Cylindrical and Spherical Coordinates

Cylindrical Coordinates, (r, θ, z) are related to Cartersian coordinates (x, y, z) by:

$x = r \cdot \cos\theta$
$y = r \cdot \sin\theta$
$z = z$

Figure 1

where (r, θ) represent the polar coodinates of the projection of P onto the xy-plane (with polar-coodinates) and z is the "signed" height (or Cartesian height) of P above or below this plane.

$$\begin{aligned} &\text{The volume of a parallelepiped (or fundamental region)} \\ &\text{in cartesian coordinates} = dxdydz \\ &= (r \cdot dr\, d\theta)\, dz \end{aligned}$$

Figure 2

Example 1 $\{r = 2, \quad 0 \le z \le 6\}$ *is a cylinder of height* 6 *units and radius* 2 *units.*

Figure 3

Remark: This can (cylinder) is empty, *i.e.,* the surface described does not include its "top","bottom" or "interior".

On the other hand, the set $\{0 \le r \le 2, \quad 0 \le z \le 6\}$ does contain the "top","bottom" and inside of the can.

Example 2 $\{0 \le r \le 1, \quad 0 \le \phi \le \frac{\pi}{4}, \quad 0 \le z \le 5\}$ *looks like a triangular brick of cheese.*

Figure 4

Example 3 *The cylinder* $x^2 + y^2 = 1\ (z \ge 0)$ *is cut by a plane* $z = x + 2$ *(in 3D). Describe the solid by means of:*

1) Cartesian/ rectangular coordinates

2) Cylindrical coordinates

Solution:

Figure 5

1) The base of this region is given by:

$$\{(x, y, z) : 0 \le x^2 + y^2 \le 1, \quad z = 0\}$$

1. With base point $P(x, y, z)$ (because $z = 0$ on base)
   {This turns out to be the projection of our solid onto the xy-plane.}
   Make a slice vertically up until it intersects the surface $z = x + 2$.
2. Find points of intersection of this slice with our solid.
3. Now write down the inequalities which describe this slice.

$$T_{x,y} = \{(x, y, z) : 0 \leq z \leq x + 2, \quad 0 \leq x^2 + y^2 \leq 1\}$$

4. Put these T's together to reconstruct the solid!

$$\begin{aligned}
\text{The solid} &= \{T_{x,y} : (x, y, 0) \ \text{“lie on base”}\} \\
&= \{T_{x,y} : 0 \leq x^2 + y^2 \leq 1\} \\
&= \{(x, y, z) : 0 \leq z \leq x + 2, \quad 0 \leq x^2 + y^2 \leq 1\} \\
&= \{(x, y, z) : 0 \leq z \leq x + 2, \quad -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}, \quad -1 \leq x \leq 1\}
\end{aligned}$$

2)

$$\{(r, \phi, z) : \quad 0 \leq z \leq r \cos\theta + 2, \quad 0 \leq r \leq 1, \ 0 \leq \theta < 2\pi\}$$

Spherical Coordinates, (r, θ, ϕ).

Spherical coordinates are based on spheres.

When to use these?

When regions are made up of spheres or sections thereof, like hemispheres, etc.

Spherical coordinates of point P in 3D are given by:

$$P(r, \theta, \phi) \text{ where } r^2 = x^2 + y^2 + z^2$$

Figure 6

$$x = r \sin\phi \cdot \cos\theta$$
$$y = r \sin\phi \sin\theta$$
$$z = r \cos\phi$$

where, in this case, the Jacobian is given by

$$\text{Jacobian} = r^2 \sin\phi.$$

element of volume in spherical coordinates $= r^2 \sin\phi dr d\phi d\theta$. Always introduce factor $r^2 \sin\phi$ when changing from cartesian to spherical coordinates.

Example 4 *How to describe an ice cream cone with or without the goods!*

Solution: Its surface only!

1) We called the angle subtended between the positive z-axis and the extremity of the cone $\phi = \phi_0$
2) $0 \leq \theta < 2 \cdot \pi$ (gives "spin" to the flat figure)

so the region is given by:

$$\begin{aligned} \text{cone surface} &= \{(r,\theta,\phi) : r = R, \quad 0 \leq \phi \leq \phi_0, \quad 0 \leq \theta < 2\pi\} \\ &\bigcup \{(r,\theta,\phi) : 0 \leq r < R, \quad \phi = \phi_0, \quad 0 \leq \theta < 2\pi\} \\ \text{therefore} &= \boxed{1} \bigcup \boxed{2} \end{aligned}$$

Empty $\{r = R,\ 0 \leq \phi \leq \phi_0, \quad 0 \leq \theta < 2\pi\}$

Full $\{0 \leq r \leq R, \quad 0 \leq \phi \leq \phi_0, \quad 0 \leq \theta < 2\pi\}$

Figure 8

Example 5 *Infinite cone (solid):*

Solution: $C = \{(r,\theta,\phi) : 0 \leq r < \infty, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \phi_0\}$

Example 6 *Infinite cone including (shell) surface only*

Solution: $C = \{(r,\theta,\phi) : 0 \leq r < \infty, \quad 0 \leq \theta < 2\pi,\ \phi = \phi_0\}$

Example 7 *Describe the region defined by a hemispherical cap whose center is at the point* $(0,0,1)$ *and having radius equal to* 1 *unit in spherical coordinates.*

Solution: In order to describe this region we need to take an arbitrary point P inside it and then estimate the range of the corresponding variables (r,θ,ϕ). Okay, so we are given a hemisphere centered at $(0,0,1)$ (cartesian coordinates of radius 1 above $z = 1$). The set $\Pi = \{(r,\theta,\phi) : \theta = \theta_0, \quad 0 \leq r < \infty, \quad 0 \leq \phi < \pi\}$ is a plane $\perp$ to the xy-plane containing the line $\theta = \theta_0$ (as viewed on the $xy-$plane).

Figure 9

Produce the line segment OP', from the origin O to a point P' ON the hemisphere. We see that the distance $|OP'|$ must be a function of ϕ *alone* (where ϕ is the usual angle measured from the $z-$axis to the radial line OP'.)

Figure 10

Now we want to determine the length, $|OP|$, as a function of ϕ, since $r = |OP|$ where r is the r-coordinate of our system of spherical coordinates. So,

1. Fix $\theta = \theta_0$ $(0 \le \theta < 2\pi)$

   This relation defines a plane that cuts the hemisphere in a semicircle which on the plane $\theta = \theta_0$ looks like a semicircle whose base is on the plane $z = 1$. Note that this picture is planar, not in space.

Figure 11

2. Now, we **need a radial slice** of our region. So, fix $\phi = \phi_0$ (where $0 \le \phi_0 \le \frac{\pi}{4}$). Let the slice intersect the semicircle's base at S and let the positive $z-$axis intersect the semicircle's base at T. Once we do this (ie. fix ϕ & θ) all "that's left" is the small section of line segment SP' (making up the slice) and lying inside our semicircle.

   Consider triangle OST. Since $|OT| = 1$ and $\angle TOS = \phi_0$ it follows that $1 = r\cos\phi_0$ from which $|OS| = \sec\phi_0$, which means that the $r-$coordinate of S is $\sec\phi_0$. **Remember that we need to estimate the variation of the coordinates of a point $P(r, \theta, \phi)$ ON the slice OP' and INSIDE the semicircle**.

3. We find the coordinates of S: At this time it is clear that O, S, P' all have the same θ and ϕ values (since they all lie on the same plane $\theta = \theta_0$ and line $\phi = \phi_0$). All that changes is their $r-$values! But the coordinates of S are

   $$S(\sec\phi_0, \theta_0, \phi_0)$$

   while the coordinates of P' are $P'(r, \theta_0, \phi_0)$ where we still have to determine the length, $|OP'|$, in order to get a value for r, the $r-$coordinate of P'. In other words, once we know $|OP'|$ we also know the $r-$coordinate of P'.

Figure 12

4. We find the coordinates of P': To find the value of $|OP'|$ we produce the line segment TP' and consider $\Delta OTP'$. Since $|OT| = |TP'| = 1$ this triangle is isosceles and so $\angle TOS = \angle TP'O = \phi_0$. This means that $\angle OTP' = \pi - 2\phi_0$. We now use the Cosine Law to find the length of OP'.

   $$\begin{aligned} |OP'|^2 &= |OT|^2 + |TP'|^2 - 2|OT|\,|TP'| \cos\angle OTP' \\ &= 1^2 + 1^2 - 2\cdot 1\cdot 1\cdot \cos(\pi - 2\phi_0) \\ &= 2 - 2\cos(\pi - 2\phi_0) \\ &= 2 + 2\cos(2\phi_0) = 4\,\frac{1+\cos(2\phi_0)}{2} \\ &= 4\cos^2\phi_0. \end{aligned}$$

   Therefore, $|\,OP'\,| = 2\cos\phi_0$. This means that $r-$coordinate of P' is given by $2\cos\phi_0$. Thus, the coordinates of P' are given by

   $$P'(2\cos\phi_0, \theta_0, \phi_0).$$

5. Now take an arbitrary point

   $$P(r, \theta_0, \phi_0)$$

   ON the line segment SP'. Comparing the coordinates of P with those of the extremities of SP' we get, $S(\sec\phi_0, \theta_0, \phi_0)$ and $P'(2\cos\phi_0, \theta_0, \phi_0)$ above we see that necessarily,

   $$\sec\phi_0 \le r \le 2\cdot\cos\phi_0.$$

6. Finally, since ϕ_0 can take on any value of ϕ where $0 \leq \phi \leq \frac{\pi}{4}$, and θ spins around a full circle (or 2π radians), we see that the set

$$\{(r, \theta, \phi) : \sec\phi \leq r \leq 2\cos\phi,\ 0 \leq \phi \leq \frac{\pi}{4},\ 0 \leq \theta < 2\pi\}$$

describes the hemisphere centered at $(0, 0, 1)$ with radius 1 unit.