3. $\displaystyle\int \frac{x^2\,dx}{x+2} = \int \left(x - 2 + \frac{4}{x+2}\right)\,dx = \frac{x^2}{2} - 2x + 4\ln|x+2| + C.$
4. $\displaystyle\int \frac{x^2\,dx}{x^2+1} = \int \left(1 - \frac{1}{x^2+1}\right)\,dx = x - \operatorname{Arctan} x + C.$
5. Since the denominator and the numerator have the same degree, we have to perform the long division first
$$\begin{aligned} I &\equiv \int \frac{x^2}{(x-1)(x+1)}\,dx = \int \frac{x^2}{x^2-1}\,dx = \int \left(1 + \frac{1}{x^2-1}\right)\,dx \\ &= \int \left(1 + \frac{1}{(x+1)(x-1)}\right)\,dx = \int \left(1 + \frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1}\right)\,dx \\ &= x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C. \end{aligned}$$
6. Put $\dfrac{2x}{(x-1)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-3}$. Then $2x = A(x-3) + B(x-1)$. Setting $x = 1$ we have $A = -1$ and setting $x = 3$ we have $B = 3$. Thus the required integral is
$$\int \left(\frac{-1}{x-1} + \frac{3}{x-3}\right)\,dx = 3\ln|x-3| - \ln|x-1| + C.$$
7. Put $\dfrac{3x^2}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$. Then
$$3x^2 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$
Setting $x = 1,\ 2,\ 3$ respectively, we have $A = 3/2$, $B = -12$ and $C = 27/2$. Thus
$$\int \frac{3x^2\,dx}{(x-1)(x-2)(x-3)} = \frac{3}{2}\ln|x-1| - 12\ln|x-2| + \frac{27}{2}\ln|x-3| + C.$$
8. We start with long division:
$$\begin{aligned} I &\equiv \int_0^1 \frac{x^3-1}{x+1}dx = \int_0^1 \left(x^2 - x + 1 - \frac{2}{x+1}\right)\,dx \\ &= \frac{x^3}{3} - \frac{x^2}{2} + x - 2\ln|x+1|\Big|_0^1 = \frac{1}{3} - \frac{1}{2} + 1 - 2\ln 2 - 0 = \frac{5}{6} - \ln 4. \end{aligned}$$
9. Here we perform a small trick on the numerator of the integrand:
$$\begin{aligned} \int \frac{3x}{(x-1)^2}\,dx &= \int \frac{3(x-1)+3}{(x-1)^2}\,dx \\ &= \int \frac{3}{(x-1)^2}\,dx + \int \frac{3}{x-1}\,dx \\ &= 3(1-x)^{-1} + 3\ln|x-1| + C. \end{aligned}$$

10. Put
$$\frac{2x-1}{(x-2)^2(x+1)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$
Then $2x-1 = A(x-2)^2 + B(x-2)(x+1) + C(x+1)$. Setting $x=-1$, we have $-3 = A(-3)^2$ and hence $A = -1/3$. Setting $x=2$, we have $3 = 3C$; so $C=1$. Comparing the coefficients of x^2 on both sides, we get $0 = A+B$, which gives $B = -A = 1/3$. Thus
$$\begin{aligned}\int \frac{2x-1\ dx}{(x-2)^2(x+1)} &= \int \left(-\frac{1}{3}\cdot\frac{1}{x+1} + \frac{1}{3}\cdot\frac{1}{x-2} + \frac{1}{(x-2)^2}\right)\ dx \\ &= \frac{1}{2-x} + \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C.\end{aligned}$$

11. By long division, we get $\dfrac{x^4+1}{x^2+1} = x^2 - 1 + \dfrac{2}{x^2+1}$. So
$$\int \frac{x^4+1}{x^2+1}\ dx = \frac{x^3}{3} - x + 2\,\text{Arctan}\ x + C.$$

12. Putting $u = x^2$, the integrand becomes $\dfrac{1}{(u+1)(u+4)} = \dfrac{1}{3}\cdot\dfrac{1}{u+1} - \dfrac{1}{3}\dfrac{1}{u+4}$. So
$$\begin{aligned}\int \frac{dx}{(x^2+1)(x^2+4)} &= \frac{1}{3}\int \frac{dx}{x^2+1} - \frac{1}{3}\int \frac{dx}{x^2+4} \\ &= \frac{1}{3}\,\text{Arctan}\ x - \frac{1}{6}\,\text{Arctan}\ \frac{x}{2} + C.\end{aligned}$$

13. Put
$$\frac{1}{x^2(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+2}.$$
The we have
$$1 = Ax(x-1)(x+2) + B(x-1)(x+2) + Cx^2(x+2) + Dx^2(x-1).$$
Setting $x = 1,\ 0,\ -2$ respectively, we have $C = 1/3$, $B = -1/2$ and $D = -1/12$. Comparing coefficients of x^3 on both sides, we have $0 = A + C + D$, or $A + \frac{1}{3} - \frac{1}{12} = 0$ and hence $A = -\frac{1}{4}$. Thus
$$\begin{aligned}\int \frac{dx}{x^2(x-1)(x+2)} &= -\frac{1}{4}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x^2} + \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{12}\int \frac{dx}{x+2} \\ &= -\frac{1}{4}\ln|x| + \frac{1}{2x} + \frac{1}{3}\ln|x-1| - \frac{1}{12}\ln|x+2| + C.\end{aligned}$$

14. Put
$$\frac{x^5+1}{x(x-2)(x-1)(x+1)(x^2+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1} + \frac{D}{x+1} + \frac{Ex+F}{x^2+1}.$$
Using the method of "covering" described in this section, we get $A = 1/2$, $B = 11/10$, $C = -1/2$ and $D = 0$. By using the "plug-in method" described in the present section we have $E = -\frac{1}{10}$ and $F = \frac{3}{10}$. Thus the partial fraction decomposition for the integrand is
$$\frac{1}{2}\cdot\frac{1}{x} + \frac{11}{10}\cdot\frac{1}{x-2} - \frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{10}\cdot\frac{x}{x^2+1} + \frac{3}{10}\cdot\frac{1}{x^2+1}.$$
Thus the required integral is
$$\frac{1}{2}\ln|x| + \frac{11}{10}\ln|x-2| - \frac{1}{2}\ln|x-1| - \frac{1}{20}\ln(x^2+1) + \frac{3}{10}\text{Arctan}\ x + C.$$

15. Putting

$$\frac{2}{x(x-1)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{Dx+E}{x^2+1},$$

we have

$$2 = A(x-1)^2(x^2+1) + Bx(x-1)(x^2+1) + Cx(x^2+1) + (Dx+E)x(x-1)^2.$$

Setting $x = 0$, we obtain $A = 2$. Setting $x = 1$, we get $C = 1$. Next we set $x = 2$. This gives us an identity relating the unknowns from A to E. Substituting $A = 2$ and $C = 1$ in this identity and then simplifying, we get a relation

$$5B + 2D + E = -9$$

between $B,\ D$ and E. Setting $x = 3$ will give us another such a relation:

$$5B + 3D + E = -9.$$

From these two relations we can deduce that $D = 0$ and $5B + E = -9$. Finally, setting $x = -1$ will give us yet another relation among $B,\ D$ and E:

$$B + D - E = -3.$$

Now it is not hard to solve for B and E: $B = -2$, $E = 1$. (Remark: if you are familiar with complex numbers, you can find D and E efficiently by setting $x = i$ to arrive at $2 = (Di + E)i(i-1)^2$, which gives $Di + E = 1$ and hence $D = 0$ and $E = 1$, in view of the fact that D and E are real numbers.) We conclude

$$\frac{2}{x(x-1)^2(x^2+1)} = \frac{2}{x} - \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x^2+1}.$$

So the required integral is equal to

$$\int \frac{2\ dx}{x(x-1)^2(x^2+1)} = 2\ln|x| - 2\ln|x-1| - \frac{1}{x-1} + \operatorname{Arctan}\ x + C.$$

7.5 Exercise Set 35 (page 373)

1. Let $u = \cos 3x$ so that $du = -3\sin 3x\ dx$ and $\sin^2 3x = 1 - u^2$.

$$\begin{aligned}\int \sin^3 3x\ dx &= \int \sin^2 3x \cdot \sin 3x\ dx \\ &= \int (1-u^2)\cdot(-1/3)du = -\frac{u}{3} + \frac{u^3}{9} + C \\ &= -\frac{\cos 3x}{3} + \frac{\cos^3 3x}{9} + C.\end{aligned}$$

2. Let $u = \sin(2x-1)$ so that $du = 2\cos(2x-1)\ dx$ and $\cos^2(2x-1) = 1 - u^2$.

$$\begin{aligned}\int \cos^3(2x-1)\ dx &= \int \cos^2(2x-1)\cdot\cos(2x-1)\ dx \\ &= \int (1-u^2)\cdot\frac{1}{2}du = \frac{u}{2} - \frac{u^3}{6} + C \\ &= \frac{\sin(2x-1)}{2} - \frac{\sin^3(2x-1)}{6} + C.\end{aligned}$$

3. Let $u = \sin x$ so that $du = \cos x\ dx$. Notice that $x = 0 \Rightarrow u = 0$ and $x = \frac{\pi}{2} \Rightarrow u = 1$. Thus

$$\int_0^{\pi/2} \sin^2 x\cos^3 x\ dx = \int_0^1 u^2(1-u^2)du = \left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_0^1 = \frac{2}{15}.$$

4. Let $u = \cos(x-2)$. Then $du = -\sin(x-2)\ dx$ and

$$\begin{aligned}\int \cos^2(x-2)\sin^3(x-2)\ dx &= \int u^2(1-u^2)(-du) = -\frac{u^3}{3} + \frac{u^5}{5} + C \\ &= -\frac{1}{3}\cos^3(x-2) + \frac{1}{5}\cos^5(x-2) + C.\end{aligned}$$

5. Let $u = \sin x$. Then $du = \cos x\ dx$. Also, $x = \pi/2 \Rightarrow u = 1$ and $x = \pi \Rightarrow u = 0$. So

$$\int_{\pi/2}^{\pi} \sin^3 x\cos x\ dx = \int_1^0 u^3\ du = \frac{u^4}{4}\Big|_1^0 = -\frac{1}{4}.$$

The negative value in the answer is acceptable because $\cos x$ is negative when $\pi/2 < x < \pi$.

6. Set $u = x^2$. Then $du = 2x\ dx$. So

$$\begin{aligned}\int x\sin^2(x^2)\cos^2(x^2)\ dx &= \frac{1}{2}\int \sin^2 u\cos^2 u\ du = \frac{1}{8}\int \sin^2 2u\ du \\ &= \frac{1}{8}\int \left(\frac{1-\cos 4u}{2}\right)\ du = \frac{u}{16} - \frac{\sin 4u}{64} + C \\ &= \frac{x^2}{16} - \frac{\sin(4x^2)}{64} + C.\end{aligned}$$

7. We use the "double angle" formulae several times:

$$\begin{aligned}\int \sin^4 x\cos^4 x\ dx &= \frac{1}{16}\int \sin^4 2x\ dx = \frac{1}{16}\int \sin^2 2x\ \sin^2 2x\ dx \\ &= \frac{1}{16}\int \left(\frac{1-\cos 4x}{2}\right)^2\ dx \\ &= \frac{1}{64}\int (1 - 2\cos 4x + \cos^2 4x)\ dx \\ &= \frac{1}{64}x - \frac{1}{128}\sin 4x + \frac{1}{64}\int \frac{1+\cos 8x}{2}\ dx \\ &= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C.\end{aligned}$$

8. Let $u = \sin x$. Then $du = \cos x\ dx$ and $\cos^2 x = 1 - u^2$. So

$$\begin{aligned}\int \sin^4 x \cos^5 x\ dx &= \int u^4(1-u^2)^2\ du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C \\ &= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C.\end{aligned}$$

9. Use the "double angle formula" twice:

$$\begin{aligned}\int \cos^4 2x\ dx &= \int \left(\frac{1+\cos 4x}{2}\right)^2 dx \\ &= \frac{1}{4}\int (1 + 2\cos 4x + \cos^2 4x)\ dx \\ &= \frac{x}{4} + \frac{\sin 4x}{8} + \frac{1}{4}\int \frac{1+\cos 8x}{2}\ dx \\ &= \frac{3x}{8} + \frac{\sin 4x}{8} + \frac{\sin 8x}{64} + C.\end{aligned}$$

10. Let $u = \sin x$. Then $du = \cos x\ dx$ and $\cos^2 x = 1 - u^2$. So

$$\int \sin^5 x \cos^3 x\ dx = \int u^5(1-u^2)du = \frac{u^6}{6} - \frac{u^8}{8} + C = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C.$$

11. Set $u = \cos x$. Then $du = -\sin x\ dx$ and $\sin^2 x = 1 - u^2$. So

$$\begin{aligned}\int \sin^5 x \cos^4 x\ dx &= \int \sin^4 x \cos^4 x \cdot \sin x\ dx = \int (1-u^2)^2 u^4\ (-du) \\ &= \int (-u^4 + 2u^6 - u^8)\ du \\ &= -\frac{1}{5}u^5 + \frac{2}{7}u^7 - \frac{1}{9}u^9 + C \\ &= -\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C.\end{aligned}$$

12. We use the "double angle formula" several times.

$$\begin{aligned}\int \sin^6 x\ dx &= \int \left(\frac{1-\cos 2x}{2}\right)^3 dx \\ &= \frac{1}{8}\int (1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x)\ dx \\ &= \frac{x}{8} - \frac{3}{16}\sin 2x + \frac{3}{8}\int \frac{1+\cos 4x}{2}\ dx - \\ &\quad \frac{1}{8}\int (1-\sin^2 2x)\cdot \frac{d}{dx}\left(\frac{\sin 2x}{2}\right)\ dx \\ &= \frac{5}{16}x - \frac{1}{4}\sin 2x + \frac{1}{48}\sin^3 2x + \frac{3}{64}\sin 4x + C.\end{aligned}$$

13. Let $u = \sin x$. Then $du = \cos x\ dx$ and $\cos^6 x = (1 - \sin^2 x)^3 = (1-u^2)^3$. So

$$\begin{aligned}\int \cos^7 x\ dx &= \int (1-u^2)^3\ du = \int (1 - 3u^2 + 3u^4 - u^6)\ du \\ &= u - u^3 + \frac{3}{5}u^5 - \frac{1}{7}u^7 + C \\ &= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C.\end{aligned}$$

7.5.1 Exercise Set 36 (page 382)

1. $-\ln|\cos x| + C = \ln|\sec x| + C$. Let $u = \cos x,\ du = -\sin x\ dx$.
2. $\frac{1}{3}\tan(3x+1) + C$. Let $u = 3x + 1$.
3. $\sec x + C$, since this function's derivative is $\sec x\ \tan x$.
4. $\frac{\tan^2 x}{2} + C$. Let $u = \tan x,\ du = \sec^2 x\ dx$.
5. $\frac{\tan^3 x}{3} + C$. Let $u = \tan x,\ du = \sec^2 x\ dx$.
6. $\frac{\tan^6 x}{6} + C$. Let $u = \tan x,\ du = \sec^2 x\ dx$.
7. $\frac{\sec^3 x}{3} - \sec x + C$. Case m, n both ODD. Use (8.59) then let $u = \sec x,\ du = \sec x\ \tan x\ dx$.
8. $\frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$. Case m, n both EVEN. Solve for one copy of $\sec^2 x$ then let $u = \tan x,\ du = \sec^2 x\ dx$, in the remaining.
9. $\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$. Case m, n both ODD. Factor out one copy of $\sec x\ \tan x$, use (8.59), then let $u = \sec x,\ du = \sec x\ \tan x\ dx$ in the remaining.
10. $\frac{\sec^7 2x}{14} - \frac{\sec^5 2x}{5} + \frac{\sec^3 2x}{6} + C$. Let $u = 2x$ and use Example 364.
11. $\frac{\tan^6 2x}{12} + C$. Let $u = 2x,\ du = 2\ dx$, and use Exercise 6, above or, more directly, let $v = \tan 2x,\ dv = 2\sec^2 2x\ dx$.
12. $\frac{\tan^2 x}{2} + \ln|\cos x|$. Solve for $\tan^2 x$ in (8.59), break up the integral into two parts, use the result in Exercise 1 for the first integral, and let $u = \tan x$ in the second integral.
13. $\frac{1}{6}\sec^5 x\ \tan x + \frac{5}{24}\sec^3 x\ \tan x + \frac{5}{16}(\sec x\ \tan x + \ln|\sec x + \tan x|)$. Use Example 362 with $k = 7$, and then apply Example 365.
14. See Example 358.
15. $\frac{1}{4}\sec^3 x\ \tan x - \frac{1}{8}(\sec x\ \tan x + \ln|\sec x + \tan x|)$. The case where m is ODD and n is EVEN. Solve for $\tan^2 x$ and use Example 362 with $k = 5$ along with Example 357.

7.6 Exercise Set 37 (page 387)

1. $\displaystyle\int_0^1 \frac{1}{1+x^2}\ dx = \text{Arctan } x\Big|_0^1 = \frac{\pi}{4}$.
2. $\displaystyle\int \frac{2\ dx}{x^2 - 2x + 2} = \int \frac{2\ dx}{(x-1)^2 + 1} = 2\,\text{Arctan }(x-1) + C$.
3. $\displaystyle I = \int \frac{dx}{(x-1)^2 + 4} = \frac{1}{2}\text{Arctan }\frac{x-1}{2} + C$.
4. There is no need to complete a square:

$$\int \frac{dx}{x^2 - 4x + 3} \quad = \quad \int \frac{dx}{(x-1)(x-3)}$$

$$= \int \frac{1}{2}\left(\frac{1}{x-3} - \frac{1}{x-1}\right)\,dx$$
$$= \frac{1}{2}\ln|x-3| - \frac{1}{2}\ln|x-1| + C.$$

5. We need to complete the square in the denominator of the integrand:

$$\int \frac{4}{4x^2+4x+5}\,dx = \int \frac{4\,dx}{(2x+1)^2+4}$$
$$= \int \frac{dx}{\left(x-\frac{1}{2}\right)^2+1} = \text{Arctan}\left(x+\frac{1}{2}\right) + C.$$

6. The minus sign in front of x^2 should be taken out first.

$$\int \frac{dx}{4x-x^2-3} = -\int \frac{dx}{(x-1)(x-3)}$$
$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x-3| + C.$$

(For the last step, see the answer to Exercise 4 above.)

7. We have

$$\int \frac{1}{\sqrt{4x-x^2}}\,dx = \int \frac{dx}{\sqrt{-(x^2-4x+4-4)}} = \int \frac{dx}{\sqrt{4-(x-2)^2}}$$
$$= \frac{1}{2}\int \frac{dx}{\sqrt{1-\left(\frac{x-2}{2}\right)^2}} = \text{Arcsin}\ \frac{x-2}{2} + C.$$

8. We have

$$\int_{-1}^{0} \frac{1}{4x^2+4x+2}\,dx = \int_{-1}^{0} \frac{1}{(2x+1)^2+1}\,dx$$
$$= \frac{1}{2}\text{Arctan}\ (2x+1)\Big|_{-1}^{0} = \pi/4.$$

9. $\displaystyle\int \frac{dx}{\sqrt{2x-x^2+1}} = \int \frac{dx}{\sqrt{2-(x-1)^2}} = \text{Arcsin}\ \frac{x-1}{\sqrt{2}} + C.$

10. $\displaystyle\int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+1/2)^2+3/4} = \frac{2}{\sqrt{3}}\text{Arctan}\ \left(\frac{2x+1}{\sqrt{3}}\right) + C.$

11. The roots of x^2+x-1 are $(-1\pm\sqrt{5})/2$; (these interesting numbers are related to the so-called Golden Ratio and the Fibonacci sequence.) We have the following partial fraction decomposition:

$$\frac{1}{x^2+x-1} = \frac{1}{\left(x-\frac{(-1+\sqrt{5})}{2}\right)\left(x-\frac{(-1-\sqrt{5})}{2}\right)} = \frac{1}{\sqrt{5}}\left(\frac{1}{x-\frac{(-1+\sqrt{5})}{2}} - \frac{1}{x-\frac{(-1-\sqrt{5})}{2}}\right).$$

So $\displaystyle\int \frac{dx}{x^2+x-1} = \frac{1}{\sqrt{5}}\left\{\ln\left|x-\frac{(-1+\sqrt{5})}{2}\right| - \ln\left|x-\frac{(-1-\sqrt{5})}{2}\right|\right\} + C.$

12. $\displaystyle I = \int \frac{dx}{(2x+1)\sqrt{(2x+1)^2-1}} = \frac{1}{2}\text{Arcsec}\ (2x+1) + C,$

since $|2x+1| = 2x+1$ for $x > -1/2$ (see Table 6.7).

7.6.1 Exercise Set 38 (page 396)

1. Set $x = 2\sin\theta$. Then $dx = 2\cos\theta\ d\theta$ and $\sqrt{4-x^2} = 2\cos\theta$. So

$$\begin{aligned}\int \sqrt{4-x^2}\ dx &= \int 2\cos\theta \cdot 2\cos\theta\ d\theta = 4\int \cos^2\theta\ d\theta \\ &= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C \\ &= 2\,\text{Arcsin}\ (x/2) + \frac{1}{2}x\sqrt{4-x^2} + C.\end{aligned}$$

2. Let $x = 3\tan\theta$. Then $\sqrt{x^2+9} = 3\sec\theta$ and $dx = 3\sec^2\theta\ d\theta$. Hence

$$\begin{aligned}\int \sqrt{x^2+9}\ dx &= \int 3\sec\theta \cdot 3\sec^2\theta\ d\theta = 9\int \sec^3\theta\ d\theta \\ &= \frac{9}{2}\left\{(\tan\theta\sec\theta + \ln|\sec\theta + \tan\theta|\right\} + C \\ &= \frac{x}{2}\sqrt{x^2+9} + \frac{9}{2}\ln\left[\sqrt{x^2+9} + x\right] + C.\end{aligned}$$

   (A constant from the ln term is absorbed by C.)

3. Let $x = \sec\theta$. Then $\sqrt{x^2-1} = \tan\theta$ and $dx = \sec\theta\cdot\tan\theta\ d\theta$. Hence

$$\begin{aligned}\int \sqrt{x^2-1}\ dx &= \int \sec\theta\ \tan^2\theta\ d\theta \\ &= \frac{1}{2}\tan\theta\ \sec\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C \\ &= \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left|x + \sqrt{x^2-1}\right| + C.\end{aligned}$$

4. Let $x - 2 = 2\sin\theta$. Then $dx = 2\cos\theta\ d\theta$ and

$$\sqrt{4x-x^2} = \sqrt{-(x^2-4x+4-4)} = \sqrt{4-(x-2)^2} = 2\cos\theta.$$

   So we have

$$\begin{aligned}\int \sqrt{4x-x^2}\ dx &= \int 2\cos\theta \cdot 2\cos\theta \cdot d\theta \\ &= 2\theta + \sin 2\theta + C \\ &= 2\theta + 2\sin\theta\cos\theta + C \\ &= 2\sin^{-1}\frac{x-2}{2} + \frac{x-2}{2}\sqrt{4x-x^2} + C.\end{aligned}$$

5. Let $x = 2\sin u$. Then $dx = 2\cos u\ du$ and $(4-x^2)^{1/2} = 2\cos u$. Thus

$$\begin{aligned}\int \frac{dx}{(4-x^2)^{3/2}} &= \int \frac{2\cos u\ du}{2^3\cos^3 u} = \frac{1}{4}\int \sec^2 u\ du \\ &= \frac{1}{4}\tan u + C = \frac{1}{4}\cdot\frac{\sin u}{\cos u} + C = \frac{1}{4}\cdot\frac{x}{\sqrt{4-x^2}} + C.\end{aligned}$$

6. Let $x = 3\sin u$. Then $dx = 3\cos u\ du$ and $(9-x^2)^{1/2} = 3\cos u$. Thus

$$\begin{aligned}\int \frac{x^2\ dx}{(9-x^2)^{3/2}} &= \int \frac{3^2\sin^2 u \cdot 3\cos u\ du}{3^3\cos^3 u} = \int \tan^2 u\ du \\ &= \int (\sec^2 u - 1)\ du = \tan u - u + C = \frac{\sin u}{\cos u} - u + C \\ &= \frac{x}{\sqrt{9-x^2}} - \text{Arcsin}\ \frac{x}{3} + C.\end{aligned}$$

7. Let $x = 2\sec\theta$. Then $\sqrt{x^2-4} = 2\tan\theta$ and $dx = 2\sec\theta\,\tan\theta\,d\theta$. Also notice that $\cos\theta = 1/\sec\theta = 2/x$.

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-4}} &= \int \frac{2\sec\theta\cdot\tan\theta\,d\theta}{4\sec^2\theta\cdot 2\tan\theta} \\ &= \frac{1}{4}\int \cos\theta\,d\theta = \frac{1}{4}\sin\theta + C = \frac{1}{4}\sqrt{1-\cos^2\theta} + C \\ &= \frac{1}{4}\sqrt{1-(2/x)^2} + C = \frac{\sqrt{x^2-4}}{4x} + C.\end{aligned}$$

8. Let $2x-1 = \tan\theta$. Then $2\,dx = \sec^2\theta\,d\theta$ and

$$\sqrt{4x^2-4x+2} = \sqrt{(2x-1)^2+1} = \sqrt{\tan^2\theta+1} = \sec\theta.$$

Therefore we have

$$\begin{aligned}\int \sqrt{4x^2-4x+2}\,dx &= \frac{1}{2}\int \sec^3\theta\,d\theta \\ &= \frac{1}{4}\left(\tan\theta\,\sec\theta + \ln|\sec\theta+\tan\theta|\right) + C \\ &= \frac{1}{4}(2x-1)\sqrt{4x^2-4x+2} + \frac{1}{4}\ln\left|2x-1+\sqrt{4x^2-4x+2}\right| + C.\end{aligned}$$

9. Let $x = 3\tan\theta$. Then $9+x^2 = 9\sec^2\theta$ and $dx = 3\sec^2\theta\,d\theta$. So

$$\begin{aligned}\int \frac{dx}{(9+x^2)^2} &= \int \frac{3\sec^2\theta\,d\theta}{81\sec^4\theta} = \frac{1}{27}\int \cos^2\theta\,d\theta \\ &= \frac{1}{27}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C \\ &= \frac{1}{27}\left\{\frac{1}{2}\text{Arctan}\,\frac{x}{3} + \frac{1}{4}\sin\left(2\text{Arctan}\,\frac{x}{3}\right)\right\} + C.\end{aligned}$$

Note. It is possible to simplify the expression for $\sin(2\,\text{Arctan}\,x/3) \equiv \sin 2\theta$:

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta = 2\frac{\sin\theta}{\cos\theta}\cdot\cos^2\theta = 2\tan\theta\cdot\frac{1}{\sec^2\theta} \\ &= \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\cdot x/3}{1+(x/3)^2} = \frac{6x}{9+x^2}.\end{aligned}$$

10. The easiest way to solve this exercise is to use the substitution $u = \sqrt{4-x^2}$. (This is highly nontrivial! At first sight one would try the trigonometric substitution $x = 2\sin\theta$. This method works, but the computation involved is rather tedious and lengthy.) Then $u^2 = 4-x^2$ and hence $2u\,du = -2x\,dx$, which gives $u\,du = -x\,dx$. Now

$$\frac{dx}{x} = \frac{x\,dx}{x^2} = \frac{-u\,du}{4-u^2} = \frac{u\,du}{u^2-4}$$

and hence

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x}\,dx &= \int u\cdot\frac{u\,du}{u^2-4} = \int\left(1+\frac{4}{u^2-4}\right)\,du \\ &= u + \int\left(\frac{1}{u-2} - \frac{1}{u+2}\right)\,du \\ &= u + \ln|u-2| - \ln|u+2| + C \\ &= u + (\ln|2-u| + \ln|2+u|) - 2\ln|2+u| + C \\ &= u + \ln|4-u^2| - 2\ln|u+2| + C \\ &= \sqrt{4-x^2} + 2\ln|x| - 2\ln|2+\sqrt{4-x^2}\,| + C.\end{aligned}$$

11. Let $x = 5\tan\theta$. Then we have $dx = 5\sec^2\theta\ d\theta$, $(x^2+25)^{1/2} = 5\sec\theta$ and $(x^2+25)^{3/2} = 5^3\sec^3\theta$. Hence

$$\begin{aligned}\int \frac{dx}{(x^2+25)^{3/2}} &= \int \frac{5\sec^2\theta\ d\theta}{5^3\sec^3\theta} \\ &= \frac{1}{25}\int \cos\theta\ d\theta \\ &= \frac{1}{25}\sin\theta + C = \frac{1}{25}\frac{x}{\sqrt{x^2+25}} + C.\end{aligned}$$

12. Let $x = 2\sin\theta$. Then $\sqrt{4-x^2} = 2\cos\theta$ and $dx = 2\cos\theta\ d\theta$. So

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x^2}\ dx &= \int \frac{2\cos\theta}{4\sin^2\theta}\cdot 2\cos\theta\ d\theta \\ &= \int \cot^2\theta\ d\theta \\ &= \int (\csc^2\theta - 1)\ d\theta = -\cot\theta - \theta + C \\ &= -\frac{\sqrt{4-x^2}}{x} - \text{Arcsin}\ \frac{x}{2} + C.\end{aligned}$$

13. Let $x = a\sin\theta$. Then $dx = a\cos\theta\ d\theta$ and hence

$$\begin{aligned}\int \frac{dx}{x^4\sqrt{a^2-x^2}} &= \int \frac{a\cos\theta\ d\theta}{a^4\sin^4\theta\cdot a\cos\theta} \\ &= a^{-4}\int \frac{d\theta}{\sin^4\theta} = a^{-4}\int \csc^4\theta\ d\theta \\ &= a^{-4}\int (\csc^2\theta + \csc^2\theta\cot^2\theta)\ d\theta \\ &= a^{-4}\left(-\cot\theta - \frac{\cot^3\theta}{3}\right) + C, \\ &= -\frac{1}{a^4}\cdot\frac{(a^2-x^2)^{1/2}}{x} - \frac{1}{3a^4}\cdot\frac{(a^2-x^2)^{3/2}}{x^3} + C.\end{aligned}$$

14. Let $x = a\sec\theta$. Then $dx = a\sec\theta\tan\theta$ and hence

$$\begin{aligned}\int \frac{dx}{x^4\sqrt{x^2-a^2}} &= \int \frac{a\sec\theta\tan\theta\ d\theta}{a^4\sec^4\theta\cdot a\tan\theta} \\ &= \frac{1}{a^4}\int \cos^3\theta\ d\theta = \frac{1}{a^4}\int (1-\sin^2\theta)\cos\theta\ d\theta \\ &= \frac{1}{a^4}\left(\sin\theta - \frac{\sin^3\theta}{3}\right) + C \\ &= \frac{1}{a^4}\left(\frac{(x^2-a^2)^{1/2}}{x} - \frac{1}{3}\frac{(x^2-a^2)^{3/2}}{x^3}\right) + C.\end{aligned}$$

Notice that

$$\sin\theta = (1-\cos^2\theta)^{1/2} = (1-\sec^{-2}\theta)^{-1/2} = \left(1-\frac{a^2}{x^2}\right)^{1/2} = \frac{(x^2-a^2)^{1/2}}{x},$$

for $x > 0$.

15. We have

$$\begin{aligned}I &\equiv \int \frac{\sqrt{x^2+2x-3}}{x+1}\ dx \\ &= \int \frac{\sqrt{(x+1)^2-4}}{x+1}\ dx = \int \frac{\sqrt{u^2-4}}{u}\ du,\end{aligned}$$

where $u = x + 1$. Use the tricky substitution similar to the one in Exercise 8 above: $v = \sqrt{u^2 - 4} \equiv \sqrt{x^2 + 2x - 3}$. Then $v^2 = u^2 - 4$ and hence $2vdv = 2udu$, or $vdv = udu$. Thus

$$\frac{du}{u} = \frac{udu}{u^2} = \frac{vdv}{v^2 + 4}.$$

Now we can complete our evaluation as follows:

$$\begin{aligned} I &= \int \frac{v^2 dv}{v^2 + 4} = \int \left(1 - \frac{4}{v^2 + 4}\right)\, dv \\ &= v - 2\text{Arctan}\, \frac{v}{2} + C \\ &= \sqrt{x^2 + 2x - 3}\, - 2\text{Arctan}\, \frac{\sqrt{x^2 + 2x - 3}}{2} + C. \end{aligned}$$

16. Let $u = x^2 + 2x + 5$. Then $du = (2x + 2)dx$ and hence

$$\begin{aligned} \int \frac{(2x + 1)\, dx}{\sqrt{x^2 + 2x + 5}} &= \int \frac{(2x + 2 - 1)\, dx}{\sqrt{x^2 + 2x + 5}} \\ &= \int \frac{du}{\sqrt{u}} - I = 2\sqrt{u} - I = 2\sqrt{x^2 + 2x + 5} - I, \end{aligned}$$

where

$$I = \int \frac{dx}{\sqrt{x^2 + 2x + 5}} \equiv \int \frac{dx}{\sqrt{(x + 1)^2 + 4}}.$$

Let $x + 1 = 2\tan\theta$. Then $dx = 2\sec^2\theta\ d\theta$ and $\sqrt{x^2 + 2x + 5} = 2\sec\theta$. So

$$\begin{aligned} I &= \int \frac{2\sec^2\theta\ d\theta}{2\sec\theta} = \int \sec\theta\ d\theta = \ln|\tan\theta + \sec\theta| + C \\ &= \ln\left|x + 1 + \sqrt{x^2 + 2x + 5}\right| + C, \end{aligned}$$

where a factor of $\frac{1}{2}$ inside the logarithm symbol is absorbed by the integral constant C. Thus our final answer is

$$\int \frac{(2x + 1)\, dx}{\sqrt{x^2 + 2x + 5}} = 2\sqrt{x^2 + 2x + 5} - \ln\left|x + 1 + \sqrt{x^2 + 2x + 5}\right| + C.$$

7.7 Exercise Set 39 (page 411)

1. Yes, $x = 0$ is an infinite discontinuity.
2. No, the integrand is continuous on $[-1, 1]$.
3. Yes, $x = 0$ is an infinite discontinuity.
4. Yes, $x = 1$ is an infinite discontinuity (and ∞ is an upper limit).
5. Yes, $x = -1$ is an infinite discontinuity.
6. No, the integrand is continuous on $[-1, 1]$.
7. Yes, $x = -\pi, \pi$ are each infinite discontinuities of the cosecant function.
8. Yes, $\pm\infty$ are the limits of integration.
9. Yes, $x = 0$ gives an indeterminate form of the type $0 \cdot \infty$ in the integrand.
10. Yes, $\pm\infty$ are the limits of integration.

11. 2. This is because $\lim_{T\to\infty} \int_0^T x^{-1.5}\, dx = \lim_{T\to\infty} \left(\frac{-2}{\sqrt{T}} + 2\right) = 2.$

12. $+\infty$. This is because $\lim_{T\to\infty}\int_2^T x^{-1/2}\,dx = \lim_{T\to\infty}\left(2T^{1/2}-2\sqrt{2}\right) = +\infty$.

13. $+\infty$. Note that $\lim_{T\to 0^+}\ \frac{1}{2}\ \int_T^2 \frac{dx}{x} = \lim_{T\to 0^+}\left(\frac{1}{2}\ln|x|\right)\Big|_T^2 = \lim_{T\to 0^+}\left(\frac{1}{2}\ln 2 - \frac{1}{2}\ln T\right) = -(-\infty) = +\infty$.

14. 2. Use Integration by Parts (with the Table Method) and L'Hospital's Rule twice.
This gives $\lim_{T\to\infty}\int_0^T x^2\ e^{-x}\,dx = \lim_{T\to\infty}\left(2-\frac{T^2+2T+2}{e^T}\right) = 2$.

15. 0. Use the substitution $u = 1+x^2$, $du = 2x\,dx$ to find an antiderivative and note that
$\int_{-\infty}^{\infty}\frac{2x}{(1+x^2)^2}\,dx = \int_{-\infty}^{0}\frac{2x}{(1+x^2)^2}\,dx + \int_0^{\infty}\frac{2x}{(1+x^2)^2}\,dx$.
$= \lim_{T\to-\infty}\int_T^0\frac{2x}{(1+x^2)^2}\,dx + \lim_{T\to\infty}\int_0^T\frac{2x}{(1+x^2)^2}\,dx$,
$= \lim_{T\to-\infty}\left(-\frac{1}{1+x^2}\right)\Big|_T^0 + \lim_{T\to\infty}\left(-\frac{1}{1+x^2}\right)\Big|_0^T = -1+0+0-(-1) = 0$.

16. -1. Note that the infinite discontinuity is at $x=-1$ only. Now, use the substitution $u = 1-x^2$, $-\frac{du}{2} = x\,dx$. Then
$\int_{-1}^0\frac{x}{\sqrt{1-x^2}}\,dx = \lim_{T\to-1}\left(-\sqrt{1-T^2}\right)\Big|_T^0 = -1-0 = -1$.

17. Diverges (or does not exist). There is one infinite discontinuity at $x=1$. First, use partial fractions here to find that
$\frac{1}{x^2-1} = \frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1}$.
Next, using the definitions, we see that
$\int_0^2\frac{1}{x^2-1}\,dx = \int_0^1\frac{1}{x^2-1}\,dx + \int_1^2\frac{1}{x^2-1}\,dx =$
$= \lim_{T\to1^-}\int_0^T\frac{1}{x^2-1}\,dx + \lim_{T\to1^+}\int_T^2\frac{1}{x^2-1}\,dx = \lim_{T\to1^-}\left(\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|\right)\Big|_0^T +$
$+\lim_{T\to1^+}\left(\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|\right)\Big|_T^2 =$
$= \lim_{T\to1^-}\left(\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right)\Big|_0^T + \lim_{T\to1^+}\left(\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right)\Big|_T^2 =,$
$= \lim_{T\to1^-}\left(\frac{1}{2}\ln\left|\frac{T-1}{T+1}\right| - \frac{1}{2}\ln|-1|\right) + \lim_{T\to1^+}\left(\frac{1}{2}\ln\left|\frac{1}{3}\right| - \frac{1}{2}\ln\left|\frac{T-1}{T+1}\right|\right) = = (-\infty-0)+(-\frac{1}{2}\ln 3-(-\infty) =$
$\infty-\infty$, and so the limit does not exist.
So, the improper integral diverges.

18. $-\infty$. See the (previous) Exercise 17 above for more details. In this case the discontinuity, $x=1$, is at an end-point. Thus, using partial fractions as before, we find that
$\int_1^2\frac{1}{1-x^2}\,dx = \lim_{T\to1^+}\int_T^2\frac{1}{1-x^2}\,dx = -\lim_{T\to1^+}\int_T^2\frac{1}{x^2-1}\,dx =$
$= -\lim_{T\to1^+}\left(\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right)\Big|_T^2 = -\left(-\frac{1}{2}\ln 3-(-\infty)\right) = -\infty$.

19. $\frac{1}{2}$. Use Integration by Parts and the Sandwich Theorem to find that
$\int_0^{\infty}e^{-x}\ \sin x\,dx = \lim_{T\to\infty}\ \int_0^T e^{-x}\ \sin x\,dx = \lim_{T\to\infty}\ \frac{1}{2}\ \left(-e^{-x}\cos x - e^{-x}\sin x\right)\Big|_0^T$
$= \lim_{T\to\infty}\left(\frac{1}{2}\ \left(-\frac{\cos T}{e^T} - \frac{\sin T}{e^T}\right) - \left(-\frac{1}{2}\right)\right) = \frac{1}{2}$.
Recall that the Sandwich Theorem tells us that, in this case,

$$0 \le \lim_{T\to\infty}\left|\frac{\cos T}{e^T}\right| \le \lim_{T\to\infty}\left|\frac{1}{e^T}\right| = 0,$$

and so the required limit is also 0. A similar argument applies to the other limit.

20. $+\infty$. The infinite discontinuity is a $x = 1$. Use the substitution $u = \ln x$, $du = \frac{dx}{x}$. Then $\int_1^2 \frac{dx}{x \ln x} = \lim_{T\to 1^+} \int_T^2 \frac{dx}{x \ln x} = \lim_{T\to 1^+} \ln(\ln x))\Big|_T^2 =$
$= \lim_{T\to 1^+} (\ln(ln2) - \ln(\ln T)) = -(-\infty) = +\infty.$

21. $\frac{10}{7}$. The integrand is the same as $\int_{-1}^1 \left(x^{2/5} + x^{-3/5}\right)\, dx$ and so the infinite discontinuity (at $x = 0$) is in the second term only. So, $\int_{-1}^1 \left(x^{2/5} + x^{-3/5}\right)\, dx = \int_{-1}^1 x^{2/5}\, dx + \int_{-1}^1 x^{-3/5}\, dx = \frac{10}{7} + \int_{-1}^0 x^{-3/5}\, dx +$
$\int_0^1 x^{-3/5}\, dx = \frac{10}{7} + \lim_{T\to 0^-} \int_{-1}^T x^{-3/5}\, dx + \lim_{T\to 0^+} \int_T^1 x^{-3/5}\, dx = \frac{10}{7} + \lim_{T\to 0^-} \left(\frac{5T^{2/5}}{2} - \frac{5}{2}\right) + \lim_{T\to 0^+} \left(\frac{5}{2} - \frac{5T^{2/5}}{2}\right) =$
$\frac{10}{7} - \frac{5}{2} + \frac{5}{2} = \frac{10}{7}.$

22. Diverges. The integrand is the same as $\int_{-1}^1 \left(x^{-2/3} + x^{-5/3}\right)\, dx$ and so the discontinuity is present in both terms. Thus, $\int_{-1}^1 \left(x^{-2/3} + x^{-5/3}\right)\, dx =$
$= \int_{-1}^0 \left(x^{-2/3} + x^{-5/3}\right)\, dx + \int_0^1 \left(x^{-2/3} + x^{-5/3}\right)\, dx$
$= \lim_{T\to 0^-} \int_{-1}^T \left(x^{-2/3} + x^{-5/3}\right)\, dx + \lim_{T\to 0^+} \int_T^1 \left(x^{-2/3} + x^{-5/3}\right)\, dx$
$= \lim_{T\to 0^-} \left(3x^{1/3} - \frac{3}{2}x^{-2/3}\right)\Big|_{-1}^T + \lim_{T\to 0^+} \left(3x^{1/3} - \frac{3}{2}x^{-2/3}\right)\Big|_T^1$
$= \lim_{T\to 0^-} \left(3T^{1/3} - \frac{3}{2}T^{-2/3}\right) - \left(-3 - \frac{3}{2}\right) + \left(3 - \frac{3}{2}\right) -$
$= \lim_{T\to 0^+} \left(3T^{1/3} - \frac{3}{2}T^{-2/3}\right) =$
$-\infty + 6 + \infty = \infty - \infty,$

and so the improper integral diverges.

23. 2. Simply rewrite this integral as $\int_{-\infty}^\infty e^{-|x|}\, dx = \int_{-\infty}^0 e^x\, dx + \int_0^\infty e^{-x}\, dx$, since for $x < 0$ we have $|x| = -x$ while for $x > 0$ we have $|x| = x$. The integrals are straightforward and so are omitted.

24. Converges for $p > 1$ only. Let $u = \ln x$, $du = \frac{dx}{x}$. Then $\int_2^\infty \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^\infty \frac{du}{u^p} = \lim_{T\to\infty} \int_{\ln 2}^T \frac{du}{u^p} =$
$\lim_{T\to\infty} \frac{u^{1-p}}{1-p}\Big|_{\ln 2}^T = \lim_{T\to\infty} \left(\frac{T^{1-p}}{1-p}\right) - \frac{(\ln 2)^{1-p}}{1-p} =$
$0 - \frac{(\ln 2)^{1-p}}{1-p} = \frac{1}{(p-1)\ (\ln 2)^{p-1}}$, only for $p > 1$. The case $p = 1$ is treted as in Exercise 20, above, while the case $p < 1$ leads to an integral which converges to $+\infty$.

25. No, this is impossible. There is no real number p such the stated integral converges to a finite number. Basically, this is because the integrand has a "bad" discontinuity at $x = 0$ whenever $p < 1$ and another discontinuity at $x = \infty$ whenever $p \geq 1$. The argument is based on a case-by-case analysis and runs like this:

If $p + 1 > 0$, then $\int_0^\infty x^p\, dx = \lim_{T\to\infty} \int_0^T x^p\, dx = \lim_{T\to\infty} \left(\frac{T^{p+1}}{p+1} - \frac{1}{p+1}\right) = +\infty$. On the other hand, if $p + 1 < 0$, then $\int_0^\infty x^p\, dx = \int_0^1 x^p\, dx + \int_1^\infty x^p\, dx = \lim_{T\to 0^+} \left(\frac{1}{p+1} - \frac{T^{p+1}}{p+1}\right) + \lim_{T\to\infty} \left(\frac{T^{p+1}}{p+1} - \frac{1}{p+1}\right) =$

$= \left(\dfrac{1}{p+1} - \infty\right) + \left(0 - \dfrac{1}{p+1}\right) = -\infty$. Finally, if $p = 1$ the integrand reduces to x, by itself and it converges to $+\infty$. Thus, we have shown that for any value of p the improper integral cannot converge to a finite value.

26. $\sqrt{\dfrac{2}{\pi}} \cdot \dfrac{2}{4+\lambda^2}$. Use the method outlined in Exercise 19, above.
27. No, the integral must converge to $+\infty$. Follow the hints.
28. Follow the hints.
29. Follow the hints.
30. $L = \sqrt{\pi}$. Simpson's Rule with $n = 22$ gives us the value 1.7725 as an estimate for the value of this integral over the interval $[-5, 5]$. Its square is about 3.1416, which is close to $\sqrt{\pi}$.

7.8 Exercise Set 40 (page 420)

1. Vertical slice area $= (0 - (x^2 - 1))\ dx = (1 - x^2)\ dx$.
2. Horizontal slice area $\sqrt{y+1}\ dy$.
3. Vertical slice area $= ((x^2 + 5x + 6) - (e^{2x}))\ dx = (x^2 + 5x + 6 - e^{2x})\ dx$. Note that e^{2x} is smaller than $x^2 + 5x + 6$ on this interval. See the figure in the margin, on the left.
4. Sketch the region bounded by these curves. You should get a region like the one below:

Now, using Newton's Method with $x_0 = 1.5$ as an initial estimate, $n = 3$, and $f(x) = x^2 + 5x + 6 - e^{2x}$, we obtain the approximate value of the zero of f as 1.3358. The common value of these curves at this point is given by $e^{2(1.3358)} \approx 14.46$. This represents the point of intersection of the curves $x^2 + 5x + 6$ and e^{2x}, in the interval $[0, 2]$. Beyond $x = 2$ we see that these curves get further apart so they cannot intersect once again. Since we are dealing with horizontal slices we need to write down the inverse function of each of these functions. For example, the inverse function of $y = x^2 + 5x + 6$ is given by solving for x in terms of y using the quadratic formula. This gives

$$x = \frac{-5 \pm \sqrt{1 + 4y}}{2}.$$

Since $x \geq 0$ here, we must choose the +-sign. On the other hand, the inverse function of the function whose values are $y = e^{2x}$ is simply given by $x = (\ln y)/2$. So, the area of a typical horizontal slice in the darker region above is given by

$$\left(\frac{\ln y}{2} - \frac{-5 + \sqrt{1 + 4y}}{2}\right)\ dy,$$

and this formula is valid provided $6 \leq y \leq 14.46$.

If the horizontal slice is in the lighter area above, then its area is given by

$$\left(\frac{\ln y}{2} - 0\right)\ dy = \left(\frac{\ln y}{2}\right)\ dy,$$

and this formula is valid whenever $0 \leq y \leq 6$.

As a check, note that both slice formulae agree when $y = 6$.

5. The horizontal line $y = 5$ intersects with the graph of $y = e^{2x}$ at the point $P \equiv (\frac{\ln 5}{2}, 5)$, approxmately $(0.8047, 5)$. Draw a vertical line through P. The area of a typical vertical slice on the left of this line is

$$((x^2 + 5x + 6) - 5)\ dx \equiv (x^2 + 5x + 1)\ dx.$$

On the right of this line we have

$$(x^2 + 5x + 6 - e^{2x})\ dx$$

instead.

7.8.1 Exercise Set 41 (page 430)

1. Area $= \displaystyle\int_{-1}^{1} (1 - x^2)\ dx = \frac{4}{3}$.
2. Area $= \displaystyle\int_{-2}^{2} (4 - x^2)\ dx = \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2} = \frac{32}{3}$.
3. Area $= \displaystyle\int_{0}^{1} (x^2 + 5x + 6 - e^{2x})dx = \left(\frac{x^3}{3} + \frac{5}{2}x^2 + 6x - \frac{e^{2x}}{2}\right)\Big|_{0}^{1} = \frac{28}{3} - \frac{e^2}{2}$ ≈ 5.63881.
4. Area $= \displaystyle\int_{0}^{1.3358} (x^2 + 5x + 6 - e^{2x})\ dx \approx 6.539$.
5. Area $= \displaystyle\int_{0}^{1} ye^y dy = (ye^y - e^y)\Big|_{0}^{1} = 1$.
6. $\pi^2 - 4$. This curve lies above the $x-$axis because $\sin x \geq 0$ for $0 \leq x \leq \pi$. It follows that $x^2 \sin x \geq 0$ for $0 \leq x \leq \pi$, and so the area is given by the definite integral

$$\text{Area } = \int_0^{\pi} x^2 \sin x\ dx = \pi^2 - 4,$$

where the Table method of Integration by Parts is used to evaluate it. In particular, we note that an antiderivative is given by

$$\int^{x} t^2 \sin t\ dt = -x^2 \cos x + 2x \sin x + 2 \cos x.$$

7. Area $= \displaystyle\int_{0}^{\pi} \cos^2 x \sin x\ dx = -\frac{\cos^3 x}{3}\Big|_{0}^{\pi} = \frac{2}{3}$.
8. Using the Table method of Integration by Parts (since this is a three-row problem), we find

$$\int \sin 3x \cdot \cos 5x\ dx = \frac{5}{16} \sin 5x \cdot \sin 3x + \frac{3}{16} \cos 3x \cdot \cos 5x + C.$$

Alternatively, this integral can be computed as follows:

$$\int \sin 3x \cdot \cos 5x\ dx = \int \frac{1}{2}(\sin 8x - \sin 2x)\ dx = -\frac{1}{16} \cos 8x + \frac{1}{4} \cos 2x + C.$$

(Don't be fooled by its different look! This is the same answer as the above.) Notice that, for x in the interval $[\pi/10, 3\pi/10]$, $3x$ is in $[3\pi/10, 9\pi/10]$ and hence $\sin 3x$ is positive. However, for the same range of x, $5x$ is in $[\pi/2, 3\pi/2]$ and hence $\cos 3x$ is negative or zero. Hence the area of the region is the absolute value of

$$\int_{\pi/10}^{3\pi/10} \sin 3x \cdot \cos 5x\ dx \quad = \quad \left(\frac{5}{16} \sin 5x \cdot \sin 3x + \frac{3}{16} \cos 3x \cdot \cos 5x\right)\Big|_{\pi/10}^{3\pi/10}$$

$$= -\frac{5}{16}\left(\sin\frac{9}{10}\pi + \sin\frac{3}{10}\pi\right) = -\frac{5\sqrt{5}}{32} \approx -0.35.$$

Here we use the facts that $\cos\frac{\pi}{2} = 0$, $\cos\frac{3\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$ and $\sin\frac{3\pi}{2} = -1$. It turns out that $\sin\frac{9}{10}\pi + \sin\frac{3}{10}\pi = \frac{\sqrt{5}}{2}$, which is very hard to prove!

9. $\dfrac{9}{2}$. Refer to the graph below:

The points of intersection of these two graphs are given by setting $y = -x$ into the expression $x + y^2 = 2$ and solving for x. This gives the two points, $x = 1$ and $x = -2$. Note that if we use vertical slices we will need two integrals. Solving for x in terms of y gives $x = -y$ and $x = 2 - y^2$ and the limits of integration are then $y = -1$ and $y = 2$. The coordinates of the endpoints of a typical horizontal slice are given by $(-y, y)$ and $(2 - y^2, y)$. So, the corresponding integral is given by

$$\text{Area} = \int_{-1}^{2} (2 - y^2 + y)\, dy = \frac{9}{2}.$$

10. The required area is

$$\int_{-2}^{2} (y^2 - (y - 5))\, dy = \left(\frac{y^3}{3} - \frac{y^2}{2} + 5y\right)\Big|_{-2}^{2} = \frac{76}{3}.$$

11. 4 units. Note the symmetry: Since, f is an *even function*, (see Chapter 5), its graph over the interval $[-\pi, \pi]$ is symmetric with respect to the $y-$axis and so, since f is "V"-shaped and positive, the area is given by

$$\text{Area} = 2 \times (\textit{area to the right of } x = 0),$$

and this gives

$$\text{Area} = 2\int_{0}^{\pi} (\sin x)\, dx = 4 \textit{ units}.$$

12. $4\sqrt{2}$. The graph on the right represents the two curves over the interval $[\frac{\pi}{4}, \frac{9\pi}{4}]$:
Using the symmetry in the graph we see that

$$\text{Area} = 2\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\, dx = 4\sqrt{2},$$

since $\cos\dfrac{5\pi}{4} = -\dfrac{\sqrt{2}}{2}$, $\sin\dfrac{5\pi}{4} = -\dfrac{\sqrt{2}}{2}$.

7.9 Exercise Set 42 (page 446)

1. Using a vertical slice: $\int_0^1 \pi x^2\, dx$;
using a horizontal slice $\int_0^1 (1 - y) \cdot 2\pi y\, dy$.
2. Using a vertical slice: $\int_0^1 (x - x^2) \cdot 2\pi x\, dx$;
using a horizontal slice: $\int_0^1 \pi(y - y^2)\, dy$.

3. Using a vertical slice: $\int_0^1 3\pi\ x^2\ dx$; (we do not use horizontal slices because this method is too complicated for the present problem.)

4. Using a vertical slice: $\int_0^2 2x \cdot 2\pi x\ dx$;

   using a horizontal slice: $\int_0^1 \pi(2^2 - (y/2)^2)\ dy$.

5. Using a vertical slice: $\int_0^1 (2x - x) \cdot 2\pi x\ dx = \int_0^1 2\pi x^2\ dx$.

   Using a horizontal slice:
   $\dfrac{3\pi}{4}\displaystyle\int_0^1 y^2\ dy + \int_1^2 \pi\left(1 - \frac{y^2}{4}\right)\ dy$.

6. $\pi/3$; $\pi/6$; 8π; $\frac{32}{3}\pi$; $\frac{2}{3}\pi$.

7.10 Exercise Set 43 (page 457)

1. 2. Since $y' = 0$, it follows that $L = \displaystyle\int_0^2 \sqrt{1}\ dx = 2$.

2. $4\sqrt{2}$. Since $y' = 1$, it follows that $L = \displaystyle\int_0^4 \sqrt{1+1}\ dx = 4\sqrt{2}$.

3. $2\sqrt{5}$. Here $y' = 2$, and so $L = \displaystyle\int_{-1}^1 \sqrt{1+4}\ dx = 2\sqrt{5}$.

4. $2\sqrt{2}$. Now $x'(y) = 1$. So, $L = \displaystyle\int_0^2 \sqrt{2}\ dy = 3\sqrt{2}$.

5. $3\sqrt{2}$. This is the same as $y = x + 3$, so $y' = 1$, and it follows that $L = \displaystyle\int_{-2}^1 \sqrt{2}\ dx = 3\sqrt{2}$.

6. $\dfrac{52}{3}$. Now $y'(x) = \sqrt{x}$ and so (if we set $u = 1 + x,\ du = dx$) we see that $L = \displaystyle\int_0^8 \sqrt{1+x}\ dx = \frac{52}{3}$.

7. $\dfrac{1}{2}\sqrt{5} + \dfrac{1}{4}\ln\left(\sqrt{5}+2\right)$. In this case, $L = \displaystyle\int_0^1 \sqrt{1+4x^2}\ dx$. Use the substitution $2x = \tan\theta,\ 2\ dx = sec^2\theta\ d\theta$, and the usual identity to obtain an

   antiderivative in the form $\dfrac{1}{2}\displaystyle\int \sec^3\theta\ d\theta$. Now, see Example 357 for this integral. We have, $\dfrac{1}{2}\displaystyle\int \sec^3\theta\ d\theta = \dfrac{1}{4}\left(\tan\theta\sec\theta + \ln|\sec\theta + \tan\theta|\right)$. But $\tan\theta = 2x$, and so $\sec\theta = \sqrt{1+4x^2}$. Thus, $L = \displaystyle\int_0^1 \sqrt{1+4x^2}\ dx = \dfrac{1}{4}\left(2x\sqrt{1+4x^2} + \ln|\sqrt{1+4x^2}+2x|\right)\Big|_0^1 = \dfrac{1}{4}\cdot\left(2\sqrt{5} + \ln(2+\sqrt{5})\right)$, and the rest follows.

8. $\sqrt{65} + \dfrac{1}{8}\ln\left(8+\sqrt{65}\right)$. Use the method of Example 7 above. The arc length is given by $L = \displaystyle\int_0^2 \sqrt{1+16x^2}\ dx$. Now use the substitution $4x = \tan\theta,\ 4\ dx = sec^2\theta\ d\theta$, and an antiderivative will look like $\dfrac{1}{4}\displaystyle\int \sec^3\theta\ d\theta$. Finally, we see that

   $L = \displaystyle\int_0^2 \sqrt{1+16x^2}\ dx = \dfrac{1}{8}\left(4x\sqrt{1+16x^2} + \ln|\sqrt{1+16x^2}+4x|\right)\Big|_0^2 = \dfrac{1}{8}\cdot\left(8\sqrt{65} + \ln(8+\sqrt{65})\right)$, and the result follows.

9. $\dfrac{1}{2}\sqrt{17} + \dfrac{1}{8}\ \ln(4+\sqrt{17})$. See Exercise 8, above. We know that $L = \displaystyle\int_0^1 \sqrt{1+16x^2}\ dx$. Use the substitution $4x = \tan\theta,\ 4\ dx = sec^2\theta\ d\theta$, and the usual identity to obtain an antiderivative in the form

   $\dfrac{1}{4}\displaystyle\int \sec^3\theta\ d\theta$. Reverting back to the original variables, we get,

   $L = \displaystyle\int_0^1 \sqrt{1+16x^2}\ dx = \dfrac{1}{8}\left(4x\sqrt{1+16x^2} + \ln|\sqrt{1+16x^2}+4x|\right)\Big|_0^1 = \dfrac{1}{8}\cdot\left(4\sqrt{17} + \ln(4+\sqrt{17})\right) = \dfrac{1}{2}\sqrt{17} + \dfrac{1}{8}\ \ln(4+\sqrt{17})$.

10. $\dfrac{181}{9}$. Note that $1 + y'(x)^2 = 1 + \left(x^6 - \dfrac{1}{2} + \dfrac{1}{16x^6}\right) = \left(x^3 + \dfrac{1}{4x^3}\right)^2$.

    It follows that the expression for the arc length is given by $L = \displaystyle\int_1^3 \left(x^3 + \frac{1}{4x^3}\right)\ dx$, giving the stated result.

11. 4π. Here, $x'(t) = -2\sin t,\ \ y'(t) = 2\cos t$ so that the length of the arc is given by $L = \displaystyle\int_0^{2\pi} \sqrt{4\sin^2 t + 4\cos^2 t}\ dt =$ $\displaystyle\int_0^{2\pi} \sqrt{4 \cdot 1}\ dt = 2 \cdot 2\pi = 4\pi.$

12. 2π. Now, $x'(t) = -\sin t,\ \ y'(t) = -\cos t$ so that the length of the arc is given by $L = \displaystyle\int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\ dt =$ $\displaystyle\int_0^{2\pi} \sqrt{1}\ dt = 1 \cdot 2\pi = 2\pi.$

13. $\sqrt{2}$. In this example, $x'(t) = 1,\ \ y'(t) = -1$ so that the length of the arc is given by $L = \displaystyle\int_0^1 \sqrt{1+1}\ dt =$ $\displaystyle\int_0^1 \sqrt{2}\ dt = \sqrt{2}.$

14. $\frac{3}{2}$. Use the Fundamental Theorem of Calculus to show that $y'(x) = \sqrt{x^2 - 1}$. Then, $\sqrt{1 + (y'(x))^2} =$ $\sqrt{x^2} = x$. So, $L = \displaystyle\int_1^2 x\ dx = \frac{3}{2}.$

15. $\sqrt{2}$. Once again, use the Fundamental Theorem of Calculus to show

    that $y'(x) = \sqrt{\cos 2x}$. Then, $\sqrt{1 + (y'(x))^2} = \sqrt{1 + \cos 2x} = \sqrt{2\cos^2 x}$, by a trig. identity (which one?). So, $L = \displaystyle\int_0^{\pi/2} \sqrt{2\cos^2 x}\ dx = \sqrt{2}\int_0^{\pi/2} \cos x\ dx = \sqrt{2}.$

16. 10.602. See Example 448 except that we solve for x in terms of $y > 0$ (because the given interval is a $y-$interval). The length L is then given by doubling the basic integral over half the curve, that is,

    $$L = 2\ \int_{-1}^{1} \sqrt{\frac{4 - 3y^2}{4(1-y^2)}}\ dy \approx \int_{-0.99}^{0.99} \sqrt{\frac{4 - 3y^2}{4(1-y^2)}}\ dy \approx 5.3010.$$

17. 3.3428. The length L is given by an integral of the form $L = \displaystyle\int_1^4 \frac{\sqrt{1+x^2}}{x}\ dx.$

    We use a trigonometric substitution $x = \tan\theta,\ dx = sec^2\theta\ d\theta$. Then, $\sqrt{1+x^2} = \sec\theta$ and an antiderivative is given by

    $$\int \frac{sec^3\theta}{\tan\theta}\ d\theta = \int \frac{sec\theta}{\tan\theta}\ \left(1 + \tan^2\theta\right)\ d\theta = \int \frac{\sec\theta}{\tan\theta}\ d\theta + \int \sec\theta\ \tan\theta\ d\theta =$$
    $$\int \csc\theta\ d\theta + \sec\theta = \ln|\csc\theta - \cot\theta| + \sec\theta.$$

    Since $x = \tan\theta$ it follows that $\csc\theta = \dfrac{\sec\theta}{\tan\theta} = \dfrac{\sqrt{1+x^2}}{x},\ \cot\theta = \dfrac{1}{x}$. So an antiderivative is given by $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\ dx = \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \sqrt{1+x^2}$. Finally, we see that $L = \displaystyle\int_1^4 \frac{\sqrt{1+x^2}}{x}\ dx = \left(\ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \sqrt{1+x^2}\right)$
    $= \left(\ln\left|\dfrac{\sqrt{17}}{4} - \dfrac{1}{4}\right| + \sqrt{17}\right) - \left(\ln\left|\sqrt{2} - 1\right| + \sqrt{2}\right) = \sqrt{17} - \sqrt{2} + \ln\left|\dfrac{\sqrt{17} - 1}{4}\right| - \ln\left|\sqrt{2} - 1\right| \approx 3.3428.$

18. $\ln(1 + \sqrt{2}) \approx 0.8813$. Here, $y'(x) = \tan x$ and so $\sqrt{1 + y'(x)^2} = \sqrt{1 + \tan^2 x} = \sqrt{\sec^2 x} = \sec x$. So, the arc length is given by

    $$\int_0^{\pi/4} \sec x\ dx = \ln|\sec x + \tan x|\Big|_0^{\pi/4} = \ln(\sqrt{2} + 1) - \ln(1 + 0) = \ln(1 + \sqrt{2}).$$

19. $\frac{1}{2}\sqrt{5} - \frac{1}{4}\ln(\sqrt{5} - 2)$. See Exercise 7, above for the evaluation of the integral. Note that $-\dfrac{1}{4}\ln(\sqrt{5} - 2) =$ $\dfrac{1}{4}\ln(\sqrt{5} + 2)$.

20. Follow the hints.

7.11 Exercise Set 44 (page 470)

1. 3.75. Set m_1 at $x_1 = 0$ and m_2 at $x_2 = 5$. Then $\overline{x} = \frac{m_1x_1+m_2x_2}{m_1+m_2} = 3.75$.
2. 1.33. Set m_1 at $x_1 = 0$, m_2 at $x_2 = 1$ and m_3 at $x_3 = 2$. Then $\overline{x} = 1.33$.
3. $(\overline{x}.\overline{y}) = \left(\frac{5}{12}, \frac{1}{3}\right)$. Note that $\overline{x} = \frac{\sum m_i x_i}{\sum m_i}$ and $\overline{y} = \frac{\sum m_i y_i}{\sum m_i}$ where (x_i, y_i) are the coordinates of m_i. In this case, $\overline{x} = \frac{4 \cdot 0 + 5 \cdot 1}{12} = \frac{5}{12}$. Similarly, $\overline{y} = \frac{4 \cdot 1 + 0}{12} = \frac{1}{3}$. Note that even though the system of masses is at the vertices of an isosceles triangle, the center of mas is not along the bisector of the right-angle (which is the line of symmetry). This doesn't contradict the Symmetry Principle since the masses are not all the same!
4. $(\overline{x}.\overline{y}) = \left(1, \frac{\sqrt{3}}{3}\right)$. As before, $\overline{x} = \frac{\sum m_i x_i}{\sum m_i}$ and $\overline{y} = \frac{\sum m_i y_i}{\sum m_i}$ where (x_i, y_i) are the coordinates of m_i. Here, $\overline{x} = \frac{0+6+3}{9} = 1$. Similarly, $\overline{y} = \frac{0+0+3\sqrt{3}}{9} = \frac{\sqrt{3}}{3}$. In this exercise the masses are all the same and the triangle is equilateral, so (by the Symmetry Principle) the center of mass must lie along the line of symmetry (which it does), that is, it must lie on the line $x = 1$ which bisects the base of the triangle.
5. $\left(0, \frac{4R}{3\pi}\right)$. The total mass $m = \frac{\pi R^2 \delta}{2}$ since we are dealing with one-half the area of a circle and δ is constant. This use of geometry saves us from actually *calculating* the mass integral which looks like $\int_{-R}^{R} \sqrt{R^2 - x^2}\ \delta\ dx$. Next, the moment about the $y-$axis is given by $M_y = \int_{-R}^{R} \overline{x}_{slice}\ \delta\ dA = \delta \int_{-R}^{R} x\sqrt{R^2 - x^2}\ dx$. Now, let $x = R\cos\theta$, etc. But even simpler is the remark that the integrand, $x\sqrt{R^2 - x^2}$, is an *odd function* defined over a symmetric interval and so its integral must be zero. Either way, this gives $M_x = \delta \int_{-R}^{R} x\sqrt{R^2 - x^2}\ dx = 0$ and so $\overline{x} = 0$, *i.e.,* the center of mass lies along the axis of symmetry (which is the $y-$axis, since δ is constant). Similarly we find the moment about the $x-$axis, $M_x = \int_{-R}^{R} \overline{y}_{slice}\ \delta\ dA = \frac{\delta}{2} \int_{-R}^{R} \left(R^2 - x^2\right)\ dx = \frac{\delta}{2}\frac{4R^3}{3} = \frac{2R^3\delta}{3}$. It follows that the $y-$coordinate, $\overline{y}$, of the center of mass is given by $\overline{y} = \frac{M_x}{m} = \frac{2R^3\delta}{3} \cdot \frac{2}{\pi R^2 \delta} = \frac{4R}{3\pi}$.
6. $\left(\frac{b}{2}, \frac{h}{2}\right)$. Use of geometry shows us that the total mass is its area times its density, that is, $m = bh\delta$. Next, $\overline{x} = \int_0^b \overline{x}_{slice}\ \delta\ dA = \frac{1}{bh\delta} \int_0^b xh\delta\ dx = \frac{b}{2}$. Similarly, $\overline{y} = \int_0^b \overline{y}_{slice}\ \delta\ dA = \frac{1}{bh\delta} \int_0^b \frac{h}{2} h\delta\ dx = \frac{h}{2}$.
7. $\left(0, \frac{2}{3}\right)$. The region is an inverted triangle with a vertex at the origin and opposite side equal to 2 units. Its total mass is its area times its density, which, in this case, is δ. So, $m = \delta$. Let $f(x) = 1$ and $g(x) = 1 - |x|$, over $[-1, 1]$. Note that the region can be described by means of these two graphs. Also, $f(x) \geq g(x)$ and so we can use the formulae already derived for the center of mass. So, $\overline{x} = \frac{1}{\delta} \int_{-1}^{1} \overline{x}_{slice}\ \delta\ dA = \frac{1}{\delta} \int_{-1}^{1} x\ (1 - |x|)\ \delta\ dx = \frac{1}{\delta} \int_{-1}^{0} x\ (1 + x)\ \delta\ dx + \frac{1}{\delta} \int_{0}^{1} x\ (1 - x)\ \delta\ dx = 0$. Next, $\overline{y} = \frac{1}{\delta} \int_{-1}^{1} \overline{y}_{slice}\ \delta\ dA = \frac{1}{\delta} \int_{-1}^{1} \left(\frac{1 + |x|}{2}\right)\ (1 - |x|)\ \delta\ dx = \frac{1}{2\delta} \int_{-1}^{1} \left(1 - x^2\right)\ \delta\ dx = \frac{2}{3}$.
8. $\left(-\frac{1}{12}, \frac{1}{3}\right)$. The total mass is $m = \int_{-1}^{1} \delta(x)\ dx = \int_{-1}^{1} (1 - x)\ dx = 2$. Next, $\overline{x} = \frac{1}{2} \int_{-1}^{1} \overline{x}_{slice}\ \delta\ dA =$

$\frac{1}{2}\int_{-1}^{1} x\,(1-|x|)\,(1-x)\,dx =$
$\frac{1}{2}\int_{-1}^{0} x\,(1+x)\,(1-x)\,dx + \frac{1}{2}\int_{0}^{1} x\,(1-x)\,(1-x)\,dx = \left(\frac{1}{2}\right)\cdot\left(\frac{-1}{6}\right) = -\frac{1}{12}$.
Similarly, $\overline{y} = \frac{1}{2}\int_{-1}^{1} \overline{y}_{slice}\;\delta\;dA = \frac{1}{2}\int_{-1}^{1}\left(\frac{1+|x|}{2}\right)\,(1-|x|)\,(1-x)\,dx = \frac{1}{4}\int_{-1}^{1}(1-x^2)\,(1-x)\,dx = \frac{1}{3}$.

9. $\left(\frac{3}{5},\frac{3}{8}\right)$. The total mass, $m = \delta\int_{0}^{1}\sqrt{x}\,dx = \frac{2\delta}{3}$. So,
$\overline{x} = \frac{3}{2\delta}\int_{0}^{1}\overline{x}_{slice}\;\delta\;dA = \frac{3}{2}\int_{0}^{1} x\,\sqrt{x}\,dx = \frac{3}{2}\int_{0}^{1} x^{3/2}\,dx = \frac{3}{5}$.
Furthermore,
$\overline{y} = \frac{3}{2\delta}\int_{0}^{1}\overline{y}_{slice}\;\delta\;dA = \frac{3}{2}\int_{0}^{1}\frac{\sqrt{x}}{2}\,\sqrt{x}\,dx = \frac{3}{4}\int_{0}^{1} x\,dx = \frac{3}{8}$.

10. $\left(\frac{3}{2},\frac{3}{10}\right)$. The total mass, $m = \delta\int_{0}^{2}\frac{x^2}{4}\,dx = \frac{2\delta}{3}$. So,
$\overline{x} = \frac{3}{2\delta}\int_{0}^{2}\overline{x}_{slice}\;\delta\;dA = \frac{3}{2\delta}\int_{0}^{2} x\;\delta\;\frac{x^2}{4}\,dx = \frac{3}{8}\int_{0}^{2} x^3\,dx = \frac{3}{2}$.
Similarly,
$\overline{y} = \frac{3}{2\delta}\int_{0}^{2}\overline{y}_{slice}\;\delta\;dA = \frac{3}{2\delta}\int_{0}^{2}\delta\;\frac{x^2}{8}\;\frac{x^2}{4}\,dx = \frac{3}{2\cdot 32}\int_{0}^{2} x^4\,dx = \frac{3}{10}$.

11. $\left(\frac{1}{2},\frac{\pi}{4}\right)$. The graph of this function is positive on $[0,1]$. The total mass, $m = \delta\int_{0}^{1} 2\,\sin(\pi x)\,dx =$
$2\delta\left(-\frac{\cos(\pi x)}{\pi}\right)\Big|_0^1 = \frac{4\delta}{\pi}$. So,
$\overline{x} = \frac{\pi}{4\delta}\int_{0}^{1}\overline{x}_{slice}\;\delta\;dA = \frac{\pi}{4\delta}\int_{0}^{1} x\;2\sin(\pi x)\;\delta\;dx$
$= \frac{\pi}{2}\int_{0}^{1} x\;\sin(\pi x)\;\;dx = \frac{1}{2}$.
Similarly,
$\overline{y} = \frac{\pi}{4\delta}\int_{0}^{1}\overline{y}_{slice}\;\delta\;dA = \frac{\pi}{4\delta}\int_{0}^{1}\frac{2\sin(\pi x)}{2}\;\delta\;2\sin(\pi x)\,dx = \frac{\pi}{2}\int_{0}^{1}\sin^2(\pi x)\,dx = \frac{\pi}{2}\int_{0}^{1}\frac{1-\cos(2\pi x)}{2}\,dx =$
$\frac{\pi}{4}$. Note the symmetry about the line $x = 1/2$ so that the center of mass must lie along this line.

12. $\left(\frac{2(e^2-1)}{1+e^2},\frac{3e^4+1}{8(1+e^2)}\right)$. The total mass is $m = \int_{0}^{2} x\;\delta\;e^x\,dx = \delta\int_{0}^{2} x\,e^x\,dx = (e^2+1)\delta$. Next,
$\overline{x} = \frac{1}{1+e^2}\int_{0}^{2} x^2\,e^x\,dx = \frac{2e^2-2}{1+e^2}$. Finally, one more application of
Integration by Parts (or the Table Method)
$\overline{y} = \frac{1}{2(1+e^2)}\int_{0}^{2} x\,e^{2x}\,dx = \frac{3e^4+1}{8(1+e^2)}$.

13. $\left(0,\frac{11}{3\sqrt{3}+2\pi}\right)$. Since $\delta = 2$, the total mass is $m = \int_{-1}^{1}\sqrt{4-x^2}\;\delta\;dx = 2\int_{-1}^{1}\sqrt{4-x^2}\,dx = 2\sqrt{3}+\frac{4\pi}{3}$, where we used the trig. substitution $x = 2\sin\theta$, etc.

The geometric area is not so easy to calculate in this case, so we return to the integral definition. Now, because of symmetry about the line $x = 0$ and since δ is constant, we must have

$\overline{x} = 0$.

Furthermore,
$\overline{y} = \dfrac{3}{6\sqrt{3}+4\pi} \int_{-1}^{1} \dfrac{\sqrt{4-x^2}}{2} \sqrt{4-x^2}\ 2\ dx = \dfrac{3}{6\sqrt{3}+4\pi} \int_{-1}^{1} (4-x^2)\ dx = \dfrac{3}{6\sqrt{3}+4\pi} \cdot \dfrac{22}{3} = \dfrac{11}{3\sqrt{3}+2\pi}$.

14. $\left(\dfrac{6}{5}, -\dfrac{2}{5}\right)$, see the solved example. The total mass is
$m = \int_0^2 (6x - 3x^2)\ 2x\ dx = 8$. Next, $\overline{x} = \dfrac{1}{8} \int_0^2 x\ (6x - 3x^2)\ 2x\ dx = \dfrac{6}{5}$. Similarly,
$\overline{y} = \dfrac{1}{8} \int_0^2 \dfrac{x^2 - 2x}{2} (6x - 3x^2)\ 2x\ dx = -\dfrac{2}{5}$.

7.12 Chapter Exercises (page 472)

1. $\dfrac{\pi}{3} = \int_0^1 \pi\ y^2\ dy = \int_0^1 2\pi\ x(1-x)\ dx$.
2. $\dfrac{\pi}{3} = \int_0^1 \pi\ x^2\ dx = \int_0^1 2\pi\ y(1-y)\ dy$.
3. $\dfrac{\pi}{3} = \int_0^1 \pi\ y^2\ dy = \int_0^{1/2} \pi\ (1-4x^2)\ dx$.
4. $\dfrac{4\pi}{3} = \int_0^1 4\pi\ x^2\ dx = \int_0^2 \pi \left(1 - \dfrac{y^2}{4}\right)\ dy$.
5. $\frac{16\pi}{3}$ (you'll need two terms if you use horizontal slices here).

$$\begin{aligned} \frac{16\pi}{3} &= \int_0^2 2\pi\ x^2\ dx \\ &= \frac{3\pi}{4} \int_0^2 y^2\ dy + \pi \int_2^4 \left(4 - \frac{y^2}{4}\right)\ dy. \end{aligned}$$

6. $\dfrac{8\pi}{5} = 4\pi \int_0^1 y^{3/2}\ dy = \int_{-1}^1 \left(1 - x^4\right)\ dx$.
7. $\dfrac{\pi^2}{2} = \pi \int_0^\pi \sin^2 x\ dx = 4\pi \int_0^1 y\ \operatorname{Arcsin} y\ dy$.
8. $\pi^2 - 2\pi = 2\pi \int_0^{\pi/2} x \cos x\ dx = \pi \int_0^1 \operatorname{Arccos}^2 y\ dy$. This last integral is very hard to evaluate! Try the substitution $y = \cos u,\ dy = -\sin u\ du$. Then use the Table method and then back-substitute. The first intergal in x is evaluated using the Table method.
9. $\dfrac{\pi}{4}(e^2 - 1) = \pi \int_0^1 x^2 e^{2x}\ dx$. You can't use horizontal slices here because it is almost impossible to solve for x in terms of y in the expression for $y = xe^x$. Use the Table method to evaluate the integral.
10. $2\pi \left(1 - \dfrac{5}{e^2}\right) = 2\pi \int_0^2 x^2 e^{-x}\ dx$. Use the Table method to evaluate the integral.
11. $\dfrac{3\pi}{10} = \pi \int_0^1 \left(x - x^4\right)\ dx = 2\pi \int_0^1 \left(y^{3/2} - y^3\right)\ dy$.

12. $\dfrac{4223\pi}{5670} = \pi \int_0^{1/3} \left(1 - (x^3 - 3x + 1)^2\right)\, dx + \pi \int_{1/3}^{1} \left(1 - x^6\right)\, dx.$

7.12.1 Chapter Exercises, continued (page 472)

Please add a constant of integration, C, after every indefinite integral!

1. $\cos^2 x - \sin^2 x = \cos 2x$. Use the identity $\cos(A + B) = \cos A \cos B - \sin A \sin B$ with $A = B = x$.
2. $\cos^4 x - \sin^4 x = \cos 2x$. This is because
$\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) = (\cos^2 x - \sin^2 x)(1) = \cos 2x$.
3. $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$. Use the same idea as the preceding one except that now, $\sec^2 x - \tan^2 x = 1$.
4. $\sqrt{1 + \cos x} = \sqrt{2} \cdot \cos\left(\dfrac{x}{2}\right)$, if $-\pi \le x \le \pi$. Replace x by $x/2$ in the identity $\frac{1+\cos 2x}{2} = \cos^2 x$, and then extract the square root. Note that whenever $-\pi/2 \le \theta \le \pi/2$, we have $\cos\theta \ge 0$. Consequently, if $-\pi \le \theta \le \pi$, then $\cos\frac{\theta}{2} \ge 0$. This explains that the positive square root of $\cos^2\frac{\theta}{2}$ is $\cos\frac{\theta}{2}$.
5. $\sqrt{1 - \cos x} = \sqrt{2} \cdot \sin\left(\dfrac{x}{2}\right)$, if $0 \le x \le 2\pi$. Replace x by $x/2$ in the identity $\frac{1-\cos 2x}{2} = \sin^2 x$.
6. $\sqrt{1 + \cos 5x} = \sqrt{2} \cdot \cos\left(\dfrac{5x}{2}\right)$, if $-\pi \le 5x \le \pi$. Replace x by $5x/2$ in the identity $\frac{1+\cos x}{2} = \cos^2 x$.
7. $\int_0^2 (2x - 1)\, dx = 2$, since the function is linear (a polynomial of degree 1). In this case, the Trapezoidal Rule always gives the Actual value.
8. $\int_0^4 (3x^2 - 2x + 6)\, dx = 72$, using Simpson's Rule with $n = 6$. Once again, since the integrand is a quadratic function, Simpson's Rule is exact and always gives the Actual value.
9. $\int_{-\pi}^{\pi} (\cos^2 x + \sin^2 x)\, dx = 2\pi$. The Trapezoidal Rule with $n = 6$ and the Actual value agree exactly, since the integrand is equal to 1.
10. $\int_{-\pi}^{\pi} (\cos^2 x - \sin^2 x)\, dx = 0$, using Simpson's Rule with $n = 6$. The exact answer, obtained by direct integration, is 0, since the integrand is equal to $\cos 2x$. Note that the two values agree!
11. $\int_0^1 e^{-x^2}\, dx \approx 1.4628$, using Simpson's Rule with $n = 6$. The Actual value is 1.462651746
12. $\int_{-1}^{2} \dfrac{1}{1 + x^6}\, dx \approx 1.82860$, using Simpson's Rule with $n = 4$. The Actual value is ≈ 1.94476. Don't try to work it out!
13. $\int_{-2}^{2} \dfrac{x^2}{1 + x^4}\, dx \approx 1.221441$, using the Trapezoidal Rule with $n = 6$. The exact answer obtained by direct integration is 1.23352.
14. $\int_1^2 (\ln x)^3\, dx \approx 0.10107$, using Simpson's Rule with $n = 6$. The Actual value is $2\ln^3 2 - 6\ln^2 2 + 12\ln 2 - 6 \approx 0.101097387$.

7.12.2 Review Exercises on Integration (page 473)

15. $\int \sqrt{3x + 2}\, dx = \dfrac{2}{9}\left(\sqrt{3x + 2}\right)^3.$

Let $u = 3x + 2$.

16. $\displaystyle\int \frac{1}{x^2 + 4x + 4}\, dx = -\frac{1}{x+2}$.

Note that $x^2 + 4x + 4 = (x+2)^2$.
Then let $u = x + 2,\ du = dx$.

17. $\displaystyle\int \frac{dx}{(2x-3)^2} = -\frac{1}{2\,(2x-3)}$.

Let $u = 2x - 3,\ du = 2dx$, and so $dx = du/2$.

18. $\displaystyle\int \frac{dx}{\sqrt{a+bx}} = 2\frac{\sqrt{a+bx}}{b}$.

Let $u = a + bx,\ du = bdx$, and $dx = du/b$, if $b \neq 0$.

19. $\displaystyle\int (\sqrt{a} - \sqrt{x})^2\, dx = ax - \frac{4\sqrt{a}}{3}\left(\sqrt{x}\right)^3 + \frac{1}{2}x^2$.

Expand the integrand and integrate term-by-term.

20. $\displaystyle\int \frac{x\, dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2}$.

Let $u = a^2 - x^2$. Then $du = -2xdx$ and $x\, dx = -du/2$.

21. $\displaystyle\int x^2\sqrt{x^3+1}\, dx = \frac{2}{9}\left(\sqrt{x^3+1}\right)^3$.

Let $u = x^3 + 1,\ du = 3x^2\, dx$, so that $x^2\, dx = du/3$.

22. $\displaystyle\int \frac{(x+1)}{\sqrt[3]{x^2+2x+2}}\, dx = \frac{3}{4}\left(\sqrt[3]{(x^2+2x+2)}\right)^2$.

Let $u = x^2 + 2x + 2,\ du = (2x+2)\, dx = 2(x+1)\, dx$. So, $(x+1)\, dx = du/2$.

23. $\displaystyle\int (x^4 + 4x^2 + 1)^2(x^3 + 2x)\, dx = \frac{1}{12}(x^4 + 4x^2 + 1)^3$

Let $u = x^4 + 4x^2 + 1,\ du = (4x^3 + 8x)\, dx = 4(x^3 + 2x)\, dx$ and so, $(x^3 + 2x)\, dx = du/4$.

24. $\displaystyle\int x^{-1/3}\sqrt{x^{2/3} - 1}\, dx = \left(\sqrt{x^{2/3} - 1}\right)^3$.

Let $u = x^{2/3} - 1$. Then $du = (2/3)x^{-1/3}\, dx$, or $x^{-1/3}\, dx = 3\, du/2$.

25. $\displaystyle\int \frac{2x\, dx}{(3x^2-2)^2} = -\frac{1}{3\,(3x^2-2)}$

Let $u = 3x^2 - 2,\ du = 6x\, dx$ and so $2x\, dx = du/3$.

26. $\displaystyle\int \frac{dx}{4x+3} = \frac{1}{4}\ln|4x+3|$

Let $u = 4x + 3,\ du = 4dx$ so that $dx = du/4$.

27. $\displaystyle\int \frac{x\, dx}{2x^2 - 1} = \frac{1}{4}\ln\left|2x^2 - 1\right|$

Let $u = 2x^2 - 1,\ du = 4x\, dx$ so that $x\, dx = du/4$.

28. $\displaystyle\int \frac{x^2\, dx}{1+x^3} = \frac{1}{3}\ln\left|1 + x^3\right|$

Let $u = 1 + x^3,\ du = 3x^2\, dx$ so that $x^2\, dx = du/3$.

29. $\displaystyle\int \frac{(2x+3)\ dx}{x^2+3x+2} = \ln\left|x^2+3x+2\right|$

Let $u = x^2+3x+2,\ du = (2x+3)\ dx$.

30. $\displaystyle\int \sin(2x+4)\ dx = -\frac{1}{2}\cos(2x+4)$

Let $u = 2x+4,\ du = 2\ dx$, and $dx = du/2$.

31. $\displaystyle\int 2\cos(4x+1)\ dx = \frac{1}{2}\sin(4x+1)$

Let $u = 4x+1,\ du = 4\ dx$, and $dx = du/4$.

32. $\displaystyle\int \sqrt{1-\cos 2x}\ dx = \sqrt{2}\cos x.$

Note that $\frac{1-\cos 2x}{2} = \sin^2 x$. The result follows upon the extraction of a square root. In actuality, we are assuming that $\sqrt{\sin^2 x} = |\sin x| = \sin x$, here (or that $\sin x \geq 0$ over the region of integration).

33. $\displaystyle\int \sin\frac{3x-2}{5}\ dx = -\frac{5}{3}\cos\left(\frac{3x-2}{5}\right)$

Let $u = \frac{3x-2}{5},\ du = \frac{3dx}{5}$. Then $dx = \frac{5du}{3}$.

34. $\displaystyle\int x\cos ax^2\ dx = \frac{1}{2}\frac{\sin ax^2}{a}$

Assume $a \neq 0$. Let $u = ax^2,\ du = 2ax\ dx$, so that $x\ dx = du/2a$.

35. $\displaystyle\int x\sin(x^2+1)\ dx = -\frac{1}{2}\cos\left(x^2+1\right)$

Let $u = x^2+1,\ du = 2x\ dx$. Then $x\ dx = du/2$.

36. $\displaystyle\int \sec^2\frac{\theta}{2}\ \mathrm{d}\theta = 2\tan\frac{1}{2}\theta$

Let $u = \theta/2,\ du = d\theta/2$. The result follows since $\int \sec^2 u\ du = \tan u$.

37. $\displaystyle\int \frac{\mathrm{d}\theta}{\cos^2 3\theta} = \frac{1}{3}\tan 3\theta$

The integrand is equal to $\sec^2 3\theta$. Now let $u = 3\theta,\ du = 3d\theta$.

38. $\displaystyle\int \frac{\mathrm{d}\theta}{\sin^2 2\theta} = -\frac{1}{2}\cot 2\theta$

The integrand is equal to $\csc^2 2\theta$. Now let $u = 2\theta,\ du = 2d\theta$, and note that $\int \csc^2 u\ du = -\cot u$.

39. $\displaystyle\int x\csc^2(x^2)\ dx = -\frac{1}{2}\cot x^2$

Let $u = x^2,\ du = 2x\ dx$, so that $x\ dx = du/2$. Note that $\int \csc^2 u\ du = -\cot u$.

40. $\displaystyle\int \tan\frac{3x+4}{5}\ dx = \frac{5}{3}\ln\left|\sec\frac{3x+4}{5}\right|$

Let $u = \frac{3x+4}{5},\ du = 3dx/5$ and so $dx = 5du/3$. The result follows since $\int \tan u\ du = -\ln|\cos u| = \ln|\sec u|$.

41. $\displaystyle\int \frac{dx}{\tan 2x} = \frac{1}{2}\ln|\sin 2x|$

The integrand is equal to $\cot 2x$. Let $u = 2x,\ du = 2dx$. Then, $dx = du/2$, and since $\int \cot u\ du = \ln|\sin u|$, the result follows.

42. $\displaystyle\int \sqrt{1+\cos 5x}\ dx = \frac{2\sqrt{2}}{5}\sin\frac{5x}{2}$

Use the identity in Exercise 6, above. Since $\sqrt{1+\cos 5x} = \sqrt{2}\cdot\cos\left(\frac{5x}{2}\right)$ we let $u = \frac{5x}{2},\ du = 5dx/2$. Then $dx = 2du/5$ and the conclusion follows.

43. $\displaystyle\int \csc(x+\frac{\pi}{2})\ \cot(x+\frac{\pi}{2})\ dx = -\sec x$

Trigonometry tells us that $\sin(x+\frac{\pi}{2}) = \cos x$, and $\cos(x+\frac{\pi}{2}) = -\sin x$. Thus, by definition, $\csc(x+\frac{\pi}{2})\ \cot(x+\frac{\pi}{2}) = -\sec x\tan x$. On the other hand, $\int \sec x\tan x\ dx = \sec x$.

44. $\displaystyle\int \cos 3x\ \cos 4x\ dx = \frac{1}{2}\sin x + \frac{1}{14}\sin 7x$

Use the identity $\cos A\cos B = \frac{1}{2}\left(\cos(A-B)+\cos(A+B)\right)$, with $A = 4x$, $B = 3x$, and integrate the terms individually. This is also a " three-row problem" using the Table method in Integration by Parts and so you can use this alternate method as well.

45. $\displaystyle\int \sec 5\theta\tan 5\theta\ \mathrm{d}\theta = \frac{1}{5}\sec 5\theta$

Let $u = 5\theta,\ du = 5d\theta$. Then $d\theta = du/5$ and since $\int \sec u\tan u\ du = \sec u$, we have the result.

46. $\displaystyle\int \frac{\cos x}{\sin^2 x}\ dx = -\frac{1}{\sin x}$

The integrand is equal to $\cot x\csc x$. The result is now clear since
$\frac{1}{\sin x} = \csc x$.

47. $\displaystyle\int x^2\cos(x^3+1)\ dx = \frac{1}{3}\sin\left(x^3+1\right)$

Let $u = x^3+1,\ du = 3x^2\ dx$. Then $x^2\ dx = du/3$ and the answer follows.

48. $\displaystyle\int \sec\theta\ (\sec\theta+\tan\theta)\ \mathrm{d}\theta = \sec\theta+\tan\theta$

Expand the integrand and integrate it term-by-term. Use the facts
$\int \sec^2 u\ du = \tan u$, and $\int \sec u\tan u\ du = \sec u$

49. $\displaystyle\int (\csc\theta-\cot\theta)\csc\theta\ \mathrm{d}\theta = \csc\theta-\cot\theta = \frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}$

Expand the integrand and integrate it term-by-term. Use the facts
$\int \csc^2 u\ du = -\cot u$, and $\int \csc u\cot u\ du = -\csc u$. Rewrite your answer using the elementary functions sine and cosine.

50. $\displaystyle\int \cos^{-4} x\sin(2x)\ dx = \frac{1}{\cos^2 x}$

Write $\sin 2x = 2\sin x\cos x$ and simplify the integrand. Put the $\cos^3 x$-term in the denominator and then use the substitution $u = \cos x$,
$du = -\sin x\ dx$. Then $-2\int u^{-3}\ du = u^{-2}$ and the result follows.

51. $\displaystyle\int \frac{\tan^2\sqrt{x}}{\sqrt{x}}\ dx = 2\tan\sqrt{x} - 2\sqrt{x}$

Let $u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\ dx$, which gives $2\sqrt{x}\ du = dx$, or $dx = 2u\ du$. The integral becomes
$\displaystyle\int \frac{2u\ \tan^2 u}{u}\ du = \int 2\tan^2 u\ du = \int 2(\sec^2 u - 1)\ du = 2\tan u - 2u$, and the result follows.

52. $\displaystyle\int \frac{1+\sin 2x}{\cos^2 2x}\,dx = \frac{1}{2\cos 2x} + \frac{1}{2}\frac{\sin 2x}{\cos 2x}$

Note that the integrand is equal to $\sec^2 2x + \sec 2x \tan 2x$. Let $u = 2x$, $du = 2dx$, or $dx = du/2$. Use the facts $\int \sec^2 u\ du = \tan u$, and
$\int \sec u \tan u\ du = \sec u$. Now reduce your answer to elementary sine and cosine functions.

53. $\displaystyle\int \frac{dx}{\cos 3x} = \frac{1}{3}\ln|\sec 3x + \tan 3x|$

Let $u = 3x,\ du = 3dx,\ dx = du/3$, and use the result from Example 356, with $x = u$.

54. $\displaystyle\int \frac{dx}{\sin(3x+2)} = \frac{1}{3}\ln|\csc(3x+2) - \cot(3x+2)|$

Note that the integrand is equal to $\csc(3x+2)$. Now let $u = 3x+2$, $du = 3dx,\ \ dx = du/3$. The integral looks like
$(1/3)\int \csc u\ du = (1/3)\ln|\csc u - \cot u|$ and the result follows. This last integral is obtained using the method described in Example 356, but applied to these functions. See also Table 8.9.

55. $\displaystyle\int \frac{1+\sin x}{\cos x}\,dx = \ln|\sec x + \tan x| - \ln|\cos x|$

Break up the integrand into two parts and integrate term-by-term. Note that $-\ln|\cos x| = \ln|\sec x|$ so that the final answer may be written in the form
$\ln|\sec x + \tan x| + \ln|\sec x| = \ln\left|\sec^2 x + \tan x\ \sec x\right|$.

56. $\displaystyle\int (1+\sec\theta)^2\ \mathrm{d}\theta = \theta + 2\ln|\sec\theta + \tan\theta| + \tan\theta$

Expand the integrand and integrate term-by-term.

57. $\displaystyle\int \frac{\csc^2 x\ dx}{1+2\cot x} = -\frac{1}{2}\ln|1+2\cot x|$

Let $u = 1+2\cot x,\ du = -2\csc^2 x\ dx$. So, $\csc^2\ dx = -\frac{du}{2}$. The integral now becomes $(-1/2)\displaystyle\int \frac{du}{u} = -(1/2)\ln|u|$.

58. $\displaystyle\int e^x\ \sec e^x\ dx = \ln|\sec(e^x) + \tan(e^x)|$

Let $u = e^x,\ du = e^x\ dx$, and use Example 356.

59. $\displaystyle\int \frac{dx}{x\ln x} = \ln|\ln x|$

Let $u = \ln x,\ du = \frac{dx}{x}$. The integral looks like $\int \frac{du}{u} = \ln|u|$ and the result follows.

60. $\displaystyle\int \frac{\mathrm{d}t}{\sqrt{2-t^2}} = \operatorname{Arcsin}\frac{1}{2}\sqrt{2}t$

The integrand contains a square root of a difference of squares of the form $\sqrt{a^2-u^2}$ where $a = \sqrt{2}$, and $u = t$. Let $t = \sqrt{2}\sin\theta,\ dt = \sqrt{2}\cos\theta\ d\theta$. Since $\sqrt{2-t^2} = \sqrt{2}\cos\theta$, the integral looks like $\int d\theta = \theta = \operatorname{Arcsin}\frac{t}{\sqrt{2}}$.

61. $\displaystyle\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\operatorname{Arcsin}\frac{2}{3}\sqrt{3}x$

The integrand contains a square root of a difference of squares of the form $\sqrt{a^2-u^2}$ where $a = \sqrt{3}$, and $u = 2x$. Let $2x = \sqrt{3}\sin\theta,\ 2dx = \sqrt{3}\cos\theta\ d\theta$. Since $\sqrt{3-4x^2} = \sqrt{3}\cos\theta$, the integral looks like $\int(1/2)d\theta = \frac{\theta}{2} = \frac{1}{2}\operatorname{Arcsin}\frac{2x}{\sqrt{3}}$, which is equivalent to the answer.

62. $\displaystyle\int \frac{(2x+3)\ dx}{\sqrt{4-x^2}} = -2\sqrt{4-x^2} + 3\operatorname{Arcsin}\frac{1}{2}x$

Break up the integrand into two parts so that the integral looks like
$\displaystyle\int \frac{2x\ dx}{\sqrt{4-x^2}} + \int \frac{3}{\sqrt{4-x^2}}\ dx.$

Let $u = 4 - x^2,\ du = -2x\ dx$ in the first integral and $x = 2\sin\theta,\ dx = 2\cos\theta\ d\theta$ in the second integral. Then $\sqrt{4-x^2} = 2\cos\theta$ and the second integral is an Arcsine. The first is a simple substitution.

63. $\displaystyle\int \frac{dx}{x^2+5} = \frac{1}{5}\sqrt{5}\operatorname{Arctan}\frac{1}{5}x\sqrt{5}$

This integrand contains a sum of two squares. So let, $x = \sqrt{5}\tan\theta$, $dx = \sqrt{5}\sec^2\theta\ d\theta$. The integral becomes $\displaystyle\int \frac{\sqrt{5}\sec^2\theta\ d\theta}{5\sec^2\theta} = \frac{\sqrt{5}}{5}\int d\theta$ and the result follows since $\theta = \operatorname{Arctan}\frac{x}{\sqrt{5}}$, and $\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$.

64. $\displaystyle\int \frac{dx}{4x^2+3} = \frac{1}{6}\sqrt{3}\operatorname{Arctan}\frac{2}{3}\sqrt{3}x$

The integrand contains a sum of two squares, $a^2 + u^2$ where $a = \sqrt{3}$ and $u = 2x$. So let $2x = \sqrt{3}\tan\theta$, $2\ dx = \sqrt{3}\sec^2\theta\ d\theta$. The integral becomes
$\displaystyle\int \frac{(1/2)\sqrt{3}\sec^2\theta\ d\theta}{3\sec^2\theta} = \frac{\sqrt{3}}{6}\int d\theta$ and the result follows since $\theta = \operatorname{Arctan}\frac{2x}{\sqrt{3}}$.

65. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}} = \frac{1}{2}\operatorname{Arcsec}\frac{x}{2}$

The integrand contains a square root of a difference of two squares, $\sqrt{u^2-a^2}$ where $a = 2$ and $u = x$. So let $x > 2$ and $x = 2\sec\theta$, $dx = 2\sec\theta\ \tan\theta\ d\theta$. Moreover, $\sqrt{x^2-4} = 2\tan\theta$. The integral becomes
$\displaystyle\int \frac{2\sec\theta\ \tan\theta\ d\theta}{(2\sec\theta)(2\tan\theta)} = \frac{1}{2}\int d\theta$ and the result follows since $\theta = \operatorname{Arcsec}\frac{x}{2}$.

66. $\displaystyle\int \frac{dx}{x\sqrt{4x^2-9}} = \frac{1}{3}\operatorname{Arcsec}\frac{2x}{3}$

The integrand contains a square root of a difference of two squares, $\sqrt{u^2-a^2}$ where $a = 3$ and $u = 2x$. So let $x > 0$ and set $2x = 3\sec\theta$, $2dx = 3\sec\theta\ \tan\theta\ d\theta$. Moreover, $\sqrt{4x^2-9} = 3\tan\theta$. The integral becomes
$\displaystyle\int \frac{(3/2)\sec\theta\ \cdot\tan\theta\ d\theta}{(3/2)\sec\theta\cdot 3\tan\theta} = \frac{1}{3}\int d\theta$ and the result follows since $\theta = \operatorname{Arcsec}\frac{2x}{3}$.

67. $\displaystyle\int \frac{dx}{\sqrt{x^2+4}} = \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right|$
$= \ln\left|\sqrt{4+x^2} + x\right|$, where the "missing" constants are absorbed by the constant of integration, C.

The integrand contains a square root of a sum of two squares, $\sqrt{u^2+a^2}$ where $a = 2$ and $u = x$. Set $x = 2\tan\theta$, $dx = 2\sec^2\theta\ d\theta$. Moreover, $\sqrt{x^2+4} = 2\sec\theta$. The integral becomes
$\displaystyle\int \frac{2\sec^2\theta\ d\theta}{(2\sec\theta)} = \int \sec\theta\ d\theta$ and the result follows from Example 356.

68. $\displaystyle\int \frac{dx}{\sqrt{4x^2+3}} = \frac{1}{2}\ln\left|\frac{\sqrt{4x^2+3}}{\sqrt{3}} + \frac{2x}{\sqrt{3}}\right|$
$= \ln\left|\sqrt{4x^2+3} + 2x\right|$, where the "missing" constants are absorbed by the constant of integration, C.

The integrand contains a square root of a sum of two squares, $\sqrt{u^2+a^2}$ where $a = \sqrt{3}$ and $u = 2x$. Set $2x = \sqrt{3}\tan\theta$, $2\ dx = \sqrt{3}\sec^2\theta\ d\theta$. Moreover, $\sqrt{4x^2+3} = \sqrt{3}\sec\theta$. The integral becomes
$\displaystyle\int \frac{(\sqrt{3}/2)\sec^2\theta\ d\theta}{\sqrt{3}\sec\theta} = (1/2)\int \sec\theta\ d\theta$ and the result follows from Example 356, once again.

69. $\displaystyle\int \frac{dx}{\sqrt{x^2-16}} = \ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right|$
$= \ln\left|x + \sqrt{x^2-16}\right|$, where the "missing" constants are absorbed by the constant of integration, C.

The integrand contains a square root of a difference of two squares, $\sqrt{u^2-a^2}$ where $a=4$ and $u=x$. Set $x=4\sec\theta$, $dx=4\sec\theta\ \tan\theta\ d\theta$. Moreover, $\sqrt{x^2-16}=4\tan\theta$. The integral becomes $\displaystyle\int\frac{4\sec\theta\ \tan\theta\ d\theta}{4\tan\theta}=\int\sec\theta\ d\theta$ and the result follows from Example 356.

70. $\displaystyle\int\frac{e^x}{1+e^{2x}}\ dx=\operatorname{Arctan}(e^x)$

Use a substitution here: Let $u=e^x,\ du=e^x\ dx$. The integral now looks like $\displaystyle\int\frac{1}{1+u^2}\ du=\operatorname{Arctan}u$, where $u=e^x$.

71. $\displaystyle\int\frac{1}{x\sqrt{4x^2-1}}\ dx=\operatorname{Arcsec}2x$

The integrand contains a square root of a difference of two squares, $\sqrt{u^2-a^2}$ where $a=1$ and $u=2x$. So let $x>0$ and set $2x=\sec\theta$, $2dx=\sec\theta\ \tan\theta\ d\theta$. Moreover, $\sqrt{4x^2-1}=\tan\theta$. The integral becomes $\displaystyle\int\frac{(1/2)\sec\theta\ \cdot\tan\theta\ d\theta}{(1/2)\sec\theta\cdot\tan\theta}=\int d\theta$ and the result follows since $\theta=\operatorname{Arcsec}2x$.

72. $\displaystyle\int\frac{dx}{\sqrt{4x^2-9}}=\frac{1}{2}\ln\left|\frac{2x}{3}+\frac{\sqrt{4x^2-9}}{3}\right|=\ln\left|2x+\sqrt{4x^2-9}\right|$, where the "missing" constants are absorbed by the constant of integration, C.

The integrand contains a square root of a difference of two squares, $\sqrt{u^2-a^2}$ where $a=3$ and $u=2x$. So let $x>0$ and set $2x=3\sec\theta$, $2\ dx=3\sec\theta\ \tan\theta\ d\theta$. Moreover, $\sqrt{4x^2-9}=3\tan\theta$. The integral becomes $\displaystyle\int\frac{(3/2)\sec\theta\cdot\tan\theta\ d\theta}{3\tan\theta}=(1/2)\int\sec\theta\ d\theta$ and the result follows since $\int\sec\theta\ d\theta=\ln|\sec\theta+\tan\theta|$.

73. $\displaystyle\int e^{-3x}\ dx=-\frac{1}{3}e^{-3x}$

Let $u=-3x,\ du=-3\ dx$. Then $dx=-du/3$.

74. $\displaystyle\int\frac{dx}{e^{2x}}=-\frac{1}{2}e^{-2x}$

Write the integrand as e^{-2x} and let $u=-2x$, $du=-2\ dx$.

75. $\displaystyle\int(e^x-e^{-x})^2\ dx=\frac{1}{2}e^{2x}-2x-\frac{1}{2}e^{-2x}$

Expand the expression and integrate term-by-term using the two preceding exercises.

76. $\displaystyle\int xe^{-x^2}\ dx=-\frac{1}{2}e^{-x^2}$

Let $u=-x^2,\ du=-2x\ dx$ so that $x\ dx=-du/2$.

77. $\displaystyle\int\frac{\sin\theta\ \mathrm{d}\theta}{\sqrt{1-\cos\theta}}=2\sqrt{1-\cos\theta}$

Let $u=1-\cos\theta,\ du=\sin\theta\ d\theta$. We now have an easily integrable form.

78. $\displaystyle\int\frac{\cos\theta\ \mathrm{d}\theta}{\sqrt{2-\sin^2\theta}}=\operatorname{Arcsin}\left(\frac{1}{2}\sqrt{2}\sin\theta\right)$

Write $\theta=x$. Let $u=\sin x,\ du=\cos x\ dx$. The integral takes the form $\displaystyle\int\frac{du}{\sqrt{2-u^2}}$. Now set $u=\sqrt{2}\sin\theta$. (This is why we changed the name of the original variable to "x", so that we wouldn't get it confused with THIS θ). Then $du=\sqrt{2}\cos\theta\ d\theta$ and $\sqrt{2-u^2}=\sqrt{2}\cos\theta$ and the rest of the integration is straightforward. (Note: If you want, you could set $u=\sin\theta$ immediately and proceed as above without first having to let $\theta=x$ etc.)

79. $\displaystyle\int \frac{e^{2x}\ dx}{1+e^{2x}} = \frac{1}{2}\ln\left(1+e^{2x}\right)$

Let $u = 1+e^{2x},\ du = 2e^{2x}\ dx$. Now, the integral gives a natural logarithm

80. $\displaystyle\int \frac{e^{x}\ dx}{1+e^{2x}} = \text{Arctan}\left(e^{x}\right)$

Let $u = e^{x},\ du = e^{x}\ dx$. Now, the integral is of the form
$\displaystyle\int \frac{du}{1+u^2}$ and this gives an Arctangent.

81. $\displaystyle\int \frac{\cos\theta\ \mathrm{d}\theta}{2+\sin^2\theta} = \frac{1}{2}\sqrt{2}\,\text{Arctan}\left(\frac{1}{2}\sqrt{2}\sin\theta\right)$

Write $\theta = x$. Let $u = \sin x,\ du = \cos x\ dx$. The integral takes the form
$\displaystyle\int \frac{du}{2+u^2}$. Now set $u = \sqrt{2}\tan\theta$. (This is why we changed the name of the original variable to "x", so that we wouldn't get it confused with THIS θ). Then $du = \sqrt{2}\sec^2\theta\ d\theta$ and $2+u^2 = 2\sec^2\theta$ and the rest of the integration is straightforward. (Note: If you want, you could set $u = \sin\theta$ immediately and proceed as above without first having to let $\theta = x$ etc.)

82. $\displaystyle\int \sin^3 x\cos x\ dx = \frac{1}{4}\sin^4 x$

Let $u = \sin x,\ du = \cos x\ dx$.

83. $\displaystyle\int \cos^4 5x\sin 5x\ dx = -\frac{1}{25}\cos^5 5x$

Let $u = \cos 5x,\ du = -5\sin 5x\ dx$ or $\sin 5x = -du/5$. The rest is straightforward.

84. $\displaystyle\int (\cos\theta+\sin\theta)^2\ \mathrm{d}\theta = \theta - \cos^2\theta$
or, this can also be rewritten as $\theta + \sin^2\theta$

Expand and use the identities $\cos^2\theta+\sin^2\theta = 1$, along with
$\sin 2\theta = 2\sin\theta\ \cos\theta$. Then use the substitution $u = 2x$, or if you prefer, let $u = \sin\theta$, etc.

85. $\displaystyle\int \sin^3 x\ dx = -\frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x$

This is the case m is even ($m = 0$) and n is odd ($n = 3$) in the text.

86. $\displaystyle\int \cos^3 2x\ dx = \frac{1}{6}\cos^2 2x\sin 2x + \frac{1}{3}\sin 2x$

Let $u = 2x$. The new integral is in the case where m is odd ($m = 3$) and n is even ($n = 0$) in the text.

87. $\displaystyle\int \sin^3 x\cos^2 x\ dx = -\frac{1}{5}\sin^2 x\cos^3 x - \frac{2}{15}\cos^3 x$

This is the case m is even ($m = 2$) and n is odd ($n = 3$) in the text. To get the polynomial in $\cos x$ simply use the identities $\sin^2 x = 1-\cos^2 x$ whenever you see the $\sin^2 x-$term and expand and simplify.

88. $\displaystyle\int \cos^5 x\ dx = \frac{1}{5}\cos^4 x\sin x + \frac{4}{15}\cos^2 x\sin x + \frac{8}{15}\sin x$

This is the case m is odd ($m = 5$) and n is even ($n = 0$) in the text. To get the polynomial in $\sin x$ simply use the identities $\cos^2 x = 1-\sin^2 x$ whenever you see a $\cos^2 x-$term and then expand and simplify.

89. $\displaystyle\int \sin^3 4\theta\cos^3 4\theta\ \mathrm{d}\theta = -\frac{1}{24}\sin^2 4\theta\cos^4 4\theta - \frac{1}{48}\cos^4 4\theta$

Let $u = 4\theta$. Then the new integral is in the case where m is odd ($m = 3$) and n is odd ($n = 3$) in the text.

90. $\displaystyle\int \frac{\cos^2 x\ dx}{\sin x} = \cos x + \ln|\csc x - \cot x|$

Write $cos^2 x = 1 - \sin^2 x$, break up the integrand into two parts, and use the fact that $\displaystyle\int \csc x\ dx = \ln|\csc x - \cot x|$.

91. $\displaystyle\int \frac{\cos^3 x\ dx}{\sin x} = \frac{1}{2}\cos^2 x + \ln|\sin x|$

Write $cos^2 x = 1 - \sin^2 x$, break up the integrand into two parts. In one, use the fact that $\displaystyle\int \cot x\ dx = \ln|\sin x|$. In the other, use the substitution $u = \sin x$ in the other.

92. $\displaystyle\int \tan^2 x \sec^2 x\ dx = \frac{1}{3}\tan^3 x$

Let $u = \tan x,\ du = \sec^2 x\ dx$.

93. $\displaystyle\int \sec^2 x \tan^3 x\ dx = \frac{1}{4}\tan^4 x$

Let $u = \tan x,\ du = \sec^2 x\ dx$.

94. $\displaystyle\int \frac{\sin x\ dx}{\cos^3 x} = \frac{1}{2\cos^2 x}$

Let $u = \cos x,\ du = -\sin x\ dx$.

95. $\displaystyle\int \frac{\sin^2 x\ dx}{\cos^4 x} = \frac{1}{3}\tan^3 x$

The integrand is equal to $\tan^2 x\ \sec^2 x$. Now let $u = \tan x$.

96. $\displaystyle\int \sec^4 x\ dx = \frac{1}{3}\tan x\ \sec^2 x + \frac{2}{3}\tan x$

This is the case $m = 4,\ n = 0$ in the text. Note that $sec^2 x = 1 + \tan^2 x$. So, this answer is equivalent to $\tan x + \frac{\tan^3 x}{3}$ with the addition of a constant.

97. $\displaystyle\int \tan^2 x\ dx = \tan x\ -\ x$

The integrand is equal to $1 - \sec^2 x$. Now break up the integrand into two parts and integrate term-by-term.

98. $\displaystyle\int (1 + \cot\theta)^2\ \mathrm{d}\theta = -\cot\theta - \ln\left(1 + \cot^2\theta\right)$

Expand the integrand, use the identity $1 + \cot^2\theta = \csc^2\theta$ and integrate using the facts that $\int \csc^2 x\ dx = -\cot x$, and $\int \cot x\ dx = \ln|\sin x|$. Note that the second term may be simplified further using the fact that $\ln\left(1 + \cot^2\theta\right) = \ln\csc^2\theta = -\ln\sin^2\theta = -2\ln\sin\theta$.

99. $\displaystyle\int \sec^4 x \tan^3 x\ dx = \frac{1}{6}\tan^4 x\ \sec^2 x + \frac{1}{12}\tan^4 x$

This is the case $m = 4,\ n = 3$ in the text.

100. $\displaystyle\int \csc^6 x\ dx = -\frac{1}{5}\csc^4 x\ \cot x - \frac{4}{15}\csc^2 x\ \cot x - \frac{8}{15}\cot x$

Use the same ideas as in the case $m = 6,\ n = 0$ in the secant/tangent case.

101. $\displaystyle\int \tan^3 x\ dx = \frac{1}{2}\tan^2 x - \frac{1}{2}\ln\left(1 + \tan^2 x\right)$

This is the case $m = 0,\ n = 3$ in the text.

102. $\displaystyle\int \frac{\cos^2 t\ dt}{\sin^6 t} = -\frac{1}{5}\csc^2 t\ \cot^3 t - \frac{2}{15}\cot^3 t$

The integrand is equal to $\cot^2 x\ \csc^4 x$, and this corresponds to the case $m = 4,\ n = 2$ in the secant/tangent case.

103. $\displaystyle\int \tan\theta \csc\theta\ \mathrm{d}\theta = \ln|\sec\theta + \tan\theta|$

The integrand is really $\sec\theta$ in disguise!

104. $\displaystyle\int \cos^2 4x\ dx = \frac{1}{8}\cos 4x \sin 4x + \frac{1}{2}x$

Use the identity $\cos^2\square = \frac{1+\cos 2\square}{2}$, with $\square = 4x$. Then use a simple substitution $u = 8x$, and simplify your answer using the identity $\sin 8x = \sin(2\cdot 4x) = 2\sin 4x\ \cos 4x$.

105. $\displaystyle\int (1+\cos\theta)^2\ \mathrm{d}\theta = \frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\cos\theta\sin\theta$

Expand the integrand, use the identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ and integrate term-by-term.

106. $\displaystyle\int (1-\sin x)^3\ dx = \frac{5}{2}x + \frac{11}{3}\cos x - \frac{3}{2}\cos x\sin x + \frac{1}{3}\sin^2 x\cos x$

Expand the integrand, and integrate term-by-term using the identity $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, and the case $m = 0,\ n = 3$ in the text.
Recall that $(1-\square)^3 = 1 - 3\square + 3\square^2 - \square^3$

107. $\displaystyle\int \sin^4 x\ dx = -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\cos x\sin x + \frac{3}{8}x$

This is the case $m = 0,\ n = 4$ in the text.

108. $\displaystyle\int \sin^2 2x\cos^2 2x\ dx = -\frac{1}{8}\sin 2x\cos^3 2x + \frac{1}{16}\cos 2x\sin 2x + \frac{1}{8}x$

Let $u = 2x$ first. Then the new integral corresponds to the case $m = 2,\ n = 2$ in the text.

109. $\displaystyle\int \sin^4\theta\cos^2\theta\ \mathrm{d}\theta = -\frac{1}{6}\sin^3\theta\cos^3\theta - \frac{1}{8}\sin\theta\cos^3\theta + \frac{1}{16}\cos\theta\sin\theta + \frac{1}{16}\theta$

This is the case $m = 2,\ n = 4$ in the text.

110. $\displaystyle\int \cos^6 x\ dx = \frac{1}{6}\cos^5 x\sin x + \frac{5}{24}\sin x\cos^3 x + \frac{5}{16}\cos x\sin x + \frac{5}{16}x$

This is the case $m = 6,\ n = 0$ in the text.

111. $\displaystyle\int \cos x\sin 2x\ dx = -\frac{1}{6}\cos 3x - \frac{1}{2}\cos x$

You can use either Table integration in a three-row problem or the identity $\cos A\sin B = \frac{1}{2}\sin(A+B) - \frac{1}{2}\sin(A-B)$ to find this integral.

112. $\displaystyle\int \sin x\cos 3x\ dx = -\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x$

You can use either Table integration in a three-row problem or the identity $\cos A\sin B = \frac{1}{2}\sin(A+B) - \frac{1}{2}\sin(A-B)$ to find this integral.

113. $\displaystyle\int \sin 2x\sin 3x\ dx = \frac{1}{2}\sin x - \frac{1}{10}\sin 5x$

You can use either Table integration in a three-row problem or the identity
$\sin A \sin B = \dfrac{1}{2}\cos(A-B) - \dfrac{1}{2}\cos(A+B)$ to find this integral.

114. $\displaystyle\int \cos 2x \cos 4x \; dx = \frac{1}{4}\sin 2x + \frac{1}{12}\sin 6x$

You can use either Table integration in a three-row problem or the identity
$\cos A \cos B = \dfrac{1}{2}\cos(A-B) + \dfrac{1}{2}\cos(A+B)$ to find this integral.

115. $\displaystyle\int \sin^2 2x \cos 3x \; dx = \frac{1}{6}\sin 3x - \frac{1}{4}\sin x - \frac{1}{28}\sin 7x$

Use the identity $\sin^2 \square = \frac{1-\cos 2\square}{2}$ with $\square = 2x$. Break up the integrand into two parts, and integrate using the substitution $u = 4x$ and the identity
$\cos A \cos B = \dfrac{1}{2}\cos(A-B) + \dfrac{1}{2}\cos(A+B)$ to find the other integral.

116. $\displaystyle\int \sec x \csc x \; dx = \ln|\tan x|$

There are two VERY different ways of doing this one:

In the first proof we note the trigonometric identity (and this isn't obvious!),

$$\frac{\sec^2 x}{\tan x} = \frac{1}{\sin x \cos x} = \sec x \csc x,$$

so the result follows after using the substitution $u = \tan x$, $du = \sec^2 x \; dx$.

In the second proof we note that (and this isn't obvious either!)

$$\frac{1}{\sin x \cos x} = \frac{2}{\sin 2x} = 2\csc 2x.$$

Now use the substitution $u = 2x$, $du = 2dx$ and this new integral becomes
$2 \cdot \dfrac{1}{2}\displaystyle\int \csc u \; du = \ln|\csc u - \cot u|$. The answer is equivalent to
$\ln|\csc 2x - \cot 2x| + C$, because of the identity $1 - \cos 2x = 2\sin^2 x$.

117. $\displaystyle\int \frac{dx}{1-\cos x} = -\frac{1}{\tan \frac{1}{2}x} = -\cot \frac{x}{2}.$

Use the identity $1 - \cos 2\square = 2\sin^2 \square$, with $\square = \frac{x}{2}$. Then $\frac{1}{1-\cos x} = \frac{1}{2}\csc^2 \frac{x}{2}$. Let $u = x/2$, $du = dx/2$ and use the integral
$\int \csc^2 u \; du = -\cot u$ and simplify.

118. $\displaystyle\int \frac{dx}{\sqrt{2+2x-x^2}} = \operatorname{Arcsin} \frac{1}{3}\sqrt{3}\,(x-1)$

First, complete the square to find $2 + 2x - x^2 = 3 - (x-1)^2$. Next, let $a = \sqrt{3}$, $u = x - 1$. This integrand has a term of the form $\sqrt{a^2 - u^2}$. So we use the trigonometric substitution
$u = x - 1 = \sqrt{3}\sin\theta$, $dx = \sqrt{3}\cos\theta \; d\theta$.
Furthermore, $\sqrt{2 + 2x - x^2} = \sqrt{3}\cos\theta$. So, the integral now takes the form
$\displaystyle\int \frac{\sqrt{3}\cos\theta \; d\theta}{\sqrt{3}\cos\theta} = \int d\theta = \theta$
where $\theta = \operatorname{Arcsin} \frac{x-1}{\sqrt{3}}$ which is equivalent to the stated answer.

119. $\displaystyle\int \frac{dx}{\sqrt{1+4x-4x^2}} = \frac{1}{2}\operatorname{Arcsin}\sqrt{2}\left(x - \frac{1}{2}\right)$

First, complete the square to find $1 + 4x - 4x^2 = 2 - (2x - 1)^2$. Next, let $a = \sqrt{2},\ u = 2x - 1$. This integrand has a term of the form $\sqrt{a^2 - u^2}$. So we use the trigonometric substitution

$u = 2x - 1 = \sqrt{2}\sin\theta,\ 2dx = \sqrt{2}\cos\theta\ d\theta$

or, $dx = \dfrac{\sqrt{2}}{2}\cos\theta\ d\theta$.

Furthermore, $\sqrt{1 + 4x - 4x^2} = \sqrt{2}\cos\theta$. So, the integral now takes the form
$\dfrac{1}{2}\displaystyle\int \frac{\sqrt{2}\cos\theta\ d\theta}{\sqrt{2}\cos\theta} = \frac{1}{2}\int d\theta = \frac{\theta}{2}$
where $\theta = \operatorname{Arcsin}\frac{2x-1}{\sqrt{2}}$ which is equivalent to the stated answer.

120. $\displaystyle\int \frac{dx}{\sqrt{2 + 6x - 3x^2}} = \frac{1}{3}\sqrt{3}\operatorname{Arcsin}\frac{1}{5}\sqrt{15}\,(x - 1)$

This one is a little tricky: First, complete the square to find $2 + 6x - 3x^2 = 5 - 3(x - 1)^2$. But this is not exactly a difference of squares, yet! So we rewrite this as

$5 - 3(x - 1)^2 = 5 - (\sqrt{3}x - \sqrt{3})^2$,

and this is a difference of squares. Now let $a = \sqrt{5},\ u = \sqrt{3}x - \sqrt{3}$. We see that the integrand has a term of the form $\sqrt{a^2 - u^2}$. So we use the trigonometric substitution
$u = \sqrt{3}x - \sqrt{3} = \sqrt{5}\sin\theta$,
$\sqrt{3}\ dx = \sqrt{5}\cos\theta\ d\theta$

or, $dx = \dfrac{\sqrt{5}}{\sqrt{3}}\cos\theta\ d\theta$.

Furthermore, $\sqrt{2 + 6x - 3x^2} = \sqrt{5}\cos\theta$. So, the integral now takes the form
$\displaystyle\int \frac{\frac{\sqrt{5}}{\sqrt{3}}\cos\theta\ d\theta}{\sqrt{5}\cos\theta} = \frac{1}{\sqrt{3}}\int d\theta = \frac{\theta}{\sqrt{3}}$
where $\theta = \operatorname{Arcsin}\frac{\sqrt{3}x-\sqrt{3}}{\sqrt{5}}$ which is equivalent to the stated answer.

121. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 6x + 13}} = \ln\left|\frac{\sqrt{x^2 + 6x + 13}}{2} + \frac{x + 3}{2}\right|$

First, complete the square to find $x^2 + 6x + 13 = (x + 3)^2 + 4$. Next, let $a = 2,\ u = x + 3$. This integrand has a term of the form $\sqrt{a^2 + u^2}$. So we use the trigonometric substitution

$u = x + 3 = 2\tan\theta$,
$dx = 2\sec^2\theta\ d\theta$.

Furthermore, $\sqrt{x^2 + 6x + 13} = 2\sec\theta$. So, the integral now takes the form

$$\int \frac{2\sec^2\theta\ d\theta}{2\sec\theta} = \int \sec\theta\ d\theta = \ln|\sec\theta + \tan\theta|,$$

where $\sec\theta = \operatorname{Arcsec}\frac{\sqrt{x^2+6x+13}}{2}$
and $\tan\theta = \frac{x+3}{2}$ which is equivalent to the stated answer.

122. $\displaystyle\int \frac{dx}{2x^2 - 4x + 6} = \frac{1}{4}\sqrt{2}\operatorname{Arctan}\frac{1}{8}\,(4x - 4)\,\sqrt{2}$

First, complete the square to find $2x^2 - 4x + 6 = 2(x - 1)^2 + 4$. The integral now looks like:

$$\int \frac{1}{2x^2-4x+6}\,dx = \int \frac{1}{2(x-1)^2+4}\,dx = \frac{1}{2}\int \frac{1}{(x-1)^2+2}\,dx.$$

Next, let $a=\sqrt{2},\ u=x-1$. The previous integrand has a term of the form a^2+u^2. So we use the trigonometric substitution

$u = x-1 = \sqrt{2}\tan\theta,$
$dx = \sqrt{2}\sec^2\theta\ d\theta.$

Furthermore, $2x^2-4x+6 = 2\sec^2\theta$. So, the original integral now takes the form

$$\frac{1}{2}\int \frac{\sqrt{2}\sec^2\theta\ d\theta}{2\sec^2\theta} = \frac{\sqrt{2}}{4}\theta = \frac{\sqrt{2}}{4}\operatorname{Arctan}\frac{x-1}{\sqrt{2}},$$

which is equivalent to the stated answer.

123. $$\int \frac{dx}{(1-x)\sqrt{x^2-2x-3}} = -\frac{1}{2}\operatorname{Arcsec}\frac{x-1}{2}$$

First we complete the square so that $x^2-2x-3=(x-1)^2-4$. A trigonometric substitution is hard here: Let's try another approach...

Let $u=x-1, du=dx$. Then the integral becomes (note the minus sign)

$$-\int \frac{du}{u\sqrt{u^2-4}}.$$

Now we incorporate the number 4 into the square by factoring it out of the expression, thus:

$$u\sqrt{u^2-4} = 2u\sqrt{\left(\frac{u}{2}\right)^2-1}.$$

Now we use the substitution $v=\frac{u}{2},\ 2dv=du$. The integral in u now becomes

$$-\int \frac{2\ dv}{4v\sqrt{v^2-1}} = -\frac{1}{2}\int \frac{dv}{v\sqrt{v^2-1}} = -\frac{1}{2}\operatorname{Arcsec} v,$$

according to Table 6.7 with $\square = v$. The answer follows after back-substitution.

124. $$\int \frac{(2x+3)\ dx}{x^2+2x-3} = \frac{3}{4}\ln|x+3| + \frac{5}{4}\ln|x-1|$$

Use partial fractions. The factors of the denominator are $(x+3)(x-1)$. You need to find two constants.

125. $$\int \frac{(x+1)\ dx}{x^2+2x-3} = \frac{1}{2}\ln\left|x^2+2x-3\right|$$

Let $u=x^2+2x-3,\ du=(2x+2)\ dx$ so that
$du = 2(x+1)\ dx$. Now the integral in u gives a natural logarithm.

Alternately, use partial fractions. The factors of the denominator are $(x+3)(x-1)$. You need to find the two constants.

126. $$\int \frac{(x-1)\ dx}{4x^2-4x+2} = \frac{1}{8}\ln\left|4x^2-4x+2\right| - \frac{1}{4}\operatorname{Arctan}(2x-1)$$

The denominator is a Type II factor (it is irreducible) since $b^2-4ac=(-4)^2-4(4)(2)<0$. So the expression is already in its partial fraction decomposition. So, the partial fractions method gives nothing.

So, complete the square in the denominator. This gives an integral of the form

$$\int \frac{(x-1)\ dx}{4x^2-4x+2} = \int \frac{(x-1)\ dx}{(2x-1)^2+1},$$

which can be evaluated using the trigonometric substitution,

$u = 2x-1,\ du = 2dx$ or $dx = du/2$. Solving for x we get
$x = \frac{u+1}{2}$, so $x-1 = \frac{u-1}{2}$. The $u-$integral looks like

$$\frac{1}{2}\int \frac{u-1}{1+u^2}\ du.$$

Break this integral into two parts and use the substitution

$$v = 1+u^2,\ dv = 2u\ du, \quad udu = dv/2$$

in the first, while the second one yields an Arctangent.

127. $$\int \frac{x\ dx}{\sqrt{x^2-2x+2}} = \sqrt{x^2-2x+2} + \ln\left|\sqrt{x^2-2x+2}+x-1\right|$$

Completing the square we see that $x^2-2x+2 = (x-1)^2+1$. Next, we set

$$x-1 = \tan\theta, \quad dx = \sec^2\theta\ d\theta$$
$$x = 1+\tan\theta,$$
$$\sqrt{x^2-2x+2} = \sqrt{(x-1)^2+1} = \sec\theta.$$

The integral becomes

$$\int \frac{x\ dx}{\sqrt{x^2-2x+2}} = \int \frac{(1+\tan\theta)\sec^2\theta}{\sec\theta} d\theta$$

and this simplifies to

$$\int (\sec\theta + \sec\theta\tan\theta)\ d\theta = \ln|\sec\theta+\tan\theta| + \sec\theta.$$

Finally, use the back-substitutions $\sec\theta = \sqrt{x^2-2x+2}$ and $\tan\theta = x-1$.

128. $$\int \frac{(4x+1)\ dx}{\sqrt{1+4x-4x^2}} = -\sqrt{1+4x-4x^2} + \frac{3}{2}\operatorname{Arcsin}\sqrt{2}\left(x-\frac{1}{2}\right)$$

Completing the square we see that $1+4x-4x^2 = 2-(2x-1)^2$. The integrand has a term of the form $\sqrt{a^2-u^2}$ where $a = \sqrt{2},\ u = 2x-1$. So, we set

$$2x-1 = \sqrt{2}\sin\theta, \quad 2dx = \sqrt{2}\cos\theta\ d\theta$$
$$x = \tfrac{1+\sqrt{2}\sin\theta}{2},$$
$$4x+1 = 3+2\sqrt{2}\sin\theta,$$
$$\sqrt{1+4x-4x^2} = \sqrt{2}\cos\theta.$$

The integral becomes

$$\int \frac{(4x+1)\ dx}{\sqrt{1+4x-4x^2}} = \int \left(\frac{3+2\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right)\frac{\sqrt{2}}{2}\cos\theta\ d\theta$$

which simplifies to

$$\frac{1}{2}\int (3+2\sqrt{2}\sin\theta)\ d\theta = \frac{3}{2}\theta - \sqrt{2}\cos\theta.$$

Finally, use the back-substitutions $\theta = \text{Arcsin}\,\frac{2x-1}{\sqrt{2}}$ and $\cos\theta = \frac{\sqrt{1+4x-4x^2}}{\sqrt{2}}$, to get it in a form equivalent to the stated answer.

129. $\displaystyle\int \frac{(3x-2)\,dx}{\sqrt{x^2+2x+3}} = 3\sqrt{x^2+2x+3} - 5\ln\left|\frac{\sqrt{x^2+2x+3}}{\sqrt{2}} + \frac{x+1}{\sqrt{2}}\right|$

Completing the square we see that $x^2+2x+3 = 2+(x+1)^2$. The integrand has a term of the form $\sqrt{a^2+u^2}$ where $a=\sqrt{2},\ u=x+1$. So, we set

$x+1 = \sqrt{2}\tan\theta, \quad dx = \sqrt{2}\sec^2\theta\ d\theta$
$x = \sqrt{2}\tan\theta - 1,$
$3x-2 = 3\sqrt{2}\tan\theta - 5 = 3\sqrt{2}\tan\theta - 5,$
$\sqrt{x^2+2x+3} = \sqrt{2}\sec\theta.$

The integral becomes

$$\int \frac{(3x-2)\,dx}{\sqrt{x^2+2x+3}} = \int \left(\frac{3\sqrt{2}\tan\theta - 5}{\sqrt{2}\sec\theta}\right)\sqrt{2}\sec^2\theta\ d\theta$$

which simplifies to

$3\sqrt{2}\displaystyle\int \sec\theta\tan\theta\ d\theta - 5\int \sec\theta\ d\theta = 3\sqrt{2}\sec\theta - 5\ln|\sec\theta+\tan\theta|.$

Finally, use the back-substitutions $\sec\theta = \frac{\sqrt{x^2+2x+3}}{\sqrt{2}}$, and $\tan\theta = \frac{x+1}{\sqrt{2}}$, to get it in a form equivalent to the stated answer.

130. $\displaystyle\int \frac{e^x\,dx}{e^{2x}+2e^x+3} = \frac{1}{2}\sqrt{2}\,\text{Arctan}\,\frac{1}{4}(2e^x+2)\sqrt{2}$

Let $u=e^x,\ du=e^x\,dx$. The integral is now a rational function in u on which we can use partial fractions. The denominator is irreducible, since $b^2-4ac = 4-4(1)(3) < 0$. You need to find two constants.

131. $\displaystyle\int \frac{x^2\,dx}{x^2+x-6} = x - \frac{9}{5}\ln|x+3| + \frac{4}{5}\ln|x-2|$

Use long division first, then use partial fractions. The factors of the denominator are $x^2+x-6 = (x+3)(x-2)$. You need to find two constants.

132. $\displaystyle\int \frac{(x+2)\,dx}{x^2+x} = 2\ln|x| - \ln|1+x|$

Use partial fractions. The factors of the denominator are $x^2+x = x(x+1)$. You need to find two constants.

133. $\displaystyle\int \frac{(x^3+x^2)\,dx}{x^2-3x+2} = \frac{1}{2}x^2 + 4x - 2\ln|x-1| + 12\ln|x-2|$

Use long division first. Then use partial fractions. The factors of the denominator are $x^2-3x+2 = (x-1)(x-2)$. You need to find two constants.

134. $\displaystyle\int \frac{dx}{x^3-x} = -\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|1+x|$

Use partial fractions. The factors of the denominator are $x^3-x = x(x^2-1) = x(x-1)(x+1)$. You need to find three constants.

135. $\displaystyle\int \frac{(x-3)\,dx}{x^3+3x^2+2x} = -\frac{3}{2}\ln|x| - \frac{5}{2}\ln|x+2| + 4\ln|1+x|$

Use partial fractions. The factors of the denominator are $x^3+3x^2+2x = x(x^2+3x+2) = x(x+1)(x+2)$. You need to find three constants.

136. $\displaystyle\int \frac{(x^3+1)\,dx}{x^3-x^2} = x + \frac{1}{x} - \ln|x| + 2\ln|x-1|$

Use partial fractions. The factors of the denominator are $x^3 - x^2 = x^2(x-1)$. You need to find three constants.

137. $\displaystyle\int \frac{x\,dx}{(x+1)^2} = \frac{1}{1+x} + \ln|1+x|$

Use partial fractions.

138. $\displaystyle\int \frac{(x+2)\,dx}{x^2-4x+4} = -\frac{4}{x-2} + \ln|x-2|$

Use partial fractions. The factors of the denominator are $x^2 - 4x + 4 = (x-2)^2$. You need to find two constants.

139. $\displaystyle\int \frac{(3x+2)\,dx}{x^3-2x^2+x} = 2\ln|x| - \frac{5}{x-1} - 2\ln|x-1|$

Use partial fractions. Note that $x^3 - 2x^2 + x = x(x^2-2x+1) = x(x-1)^2$. There are four constants to be found here!

140. $\displaystyle\int \frac{8\,dx}{x^4-2x^3} = \frac{2}{x^2} + \frac{2}{x} - \ln|x| + \ln|x-2|$

Use partial fractions. Note that $x^4 - 2x^3 = x^3(x-2)$. There are four constants to be found here!

141. $\displaystyle\int \frac{dx}{(x^2-1)^2} = -\frac{1}{4\,(x-1)} - \frac{1}{4}\ln|x-1| - \frac{1}{4\,(1+x)} + \frac{1}{4}\ln|1+x|$

Use partial fractions. Note that $(x^2-1)^2 = (x-1)^2(x+1)^2$.

142. $\displaystyle\int \frac{(1-x^3)\,dx}{x(x^2+1)} = -x + \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + \operatorname{Arctan} x$

Use long division first, then use partial fractions.

143. $\displaystyle\int \frac{(x-1)\,dx}{(x+1)(x^2+1)} = -\ln|1+x| + \frac{1}{2}\ln\left(x^2+1\right)$

Use partial fractions.

144. $\displaystyle\int \frac{4x\,dx}{x^4-1} = \ln|x-1| + \ln|1+x| - \ln\left(x^2+1\right)$

Note that $x^4 - 1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$. Use partial fractions.

145. $\displaystyle\int \frac{3(x+1)\,dx}{x^3-1} = 2\ln|x-1| - \ln\left(x^2+x+1\right)$

Note that $x^3 - 1 = (x-1)(x^2+x+1)$. Use partial fractions.

146. $\displaystyle\int \frac{(x^4+x)\,dx}{x^4-4} = \frac{1}{4}\ln|x-2| - \frac{1}{12}\ln|x+2| - \frac{1}{12}\ln(x^2+2) + \frac{\sqrt{2}}{3}\operatorname{Arctan}\frac{x\sqrt{2}}{2}$

Use long division first, then use partial fractions.

147. $\displaystyle\int \frac{x^2\,dx}{(x^2+1)(x^2+2)} = -\operatorname{Arctan} x + \sqrt{2}\operatorname{Arctan}\frac{1}{2}\sqrt{2}x$

The factors are $(x^2+2)(x^2+1)$, both irreducible. Four constants need to be found. This is where the Arctangents come from!

148. $\displaystyle\int \frac{3\ dx}{x^4+5x^2+4} = -\frac{1}{2}\operatorname{Arctan}\frac{1}{2}x + \operatorname{Arctan} x$

The factors are $(x^2+4)(x^2+1)$, both irreducible. Four constants need to be found. This is where the Arctangents come from!

149. $\displaystyle\int \frac{(x-1)\ dx}{(x^2+1)(x^2-2x+3)} = -\frac{1}{2}\operatorname{Arctan} x + \frac{1}{4}\sqrt{2}\operatorname{Arctan}\frac{1}{4}(2x-2)\sqrt{2}$

Use partial fractions. Watch out, as both factors in the denominator are Type II.

150. $\displaystyle\int \frac{x^3\ dx}{(x^2+4)^2} = \frac{2}{x^2+4} + \frac{1}{2}\ln\left(x^2+4\right)$

Use partial fractions.

151. $\displaystyle\int \frac{(x^4+1)\ dx}{x(x^2+1)^2} = \ln|x| + \frac{1}{x^2+1}$

Use partial fractions.

152. $\displaystyle\int \frac{(x^2+1)\ dx}{(x^2-2x+3)^2} = -\frac{1}{x^2-2x+3} + \frac{1}{2}\sqrt{2}\operatorname{Arctan}\frac{1}{4}(2x-2)\sqrt{2}$

Use partial fractions. Note that $(x^2-2x+3)^2$ is irreducible (Type II). Now you have to find the four constants!

153. $\displaystyle\int \frac{x\ dx}{\sqrt{x+1}} = -2\sqrt{x+1} + \frac{2}{3}\left(\sqrt{x+1}\right)^3$

Let $u = x+1,\ du = dx$. Then $x = u-1$, and the integral becomes easy.

154. $\displaystyle\int x\sqrt{x-a}\ dx = \frac{2}{5}\left(\sqrt{x-a}\right)^5 + \frac{2}{3}\left(\sqrt{x-a}\right)^3 a$

Let $u = x-a,\ du = dx$. Then $x = u+a$, and the integral becomes easy.

155. $\displaystyle\int \frac{\sqrt{x+2}}{x+3}\ dx = 2\sqrt{x+2} - 2\operatorname{Arctan}\sqrt{x+2}$

Let $u = \sqrt{x+2},\ u^2 = x+2$. Then $2u\ du = dx$ and $x = u^2-2$ which means that $x+3 = u^2+1$. The integral takes the form $\int \frac{2u^2\ du}{1+u^2}$. This one can be evaluated using a long division and two simple integrations.

156. $\displaystyle\int \frac{dx}{x\sqrt{x-1}} = 2\operatorname{Arctan}\sqrt{x-1}$

Let $u = \sqrt{x-1},\ u^2 = x-1$. Then $2u\ du = dx$ and so $x = 1+u^2$. The integral takes the form $\int \frac{2u\ du}{u(1+u^2)}$ which is an arctangent function...

157. $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\ln\left|\frac{a}{x} - \frac{\sqrt{a^2-x^2}}{x}\right|.$

Let $x = a\sin\theta,\ dx = a\cos\theta\ d\theta$. Then $\sqrt{a^2-x^2} = a\cos\theta$. After some simplification we find $a^{-1}\int \csc\theta\ d\theta = a^{-1}\ln|\csc\theta - \cot\theta|$. Finally, $\csc\theta = \dfrac{a}{x},\ \cot\theta = \dfrac{\sqrt{a^2-x^2}}{x}$.

158. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{1}{a^2x}\sqrt{a^2-x^2}$

Let $x = a\sin\theta,\ dx = a\cos\theta\ d\theta$. Then $sqrta^2-x^2 = a\cos\theta$. After some simplification we find $a^{-2}\int \csc^2\theta\ d\theta = -a^{-2}\cot\theta$.

159. $\displaystyle\int x^3\sqrt{x^2+a^2}\,dx = \frac{1}{5}x^2\left(\sqrt{x^2+a^2}\right)^3 - \frac{2}{15}a^2\left(\sqrt{x^2+a^2}\right)^3$

Let $x = a\tan\theta,\ dx = a\sec^2\theta\,d\theta$. Then $\sqrt{x^2+a^2} = a\sec\theta$. After some simplification you're left with an integral with an integrand equal to $\sec^2\theta\ \tan^3\theta$. Use Example 360.

160. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{1}{a^2x}\sqrt{x^2+a^2}$

Let $x = a\tan\theta,\ dx = a\sec^2\theta\,d\theta$. Then $x^2+a^2 = a^2\sec^2\theta$. After some simplification you're left with an integral with an integrand equal to $\csc\theta\ \cot\theta$. Its value is a cosecant function. Finally, use the fact that, in this case, $\csc\theta = \dfrac{\sqrt{x^2+a^2}}{x}$

161. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right|$

Let $x = a\tan\theta,\ dx = a\sec^2\theta\,d\theta$. Then $x^2+a^2 = a^2\sec^2\theta$. After some simplification you're left with an integral of the form in Example 356.

162. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2+a^2}} = \frac{1}{2}x\sqrt{x^2+a^2} - \frac{1}{2}a^2\ln\left|x+\sqrt{x^2+a^2}\right|$

Let $x = a\tan\theta,\ dx = a\sec^2\theta\,d\theta$. Then $x^2+a^2 = a^2\sec^2\theta$. After some simplification you're left with an integral of the form in Example 358.

163. $\displaystyle\int \frac{x^2\,dx}{(x^2+a^2)^2} = -\frac{1}{2}\frac{x}{x^2+a^2} + \frac{1}{2a}\operatorname{Arctan}\frac{x}{a}$

Let $x = a\tan\theta,\ dx = a\sec^2\theta\,d\theta$. Then $x^2+a^2 = a^2\sec^2\theta$. After some simplification you're left with the integral of the square of a sine function...

164. $\displaystyle\int x\cos x\,dx = \cos x + x\sin x$

Use Table integration

165. $\displaystyle\int x\sin x\,dx = \sin x - x\cos x$

Use Table integration

166. $\displaystyle\int x\sec^2 x\,dx = x\tan x + \ln|\cos x|$

Use Integration by Parts: Let $u = x,\ dv = \sec^2 x\,dx$. No need to use Table integration here.

167. $\displaystyle\int x\sec x\tan x\,dx = \ x\sec x - \ln|\sec x + \tan x|$

Use Integration by Parts: Let $u = x,\ dv = \sec x\tan x\,dx$. No need to use Table integration here.

168. $\displaystyle\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x$

Use Table integration

169. $\displaystyle\int x^4\ln x\,dx = \frac{1}{5}x^5\ln x - \frac{1}{25}x^5$

Use Integration by Parts: Let $u = \ln x,\ dv = x^4\,dx$. No need to use Table integration here.

170. $\displaystyle\int x^3e^{x^2}\,dx = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2}$

Write the integrand as $x^3e^{x^2} = x^2\cdot xe^{x^2}$. Then use Integration by Parts with $u = x^2,\ dv = xe^{x^2}\,dx$. Use the substitution $v = x^2$ in the remaining integral.

171. $\displaystyle\int \sin^{-1} x \; dx = x \operatorname{Arcsin} x + \sqrt{(1-x^2)}$

Use Integration by Parts: Let $u = \operatorname{Arctan} x, \; dv = dx$, followed by the substitution $u = 1 + x^2$, etc.

172. $\displaystyle\int \tan^{-1} x \; dx = x \operatorname{Arctan} x - \frac{1}{2} \ln\left(x^2 + 1\right)$

Use Integration by Parts: Let $u = \operatorname{Arctan} x, \; dv = dx$, followed by the substitution $u = 1 + x^2$, etc.

173. $\displaystyle\int (x-1)^2 \sin x \; dx = \cos x - 2\sin x + 2x\cos x - x^2 \cos x + 2x \sin x$

Use Table integration

174. $\displaystyle\int \sqrt{x^2 - a^2} \; dx = \frac{1}{2} x\sqrt{x^2 - a^2} - \frac{1}{2} a^2 \ln\left|x + \sqrt{x^2 - a^2}\right|$

Let $x = a\sec\theta, \; dx = a\sec\theta \; \tan\theta \; d\theta$. Then $\sqrt{x^2 - a^2} = a\tan\theta$, etc.

175. $\displaystyle\int \sqrt{x^2 + a^2} \; dx = \frac{1}{2} x\sqrt{x^2 + a^2} + \frac{1}{2} a^2 \ln\left|x + \sqrt{x^2 + a^2}\right|$

Let $x = a\tan\theta, \; dx = a\sec^2\theta \; d\theta$. Then $\sqrt{x^2 + a^2} = a\sec\theta$, etc.

176. $\displaystyle\int \frac{x^2 \; dx}{\sqrt{x^2 - a^2}} = \frac{1}{2} x\sqrt{x^2 - a^2} + \frac{1}{2} a^2 \ln\left|x + \sqrt{x^2 - a^2}\right|$

Let $x = a\sec\theta, \; dx = a\sec\theta \; \tan\theta \; d\theta$. Then $\sqrt{x^2 - a^2} = a\tan\theta$, etc.

177. $\displaystyle\int e^{2x} \sin 3x \; dx = -\frac{3}{13} e^{2x} \cos 3x + \frac{2}{13} e^{2x} \sin 3x$

Use Table integration

178. $\displaystyle\int e^{-x} \cos x \; dx = -\frac{1}{2} e^{-x} \cos x + \frac{1}{2} e^{-x} \sin x$

Use Table integration

179. $\displaystyle\int \sin 3x \cos 2x \; dx = -\frac{1}{10} \cos 5x - \frac{1}{2} \cos x$

Use a trig.. identity ... the one for $\sin A \cos B$, with $A = 3x, \; B = 2x$.

180. $\displaystyle\int_0^{\frac{\pi}{8}} \cos^3(2x) \sin(2x) \; dx = \frac{3}{32}$

Let $u = 2x$ first, $du = 2dx$, and follow this by the substitution $v = \cos u, dv = -\sin u \; du$ which allows for an easy calculation of an antiderivative.

181. $\displaystyle\int_1^4 \frac{2\sqrt{x}}{2\sqrt{x}} \; dx = \frac{2}{\ln 2}$

Let $u = \sqrt{x}$. The result follows easily.

182. $\displaystyle\int_0^{\infty} x^3 e^{-2x} \; dx = \frac{3}{8}$

Use Table integration to find an antiderivative and then use L'Hospital's Rule (three times!).

183. $\displaystyle\int_{-\infty}^{+\infty} e^{-|x|} \; dx = 2$

Divide this integral into two parts, one where $x \geq 0$ (so that $|x| = x$), and one where $x < 0$ (so that $|x| = -x$). Then
$\int_{-\infty}^{+\infty} e^{-|x|}\ dx = \int_{-\infty}^{0} e^{x}\ dx + \int_{0}^{\infty} e^{-x}\ dx$ and the integrals are defined by a limit.

184. $\int_0^{\infty} \dfrac{4x}{1+x^4}\ dx = \pi$

Let $u = x^2,\ du = 2x\ dx$. The integral becomes an Arctangent.

185. $\int_{-1}^{1} x^2 \cos(n\pi x)\ dx = \dfrac{4\cos n\pi}{n^2\pi^2}$, when $n \geq 1$, is an integer. Use Table integration.

186. $\frac{1}{2}\int_{-2}^{2} x^2 \sin\left(\frac{n\pi x}{2}\right)\ dx = 0$, when $n \geq 1$, is an integer. Use Table integration.

187. $\frac{1}{L}\int_{-L}^{L} (1-x)\sin\left(\frac{n\pi x}{L}\right)\ dx = 2L\frac{\cos n\pi}{n\pi}$,
when $n \geq 1$, $L \neq 0$. Use Table integration.

188. $\int_0^2 (x^3+1)\cos\left(\dfrac{n\pi x}{2}\right)\ dx = 6\dfrac{8n^2\pi^2\cos n\pi - 16\cos n\pi + 16}{n^4\pi^4}$,
when $n \geq 1$, is an integer. Use Table integration.

189. $\int_{-1}^{1} (2x+1)\cos(n\pi x)\ dx = \dfrac{2}{n\pi}\sin n\pi = 0$,
when $n \geq 1$, is an integer. Use Table integration.

190. $\dfrac{1}{L}\int_{-L}^{L} \sin x\ \cos\left(\dfrac{n\pi x}{L}\right)\ dx = 0$,
when $n \geq 1$, is an integer and $L \neq 0$. Use Table integration.

191. Total demand over 10 years is

$$\int_0^{10} 500\left(20 + t\ e^{-0.1t}\right)\ dt = \int_0^{10} 10000\ dt + 500\int_0^{10} t\ e^{-0.1t}\ dt.$$

Now integrating by parts

$$\int t\ e^{-0.1t}\ dt = -10t\ e^{-0.1t} + 10\int e^{-0.1t}\ dt = -10t\ e^{-0.1t} + 10\left(-10e^{-0.1t}\right).$$

Thus total demand $= \left[10,000t + 500\{-10t\ e^{-0.1t} + 10(-10e^{-0.1t})\}\right]_0^{10} = \left[10,000t - 5000te^{-0.1t} - 50,000e^{-0.1t}\right]_0^{10} =$
$100,000 - 50,000e^{-1} - 50,000e^{-1} - (0 - 0 - 50,000) = 150,000 - 100,000e^{-1} = 113212.1 \approx 113212$ units.

192. (a) Use partial fractions.

$$\frac{1}{y(y-10)} = \frac{A}{y} + \frac{B}{10-y} = \frac{A(10-y)+By}{y(10-y)}$$

If $y = 0$, then $10A = 1$, so $A = \frac{1}{10}$. If $y = 10$, then $10B = 1$, and $B = \frac{1}{10}$. Therefore,

$$\int \frac{1}{y(y-10)}dy = \frac{1}{10}\int \frac{dy}{y} + \frac{1}{10}\int \frac{dy}{10-y}dy$$

$$= \frac{1}{10}\ ln|y| - \frac{1}{10}\ ln|10-y| + C = \frac{1}{10}\ ln\left|\frac{y}{10-y}\right| + C$$

Thus

$$t = \frac{25}{10}\ ln\left|\frac{y}{10-y}\right| + C$$

When $t = 0$, $y = 1$, so $0 = 2.5ln\frac{1}{9} + C = 2.5(ln\ 1 - ln\ 9) + C = -2.5ln\ 9 + C$. Thus $C = 2.5ln\ 9$ and

$$t = 2.5ln\left|\frac{y}{10-y}\right| + 2.5ln\ 9 = 2.5ln\left|\frac{9y}{10-y}\right|$$

(b) When $y = 4$, $t = 2.5ln\ \frac{4\times 9}{6} = 4.479$ hours.

(c) From (a), $\frac{t}{2.5} = ln\ \frac{9y}{10-y}$, so $e^{\frac{t}{2.5}} = \frac{9y}{10-y}$, and $(10-y)e^{0.4t} = 9y$, so $10e^{0.4t} = 9y + ye^{0.4t} = y(9+e^{0.4t})$. Thus

$$y = \frac{10e^{0.4t}}{9+e^{0.4t}} = \frac{10}{1+e^{-0.4t}}$$

(d) At $t = 10$, $y = \frac{10}{1+9e^{-4}} = 8.58$ gm.

193. Let I denote an antiderivative. Now let $u = 3+\sin t$, $du = \cos t\, dt$. Then I is of the form $\int du/u = \ln|u| + C$ or in terms of the original variables, $I = \ln|3+\sin t| + C$.

194. Let I denote the integral. Now let $x = z^2$, $dx = 2z\, dz$. Then use the trig. subs. $z = \tan\theta/\sqrt{2}$ to get $I = \sqrt{x} - (1/\sqrt{2})\tan^{-1}(\sqrt{2x}) + C$. (Note that another identical answer is given by $I = x/2 - \sqrt{x}/2 + (1/2)\ln|1+2\sqrt{x}| + C$. Of course they have to differ by a constant)

195. I is as before. Let $t = z^3$, $dt = 3z^2\, dz$. Use long division to simplify the rational function and the method of partial fractions to get $I = 3t^{2/3}/2 + \ln|t^{1/3}+1| - (1/2)\ln|t^{2/3}-t^{1/3}+1| - \sqrt{3}\tan^{-1}\left((2t^{1/3}-1)\sqrt{3}/3\right) + C$.

196. Use the identity $\sin 2t = 2\sin t\cos t$ first, then the substitution $u = \cos t$, $du = -\sin t\, dt$ to get an antiderivative $I = -2\cos t + 4\ln|2+\cos t| + C$. The definite integral is now given by $2 - 4\ln 3 + 4\ln 2$.

197. Let I denote the integral and $x = z^2$, $dx = 2z\, dz$. Then use long division to simplify the rational function and the method of partial fractions to get $I = 2\sqrt{x}/3 - (4/9)\ln|2+3\sqrt{x}| + C$.

198. Write $\tan x = \sin x/\cos x$ and simplify the resulting expression of sines and cosines. Next use the identity $\cos^2 x = 1 - \sin^2 x$ in the denominator and the substitution $u = \sin x$ to reduce the integral into a rational function of u. Now use the method of partial fractions to get

$$I = \frac{\alpha}{\beta-\alpha}\ln|\sin x - \alpha| + \frac{\beta}{\alpha-\beta}\ln|\sin x - \beta| + C,$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

199. Reduce the integrand to sines and cosines. Next, take a common denominator in both numerator and denominator and simplify. Recombine the terms in the denominator so as to use the identity $\sin^2 t = 1 - \cos^2 t$. Now use the substitution $u = \cos t$, $du = -\sin t\, dt$. The denominator can now be written as a difference of two squares. Use partial fractions as before to get $I = (1/\sqrt{6})\left(\ln|\cos t - \sqrt{3/2}| - \ln|\cos t + \sqrt{3/2}|\right) + C$.

200. Let I denote the integral and $x = z^2$, $dx = 2z\, dz$. Then use long division to simplify the rational function and the method of partial fractions to get $I = -x + 8\sqrt{x} - 16\ln|2+\sqrt{x}| + C$.

201. Multiply both numerator and denominator by $\sqrt{1+u}$ and simplfy. Now let $u = \sin x$, $du = \cos x\, dx$. The denominator simplifies because of a basic identity and the rest is easily integrated to give $I = \sin^{-1}(u) - \sqrt{1-u^2} + C$.

202. Let $u = \sqrt{x}$, $dx = 2u\, du$ so that $I = -2\cos\sqrt{x} + C$.

203. Let $x = z^4$, $dx = 4z^3\, dz$ and use partial fractions. Then $I = (1/2)\ln|x| - 2\ln|x^{1/4} - 1| + C$.

204. Let $x^{1/6} = z$, $dx = 6z^5\, dz$. The resulting rational funcion of z has a denominator of degree 9 so the method of partial fractions will be tedious. The answer, when simplified, is $I = 3\ln|1+x^{1/3}| + 3/(2x^{2/3}) - \ln|x| - 3x^{-1/3} - 1/x + C$.

205. Let $x^{1/6} = z$, $dx = 6z^5\, dz$. Now use long division to simplify the rational function in z and integrate term by term. Then $I = 6x^{7/6}/7 - 6x^{5/6}/5 + 2\sqrt{x} - 6x^{1/6} + 6\tan^{-1}(x^{1/6}) + C$.

206. Let $x^{1/6} = z$, $dx = 6z^5\, dz$. As before use long division to simplify the rational function in z and integrate term by term. Then

$$I = x + \frac{6x^{5/6}}{5} + \frac{3x^{2/3}}{2} + 2\sqrt{x} + 3x^{1/3} + 6x^{1/6} + 6\ln|x^{1/6} - 1| + C.$$

207. Let $x^{1/3} = z$, $dx = 3z^2\,dz$. Changing the limits we get the same limits in the z variables. Using long division and simplifying we get an antiderivative $I = 3z^2/2 - 3z + 3\ln|z+1|$ and this gives us the answer $3(\ln 2 - 1/2)$.

208. Let $x^{1/12} = z$, $dx = 12z^{11}\,dz$. Another long division, simplification, integration and back-substitution gives

$$I = \frac{3}{2}x^{2/3} + \frac{12}{7}x^{7/12} + 2\sqrt{x} + \frac{12}{5}x^{5/12} + 3x^{1/3} + 4x^{1/4} + 6x^{1/6} + 12x^{1/12} + 12\ln|x^{1/12} - 1| + C.$$

209. Let $x^{1/2} = z$, $dx = 2z\,dz$. Changing the limits we get the same limits in the z variables. Using long division and simplifying we get an antiderivative $I = 2z - 2\ln|z+1|$ and this gives us the answer $2(1 - \ln 2)$.

210. Let $x^{1/2} = z$, $dx = 2z\,dz$. Now use the Table Method to integrate the resulting z integral. We get $I = -2x\cos\sqrt{x} + 4\cos\sqrt{x} + 4\sqrt{x}\sin\sqrt{x} + C$.

211. Let $x^{1/5} = u$, $dx = 5u^4\,du$. Changing the limits we get the same limits in the u variables. Using long division and simplifying we get $5\ln 2 - 35/12$.

212. Let $z = \tan(x/2)$ etc. The resulting z integrand looks like $-2/(z^2 - 4z - 1)$. Now complete the square in the denominator and use partial fractions. We get an antiderivative that looks like

$$I = -\frac{1}{\sqrt{5}}\left(\ln|\tan(x/2) - 2 - \sqrt{5}| - \ln|\tan(x/2) - 2 + \sqrt{5}|\right).$$

Putting in the limits we get the answer

$$\frac{\sqrt{5}}{5}\left(\ln\frac{\sqrt{5}-1}{\sqrt{5}+1} + \ln\frac{\sqrt{5}+2}{\sqrt{5}-2}\right).$$

213. Let $x^{1/2} = z$, $dx = 2z\,dz$. Changing the limits we get the same limits in the z variables. The cubic $1 + z^3$ in the denominator is easily factored. Now use partial fractions. Then (with coffees) evaluate the limit of the antiderivative at infinity. Done correctly you'll get the answer $4\pi\sqrt{3}/9$.

214. Let $2x = u$, $dx = du/2$. The new u limits become 0 and $\pi/2$. Now let $z = \tan(x/2)$ etc. The new z limits now become 0 and 1 and the new integrand looks like $1/(1 + 2z - z^2)$. Factor the denominator using the quadratic formula and using partial fractions you'll find the answer

$$\frac{\sqrt{2}}{4}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right).$$

215. Another tedious one! Let $z = \tan(x/2)$ etc. The new z limits now become 0 and 1. The resulting rational function can be integrated using partial fractions to find the simple answer of $\pi/4$.

216. Let $z = \tan(x/2)$ etc. The new z limits now become 0 and $\tan\pi/8$. The resulting integral looks like

$$\int_0^{\tan\pi/8} \frac{2(1+z^2)}{z^4 + 6z^2 + 1}\,dz$$

which can be integrated using partial fractions (first complete the "square" in the denominator by rewriting it as $(z^2+3)^2 - (\sqrt{8})^2$ and then factor the difference of squares as usual. Another few coffees should do the trick! The answer is

$$\frac{\sqrt{2}}{2}\tan^{-1}\sqrt{2}.$$

217. Let $z = \tan(x/2)$ etc. An antiderivative is then found to be $(\sqrt{2}/2)\arctan((\sqrt{2}/2)\tan(x/2))$. Evaluating this between the limits 0 and 4π we get $(\sqrt{2}/2)(0 + 2\pi) - (\sqrt{2}/2)(0) = \pi\sqrt{2}$.

218. Let $z = \tan(x/2)$ etc. An antiderivative is given by $(2\sqrt{3}/2)\arctan((\sqrt{3}/3)(2\tan(x/2) - 1))$. Evaluating this between the limits $-\pi$ and π we get $(2\sqrt{3}/3)\,\mathrm{Arctan}(+\infty) - (2\sqrt{3}/3)\,\mathrm{Arctan}(-\infty) = (2\sqrt{3}/3)(\pi/2) + (2\sqrt{3}/3)(\pi/2) = 2\pi\sqrt{3}/3$.

219. This time we let $z = \tan(x)$ etc. An antiderivative is $(\sqrt{21}/12)\arctan((\sqrt{21}/3)(\tan(x)) - x/4$. Evaluating this between the limits 2π and 5π we get $(\sqrt{21}/12)(\text{Arctan}(0) + 5\pi) - 5\pi/4 - (\sqrt{21}/12)(\text{Arctan}(0) + \pi/2)$ $= \pi(\sqrt{21} - 3)/4$.

220. Use the Table method and Rodriguez's formula to show that

$$\int g(x)P_n(x)\,dx = g(x)\frac{1}{n!2^n}D^{n-1}((x^2-1)^n) - g'(x)\frac{1}{n!2^n}D^{n-2}((x^2-1)^n) + \cdots$$
$$+(-1)^{n-1}g^{(n-1)}(x)\frac{1}{n!2^n}D^{n-n}((x^2-1)^n) + \int (-1)^n g^{(n)}(x)\frac{1}{n!2^n}(x^2-1)^n\,dx.$$

Using this last equation we can evaluate the integrated terms over $[-1, 1]$ and note that in every boundary term there is always a term of the form $(x^2 - 1)^m$ left-over and this term becomes zero at the end-points. So, only the final integral term on the right remains.

Chapter 8

APPENDIX A

8.1 Exercise Set 45 (page 487)

1. 32
2. $\frac{1}{25}$
3. $2^6 = 64$
4. 12
5. 45
6. $4x^2 - y^2$
7. x^2
8. $\frac{2}{25}$
9. x^2y^3
10. $1 - r^4$
11. a^3b^5
12. $8a^9$
13. $\frac{1}{x^{\frac{7}{12}}}$
14. 64
15. 6
16. $\frac{x^{12}}{27}$
17. 3
18. 8
19. 3^{n+1}
20. 2
21. Expand the right side, collect terms and compare the coefficients. You'll find that $1 = -a^2$ which is an impossibility since the right side is always negative or zero and the left side is positive.
22. See Example 465 where you set $r = x^2$.

23. $1 + x^2 + x^4 + x^6$; see the previous exercise.
24. Expand the left-side and simplify.
25. Use the Power Laws and simplify
26. Since $a^{r+s} = a^r a^s$ we can set $s = -r$. Then $a^0 = a^r a^{-r}$ and since $a^0 = 1$ we get $1 = a^r a^{-r}$ and the result follows.
27. See the Introduction to this section for a similar argument.
28. Let $x = 2$, $y = 3$. Then $2^8 \neq 2^6$.
29. Replace r by $-r$ in Example 465 and then set $r = \dfrac{x}{2}$ and simplify.
30. Write x as $x = \dfrac{a^{\frac{1}{2}} + a^{-\frac{1}{2}}}{2}$. Square both sides of this equality, use the Powers Laws, and then subtract 1 from the result. Simplify.

Chapter 9

APPENDIX B

9.1 Exercise Set 46 (page 495)

1. $\frac{2}{3}$
2. $\frac{2}{3}$
3. 3
4. $y = 3x - 10$
5. $y = x - 1$
6. $y = -2x - 5$
7. $y = \frac{4}{3}x - 10$
8. $y = -\frac{3}{2}x + 3$
9. $y = -\frac{2}{3}x + \frac{1}{3}$
10. $\left(-\frac{1}{3}, \frac{2}{3}\right)$
11. There is no intersection whatsoever since the lines are parallel or have the same slope $(= 1)$
12. There is no intersection point either since the lines are parallel or have the same slope $(= 1)$
13. a) $y = x - 2$
    b) The altitude has length $\sqrt{32} = 4\sqrt{2}$. First, we find the equation of the line through $(4, -6)$ having slope -1 as it must be perpendicular to the line through $(2, 0)$ and $(6, 4)$ (*i.e.,* $y = x - 2$). This line is given by $y = -x - 2$. The find the point of intersection of this line with $y = x - 2$. We get the point $(0, -2)$. The base of the triangle has length given by the distance formula in the exercise applied to the points A and B. Its value is $\sqrt{32}$. The altitude has height given by the same distance formula, namely, the distance between the points C$(4, -6)$ and $(0, -2)$; its value is $\sqrt{32}$ as well. The rest follows.
    c) Area $= (1/2)\sqrt{(}32)\sqrt{(}32) = 16$
14. $y = x$
15. $y = -x + 4$

Chapter 10

APPENDIX C

10.1 Exercise Set 47 (page 509)

1. $1/2$
2. $\sqrt{3}/2$
3. $\sqrt{3}/3$: $\tan \pi/6 = \sin \pi/6 / \cos \pi/6 = (1/2)/(\sqrt{3}/2) = 1/\sqrt{3} = \sqrt{3}/3$
4. $-\sqrt{2}/2$: $\cos(-5\pi/4) = \cos(5\pi/4) = \cos(\pi+\pi/4) = \cos(\pi)\cos(\pi/4) - \sin(\pi)\sin(\pi/4) = -\cos(\pi/4) = -\sqrt{2}/2$
5. $\sqrt{2}/2$: $\cos(7\pi/4) = \cos(8\pi/4 - \pi/4) = \cos(2\pi - \pi/4) = \cos(-\pi/4) = \cos(\pi/4) = \sqrt{2}/2$
6. $-\sqrt{2}/2$: $\sin(5\pi/4) = \sin(\pi + \pi/4) = \sin(\pi)\cos(\pi/4) + \cos(\pi)\sin(\pi/4) = -\sin(\pi/4) = -\sqrt{2}/2$
7. $-\sqrt{3}/2$: $\cos(7\pi/6) = \cos(\pi + \pi/6) = \cos(\pi)\cos(\pi/6) - 0 = -\cos(\pi/6) = -\sqrt{3}/2$
8. $-\sqrt{2}/2$: $\sin(-3\pi/4) = -\sin(3\pi/4) = -\sin(\pi - \pi/4) =$
   $(-1)(-1)\cos(\pi)\sin(\pi/4) = (-1)(-1)(-1)\sin(\pi/4) = -\sqrt{2}/2$
9. $-\sqrt{2}/2$: $\cos(3\pi/4) = \cos(\pi - \pi/4) = \cos(\pi)\cos(\pi/4) + 0 = (-1)\cos(\pi/4) = -\sqrt{2}/2$
10. $-\sqrt{3}/2$: $\sin(5\pi/3) = \sin(2\pi - \pi/3) = 0 - \cos(2\pi)\sin(\pi/3) = -\sin(\pi/3) = -\sqrt{3}/2$
11. 0: $\cos(3\pi/2) = \cos(2\pi - \pi/2) = \cos(2\pi)\cos(\pi/2) + \sin(2\pi)\sin(\pi/2) = 0 + 0 = 0$
12. -1: $\sin(3\pi/2) = -1$
13. Undefined. $\tan(3\pi/2) = \sin(3\pi/2)/\cos(3\pi/2) = (-1)/(0) = -\infty$ or is undefined.
14. -1: Use No. 5 above; $\tan(7\pi/4) = \sin(7\pi/4)/\cos(7\pi/4) = \sin(2\pi-\pi/4)/(\sqrt{2}/2) = -\cos(2\pi)\sin(\pi/4)/(\sqrt{2}/2) =$
    $-(\sqrt{2}/2)/(\sqrt{2}/2) = -1$
15. $-1/2$: $\sin(7\pi/6) = \sin(\pi + \pi/6) = \sin(\pi)\cos(\pi/6) + \cos(\pi)\sin(\pi/6) = 0 - \sin(\pi/6) = -1/2$
16. $\sqrt{2}/2$: Use No. 5 above again; $\cos(-7\pi/4) = \cos(7\pi/4)$
17. $\sqrt{2}/2$: $\cos(17\pi/4) = \cos(16\pi/4 + \pi/4) = \cos(4\pi)\cos(\pi/4) - 0 = \sqrt{2}/2$
18. 0: $\cos(5\pi/2) = \cos(2\pi + \pi/2) = \cos(2\pi)\cos(\pi/2) - \sin(2\pi)\sin(\pi/2) = 0 - 0 = 0$
19. $\sqrt{3}/2$: $\cos(11\pi/6) = \cos(12\pi/6 - \pi/6) = \cos(2\pi)\cos(\pi/6) + 0 = \cos(\pi/6) = \sqrt{3}/2$
20. $\sqrt{3}/2$: $\cos(-13\pi/6) = \cos(13\pi/6) = \cos(12\pi/6 + \pi/6) = \cos(2\pi)\cos(\pi/6) = \cos(\pi/6) = \sqrt{3}/2$
21. $-\sqrt{2}/2$: $\cos(225^\circ) = \cos(5\pi/4)$ and use No. 4.
22. $\sqrt{2}/2$: $\cos(405^\circ) = \cos(9\pi/4) = \cos(2\pi + \pi/4) = \cos(\pi/4) = \sqrt{2}/2$
23. $-1/2$: $\cos(960^\circ) = \cos(16\pi/3) = \cos(6\pi - 2\pi/3) = \cos(2\pi/3)$

24. $1/2$: $\sin(-210^\circ) = \sin(-7\pi/6) = -\sin(7\pi/6) = 1/2$, see No. 15

25. -1: $\tan(-1125^\circ) = \tan(-25\pi/4) = -\tan(25\pi/4) = -\tan(6\pi + \pi/4) = -\tan(\pi/4) = -1$

26. Draw a picture. Note that $\sin\phi = 4/5$ and $\cos\phi = 3/5$. Thus, $\sec\phi = 1/\cos\phi = 5/3$

27. Draw a picture. Observe that $\sin u = \sqrt{15}/4 > 0$ in Quadrant II. So, $\csc u = 1/\sin u = 4/\sqrt{15}$ and $\tan u = \sin u/\cos u = (\sqrt{15}/4)/(-1/4) = -\sqrt{15}$

28. Draw a picture. Here, $\cos\phi = \sqrt{8}/3$ by Pythagoras. Thus, $\tan\phi = (1/3)/(\sqrt{8}/3) = 1/\sqrt{8}$

29. Draw a picture. Now, $\tan v = -3/4$ and v in Quadrant IV means that $\sin v = -3/5$ and $\cos v = 4/5$.

30. Draw a picture. This time $\sec\phi = 2$ and ϕ acute means that ϕ is in Quadrant I. Hence, $\cos\phi = 1/2$ and $\sin\phi = \sqrt{3}/2$. It follows that $\tan\phi = \sqrt{3}$.

31. Draw a picture. Well, $\csc w = -3$ means that $\sin w = -1/3$ and so $\cos w = -\sqrt{2}/3$. Therefore, $\cot w = \cos w/\sin w = \sqrt{2}$

32.
$$\begin{aligned}\text{LHS} = (\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x})^2 &= (\frac{\sin^2 x + \cos^2 x}{\cos x \sin x})^2 \\ &= (\frac{1}{\cos x \sin x})^2 = (\frac{1}{\cos x})^2(\frac{1}{\sin x})^2 = \sec^2 x \csc^2 x \\ &= \sec^2 x(1 + \cot^2 x) = \sec^2 x + \sec^2 x \cot^2 x \\ &= \sec^2 x + \csc^2 x.\end{aligned}$$

33.
$$\begin{aligned}\text{LHS} = \sin\theta + \cot\theta\cos\theta &= \sin\theta + \frac{\cos^2\theta}{\sin\theta} \\ &= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta} = \frac{1}{\sin\theta} = \csc\theta\end{aligned}$$

34.
$$\begin{aligned}\text{LHS} = \frac{\cos x + \tan x(1 + \sin x)}{1 + \sin x} &= \frac{\cos x + \tan x + \dfrac{\sin^2 x}{\cos x}}{1 + \sin x} \\ &= \frac{\cos^2 x + \tan x\cos x + \sin^2 x}{\cos x(1 + \sin x)} = \frac{1 + \sin x}{\cos x(1 + \sin x)} = \sec x\end{aligned}$$

35.
$$\begin{aligned}\text{LHS} = \frac{\sin^2 y}{\cos^2 y} - \sin^2 y &= \sin^2 y(\frac{1}{\cos^2 y} - 1) \\ &= \sin^2 y(\sec^2 y - 1) = \sin^2 y \tan^2 y\end{aligned}$$

36.
$$\begin{aligned}\text{LHS} = \frac{1 + \dfrac{\cos x}{\sin x}}{1 + \dfrac{\sin x}{\cos x}} &= \frac{\dfrac{\sin x + \cos x}{\sin x}}{\dfrac{\cos x + \sin x}{\cos x}} \\ &= \frac{\sin x + \cos x}{\sin x}\frac{\cos x}{\cos x + \sin x} \\ &= \cot x\end{aligned}$$

37.

$$\begin{aligned}\text{LHS} = \frac{1}{\frac{\sin\phi}{\cos\phi}+\frac{\cos\phi}{\sin\phi}} &= \frac{1}{\frac{\sin^2\phi+\cos^2\phi}{\cos\phi\sin\phi}} \\ &= \frac{1}{\frac{1}{\cos\phi\sin\phi}} \\ &= \cos\phi\sin\phi\end{aligned}$$

38.

$$\text{LHS} = \sin^2 x\frac{\cos^2 x}{\sin^2 x}+\cos^2 x\frac{\sin^2 x}{\cos^2 x} = \cos^2 x+\sin^2 x = 1$$

39.

$$\begin{aligned}\text{LHS} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x\sin^2 x} &= \frac{1}{\cos^2 x\sin^2 x} \\ &= \frac{1}{\sin^2 x(1-\sin^2 x)} = \frac{1}{\sin^2 x-\sin^4 x}\end{aligned}$$

40.

$$\begin{aligned}\text{LHS} = (\sin^2\phi-\cos^2\phi)(\sin^2\phi+\cos^2\phi) &= (\sin^2\phi-\cos^2\phi)(1) \\ &= \sin^2\phi-(1-\sin^2\phi) = 2\sin^2\phi-1\end{aligned}$$

41.

$$\begin{aligned}\text{LHS} = (1+\frac{\sin^2 u}{\cos^2 u})(1-\sin^2 u) &= (\frac{\cos^2 u+\sin^2 u}{\cos^2 u})(1-\sin^2 u) \\ &= (\frac{1}{\cos^2 u})(\cos^2 u) = 1\end{aligned}$$

42. $\cos(\pi/2+\theta) = \cos(\pi/2)\cos\theta-\sin(\pi/2)\sin\theta = -\sin\theta$

43. $\sin(\pi+x) = \sin\pi\cos x+\cos\pi\sin x = -\sin x$

44. $\cos(3\pi/2+\theta) = \cos(3\pi/2)\cos\theta-\sin(3\pi/2)\sin\theta = \sin\theta$

45. $\sin(\pi-x) = \sin\pi\cos x-\cos\pi\sin x = \sin x$

46. $\cos(\pi-x) = \cos\pi\cos x+\sin\pi\sin x = -\cos x$

47. $\sin(3\pi/2+\theta) = \sin(3\pi/2)\cos\theta+\cos(3\pi/2)\sin\theta = -\cos\theta$

48. $\cos(3\pi/2-\theta) = \cos(3\pi/2)\cos\theta+\sin(3\pi/2)\sin\theta = -\sin\theta$

49. Using No. (43) we see that

$$\begin{aligned}\text{LHS} = \tan(\pi+x) &= \frac{\sin(\pi+x)}{\cos(\pi+x)} = \frac{-\sin x}{\cos\pi\cos x-\sin\pi\sin x} \\ &= \frac{-\sin x}{-\cos x} = \tan x\end{aligned}$$

50. Using Nos. (45) and (46) we get

$$\begin{aligned}\text{LHS} = \tan(\pi-x) &= \frac{\sin(\pi-x)}{\cos(\pi-x)} = \frac{\sin x}{-\cos x} \\ &= -\tan x\end{aligned}$$

Chapter 11

APPENDIX D

11.1 Exercise Set 48 (page 515)

1. $(-\infty, \infty)$.
2. All reals except $\pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2}$, ... (these are the points where $\cos t = 0$.)
3. $z \geq -\frac{2}{3}$.
4. $|x| < 2$. (You can also write this as $-2 < x < 2$.)
5. All reals except $0,\ \pm\pi,\ \pm 2\pi,\ ....$ (these are the points where $\sin x = 0$.)
6. $(-\infty, \infty)$.
7. $(-\infty, \infty)$.
8. $(-\infty, \infty)$. Note that $t^{4/5}$ is defined for all values of t.
9. $x \neq 1$.
10. $z \geq -\frac{11}{3}$
11. $|x| > 1$. (That is, either $x > 1$ or $x < -1$.)
12. $\square^2 \leq 1$. (You can also write this as $|\square| \leq 1$, or $-1 \leq \square \leq 1$.)
13. The natural domain of f is $-\infty < \triangle < \infty$.
14. $\heartsuit \geq 0$.
15. $x^2 < 1$. (You can also write this as $|x| < 1$, or $-1 < x < 1$.)

11.2 Exercise Set 49 (page 523)

;

1. Correction: If $A < 0$, then $-A < B$ implies $-1/A > 1/B$.
2. This is false. To see this, let $A = 1$ and $B = 0$.
3. Correction: $0 \leq A < B$ implies $A^2 < B^2$.
4. Correction: $A > B > 0$ implies $1/A < 1/B$.
5. Correction: $A < B$ implies $-A > -B$.

6. Correction: If $A^2 < B^2$ and $B > 0$, then $A < B$.

7. This statement is correct. There is nothing wrong!

8. $(0, \pi)$. (Note: To complete our argument we need $\sin x > 0$, which is guaranteed by $0 < x < \pi$.)

9. It's values are less than or equal to 6. Actually, its largest value occurs when $x = 2$ in which case $f(2) \approx 5.8186$.

10. g is unbounded: This means that it can be greater than (resp. less than) any given number. The problem occurs at $x = 0$.

11. From $x > 1$ we see that both x and $x - 1$ are positive. Hence we can square both sides of the inequality $x > x - 1$ to arrive at $x^2 > (x-1)^2$. (Alternately, since both x and $x-1$ are positive, $x^2 > x^2 - x - (x-1) = x^2 - 2x + 1 = (x-1)^2$.)

12. From $p \leq 1$ we see that $1 - p \geq 0$. Since $x \geq 1$ (certainly this implies the positivity of x), we have $x^{1-p} \geq 1^{1-p}$, or $x^{1-p} \geq 1$. Now $x^{1-p} = x^{-(p-1)} = \frac{1}{x^{p-1}}$. So the last inequality can be rewritten as $\frac{1}{x^{p-1}} \geq 1$. We can multiply both sides of this inequality by $\sin x$ because $1 \leq x \leq \pi$ guarantees that $\sin x$ is positive.

13. Since both x and x^2 are ≥ 0, we can apply the AG-inequality to get $\frac{x+x^2}{2} \geq \sqrt{x \cdot x^2} = \sqrt{x^3}$. Since $x + x^2 \geq 0$, we have $x + x^2 \geq \frac{x+x^2}{2}$. So $x + x^2 \geq \sqrt{x^3}$. Yes, we can square both sides since $x \geq 0$, and so both terms in the inequality are greater than or equal to 0.

14. Yes. Under no further conditions on the symbol, since it is true that $(\square - 1)^2 \geq 0$ for any symbol, $\square$. Expanding the square and separating terms we get that $\square^2 \geq 2\square - 1$.

15. Since $1 - p \geq 0$ and $|x| \geq 1$, we have $|x|^{1-p} \geq 1^{1-p} = 1$, or $|x|\ |x|^{-p} \geq 1$, which gives $|x| \geq |x|^p$. Taking reciprocals, we get $\frac{1}{|x|} \leq \frac{1}{|x|^p}$. (The last step is legitimate because both $|x|^p$ and $|x|$ are positive.)

16. $|v| < c$. This is because we need $1 - v^2/c^2 > 0$. Now solve this inequality for v.

17. If $n = 2$, the result is clear, because $2 < (1.5)^2 < 3$. So let's assume that $n > 2$, now. We use (11.9) with the quantity "$1/n$" inside the box symbol (or replacing the box by $1/n$, if you like). We'll see that

$$\begin{aligned}\left(1 + \boxed{\tfrac{1}{n}}\right)^n &= 1 + n\boxed{\tfrac{1}{n}} + \tfrac{n(n-1)}{2!}\boxed{\tfrac{1}{n}}^2 + \cdots + \tfrac{n(n-1)\cdots(2)(1)}{n!}\boxed{\tfrac{1}{n}}^n \\ &= 1 + n(\tfrac{1}{n}) + \tfrac{n(n-1)}{2!}(\tfrac{1}{n})^2 + \cdots + \tfrac{n(n-1)(n-2)\cdots(2)(1)}{n!}(\tfrac{1}{n})^n. \\ &= 1 + n(\tfrac{1}{n}) + \tfrac{n(n-1)}{n^2}(\tfrac{1}{2!}) + \cdots + \tfrac{n(n-1)(n-2)\cdots(2)(1)}{n^n}(\tfrac{1}{n!}).\end{aligned}$$

Now, we regroup all the terms in the above display in the following way .... Note that the following term is not apparent in the display above, but it IS there! See Equation (11.9).

$$\begin{aligned}\tfrac{n(n-1)(n-2)}{n^3} &= \left(\tfrac{n}{n}\right)\left(\tfrac{n-1}{n}\right)\left(\tfrac{n-2}{n}\right) \\ &= (1)\left(\tfrac{n-1}{n}\right)\left(\tfrac{n-2}{n}\right) \\ &= \left(\tfrac{n-1}{n}\right)\left(\tfrac{n-2}{n}\right), \\ &= \left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right).\end{aligned}$$

A similar idea is used for the other terms. Okay, so using this rearrangement of terms we can rewrite $(1 + \frac{1}{n})^n$ as

$$\begin{aligned}\left(1 + \tfrac{1}{n}\right)^n &= \ldots \\ &= 1 + 1 + \left(1 - \tfrac{1}{n}\right)\tfrac{1}{2!} + \left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right)\tfrac{1}{3!} + \ldots \\ &+ \left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right)\left(1 - \tfrac{3}{n}\right)\cdots\left(1 - \tfrac{n-1}{n}\right)\tfrac{1}{n!}.\end{aligned}$$

(where there are $(n+1)$ terms in the right hand side). Now, notice that for every integer $n > 2$, each term of the form "$1 - (something)/n$" is less than 1 and bigger than zero, because we're subtracting something positive from 1. So,

$$\begin{aligned} \left(1-\frac{1}{n}\right) &< 1 \\ \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) &< (1)\left(1-\frac{2}{n}\right) < 1, \ldots \end{aligned}$$

where we have used Figure 242 with $A = 1 - 2/n$, $\square = 1 - 1/n$ (or with the symbols "$1 - 1/n$" inside the box), and $\triangle = 1$ (or with "1" inside the triangle). Using these estimates we can see that we can replace every term inside the "large brackets" by 1 so that

$$\begin{aligned} \left(1+\tfrac{1}{n}\right)^n &= \ldots \\ &= 1 + 1 + \left(1-\tfrac{1}{n}\right)\tfrac{1}{2!} + \cdots + \left(1-\tfrac{1}{n}\right)\left(1-\tfrac{2}{n}\right)\left(1-\tfrac{3}{n}\right)\cdots\left(1-\tfrac{n-1}{n}\right)\tfrac{1}{n!} \\ &< 1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \cdots + \tfrac{1}{n!} \end{aligned} \tag{11.1}$$

We're almost done! Now we use the following inequalities ...

$$\begin{aligned} 3! = 3\times 2\times 1 &> 2\times 2\times 1 = 2^2 \\ 4! = 4\times 3\times 2\times 1 &> 2\times 2\times 2\times 1 = 2^3 \\ 5! = 5\times 4\times 3\times 2\times 1 &> 2\times 2\times 2\times 2\times 1 = 2^4 \\ &\cdots \\ n! &> 2^{n-1} \end{aligned}$$

Now since we must "reverse the inequality when we take reciprocals of positive numbers" (Table 1.1, Table 1.2) we get that for every integer $n > 2$,

$$n! > 2^{n-1} \qquad \text{implies} \qquad \frac{1}{n!} < \frac{1}{2^{n-1}}$$

Combining this estimate with Equation (11.1) we get a new estimate, namely,

$$\begin{aligned} \left(1+\tfrac{1}{n}\right)^n &< 1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \cdots + \tfrac{1}{n!} \\ &< 1 + 1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \cdots + \tfrac{1}{2^{n-1}}. \end{aligned} \tag{11.2}$$

Now, the sum on the right above is a **finite geometric series** and we know that, if $n > 2$,

$$1 + \tfrac{1}{2^1} + \tfrac{1}{2^2} + \cdots + \tfrac{1}{2^n} = \tfrac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}} < \tfrac{1}{1-\frac{1}{2}} = 2.$$

Now you can see that, when we combine this latest estimate with (11.2) we find

$$\begin{aligned} \left(1+\tfrac{1}{n}\right)^n &< 1 + 1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \cdots + \tfrac{1}{2^{n-1}} \\ &< 1 + 2 = 3 \end{aligned}$$

which is what we wanted to show. Okay, this looks a bit long, but we did include all the details, right? Eventually, you'll be able to skip many of the details and do them in your head, so to speak, and the whole thing will get shorter and faster, you'll see.

NOTE: This looks tough, but we'll be using this 200 yr old inequality later on, in Chapter 4, when we define Euler's number, 2.7182818284590.

Chapter 12

APPENDIX E

12.1 Special Exercise Set (page 530)

1. The car's final (metric) speed is

$$60\frac{\text{miles}}{\text{hr}}\frac{1}{60}\frac{\text{hr}}{\text{min}}\frac{1}{60}\frac{\text{min}}{\text{sec}}1.61\frac{\text{km}}{\text{miles}}1000\frac{\text{m}}{\text{km}} = 26.83\frac{m}{sec}.$$

   Its acceleration is therefore $26.83/3.05 = 8.8\text{m/sec}^2$. Its velocity, $v(t)$, is therefore given by the formula $v(t) = 8.8t$ (basic physics). Its linear distance, $x(t)$, is therefore given by $x(t) = 8.8t^2/2 + v_0t + x_0$, where $x_0 = 0$, $v_0 = 0$ is its initial velocity at $t = 0$. Since $x(t) = 4.4t^2$, its distance after 3.05 seconds is $x(3.05) = (4.4)(3.05)^2 = 40.9$ meters. Since this exceeds 25 meters and the motion is necessarily a continuous function of time, the result follows.

2. This is easy by an immediate application of the IVT because every number between 0 and 8611 must lie somewhere on the graph of every curve that starts at sea level 0, lies on the mountain, and ends at the peak!

3. Recall that any three points determine a unique plane. Let's label the legs A, B, C, D in a clockwise fashion. In this case, three of the table's legs will rest on a plane, say, A, B, C, and the plane must be the floor itself,- the end of the fourth leg, D, must then be above the floor (or else there wouldn't be a problem). Now start rotating the table clockwise while always keeping A, B, and C touching the floor at the same time. What can happen? Well, at some point one of the legs A, B, C, may be the ones that are above the floor! In this case, it's because you can think of leg D as having gone "into" the floor. So, if D is above the floor at the beginning of the rotation and "below" the floor at some other point during the rotation there must be some rotation that will make leg D lie on the floor exactly, and so the table balances! This is a neat application of the IVT.

4. We know from basic physics that temperature varies continuously at any point of the room. Let A be the point at which the temperature is 36^o and let B be a point where the temperature is 14^o. Take the straight line joining A to B as our function (we'll assume there are no obstructions between A and B). Then this is a continuous curve in the room, i.e., the function defined by the line is continuous. By the IVT there must be some point along this line where the temperature is 20^o (because 20 is between 14 and 36). That does it. (There may be many such points but we only need one.)

5. Had you traveled from TO to NY at 65 mph ALL the time (and non-stop) then it would have taken you roughly 7.5 hours to get there. But it only took you 6 hours. So, of course you had to have gone faster than 65 mph and so broken the law. If you had never reached 81 mph and had gone, say, at speeds of only up to 80 mph, then you couldn't have reached either city from the other in exactly 6 hours (in fact it would have taken you at least $491/80 = 6.13$ hours to get to your destination.

6. Let f be a continuous function on an interval $[a, b]$ with $f(a)f(b) < 0$. Let's assume that $f(a) > 0$ and $f(b) < 0$. Then by the IVT if we are given a point $f(a) < z < f(b)$ then there is a point c between a and b such that $f(c) = z$. Let $z = 0$ be given (clearly $z = 0$ lies between $f(a)$ and $f(b)$). Then by the IVT there must be a point c such that $f(c) = 0$, which is what what we wanted to show. (Bolzano's theorem is useful in finding roots of equation by approximation.)

12.2 Exercise Set 50 (page 538)

1. $y(x) = 3x - 2$ is continuous on $[0,\ 2]$ and $y(0) = -2 < 0$, $y(2) = 4 > 0$.
2. $y(x) = x^2 - 1$ is continuous, $y(-2) = 3 > 0$ and $y(0) = -1 < 0$.
3. $y(x) = 2x^2 - 3x - 2$ is continuous, $y(0) = -2 < 0$ and $y(3) = 7 > 0$.
4. $y(x) = \sin x + \cos x$ is continuous on $[0,\ \pi]$, $y(0) = 1 > 0$ and $y(\pi) = -1 < 0$.
5. $y(\pi) = -\pi < 0$. But $y(0) = 0$; so 0 is already a root. Try another point instead of 0, say $\frac{\pi}{2}$: $y(\frac{\pi}{2}) = \frac{\pi}{2} \cdot 0 + 1 = 1 > 0$. So there is a root in $[\frac{\pi}{2}, \pi]$ and hence in $[0, \pi]$ (besides the root 0.)
6. (This is hard.) In the proof we use several times the following basic fact in differential calculus: if the derivative of a function is identically zero, then this function must be a constant. Let's begin by applying this fact to the function y'': its derivative $y''' = 0$ implies that y'' is a constant, say $y'' = a$. Let $u = y' - ax$. Then $u' = y'' - a = 0$ and hence u is a constant, say $u = b$, that is, $y' - ax = b$. Let $v = y - \frac{a}{2}x^2 - bx$. Then $v' = y' - \frac{a}{2} \cdot 2x - b = 0$ and hence v is a constant, say $v = c$. Thus $y - \frac{a}{2}x^2 - bx = c$, or $y = \frac{a}{2}x^2 + bx + c$. We can finish the proof by setting $A = \frac{a}{2}$, $B = b$ and $C = c$.
7. From the assumption that $\frac{dy}{dx} + y(x)^4 + 2 = 0$, we know that $\frac{dy}{dx}$ exists on $(a,\ b)$, and $y(x)$ is continuous on $[a, b]$. Assume the contrary that there are two zeros in $[a, b]$, say x_1, x_2. Using the **Mean Value Theorem,** we see that there exists some c between x_1 and x_2 (*a fortiori*, between a and b), such that $\frac{dy}{dx}(c) = 0$. Thus $y(c)^4 + 2 = 0$. **Impossible!** So there cannot be two zeros for $y(x)$.
8. Consider the function $y(x) = x - \sin x$. By the Mean Value Theorem we see that, for each $x > 0$, there exists some c between 0 and x such that $y(x) - y(0) = y'(c)(x - 0)$, or $x - \sin x = y'(c)x$; (notice that $y(0) = 0$.) Now $y'(x) = 1 - \cos x$, which is always ≥ 0. So, from $x > 0$ and $y'(c) \geq 0$ we see that $y'(c)x \geq 0$. Thus $x - \sin x \geq 0$, or $\sin x \leq x$.
9. Use Rolle's Theorem on $[0, \pi]$ applied to the function $f(x) = \sin x$. Since $f(0) = f(\pi) = 0$, we are guaranteed that there exists a point c inside the interval $(0, \pi)$ such that $f'(c) = \cos c = 0$. This point c is the root we seek.
10. Note that $(\sin x)' = \cos x \leq 1$. For any x in $[0, \frac{\pi}{2}]$, the function $\sin x$ satisfies all the conditions of the **Mean Value Theorem** on $[x,\ \frac{\pi}{2}]$. So, there exists c in $(x,\ \frac{\pi}{2})$ such that

$$\frac{\sin\frac{\pi}{2} - \sin x}{\frac{\pi}{2} - x} = \cos c \leq 1.$$

This statement is equivalent to the stated inequality, since $\sin(\pi/2) = 1$.

11. (a) $\frac{f(2)-f(0)}{2-0} = \frac{5-(-1)}{2} = 3$ and $f'(c) = 2c + 1 = 3$ give $c = 1$.
(b) $\frac{g(1)-g(0)}{1-0} = \frac{4-3}{1} = 1$ and $g'(c) = 2c = 1$ give $c = \frac{1}{2}$.
12. Let $x(t)$ denote the distance travelled (in meters) by the electron in time t. We assume that $x(0) = 0$ and we are given that $x(0.3 \times 10^{-8}) = 1$. Now apply the MVT to the time interval $[0, 0.3 \times 10^{-8}]$. Then,

$$\frac{x(0.3 \times 10^{-8}) - x(0)}{0.3 \times 10^{-8} - 0} = x'(c),$$

for some time $t = c$ in between. But this means that the speed of the electron at this time $t = c$ is $\frac{1}{0.3\times10^{-8}} = 3.3 \times 10^8$ m/sec, which is greater than 2.19×10^8 m/sec, or the speed of light in that medium!