Space Vectors, Lines, and Planes

In many applications to the sciences and engineering we come across quantities that have both *magnitude* (or size) and *direction.* For example, velocity, acceleration, force, torque, etc. are called vectors. Then there are concepts that involve size or magnitude only. For example, mass, pressure, length, temperature, etc. are such examples and we call these **scalars**. It follows that scalars are just represented by ordinary numbers. Once and for all we identify vectors with points whether in two or in three-dimensional space. In engineering applications our vectors are thus called **free vectors** (as opposed to **rigid vectors**, that is, vectors actually tied down to a point along a beam etc.).

Geometrically, a vector is a *directed line segment.* The symbol $\mathbf{r}$ will usually denote the **position vector**, *e.g.,* The point (x, y, z) is associated with the position vector (a directed line segment) whose **initial point** is the origin of the space we are dealing with and whose **terminal point** is the point (x, y, z) itself. The coordinates x, y, z are called the **components of the vector**. Two vectors are equal if and only if they have the same length and direction. Thus, the vector $\mathbf{PQ}$, originating at $P(-1, 2, 5)$ and ending at $Q(-3, -7, 0)$ is equivalent (or equal to) to the position vector $\mathbf{r} = (0 - 5, -7 - 2, -3 - (-1)) = (-5, -9, -2)$. We can then write $\mathbf{PQ} = \mathbf{r}$.

The vectors $\mathbf{a}$ and $-\mathbf{a}$ differ only in their direction (they must have the same length), and they have opposite direction (sometimes called *antiparallel*). The **length of a vector** is simply its length as a line segment. Once the vector is represented by a point (x, y, z), say, then its length is given by the distance from that point to the origin, that is: The length of the position vector (x, y, z) is then $\sqrt{x^2 + y^2 + z^2}$. On the other hand, if the vector is from the point (x_1, y_1, z_1) to (x_2, y_2, z_2) then its length is given by the length of its corresponding position vector $\mathbf{r} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. It follows that this length, denoted by the symbol $|\mathbf{r}|$, is given by $|\mathbf{r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. A **unit vector** is, by definition, a vector having its length equal to one unit. The zero vector is denoted by $\mathbf{0}$. Our vectors satisfy the axioms of a **real vector space** as defined in texts on *Linear Algebra.*

Thus, the **sum** (or *resultant*) of two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by the usual **parallelogram law** (see the margin). Vectors are said to be **parallel** if they have the same direction and they are **perpendicular** (or orthogonal) if they are at right angles to one another (or cross each other at a right angle). For example, the vectors $(0, 0, 1)$ and $(0, 1, 0)$ are orthogonal to each other. On the other hand, the vector $\mathbf{PQ}$, originating at $P(0, 1, 0)$ and ending at $Q(1, 1, 0)$ is parallel to the vector $(0, 0, 1)$.

In order to multiply a vector $\mathbf{a}$ by a scalar c, (the result being denoted by the symbol $c\mathbf{a}$) we simply stretch or shrink the vector $\mathbf{a}$ by a factor of c, its new direction being the same if $c > 0$ or being the opposite direction if $c < 0$. By definition, $0\mathbf{a} = \mathbf{0}$. For example, if $\mathbf{a} = (-5, -9, -2)$ and $c = 0.2$ then $c\mathbf{a} = (-5(0.2), -9(0.2), -2(0.2)) = (-0.1, -1.8, -0.4)$. Any non-zero vector $\mathbf{a}$

having length $|\mathbf{a}|$, has a corresponding unit vector $\mathbf{u}$ found by setting

$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

This vector $\mathbf{u}$ has length equal to one unit and is in the same direction as $\mathbf{a}$.

In order to simplify the operations between vectors we introduce the **standard basis vectors** or **rectangular unit vectors** in three-dimensional space: They are denoted by $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, and $\mathbf{k} = (0,0,1)$. Sometimes they are also represented by the symbols $\mathbf{e_1} = (1,0,0)$, $\mathbf{e_2} = (0,1,0)$, and $\mathbf{e_3} = (0,0,1)$. The advantage of this notation is that we can write any point/vector $\mathbf{A} = (A_1, A_2, A_3)$ as

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}.$$

Example 1 *The vectors* $\mathbf{a} = (1,3,-5)$ *and* $\mathbf{b} = (-1,-3,5)$ *have the same length and opposite directions. Their sum* $\mathbf{a}+\mathbf{b} = \mathbf{0}$. *Their difference is given by the vector* $2\mathbf{a} = (2,6,-10)$. *The vector* $(-0.3)\mathbf{a} = (-0.3, 0.9, -1.5)$. *The length of* $\mathbf{a}$, *(or even* $-\mathbf{a}$*), that is* $|\mathbf{a}|$, *is given by* $|\mathbf{a}| = \sqrt{35}$. *Finally,* $\mathbf{a}$ *may be represented by* $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$.

The main operations between vectors are called the **dot product** which is a scalar/number, (or *scalar product*) and the **cross product** which is a vector. Their importance lies in the geometrical consequences of some of their values.

For instance, if $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ then

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

is the *dot product* of a and b. Note that $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ since multiplication of scalars is commutative. On the other hand the vector, $\vec{a}\times\vec{b}$, defined by the determinant relation

$$\vec{a} \times \vec{b} = \begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = (a_2b_3 - a_3b_2)\vec{\imath} - (a_1b_3 - b_1a_3)\vec{\jmath} + (a_1b_2 - b_1a_2)\vec{k}$$

is called the *cross product* of $\vec{a}$ and $\vec{b}$. Since the interchange of any two rows in a matrix changes the sign of the determinant it follows that $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$. We call this relationship of the cross product *anticommutative.*

Figure 1

The fundamental geometric relationships of the dot and cross products follow. The symbols $\vec{a} \perp \vec{b}$ means that $\vec{a}$ is orthogonal to $\vec{b}$. Similarly, the symbols $\vec{a} \parallel \vec{b}$ means that $\vec{a}$ is parallel to $\vec{b}$ (but not necessarily in the same direction). One can show that

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\,\cos\theta,\ \ 0 \le \theta \le \pi$$

$$\vec{a}\cdot\vec{b} = 0 \quad \text{is equivalent to} \quad \vec{a}\perp\vec{b}$$

$$\vec{a}\times\vec{b} = \Big(|\vec{a}|\,|\vec{b}|\,\sin\theta\Big)\,\vec{u},\ \ 0 \le \theta \le \pi$$

where $\vec{u}$ is a unit vector in the direction of $\vec{a}\times\vec{b}$

$$\vec{a}\times\vec{b} = 0 \quad \text{is equivalent to} \quad \vec{a}\parallel\vec{b}$$

Figure 2

The scalar **triple product** (or *box product*) of the vectors $\vec{A},\vec{B},\vec{C}$ is a scalar (read as "A dot B cross C") given by

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \det\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

whose numerical value (absolute value) is represented geometrically by

$$\underbrace{|\ \vec{A}\cdot(\vec{B}\times\vec{C})\ |}_{\text{magnitude or absolute value of this scalar}} \qquad = \text{volume of a ``tilted box''}$$

see the margin.

Parametric Equations of a Straight Line in Space

Let $P(x_0, y_0, z_0)$ be a given point in space represented by the position vector $\vec{r_0}$. Let $Q(x_1, y_1, z_1)$ be another such point and represented by the position vector $\vec{r_1}$. We will determine the equation of the straight line, denoted by $\mathcal{L}$, through the points P, Q.

To this end let $R(x, y, z)$ be an arbitrary point on the line, $\mathcal{L}$. The vector, $\vec{PQ}$, whose initial point is at P and whose terminal point is at Q then lies <u>on</u> $\mathcal{L}$ (see the margin). Now the vector, let's call it $\vec{PR}$, whose initial point is at P and whose terminal point is now at R also lies on $\mathcal{L}$. It follows that the vectors $\vec{PQ}$ and $\vec{PR}$ are parallel (and so they have the same direction). Thus, they can only differ in their magnitude. This means that there is a <u>scalar</u>, call it t, such that $\vec{PR} = t\,\vec{PQ}$. Expanding this expression in terms of the components of the vectors involved we get,

$$(x - x_0, y - y_0, z - z_0) = t\,(x_1 - x_0, y_1 - y_0, z_1 - z_0).$$

This quantity t has a special name: We call it a **parameter**. The point is that it changes when we change the point R on the line (in this sense it is a *variable* too). Equating both sides in the last display and rearranging terms we find that a point $R(x, y, z)$ lies on $\mathcal{L}$ only if its coordinates (x, y, z) all satisfy the equations

$$\begin{aligned} x &= x_0 + t\,(x_1 - x_0) && (1) \\ y &= y_0 + t\,(y_1 - y_0) && (2) \\ z &= z_0 + t\,(z_1 - z_0), && (3) \end{aligned}$$

where (x_0, y_0, z_0) and (x_1, y_1, z_1) are any two points on the line $\mathcal{L}$ and t is a (scalar) parameter satisfying $-\infty < t < \infty$.

As a check . . . we note that setting $t = 0$ in (0.1-0.3) gives us that the point (x_0, y_0, z_0) is on $\mathcal{L}$. Similarly, if we set $t = 1$ in (0.1-0.3) we get that point (x_1, y_1, z_1) is also on $\mathcal{L}$. This check assures us that we didn't incorrectly calculate the equation of the line. Any other value of t will then give us new points on this line, that's how it works.

Remark: Note that the parametric equations (0.1-0.3) above are an extension of the familiar *slope-intercept formula* for the equation of a straight line in the plane. To see this, we simply note that if our line $\mathcal{L}$ is on the $xy-$plane then the $z-$coordinates of any point on it are all zero and it follows that (0.3) is automatically satisfied (since z_0, z_1, z are all zero). On the other hand, eliminating the $t-$variable from (0.1) and inserting this $t-$value into (0.2) we get

$$y = y_0 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0) = b + mx$$

provided we set

$$m = \frac{y_1 - y_0}{x_1 - x_0}, \quad b = y_0 - \left(\frac{y_1 - y_0}{x_1 - x_0}\right)x_0.$$

This last formula is the slope-intercept formula, of course. We can summarize this by saying that **the equation of any line in the xy−plane may be written in the form (0.1-0.2)**.

Example 2 *Find the parametric equations of the line through the points* $(1, -1, 6)$ *and* $(2, 3, -1)$.

Solution: Write $(x_0, y_0, z_0) = (1, -1, 6)$ and $(x_1, y_1, z_1) = (2, 3, -1)$. It doesn't matter which one is labeled $(x_0 y_0, z_0)$: You will still get the same formula! It follows from (0.1-0.3) that the parametric equations of the line are given by

$$\begin{aligned} x &= 1 + t \\ y &= -1 + 4t \\ z &= 6 - 7t. \end{aligned}$$

As a check we note that if we set $t = 0$ we get the point $(1, -1, 6)$ while if $t = 1$ we get $(2, 3, -1)$. If we set $t = -2$ we get a new point $(-1, -9, 20)$ on the line joining the two given points.

Eliminating the $t-$ variable from the equations (0.1-0.3) and rearranging terms we get **the symmetric equations of a line** defined by

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0},$$

provided all the denominators are non-zero.

Example 3 *Find the symmetric equations of the line through the points* $(1, -1, 6)$ *and* $(2, 3, -1)$.

Solution: Referring to Example 3 above and using the definition we easily find that the symmetric equations are given by

$$x - 1 = \frac{y+1}{4} = \frac{6-z}{7}.$$

Finally, we note that the parametric representation of a straight line may be rewritten in vector form as

$$\vec{r} = \vec{r_0} + t\,\vec{a}$$

where $\vec{r_0} = (x_0, y_0, z_0)$, $\vec{a} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$ is a vector that specifies the direction of the line and t is a parameter, $-\infty < t < +\infty$.

Figure 3

Planes in Space

A plane in space can be thought of as the result of sliding a line in a motion parallel to itself and without any rotation. Thus, a sheet of paper is a section of a plane, the corner of a room is really the intersection of three planes, the meeting of a floor and a wall is the intersection of two planes, etc. It is easy to see that any three points in space in general position (meaning not collinear, or on the same line) lie on one and only one plane. In other words, three points determine a unique plane. Why? It is easiest to consider the triangle whose vertices are each of these three points. This triangle has a *face*, the region inside surrounded by the edges, and this face lies on a plane and this plane must be unique.

Figure 4

Summary: From this we see that 3 points in general position in $\mathcal{R}^3$ (another symbol for three-dimensional space) determine a unique plane because the 3 points determine a unique triangle whose face is a part of the plane it defines.

One can also define a plane via its "normal", denoted by $\vec{n}$, or "a vector perpendicular to it". Think of it: A plane in $\mathcal{R}^3$ can be represented by all those vectors which are perpendicular to a given line (called the **normal to the plane**,- see the margin).

This means that if $\vec{r_0} = (x_0, y_0, z_0)$ is on a plane Π (this is the Greek capital letter "Pi" that we use to denote a generic plane) and $\vec{r} = (x, y, z)$ is any other point on the plane Π, then the vector $\vec{r} - \vec{r_0}$ is on Π also. In other words, the vector $(x - x_0, y - y_0, z - z_0)$ is on Π.

The plane Π can now be thought of as consisting of the collection of all vectors that are perpendicular to the given "normal vector", $\vec{n}$. Since every vector on the plane is of the form $\vec{r} - \vec{r_0}$ for varying points (x_0, y_0, z_0) on the plane, it follows that the plane can be defined by the relation

$$\boxed{\underbrace{(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0}_{\text{Defines a plane containing the vector } \vec{r_0} \text{ and "normal" } \vec{n} \text{ to it.}}}$$

where $\vec{n} = (n_1, n_2, n_3)$ is the vector normal to the plane. In cartesian coordinates, x, y, z, this last boxed display becomes

$$\underbrace{(x - x_0, y - y_0, z - z_0)}_{\vec{r}-\vec{r_0}} \cdot \underbrace{(n_1, n_2, n_3)}_{\vec{n}} = 0$$

or

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) \quad = \quad 0. \tag{4}$$

Expanding and simplifying this latest equation we get the equation of a plane Π in cartesian coordinates in the form,

$$\boxed{\begin{aligned} n_1x + n_2y + n_3z &= \mathit{constant} && (5)\\ &= n_1x_0 + n_2y_0 + n_3z_0. && (6)\end{aligned}}$$

where n_1, n_2, n_3, are the components of $\boxed{\text{a}}$, or *any*, normal vector to Π. The "constant" in (0.5) is given by the quantity $\mathit{constant} = n_1x_0+n_2y_0+n_3z_0$ found by expanding and collecting the constant terms in (0.4), on the right.

Example 4 *Find the equation of the plane* Π *through the points* $A(3, 1, -2)$*,* $B(-1, 2, 4)$ *and* $C(2, -1, 1)$*.*

Solution: All we need is a normal vector ($\vec{n}$) to this plane and a vector ($\vec{r_0}$) on the plane. The normal vector is simply obtained by finding the cross product of *any two* vectors in the list. Which ones? Let's choose the vectors $\vec{BC}$ and $\vec{BA}$. Note that the vectors need to have the same initial point, in which case that point (here B) becomes (x_0, y_0, z_0) in our analysis.

So we find that $\vec{n} = \vec{BC} \times \vec{BA}$ will be orthogonal to both these two vectors (because it is a cross-product). But $\vec{BC} = (2, -1, 1) - (-1, 2, 4) = (3, -3, -3)$ and $\vec{BA} = (-1, 2, 4) - (3, 1, -2) = (-4, 1, 6)$. So,

$$\vec{BC} \times \vec{BA} = \begin{pmatrix} i & j & k \\ 3 & -3 & -3 \\ -4 & 1 & 6 \end{pmatrix} = -15\vec{i} - 6\vec{j} - 9\vec{k}$$

and so we can choose $\vec{n} = (-15, -6, -9)$. The equation of the plane Π is therefore of the form

$$-15x - 6y - 9z = \mathit{constant}$$

where $\mathit{constant} = n_1x_0 + n_2y_0 + n_3z_0 = (-15)(-1) + (-6)(2) + (-9)(4) = -33$. Hence the plane has equation $-15x - 6y - 9z = -33$ or, equivalently,

$$5x + 2y + 3z = 11.$$

As a final check we note that the point $A(3, 1, -2)$ is on this plane since $5(3) + 2(1) + 3(-2) = 11$ as required. The same must be true for the other two given points B, C too.

Example 5 *Find the vector equation of the line of intersection of the two planes* $x + y - z - 5 = 0$ *and* $4x - y - z + 2 = 0$*.*

Solution: Idea: We know that two generic planes in space intersect in a straight line (unless they are parallel, which is not the case here). Since $z = x + y - 5$ and $z = 4x - y + 2$ it follows that on the line of intersection, the z's must be equal, *i.e.,* $x + y - 5 = 4x - y + 2$ from which we find that

$$3x - 2y + 7 = 0.$$

In other words, on the line of intersection, a point (x, y, z) is such that $z = x + y - 5$, say (since the line also belongs to either plane), *and* that these x, y are related by the relation $3x - 2y + 7 = 0$.

Now the vector equation of a line is given by $\vec{r} = \vec{r_0} + t\,\vec{a}$ where $\vec{r_0}$ is on the line and $\vec{a}$ gives its direction. By inspection we choose $\vec{r_0} = (-3, -1, -9) = (x_0, y_0, z_0)$. Note that this point satisfies both $3x - 2y + 7 = 0$ and $z = x + y - 5$. Next, $\vec{a}$ is simply a vector in the direction of the line $3x - 2y + 7 = 0$, *i.e.,* $\vec{a}$ is parallel to this line. To get $\vec{a}$ we just need two points on this line of intersection. We already have one point, $(-3, -1, -9)$, on this line and since a line is uniquely determined by two points we need another point on it, say, $(1, 5, 1)$ (also found by inspection). It follows that

$$\begin{aligned} \vec{a} &= (-3, -1, -9) - (1, 5, 1) \\ &= (-4, -6, -10), \end{aligned}$$

so that the vector equation looks like: $\vec{r} - \vec{r_0} = t\,\vec{a}$ or $(x, y, z) - (-3, -1, -9) = t\,(-4, -6, -10)$. From this we see that the parametric equations of the line are given by

$$\begin{cases} x + 3 &= -4t \\ y + 1 &= -6t \\ z + 9 &= -10t. \end{cases}$$

Eliminating t as usual gives us the symmetric equations of this line, that is,

$$\frac{x + 3}{-4} = \frac{y + 1}{-6} = \frac{z + 9}{-10}.$$

Example 6 *Find the equation of the plane through* $P_0(1, 2, 3)$ *and parallel to* $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$, $\vec{b} = 3\vec{i} + 6\vec{j} - 2\vec{k}$.

Figure 5

Solution: The vector equation of the plane, let's call it Π, is given by

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$$

where $\vec{n}$ is some normal vector and $\vec{r_0}$ is the position vector of some point on the plane. Here we are given $P_0(1, 2, 3)$ so we may as well choose $\vec{r_0} = (1, 2, 3)$. Now for Π to be parallel to *both* $\vec{a}$ and $\vec{b}$ means that the normal vector to Π, that is, $\vec{n}$, is perpendicular to *both* $\vec{a}$ and $\vec{b}$ (think about this). We can proceed in two different ways:

Method 1:

$$\text{If } \vec{n} = (n_1, n_2, n_3) \Rightarrow \quad \begin{aligned} 2n_1 + n_2 - n_3 &= 0 \quad (\text{since } \vec{n} \cdot \vec{a} = 0) \\ 3n_1 + 6n_2 - 2n_3 &= 0 \quad (\text{since } \vec{n} \cdot \vec{b} = 0) \end{aligned}$$

Solving this system of two equations in the three unknowns n_1, n_2, n_3, we get: ($-2\times$ the first equation + the second equation)

$$\begin{array}{rcl} -n_1 + 4n_2 = & 0 & \Rightarrow n_1 = 4n_2 \\ n_3 = & 2n_1 + n_2 & = 8n_2 + n_2 = 9n_2 \end{array}$$

It follows that

$$\begin{array}{rcl} \vec{n} & = & (n_1, n_2, n_3) \\ & = & (4n_2, n_2, 9n_2) \\ & = & (4,1,9)\cdot n_2 \quad \text{where } n_2 \text{ is arbitrary.} \\ \text{So } \vec{n} & = & (4,1,9) \quad \text{is a normal vector to both } \vec{a} \text{ and } \vec{b} \end{array}$$

Finally, $(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$ becomes

$$\begin{array}{rcl} ((x,y,z) - (1,2,3)) \cdot (4,1,9) & = & 0 \\ & \text{or} & \\ \boxed{4x + y + 9z} & = & \boxed{33.} \end{array}$$

Method 2: We choose $\vec{n}$ by finding the cross-product of the two vectors $\vec{a}$, $\vec{b}$ (why?). Thus,

$$\vec{a} \times \vec{b} = \begin{pmatrix} i & j & k \\ 2 & 1 & -1 \\ 3 & 6 & -2 \end{pmatrix} = 4\vec{i} + \vec{j} + 9\vec{k}$$

so that $\vec{n} = (4,1,9)$ and the plane Π must have the form

$$4x + y + 9z = \mathit{constant},$$

where the constant is found, as usual, by substituting the given point $P_0 = (1,2,3)$ which must lie on the plane,in the equation. Thus *constant* $= 4(1) + 1(2) + 9(3) = 33$, and we are done.