Space curves and surfaces

A space curve is defined parametrically by:

$$a \leq t \leq b = \begin{cases} x = & f(t) \\ y = & g(t) \\ z = & h(t) \end{cases}$$

Figure 1

For each t in some interval, there is a point on the curve.

The tangent line at a point $P(x_0, y_0, z_0) = (f(t_0), g(t_0), h(t_0))$ or at $t = t_0$ is given by:

$$\frac{x - x_0}{f'(t_0)} = \frac{y - y_0}{g'(t_0)} = \frac{z - z_0}{h'(t_0)}$$

The equation of the normal plane (through $P_0 \perp$ tangent line then) is:

$$f'(t_0)(x - x_0) + g'(t_0)(y - y_0) + h'(t_0)(z - z_0) = 0$$

(because the vector $(f'(t_0), g'(t_0), h'(t_0))$ is the normal vector to this plane (and the tangent vector to C).

A surface is a collection of points (x, y, z) where $F(x, y, z) = 0$ for some function F defined on $\mathcal{R}^3$ (3-dimensional space) with values in $\mathcal{R}$.

The tangent plane to the surface $F(x, y, z) = 0$ at a point $P_0(x_0, y_0, z_0)$ is:

$$\left.\frac{\partial F}{\partial x}\right|_{P_0} (x - x_0) + \left.\frac{\partial F}{\partial y}\right|_{P_0} (y - y_0) + \left.\frac{\partial F}{\partial z}\right|_{P_0} (z - z_0) = 0$$

and the normal line at P_0 is given by:

$$\boxed{\frac{x - x_0}{\frac{\partial F}{\partial x}(P_0)} = \frac{y - y_0}{\frac{\partial F}{\partial y}(P_0)} = \frac{z - z_0}{\frac{\partial F}{\partial z}(P_0)}}$$

The normal vector to the tangent plane given above is usually called the **gradient of F** and is denoted by the symbol

$$\nabla F(P_0) = \left(\frac{\partial F}{\partial x}(P_0), \frac{\partial F}{\partial y}(P_0), \frac{\partial F}{\partial z}(P_0) \right)$$

Example 1 *Find the tangent line and the normal plane if:*

$$x = 2t, \quad y = t^2, \quad z = t^3, \quad t = 1$$

Solution:

1) <u>Tangent line</u>: At $t = 1$, $P_0 = (2, 1, 1)$ is on the curve. Also $f'(t_0) = f'(1) = 2$, $g'(t_0) = g'(1) = 2$, $h'(t_0) = h'(1) = 3$

   Therefore the equation is:

$$\frac{x-2}{2} = \frac{y-1}{2} = \frac{z-1}{3}$$

2) <u>Normal Plane</u>: $2(x-2) + 2(y-1) + 3(x-1) = 0$ or

$$2x + 2y + 3z = 9$$

Example 2 *Find the equation of the tangent line and the normal plane to the curve defined by the intersection of the curve defined by the intersection of the surfaces $x^2 + z^2 = 5$ and $y^2 + z^2 = 8$ at $P(1, 2, 2)$.*

Solution: The curve consists of all points $P(x, y, z)$ that are common to *both* the surfaces defined. It follows that such a point, in this case the point $P(1, 2, 2)$, must satisfy both the relations $x^2 + z^2 = 5$ and $y^2 + z^2 = 8$ (and it does, as one can check easily). Okay, but what does the curve look like, or, how does one describe it? We only need to parametrize this curve ...

In order to parametrize the curve we set $x = t$, the basic value of the parameter. If $x = t$ then it follows that $z = \sqrt{5 - x^2}$, or $z = \sqrt{5 - t^2}$ (we use the positive square root because we are dealing with the point P where the $z-$coordinate is positive). What about y? On the other hand, since $y^2 + z^2 = 8$, we know that $y = \sqrt{8 - z^2}$ (once again, we use the positive square root) and so, since $z = \sqrt{5 - t^2}$, we get $y = \sqrt{8 - (5 - t^2)} = \sqrt{3 + t^2}$. The curve is therefore parametrized by

$$\mathbf{r}(t) = \left(t, \sqrt{3 + t^2}, \sqrt{5 - t^2}\right).$$

Note that when $t = 1$ we get the point $P(1, 2, 2)$. The tangent vector to this curve is now given by

$$\mathbf{r}'(t) = \left(1, \frac{t}{\sqrt{3 + t^2}}, -\frac{t}{\sqrt{5 - t^2}}\right),$$

whose value at $t = 1$ gives the vector $(1, \frac{1}{2}, -\frac{1}{2})$. The equation of the tangent line is therefore given by the symmetric equations

$$\frac{x-1}{1} = \frac{y-2}{\frac{1}{2}} = \frac{z-2}{-\frac{1}{2}},$$

or, equivalently, by the equations

$$\frac{x-1}{2} = y - 2 = 2 - z.$$

Now, the normal plane has its normal vector given by the *tangent vector* to our curve of intersection, that is, $\vec{n} = (1, \frac{1}{2}, -\frac{1}{2})$. Since the plane must contain the point $P(1, 2, 2)$ it follows that its equation is given by
$1(x-1) + \frac{1}{2}(y-2) - \frac{1}{2}(z-2) = 0$, that is,

$$2(x-1) + (y-2) - (z-2) = 0,$$

or

$$2x + y - z = 2.$$

Example 3 *Find the tangent plane and the normal line to the surface given by:*

$$x^2 + y^2 + z^2 = 14 \quad at \quad (1, -2, 3).$$

Solution: Here $F(x, y, z) = x^2 + y^2 + z^2 - 14$

1) Tangent plane:

$$\frac{\partial F}{\partial x}|_{P_0} = 2x|_{(1,-2,3)} = 2$$

$$\frac{\partial F}{\partial y}|_{P_0} = 2y|_{(1,-2,3)} = -4$$

$$\frac{\partial F}{\partial z}|_{P_0} = 2z|_{(1,-2,3)} = 6$$

Which gives:

$$2(x-1) - 4(y+2) + 6(z-3) = 0 \ \text{or} \ \boxed{2x - 4y + 6z - 28 = 0}$$

2) Normal line:

$$\frac{x-1}{2} = \frac{y-(-2)}{-4} = \frac{z-3}{6}$$

OR

$$\boxed{x - 1 = -\frac{y+2}{2} = \frac{z-3}{3}}$$