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Representations by quaternary quadratic forms whose coefficients are 1, 4, 9 and 36

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ABSTRACT

Explicit formulae are determined for the number of representations of a positive integer by the quadratic forms $ax^2 + by^2 + cz^2 + dt^2$ with $a, b, c, d \in \{1, 4, 9, 36\}$, $\gcd(a, b, c, d) = 1$ and $a \leq b \leq c \leq d$.

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1. Introduction

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Set

$$\sigma(n) = \sum_{\substack{d \in \mathbb{N} \\ d|n}} d, \quad n \in \mathbb{N}.$$

If $n \notin \mathbb{N}$ we set $\sigma(n) = 0$.

For $a, b, c, d \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define

$$N(a, b, c, d; n) = \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = ax^2 + by^2 + cz^2 + dt^2\}.$$

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Clearly $N(a, b, c, d; 0) = 1$. For $q \in \mathbb{C}$ with $|q| < 1$ we have

$$\sum_{n=0}^{\infty} N(a, b, c, d; n)q^n = \varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d), \tag{1.1}$$

where $\varphi(q)$ denotes Ramanujan’s theta function, namely

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}.$$

There are twenty-six quaternary quadratic forms $ax^2 + by^2 + cz^2 + dt^2$ with $a, b, c, d \in \{1, 4, 9, 36\}$, $\gcd(a, b, c, d) = 1$ and $a \leq b \leq c \leq d$. Formulae for $N(a, b, c, d; n)$ ($n \in \mathbb{N}$) for the seven forms $(a, b, c, d) = (1, 1, 1, 1), (1, 1, 1, 4), (1, 1, 1, 9), (1, 1, 4, 4), (1, 1, 9, 9), (1, 4, 4, 4), (1, 9, 9, 9)$ appear in the literature, [1,2]. In this paper we treat the remaining nineteen forms (Theorems 2.1–2.19). The form $(1, 4, 9, 36)$ was treated by Gogišvilii [9, p. 101].

We also require the theta function $\psi(q)$ defined by

$$\psi(q) = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2}.$$

Definition 1.1. For $k \in \mathbb{N}$ and $q \in \mathbb{C}$ with $|q| < 1$, we define

$$E_k = E_k(q) := \prod_{n=1}^{\infty} (1 - q^{kn}).$$

The infinite product representations of $\varphi(q), \psi(q), \varphi(-q)$ are due to Jacobi

$$\varphi(q) = \frac{E_2^5}{E_1^2 E_4^2}, \quad \psi(q) = \frac{E_2^2}{E_1}, \quad \varphi(-q) = \frac{E_1^2}{E_2}. \tag{1.2}$$

From [8, p. 357] we have

$$(3\varphi(-q^9) - \varphi(-q))^3 = 8 \frac{\psi^3(q)}{\psi(q^3)} \varphi(-q^3). \tag{1.3}$$

Then, immediately from (1.2) and (1.3), we obtain

$$3\varphi(-q^9) - \varphi(-q) = 2 \frac{E_2^2 E_3}{E_1 E_6}. \tag{1.4}$$

It is convenient to define integers $c(n)$ ($n \in \mathbb{N}$) and $t(n)$ ($n \in \mathbb{N}_0$) by

$$C_{1,6}(q) := qE_6^4 = \sum_{n=1}^{\infty} c(n)q^n, \tag{1.5}$$

$$T(q) := \frac{E_2^{17} E_3}{E_1^7 E_4^6 E_6} = \sum_{n=0}^{\infty} t(n)q^n. \tag{1.6}$$

The identity (1.4) is used to express $T(q)$ in terms of theta functions, see (6.1). It is clear from (1.5) that $c(n) = 0$ if $n \not\equiv 1 \pmod{6}$. We set

$$C_{1,12}(q) := \sum_{\substack{n=1 \\ n \equiv 1 \pmod{12}}}^{\infty} c(n)q^n, \quad C_{7,12}(q) := \sum_{\substack{n=1 \\ n \equiv 7 \pmod{12}}}^{\infty} c(n)q^n, \tag{1.7}$$

$$T_r(q) := \sum_{\substack{n=0 \\ n \equiv r \pmod{12}}}^{\infty} t(n)q^n, \quad r \in \{0, 1, 2, \dots, 11\}. \tag{1.8}$$

Klein and Fricke [10, Vol. II, p. 374] have given an arithmetic formulation of $c(n)$.

2. Statements of main results

Theorem 2.1. For $n \in \mathbb{N}$

$$N(1, 1, 1, 36; n) = \begin{cases} \frac{5}{3}\sigma(n) + 2c(n) + \frac{1}{3}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 2\sigma(n) + \frac{1}{3}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ \frac{1}{3}\sigma(n) + 3\sigma(n/9) + \frac{1}{3}t(n), & \text{if } n \equiv 3 \pmod{12}, \\ 2\sigma(n/4) - 8\sigma(n/16) + 4c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{8}{3}\sigma(n) + \frac{1}{3}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ \sigma(n) + 3\sigma(n/9) + \frac{1}{3}t(n), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{1}{3}\sigma(n) + 2c(n) + \frac{1}{3}t(n), & \text{if } n \equiv 7 \pmod{12}, \\ 4\sigma(n/4) - 16\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ \frac{5}{3}\sigma(n) + 3\sigma(n/9) + \frac{1}{3}t(n), & \text{if } n \equiv 9 \pmod{12}, \\ \sigma(n) + \frac{1}{3}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ \frac{4}{3}\sigma(n) + \frac{1}{3}t(n), & \text{if } n \equiv 11 \pmod{12}, \\ 8\sigma(n/12) - 32\sigma(n/48), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.2. For $n \in \mathbb{N}$

$$N(1, 1, 4, 9; n) = \begin{cases} \frac{13}{9}\sigma(n) + \frac{10}{3}c(n) - \frac{1}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 2\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ \frac{5}{9}\sigma(n) + \sigma(n/9) - \frac{1}{9}t(n), & \text{if } n \equiv 3 \pmod{12}, \\ 2\sigma(n/4) - 8\sigma(n/16) + 4c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{28}{9}\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ \sigma(n) + 3\sigma(n/9) - \frac{1}{9}t(n), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{5}{9}\sigma(n) + \frac{2}{3}c(n) - \frac{1}{9}t(n), & \text{if } n \equiv 7 \pmod{12}, \\ 4\sigma(n/4) - 16\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ \frac{13}{9}\sigma(n/3) + 5\sigma(n/9) - \frac{1}{9}t(n), & \text{if } n \equiv 9 \pmod{12}, \\ \sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ \frac{8}{9}\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 11 \pmod{12}, \\ 8\sigma(n/12) - 32\sigma(n/48), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.3. For $n \in \mathbb{N}$

$$N(1, 1, 4, 36; n) = \begin{cases} \frac{10}{9}\sigma(n) + \frac{4}{3}c(n) + \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ \frac{2}{3}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ 0, & \text{if } n \equiv 3, 7, 11 \pmod{12}, \\ 2\sigma(n/4) - 8\sigma(n/16) + 4c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{16}{9}\sigma(n) + \frac{2}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ 4\sigma(n/6) + \frac{1}{9}t(n), & \text{if } n \equiv 6 \pmod{12}, \\ 4\sigma(n/4) - 16\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ \frac{10}{9}\sigma(n) + 2\sigma(n/9) + \frac{2}{9}t(n), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{1}{3}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/12) - 32\sigma(n/48), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.4. For $n \in \mathbb{N}$

$$N(1, 1, 9, 36; n) = \begin{cases} \frac{10}{9}\sigma(n) + \frac{4}{3}c(n) + \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ \frac{2}{3}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 3 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{8}{9}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{2}{9}\sigma(n) + \frac{4}{3}c(n) + \frac{2}{9}t(n), & \text{if } n \equiv 7 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ 6\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{2}{3}\sigma(n) + \frac{2}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ \frac{4}{9}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 11 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.5. For $n \in \mathbb{N}$

$$N(1, 1, 36, 36; n) = \begin{cases} \frac{8}{9}\sigma(n) + \frac{4}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ \frac{2}{3}\sigma(n/2) + \frac{4}{3}c(n/2) + \frac{1}{9}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ 0, & \text{if } n \equiv 3, 7, 11 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{9}\sigma(n) + \frac{2}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ \frac{4}{3}\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{2}{9}\sigma(n) + \frac{2}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.6. For $n \in \mathbb{N}$

$$N(1, 4, 4, 9; n) = \begin{cases} \frac{8}{9}\sigma(n) + \frac{8}{3}c(n) - \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ \frac{2}{3}\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ 0, & \text{if } n \equiv 3, 7, 11 \pmod{12}, \\ 2\sigma(n/4) - 8\sigma(n/16) + 4c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{20}{9}\sigma(n) - \frac{2}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ 4\sigma(n/6) - \frac{1}{9}t(n), & \text{if } n \equiv 6 \pmod{12}, \\ 4\sigma(n/4) - 16\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ \frac{8}{9}\sigma(n) + 4\sigma(n/9) - \frac{2}{9}t(n), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{1}{3}\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/12) - 32\sigma(n/48), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.7. For $n \in \mathbb{N}$

$$N(1, 4, 4, 36; n) = \begin{cases} \frac{5}{9}\sigma(n) + \frac{2}{3}c(n) + \frac{1}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2, 3, 6, 7, 10, 11 \pmod{12}, \\ 2\sigma(n/4) - 8\sigma(n/16) + 4c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{8}{9}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ 4\sigma(n/4) - 16\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ \frac{5}{9}\sigma(n) + \sigma(n/9) + \frac{1}{9}t(n), & \text{if } n \equiv 9 \pmod{12}, \\ 8\sigma(n/12) - 32\sigma(n/48), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.8. For $n \in \mathbb{N}$

$$N(1, 4, 9, 9; n) = \begin{cases} \frac{8}{9}\sigma(n) + \frac{8}{3}c(n) - \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ \frac{2}{3}\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 3 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{10}{9}\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{4}{9}\sigma(n) - \frac{2}{9}t(n), & \text{if } n \equiv 7 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ 6\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{2}{3}\sigma(n) - \frac{2}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ \frac{2}{9}\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 11 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.9. For $n \in \mathbb{N}$

$$N(1, 4, 9, 36; n) = \begin{cases} \frac{2}{3}\sigma(n) + \frac{4}{3}c(n), & \text{if } n \equiv 1 \pmod{12}, \\ \frac{2}{3}\sigma(n/2) - \frac{2}{3}c(n/2), & \text{if } n \equiv 2 \pmod{12}, \\ 0, & \text{if } n \equiv 3, 7, 11 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{2}{3}\sigma(n), & \text{if } n \equiv 5 \pmod{12}, \\ \frac{4}{3}\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{2}{9}\sigma(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.10. For $n \in \mathbb{N}$

$$N(1, 4, 36, 36; n) = \begin{cases} \frac{4}{9}\sigma(n) + \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2, 3, 6, 7, 10, 11 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{2}{9}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.11. For $n \in \mathbb{N}$

$$N(1, 9, 9, 36; n) = \begin{cases} \frac{5}{9}\sigma(n) + \frac{2}{3}c(n) + \frac{1}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2, 5, 8, 11 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 3 \pmod{12}, \\ \frac{2}{3}\sigma(n/4) - \frac{8}{3}\sigma(n/16) + \frac{4}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{1}{9}\sigma(n) + \frac{2}{3}c(n) + \frac{1}{9}t(n), & \text{if } n \equiv 7 \pmod{12}, \\ 6\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{1}{3}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.12. For $n \in \mathbb{N}$

$$N(1, 9, 36, 36; n) = \begin{cases} \frac{4}{9}\sigma(n) + \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2, 3, 5, 7, 8, 11 \pmod{12}, \\ \frac{2}{3}\sigma(n/4) - \frac{8}{3}\sigma(n/16) + \frac{4}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{3}\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{1}{9}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.13. For $n \in \mathbb{N}$

$$N(1, 36, 36, 36; n) = \begin{cases} \frac{1}{3}\sigma(n) - \frac{2}{3}c(n) + \frac{1}{3}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2, 3, 5, 6, 7, 8, 10, 11 \pmod{12}, \\ \frac{2}{3}\sigma(n/4) - \frac{8}{3}\sigma(n/16) + \frac{4}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.14. For $n \in \mathbb{N}$

$$N(4, 4, 4, 9; n) = \begin{cases} \frac{1}{3}\sigma(n) + 2c(n) - \frac{1}{3}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2, 3, 6, 7, 10, 11 \pmod{12}, \\ 2\sigma(n/4) - 8\sigma(n/16) + 4c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{3}\sigma(n) - \frac{1}{3}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ 4\sigma(n/4) - 16\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ \frac{1}{3}\sigma(n) + 3\sigma(n/9) - \frac{1}{3}t(n), & \text{if } n \equiv 9 \pmod{12}, \\ 8\sigma(n/12) - 32\sigma(n/48), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.15. For $n \in \mathbb{N}$

$$N(4, 4, 9, 9; n) = \begin{cases} \frac{4}{9}\sigma(n) + \frac{8}{3}c(n) - \frac{4}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ \frac{2}{3}\sigma(n/2) + \frac{4}{3}c(n/2) - \frac{1}{9}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ 0, & \text{if } n \equiv 3, 7, 11 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{8}{9}\sigma(n) - \frac{2}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ \frac{4}{3}\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{2}{9}\sigma(n) - \frac{2}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.16. For $n \in \mathbb{N}$

$$N(4, 4, 9, 36; n) = \begin{cases} \frac{2}{9}\sigma(n) + \frac{4}{3}c(n) - \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2, 3, 6, 7, 10, 11 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{9}\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.17. For $n \in \mathbb{N}$

$$N(4, 9, 9, 9; n) = \begin{cases} \frac{1}{3}\sigma(n) + 2c(n) - \frac{1}{3}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2, 5, 8, 11 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 3 \pmod{12}, \\ \frac{2}{3}\sigma(n/4) - \frac{8}{3}\sigma(n/16) + \frac{4}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{1}{3}\sigma(n) - \frac{2}{3}c(n) - \frac{1}{3}t(n), & \text{if } n \equiv 7 \pmod{12}, \\ 6\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{1}{3}\sigma(n) - \frac{1}{3}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.18. For $n \in \mathbb{N}$

$$N(4, 9, 9, 36; n) = \begin{cases} \frac{2}{9}\sigma(n) + \frac{4}{3}c(n) - \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2, 3, 5, 7, 8, 11 \pmod{12}, \\ \frac{2}{3}\sigma(n/4) - \frac{8}{3}\sigma(n/16) + \frac{4}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{3}\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{1}{9}\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.19. For $n \in \mathbb{N}$

$$N(4, 9, 36, 36; n) = \begin{cases} \frac{1}{9}\sigma(n) + \frac{2}{3}c(n) - \frac{1}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2, 3, 5, 6, 7, 8, 10, 11 \pmod{12}, \\ \frac{2}{3}\sigma(n/4) - \frac{8}{3}\sigma(n/16) + \frac{4}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

A numerical study showed that there do not exist rational numbers u, v, w (not all zero) with $u\sigma(n) + vc(n) + wt(n) = 0$ for all $n \equiv 1 \pmod{12}$ so that the formulation of each of the above theorems cannot be simplified in the case $n \equiv 1 \pmod{12}$. Similarly for the remaining congruence classes.

3. Theta function identities

We state nineteen theta function identities (Theorems 3.1–3.19) from which the theorems of Section 2 follow.

Definition 3.1. Let $k \in \mathbb{N}$ and $i \in \mathbb{N}_0$ with $0 \leq i \leq k - 1$. Then we define $L_{i,k}(q)$ and the Eisenstein series $L(q)$ by

$$L_{i,k}(q) := \sum_{\substack{n=1 \\ n \equiv i \pmod{k}}}^{\infty} \sigma(n)q^n, \quad L(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n.$$

It is convenient to define $\alpha(q)$, $\beta(q)$ and $\gamma(q)$ by

$$\alpha(q) := L_{0,1}(q) - 4L_{0,1}(q^4), \quad (3.1)$$

$$\beta(q) := L_{1,3}(q) - 4L_{1,3}(q^4) + 2C_{1,6}(q), \quad (3.2)$$

$$\gamma(q) := L_{2,3}(q) - 4L_{2,3}(q^4). \quad (3.3)$$

Theorem 3.1.

$$\begin{aligned} \varphi^3(q)\varphi(q^{36}) &= 1 + 8\alpha(q^{12}) + 2\beta(q^4) + 4\gamma(q^4) + 3L_{1,4}(q^9) + 3L_{2,4}(q^9) \\ &+ 3L_{3,4}(q^9) + \frac{5}{3}L_{1,12}(q) + 2L_{2,12}(q) + \frac{1}{3}L_{3,12}(q) + \frac{8}{3}L_{5,12}(q) + L_{6,12}(q) \\ &+ \frac{1}{3}L_{7,12}(q) + \frac{5}{3}L_{9,12}(q) + L_{10,12}(q) + \frac{4}{3}L_{11,12}(q) + 2C_{1,6}(q) \\ &+ \frac{1}{3}(T_1(q) + T_2(q) + T_3(q) + T_5(q) + T_6(q) + T_7(q) + T_9(q) \\ &+ T_{10}(q) + T_{11}(q)). \end{aligned}$$

Theorem 3.2.

$$\begin{aligned} \varphi^2(q)\varphi(q^4)\varphi(q^9) &= 1 + 8\alpha(q^{12}) + 2\beta(q^4) + 4\gamma(q^4) + 5L_{1,4}(q^9) \\ &+ 3L_{2,4}(q^9) + L_{3,4}(q^9) + \frac{13}{9}L_{1,12}(q) + 2L_{2,12}(q) + \frac{5}{9}L_{3,12}(q) \\ &+ \frac{28}{9}L_{5,12}(q) + L_{6,12}(q) + \frac{5}{9}L_{7,12}(q) + \frac{13}{9}L_{9,12}(q) + L_{10,12}(q) \\ &+ \frac{8}{9}L_{11,12}(q) + \frac{10}{3}C_{1,12}(q) + \frac{2}{3}C_{7,12}(q) - \frac{1}{9}(T_1(q) + T_2(q) \\ &+ T_3(q) + T_5(q) + T_6(q) + T_7(q) + T_9(q) + T_{10}(q) + T_{11}(q)). \end{aligned}$$

Theorem 3.3.

$$\begin{aligned} \varphi^2(q)\varphi(q^4)\varphi(q^{36}) &= 1 + 8\alpha(q^{12}) + 2\beta(q^4) + 4\gamma(q^4) \\ &+ 2L_{1,4}(q^9) + L_{2,4}(q^9) + \frac{10}{9}L_{1,12}(q) + \frac{2}{3}L_{2,12}(q) + \frac{16}{9}L_{5,12}(q) \\ &+ \frac{1}{3}L_{6,12}(q) + \frac{10}{9}L_{9,12}(q) + \frac{1}{3}L_{10,12}(q) + \frac{4}{3}C_{1,12}(q) \\ &+ \frac{1}{9}(2T_1(q) + T_2(q) + 2T_5(q) + T_6(q) + 2T_9(q) + T_{10}(q)). \end{aligned}$$

Theorem 3.4.

$$\begin{aligned} \varphi^2(q)\varphi(q^9)\varphi(q^{36}) &= 1 + 8\alpha(q^{36}) + \frac{4}{3}\beta(q^4) + \frac{4}{3}\gamma(q^4) + 6L_{1,4}(q^9) \\ &+ 4L_{2,4}(q^9) + 2L_{3,4}(q^9) + \frac{10}{9}L_{1,12}(q) + \frac{2}{3}L_{2,12}(q) + \frac{8}{9}L_{5,12}(q) \end{aligned}$$

$$\begin{aligned} &+ \frac{2}{9}L_{7,12}(q) + \frac{2}{3}L_{10,12}(q) + \frac{4}{9}L_{11,12}(q) + \frac{4}{3}C_{1,6}(q) \\ &+ \frac{1}{9}(2T_1(q) + T_2(q) + T_5(q) + 2T_7(q) + 2T_{10}(q) + T_{11}(q)). \end{aligned}$$

Theorem 3.5.

$$\begin{aligned} \varphi^2(q)\varphi^2(q^{36}) &= 1 + 8\alpha(q^{36}) + \frac{4}{3}\beta(q^4) + \frac{4}{3}\gamma(q^4) + 4L_{1,4}(q^9) + \frac{4}{3}L_{2,4}(q^9) \\ &+ \frac{8}{9}L_{1,12}(q) + \frac{2}{9}L_{2,12}(q) + \frac{4}{9}L_{5,12}(q) + \frac{2}{9}L_{10,12}(q) + \frac{4}{3}C_{1,6}(q^2) \\ &+ \frac{1}{9}(4T_1(q) + T_2(q) + 2T_5(q) + 2T_{10}(q)). \end{aligned}$$

Theorem 3.6.

$$\begin{aligned} \varphi(q)\varphi^2(q^4)\varphi(q^9) &= 1 + 8\alpha(q^{12}) + 2\beta(q^4) + 4\gamma(q^4) \\ &+ 4L_{1,4}(q^9) + L_{2,4}(q^9) + \frac{8}{9}L_{1,12}(q) + \frac{2}{3}L_{2,12}(q) + \frac{20}{9}L_{5,12}(q) \\ &+ \frac{1}{3}L_{6,12}(q) + \frac{8}{9}L_{9,12}(q) + \frac{1}{3}L_{10,12}(q) + \frac{8}{3}C_{1,12}(q) \\ &- \frac{1}{9}(2T_1(q) + T_2(q) + 2T_5(q) + T_6(q) + 2T_9(q) + T_{10}(q)). \end{aligned}$$

Theorem 3.7.

$$\begin{aligned} \varphi(q)\varphi^2(q^4)\varphi(q^{36}) &= 1 + 8\alpha(q^{12}) + 2\beta(q^4) + 4\gamma(q^4) + L_{1,4}(q^9) + \frac{5}{9}L_{1,12}(q) \\ &+ \frac{8}{9}L_{5,12}(q) + \frac{5}{9}L_{9,12}(q) + \frac{2}{3}C_{1,12}(q) + \frac{1}{9}(T_1(q) + T_5(q) + T_9(q)). \end{aligned}$$

Theorem 3.8.

$$\begin{aligned} \varphi(q)\varphi(q^4)\varphi^2(q^9) &= 1 + 8\alpha(q^{36}) + \frac{4}{3}\beta(q^4) + \frac{4}{3}\gamma(q^4) + 6L_{1,4}(q^9) \\ &+ 4L_{2,4}(q^9) + 2L_{3,4}(q^9) + \frac{8}{9}L_{1,12}(q) + \frac{2}{3}L_{2,12}(q) + \frac{10}{9}L_{5,12}(q) \\ &+ \frac{4}{9}L_{7,12}(q) + \frac{2}{3}L_{10,12}(q) + \frac{2}{9}L_{11,12}(q) + \frac{8}{3}C_{1,12}(q) - \frac{1}{9}(2T_1(q) + T_2(q) \\ &+ T_5(q) + 2T_7(q) + 2T_{10}(q) + T_{11}(q)). \end{aligned}$$

Theorem 3.9.

$$\begin{aligned} \varphi(q)\varphi(q^4)\varphi(q^9)\varphi(q^{36}) &= 1 + 8\alpha(q^{36}) + \frac{4}{3}\beta(q^4) + \frac{4}{3}\gamma(q^4) + 4L_{1,4}(q^9) \\ &+ \frac{4}{3}L_{2,4}(q^9) + \frac{2}{3}L_{1,12}(q) + \frac{2}{9}L_{2,12}(q) + \frac{2}{3}L_{5,12}(q) + \frac{2}{9}L_{10,12}(q) \\ &+ \frac{4}{3}C_{1,12}(q) - \frac{2}{3}C_{1,6}(q^2). \end{aligned}$$

Theorem 3.10.

$$\begin{aligned}\varphi(q)\varphi(q^4)\varphi^2(q^{36}) &= 1 + 8\alpha(q^{36}) + \frac{4}{3}\beta(q^4) + \frac{4}{3}\gamma(q^4) + 2L_{1,4}(q^9) \\ &\quad + \frac{4}{9}L_{1,12}(q) + \frac{2}{9}L_{5,12}(q) + \frac{2}{9}T_1(q) + \frac{1}{9}T_5(q).\end{aligned}$$

Theorem 3.11.

$$\begin{aligned}\varphi(q)\varphi^2(q^9)\varphi(q^{36}) &= 1 + 8\alpha(q^{36}) + \frac{2}{3}\beta(q^4) + 6L_{1,4}(q^9) + 4L_{2,4}(q^9) \\ &\quad + 2L_{3,4}(q^9) + \frac{5}{9}L_{1,12}(q) + \frac{1}{9}L_{7,12}(q) + \frac{1}{3}L_{10,12}(q) + \frac{2}{3}C_{1,6}(q) \\ &\quad + \frac{1}{9}(T_1(q) + T_7(q) + T_{10}(q)).\end{aligned}$$

Theorem 3.12.

$$\begin{aligned}\varphi(q)\varphi(q^9)\varphi^2(q^{36}) &= 1 + 8\alpha(q^{36}) + \frac{2}{3}\beta(q^4) + 4L_{1,4}(q^9) + \frac{4}{3}L_{2,4}(q^9) \\ &\quad + \frac{4}{9}L_{1,12}(q) + \frac{1}{9}L_{10,12}(q) + \frac{2}{9}T_1(q) + \frac{1}{9}T_{10}(q).\end{aligned}$$

Theorem 3.13.

$$\varphi(q)\varphi^3(q^{36}) = 1 + 8\alpha(q^{36}) + \frac{2}{3}\beta(q^4) + 2L_{1,4}(q^9) + \frac{1}{3}L_{1,12}(q) - \frac{2}{3}C_{1,12}(q) + \frac{1}{3}T_1(q).$$

Theorem 3.14.

$$\begin{aligned}\varphi^3(q^4)\varphi(q^9) &= 1 + 8\alpha(q^{12}) + 2\beta(q^4) + 4\gamma(q^4) + 3L_{1,4}(q^9) + \frac{1}{3}L_{1,12}(q) \\ &\quad + \frac{4}{3}L_{5,12}(q) + \frac{1}{3}L_{9,12}(q) + 2C_{1,12}(q) - \frac{1}{3}(T_1(q) + T_5(q) + T_9(q)).\end{aligned}$$

Theorem 3.15.

$$\begin{aligned}\varphi^2(q^4)\varphi^2(q^9) &= 1 + 8\alpha(q^{36}) + \frac{4}{3}\beta(q^4) + \frac{4}{3}\gamma(q^4) + 4L_{1,4}(q^9) \\ &\quad + \frac{4}{3}L_{2,4}(q^9) + \frac{4}{9}L_{1,12}(q) + \frac{2}{9}L_{2,12}(q) + \frac{8}{9}L_{5,12}(q) + \frac{2}{9}L_{10,12}(q) \\ &\quad + \frac{8}{3}C_{1,12}(q) + \frac{4}{3}C_{1,6}(q^2) - \frac{1}{9}(4T_1(q) + T_2(q) + 2T_5(q) + 2T_{10}(q)).\end{aligned}$$

Theorem 3.16.

$$\begin{aligned}\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) &= 1 + 8\alpha(q^{36}) + \frac{4}{3}\beta(q^4) + \frac{4}{3}\gamma(q^4) + 2L_{1,4}(q^9) \\ &\quad + \frac{2}{9}L_{1,12}(q) + \frac{4}{9}L_{5,12}(q) + \frac{4}{3}C_{1,12}(q) - \frac{2}{9}T_1(q) - \frac{1}{9}T_5(q).\end{aligned}$$

Theorem 3.17.

$$\begin{aligned} \varphi(q^4)\varphi^3(q^9) &= 1 + 8\alpha(q^{36}) + \frac{2}{3}\beta(q^4) + 6L_{1,4}(q^9) + 4L_{2,4}(q^9) \\ &\quad + 2L_{3,4}(q^9) + \frac{1}{3}(L_{1,12}(q) + L_{7,12}(q) + L_{10,12}(q)) \\ &\quad + 2C_{1,12}(q) - \frac{2}{3}C_{7,12}(q) - \frac{1}{3}(T_1(q) + T_7(q) + T_{10}(q)). \end{aligned}$$

Theorem 3.18.

$$\begin{aligned} \varphi(q^4)\varphi^2(q^9)\varphi(q^{36}) &= 1 + 8\alpha(q^{36}) + \frac{2}{3}\beta(q^4) + 4L_{1,4}(q^9) + \frac{4}{3}L_{2,4}(q^9) \\ &\quad + \frac{2}{9}L_{1,12}(q) + \frac{1}{9}L_{10,12}(q) + \frac{4}{3}C_{1,12}(q) - \frac{2}{9}T_1(q) - \frac{1}{9}T_{10}(q). \end{aligned}$$

Theorem 3.19.

$$\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}) = 1 + 8\alpha(q^{36}) + \frac{2}{3}\beta(q^4) + 2L_{1,4}(q^9) + \frac{1}{9}L_{1,12}(q) + \frac{2}{3}C_{1,12}(q) - \frac{1}{9}T_1(q).$$

4. The (p, k)-parametrization

Following [6, p. 178] we set

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, \quad k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{4.1}$$

Duplication and change of sign principles. (See [6, Theorems 9 and 11].)

$$\begin{aligned} p(q^2) &= \frac{1 + p - p^2 - ((1 - p)(1 + p)(1 + 2p))^{1/2}}{p^2}, \\ k(q^2) &= \frac{(1 + p - p^2 + ((1 - p)(1 + p)(1 + 2p))^{1/2})k}{2}, \\ p(-q) &= \frac{-p}{1 + p}, \quad k(-q) = (1 + p)^2k. \end{aligned}$$

Proposition 4.1. Define p and k by (4.1). Then

- (a) $\varphi(q) = (1 + 2p)^{3/4}k^{1/2}$,
- (b) $\varphi(q^3) = (1 + 2p)^{1/4}k^{1/2}$,
- (c) $\varphi(q^9) = \frac{1}{3}(1 + 2p)^{1/12}((1 + 2p)^{2/3} + 2^{2/3}(1 - p)^{1/3}(2 + p)^{1/3})k^{1/2}$.

Proof. (a)(b) See [4, p. 222]. (c) See [1, p. 156]. \square

By using Proposition 4.1, we obtain the following result.

Proposition 4.2.

- (a) $\varphi(q) - \varphi(q^9) = \frac{2}{3}(1 + 2p)^{3/4}k^{1/2} - \frac{2^{2/3}}{3}(1 + 2p)^{1/12}(1 - p)^{1/3}(2 + p)^{1/3}k^{1/2},$
- (b) $\varphi(q) - 3\varphi(q^9) = -2^{2/3}(1 + 2p)^{1/12}(1 - p)^{1/3}(2 + p)^{1/3}k^{1/2}$
 $\quad + \frac{k}{3}2^{5/3}(1 + 2p)^{5/6}(1 - p)^{1/3}(2 + p)^{1/3},$
- (c) $\varphi^2(q^9) = \frac{k}{9}[(1 + 2p)^{3/2} + 2^{4/3}(1 + 2p)^{1/6}(1 - p)^{2/3}(2 + p)^{2/3}$
 $\quad + 2^{5/3}(1 + 2p)^{5/6}(1 - p)^{1/3}(2 + p)^{1/3}].$

We use the following basic properties of $\varphi(q)$, see for example [5]:

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{4.2}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \tag{4.3}$$

$$\varphi(q) - \varphi(-q) = 2(\varphi(q) - \varphi(q^4)), \tag{4.4}$$

$$\varphi^2(q^2) = \varphi^2(q) - 2\varphi(q)\varphi(q^4) + 2\varphi^2(q^4), \tag{4.5}$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2). \tag{4.6}$$

The change of sign principle and Proposition 4.1 yield Proposition 4.3.

Proposition 4.3.

- (a) $\varphi(-q) = (1 - p)^{3/4}(1 + p)^{1/4}k^{1/2},$
- (b) $\varphi(-q^3) = (1 - p)^{1/4}(1 + p)^{3/4}k^{1/2},$
- (c) $\varphi(-q^9) = \frac{1}{3}(2^{2/3}(1 + 2p)^{1/3}(1 - p)^{1/12}(1 + p)^{1/4}(2 + p)^{1/3}$
 $\quad + (1 - p)^{3/4}(1 + p)^{1/4})k^{1/2},$
- (d) $\varphi(-q) - \varphi(-q^9) = \frac{2}{3}(1 - p)^{3/4}(1 + p)^{1/4}k^{1/2}$
 $\quad - \frac{1}{3}2^{2/3}(1 + 2p)^{1/3}(1 - p)^{1/12}(1 + p)^{1/4}(2 + p)^{1/3}k^{1/2},$
- (e) $\varphi(-q) - 3\varphi(-q^9) = -2^{2/3}(1 + 2p)^{1/3}(1 - p)^{1/12}(1 + p)^{1/4}(2 + p)^{1/3}k^{1/2}.$

It was shown in [1, p. 156] that

$$\varphi^4(q^3) = 3\varphi(q)\varphi^3(q^9) - 3\varphi^2(q)\varphi^2(q^9) + \varphi^3(q)\varphi(q^9), \tag{4.7}$$

$$\varphi^2(q)\varphi^2(q^9) = \frac{1}{6}\varphi^4(q) - \frac{2}{3}\varphi^4(q^3) + \frac{3}{2}\varphi^4(q^9) + \frac{8}{3}C_{1,6}(q). \tag{4.8}$$

By appealing to (4.7) and (4.8), we obtain

$$C_{1,6}(q) = -\frac{1}{16}[\varphi(q) - \varphi(q^9)][\varphi(q) - 3\varphi(q^9)][\varphi^2(q) + 3\varphi^2(q^9)]. \tag{4.9}$$

Theorem 4.1.

$$\begin{aligned} & [\varphi(q) - \varphi(q^9)][\varphi(q) - 3\varphi(q^9)][\varphi^2(q) + 3\varphi^2(q^9)] \\ &= -[\varphi(-q) - \varphi(-q^9)][\varphi(-q) - 3\varphi(-q^9)][\varphi^2(-q) + 3\varphi^2(-q^9)]. \end{aligned}$$

Proof. The assertion follows immediately from (1.5) and (4.9). \square

Theorem 4.2.

$$\begin{aligned} & 2\varphi(q)\varphi(q^9)[\varphi^2(-q) - 2\varphi(-q)\varphi(-q^9) + 3\varphi^2(-q^9)] \\ &= \varphi^2(q)[\varphi(-q) - \varphi(-q^9)]^2 + \varphi^2(q^9)[\varphi(-q) - 3\varphi(-q^9)]^2. \end{aligned}$$

Proof. Appealing to Propositions 4.1(a)(c), 4.2(c) and 4.3(a)(c), we see that each side of asserted equation is

$$\begin{aligned} & \frac{4k^2}{9}[(1 + 2p)^{3/2}(1 - p)^{3/2}(1 + p)^{1/2} \\ &+ 2^{2/3}(1 + 2p)^{5/6}(1 - p)^{11/6}(1 + p)^{1/2}(2 + p)^{1/3} \\ &+ 2^{1/3}(1 + 2p)^{13/6}(1 - p)^{1/6}(1 + p)^{1/2}(2 + p)^{2/3} \\ &+ 2(1 + 2p)^{3/2}(1 - p)^{1/2}(1 + p)^{1/2}(2 + p)]. \quad \square \end{aligned}$$

Theorem 4.3.

- (a) $L(q^6) - 2L(q^{12}) = -\left(1 + 2p - p^3 - \frac{1}{2}p^4\right)k^2,$
- (b) $L_{1,2}(q^3) = \left(\frac{1}{8}p^3 + \frac{1}{16}p^4\right)k^2,$
- (c) $8L_{1,2}(q^3) - L(q^6) + 2L(q^{12}) = \varphi^4(q^3).$

Proof. (a) See [7, Eq. (3.11), p. 33]. (b) See [6, Eq. (14.6), p. 189].
 (c) Follows from (a), (b) and Proposition 4.1(b). \square

Theorem 4.4.

$$\begin{aligned} \text{(a)} \quad L_{2,3}(q) &= \left(\frac{1}{27} + \frac{8}{27}p + \frac{2}{3}p^2 + \frac{8}{27}p^3 + \frac{1}{27}p^4\right)k^2 \\ &+ \left(\frac{-1}{27} - \frac{1}{9}p + \frac{1}{9}p^2 + \frac{1}{27}p^3\right)2^{2/3}((1 - p)(1 + 2p)(2 + p))^{1/3}k^2 \\ &+ \left(\frac{1}{54} - \frac{1}{27}p + \frac{1}{54}p^2\right)2^{1/3}((1 - p)(1 + 2p)(2 + p))^{2/3}k^2, \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad L_{2,3}(q^4) &= \left(\frac{1}{27} + \frac{2}{27}p - \frac{1}{27}p^3 + \frac{1}{108}p^4 \right) k^2 \\
 &+ \left(-\frac{1}{27} - \frac{1}{18}p + \frac{1}{108}p^3 \right) 2^{2/3} ((1-p)(1+2p)(2+p))^{1/3} k^2 \\
 &+ \left(\frac{1}{54} + \frac{1}{54}p + \frac{1}{216}p^2 \right) 2^{1/3} ((1-p)(1+2p)(2+p))^{2/3} k^2.
 \end{aligned}$$

Proof. (a) See [6, Eq. (15.9), p. 190].

(b) Apply the duplication principle to part (a) twice. \square

Theorem 4.5.

$$\gamma(q) = \frac{3}{4} \varphi^2(q^9) [\varphi(q) - \varphi(q^9)]^2, \quad \gamma(q^4) = \frac{3}{4} \varphi^2(q^{36}) [\varphi(q^4) - \varphi(q^{36})]^2.$$

Proof. Appealing to (3.3) and Theorem 4.4, we obtain

$$\begin{aligned}
 \gamma(q) &= \frac{1}{9} (-1 + 6p^2 + 4p^3) k^2 - \frac{2^{1/3}}{18} (1+2p)((1+2p)(1-p)(2+p))^{2/3} k^2 \\
 &+ \frac{2^{2/3}}{9} (1+p+p^2)((1+2p)(1-p)(2+p))^{1/3} k^2.
 \end{aligned}$$

Set $X = (1+2p)^{3/2}$, $Y = (1+2p)^{1/6}(1-p)^{2/3}(2+p)^{2/3}$ and $Z = (1+2p)^{5/6}(1-p)^{1/3}(2+p)^{1/3}$. By Proposition 4.2(a)(c), we obtain

$$\begin{aligned}
 \frac{3}{4} \varphi^2(q^9) [\varphi(q) - \varphi(q^9)]^2 &= \frac{k^2}{108} (X + 2^{4/3}Y + 2 \cdot 2^{2/3}Z)(4X + 2^{4/3}Y - 4 \cdot 2^{2/3}Z) \\
 &= \frac{k^2}{108} (4X^2 + 5 \cdot 2^{4/3}XY + 4 \cdot 2^{2/3}XZ - 8YZ + 4 \cdot 2^{2/3}Y^2 - 16 \cdot 2^{1/3}Z^2) \\
 &= \frac{k^2}{9} (-1 + 6p^2 + 4p^3) - \frac{k^2}{18} \cdot 2^{1/3} (1+2p)((1+2p)(1-p)(2+p))^{2/3} \\
 &+ \frac{k^2}{9} \cdot 2^{2/3} (1+p+p^2)((1+2p)(1-p)(2+p))^{1/3}.
 \end{aligned}$$

The second assertion follows on replacing q by q^4 in the first one. \square

Theorem 4.6.

$$\beta(q) = \frac{3}{2} \varphi^3(q^9) [\varphi(q) - \varphi(q^9)], \quad \beta(q^4) = \frac{3}{2} \varphi^3(q^{36}) [\varphi(q^4) - \varphi(q^{36})].$$

Proof. It was proved in [1, Theorem 2.5, p. 158] that

$$\begin{aligned}
 \varphi^3(q)\varphi(q^9) &= 2L_{1,3}(q) + 4L_{2,3}(q) - 8L_{1,3}(q^4) - 16L_{2,3}(q^4) \\
 &- L(q^6) + 2L(q^{12}) + 8L_{1,2}(q^3) + 4C_{1,6}(q). \tag{4.10}
 \end{aligned}$$

Substituting (3.2), (3.3) and Theorem 4.3(c) into (4.10), we obtain

$$\varphi^3(q)\varphi(q^9) = 2\beta(q) + 4\gamma(q) + \varphi^4(q^3). \tag{4.11}$$

Appealing to (4.11), Theorem 4.5 and (4.7), we obtain

$$\beta(q) = \frac{1}{2} [\varphi^3(q)\varphi(q^9) - 4\gamma(q) - \varphi^4(q^3)] = \frac{3}{2} \varphi^3(q^9) [\varphi(q) - \varphi(q^9)].$$

The second assertion follows on replacing q by q^4 in the first one. \square

5. Identities involving $\varphi(q^i)$, $i \in \{1, 4, 9, 36\}$

We give four identities, (5.1)–(5.4), of degree 4 of the form

$$\sum_{\substack{i,j,k,l=0 \\ i+j+k+l=4}}^4 a(i, j, k, l) \varphi^i(q) \varphi^j(q^4) \varphi^k(q^9) \varphi^l(q^{36}) = 0.$$

We list the coefficients $a(i, j, k, l)$ of identities (5.1)–(5.4) in Table 5.1.

Table 5.1

$\varphi^i(q)\varphi^j(q^4)\varphi^k(q^9)\varphi^l(q^{36})$	$a(i, j, k, l)$			
	(5.1)	(5.2)	(5.3)	(5.4)
$\varphi^4(q)$	1	1	1	0
$\varphi^3(q)\varphi(q^4)$	-4	-4	-4	0
$\varphi^3(q)\varphi(q^9)$	-4	-4	-4	0
$\varphi^3(q)\varphi(q^{36})$	4	4	4	0
$\varphi^2(q)\varphi^2(q^4)$	12	4	0	1
$\varphi^2(q)\varphi(q^4)\varphi(q^9)$	12	12	12	0
$\varphi^2(q)\varphi(q^4)\varphi(q^{36})$	-24	-8	0	-2
$\varphi^2(q)\varphi^2(q^9)$	6	6	6	0
$\varphi^2(q)\varphi(q^9)\varphi(q^{36})$	-12	-12	-12	0
$\varphi^2(q)\varphi^2(q^{36})$	12	4	0	1
$\varphi(q)\varphi^3(q^4)$	-16	0	8	-2
$\varphi(q)\varphi^2(q^4)\varphi(q^9)$	-24	-8	0	-2
$\varphi(q)\varphi^2(q^4)\varphi(q^{36})$	48	0	-24	6
$\varphi(q)\varphi(q^4)\varphi^2(q^9)$	-12	-12	-12	0
$\varphi(q)\varphi(q^4)\varphi(q^9)\varphi(q^{36})$	48	16	0	4
$\varphi(q)\varphi(q^4)\varphi^2(q^{36})$	-48	0	24	-6
$\varphi(q)\varphi^3(q^9)$	-12	-12	-12	0
$\varphi(q)\varphi^2(q^9)\varphi(q^{36})$	36	36	36	0
$\varphi(q)\varphi(q^9)\varphi^2(q^{36})$	-72	-24	0	-6
$\varphi(q)\varphi^3(q^{36})$	48	0	-24	6
$\varphi^4(q^4)$	8	0	-4	1
$\varphi^3(q^4)\varphi(q^9)$	16	0	-8	2
$\varphi^3(q^4)\varphi(q^{36})$	-32	0	16	-4
$\varphi^2(q^4)\varphi^2(q^9)$	12	4	0	1
$\varphi^2(q^4)\varphi(q^9)\varphi(q^{36})$	-48	0	24	-6
$\varphi^2(q^4)\varphi^2(q^{36})$	48	0	-24	6
$\varphi(q^4)\varphi^3(q^9)$	12	12	12	0
$\varphi(q^4)\varphi^2(q^9)\varphi(q^{36})$	-72	-24	0	-6
$\varphi(q^4)\varphi(q^9)\varphi^2(q^{36})$	144	0	-72	18
$\varphi(q^4)\varphi^3(q^{36})$	-96	0	48	-12
$\varphi^4(q^9)$	9	9	9	0
$\varphi^3(q^9)\varphi(q^{36})$	-36	-36	-36	0
$\varphi^2(q^9)\varphi^2(q^{36})$	108	36	0	9
$\varphi(q^9)\varphi^3(q^{36})$	-144	0	72	-18
$\varphi^4(q^{36})$	72	0	-36	9

Identities (5.1) and (5.2) are consequences of Theorems 4.1 and 4.2 respectively using (4.2). Identities (5.3) and (5.4) are linear combinations of (5.1) and (5.2).

6. $T_r(q)$ in terms of $\varphi(q^i)$, $i \in \{1, 4, 9, 36\}$

By (1.2), (1.4), (1.6) and (4.2), we obtain

$$T(q) = \frac{1}{2} \varphi^3(q) (\varphi(q) - 2\varphi(q^4) - 3\varphi(q^9) + 6\varphi(q^{36})). \tag{6.1}$$

Let ω denote any 12th root of unity. Then

$$\varphi(\omega q) = \sum_{\substack{n=-\infty \\ n \equiv 0 \pmod{6}}}^{\infty} q^{n^2} + \omega \sum_{\substack{n=-\infty \\ n \not\equiv 0 \pmod{2} \\ n \not\equiv 0 \pmod{3}}}^{\infty} q^{n^2} + \omega^4 \sum_{\substack{n=-\infty \\ n \equiv 0 \pmod{2} \\ n \not\equiv 0 \pmod{3}}}^{\infty} q^{n^2} + \omega^9 \sum_{\substack{n=-\infty \\ n \not\equiv 0 \pmod{2} \\ n \equiv 0 \pmod{3}}}^{\infty} q^{n^2}.$$

Thus we obtain

$$\varphi(\omega q) = \omega \varphi(q) + (\omega^4 - \omega) \varphi(q^4) + (\omega^9 - \omega) \varphi(q^9) + (1 + \omega - \omega^4 - \omega^9) \varphi(q^{36}). \tag{6.2}$$

In particular we have

$$\varphi(\omega^4 q^4) = \varphi(q^{36}) + \omega^4 (\varphi(q^4) - \varphi(q^{36})), \tag{6.3}$$

$$\varphi(\omega^9 q^9) = \varphi(q^{36}) + \omega^9 (\varphi(q^9) - \varphi(q^{36})). \tag{6.4}$$

Replacing q by ωq in (6.1), and appealing to (6.3) and (6.4), we obtain

$$T(\omega q) = \frac{1}{2} \varphi^3(\omega q) (\varphi(\omega q) - 2\varphi(\omega^4 q^4) - 3\varphi(\omega^9 q^9) + 6\varphi(q^{36})). \tag{6.5}$$

Next for $r \in \{0, 1, \dots, 11\}$ we have with ω chosen to be $e^{2\pi i/12}$

$$T_r(q) = \frac{1}{12} \sum_{n=0}^{\infty} t(n) q^n \sum_{k=0}^{11} \omega^{(n-r)k} = \frac{1}{12} \sum_{k=0}^{11} \omega^{-rk} \sum_{n=0}^{\infty} t(n) (\omega^k q)^n,$$

that is

$$T_r(q) = \frac{1}{12} \sum_{k=0}^{11} \omega^{-rk} T(\omega^k q). \tag{6.6}$$

Appealing to (6.2)–(6.5), using MAPLE, we calculate $T(\omega^k q)$, and then by substituting the values of $T(\omega^k q)$ in (6.6), we obtain $T_r(q)$ ($r \in \{0, 1, \dots, 11\}$) in the form

$$T_r(q) = \sum_{\substack{i,j,k,l=0 \\ i+j+k+l=4}}^4 b_r(i, j, k, l) \varphi^i(q) \varphi^j(q^4) \varphi^k(q^9) \varphi^l(q^{36}).$$

The coefficients $b_r(i, j, k, l)$ are given in Table 6.1. (We note that we used the identities (5.2) and (5.4) to simplify $T_4(q)$ to the form given.)

Table 6.1

$\varphi^i(q)\varphi^j(q^4)\varphi^k(q^9)\varphi^l(q^{36})$	$b_r(i, j, k, l) \ (r = 0, 1, \dots, 11)$											
	$T_0(q)$	$T_1(q)$	$T_2(q)$	$T_3(q)$	$T_4(q)$	$T_5(q)$	$T_6(q)$	$T_7(q)$	$T_8(q)$	$T_9(q)$	$T_{10}(q)$	$T_{11}(q)$
$\varphi^4(q)$	0	0	0	0	0	0	0	0	0	0	0	0
$\varphi^3(q)\varphi(q^4)$	0	0	0	0	0	0	0	1	0	0	0	0
$\varphi^3(q)\varphi(q^9)$	1/2	0	0	0	0	0	0	0	0	0	0	0
$\varphi^3(q)\varphi(q^{36})$	-1/2	0	0	5/2	0	0	0	-1	0	0	0	0
$\varphi^2(q)\varphi^2(q^4)$	0	0	0	0	0	0	0	-3	0	0	0	0
$\varphi^2(q)\varphi(q^4)\varphi(q^9)$	-3/2	0	0	-3/2	0	0	0	-3	0	0	0	0
$\varphi^2(q)\varphi(q^4)\varphi(q^{36})$	3/2	0	0	-6	0	0	9/2	6	0	0	0	0
$\varphi^2(q)\varphi^2(q^9)$	-3/2	0	0	0	0	0	0	0	-3/2	0	0	0
$\varphi^2(q)\varphi(q^9)\varphi(q^{36})$	3	0	0	-6	0	0	0	3	3	0	0	3
$\varphi^2(q)\varphi^2(q^{36})$	-3/2	0	9/2	6	0	0	-9/2	-3	-3/2	0	0	-3
$\varphi(q)\varphi^3(q^4)$	0	-1	0	0	0	0	0	3	0	0	0	0
$\varphi(q)\varphi^2(q^4)\varphi(q^9)$	3/2	0	0	3	0	0	-9/2	6	0	0	0	0
$\varphi(q)\varphi^2(q^4)\varphi(q^{36})$	-3/2	3	0	9/2	0	0	-9/2	-9	0	3/2	0	0
$\varphi(q)\varphi(q^4)\varphi^2(q^9)$	3	0	0	3	0	0	0	3	3	0	0	-6
$\varphi(q)\varphi(q^4)\varphi(q^9)\varphi(q^{36})$	-6	0	0	6	0	0	0	-12	-6	0	0	6
$\varphi(q)\varphi(q^4)\varphi^2(q^{36})$	3	-3	-9	-9	0	6	9	9	3	-3	0	0
$\varphi(q)\varphi^3(q^9)$	3/2	0	0	0	3/2	0	0	0	3	0	0	0
$\varphi(q)\varphi^2(q^9)\varphi(q^{36})$	-9/2	0	0	9/2	-9/2	0	0	-9/2	-9	0	0	0
$\varphi(q)\varphi(q^9)\varphi^2(q^{36})$	9/2	0	-9	-9	9/2	0	9/2	9	9	0	9/2	0
$\varphi(q)\varphi^3(q^{36})$	-3/2	9/2	9	9/2	-3/2	-6	-9/2	-9/2	-3	3/2	-9/2	0
$\varphi^4(q^4)$	0	1	0	0	-1	0	0	-1	0	0	0	0
$\varphi^3(q^4)\varphi(q^9)$	-1/2	1	0	-3/2	0	0	9/2	-3	0	-5/2	0	0
$\varphi^3(q^4)\varphi(q^{36})$	0	-4	0	-1	4	0	0	4	0	1	0	0
$\varphi^2(q^4)\varphi^2(q^9)$	-3/2	0	-9/2	-3	0	0	9/2	-3	-3/2	0	0	6
$\varphi^2(q^4)\varphi(q^9)\varphi(q^{36})$	3	-3	9	0	0	-3	-9	9	3	6	0	-9
$\varphi^2(q^4)\varphi^2(q^{36})$	0	6	0	3	-6	-3	0	-6	0	-3	0	3
$\varphi(q^4)\varphi^3(q^9)$	-3/2	0	0	-3/2	-3/2	0	0	-9/2	-3	0	0	6
$\varphi(q^4)\varphi^2(q^9)\varphi(q^{36})$	9/2	0	9	0	9/2	0	-9/2	18	9	0	-9/2	-18
$\varphi(q^4)\varphi(q^9)\varphi^2(q^{36})$	-9/2	9/2	-9	9/2	-9/2	0	9/2	-45/2	-9	-9/2	9/2	18
$\varphi(q^4)\varphi^3(q^{36})$	0	-9	0	-3	12	6	0	9	0	3	0	-6
$\varphi^4(q^9)$	-3/2	0	0	0	-3/2	0	0	0	-3/2	0	0	0
$\varphi^3(q^9)\varphi(q^{36})$	6	0	0	-3	6	0	0	6	6	0	0	-3
$\varphi^2(q^9)\varphi^2(q^{36})$	-9	0	0	9	-9	0	0	-18	-9	0	0	9
$\varphi(q^9)\varphi^3(q^{36})$	6	-6	0	-9	6	3	0	18	6	3	0	-9
$\varphi^4(q^{36})$	0	6	0	3	-9	-3	0	-6	0	-3	0	3

7. $C_{1,12}(q)$ and $C_{7,12}(q)$ in terms of $\varphi(q^i)$, $i \in \{1, 4, 9, 36\}$

It was shown in [5, Lemma 2.3, p. 16] that

$$\begin{aligned} \varphi(iq) &= \varphi(q^4) + i(\varphi(q) - \varphi(q^4)), \\ \varphi(-iq) &= \varphi(q^4) - i(\varphi(q) - \varphi(q^4)). \end{aligned} \tag{7.1}$$

Theorem 7.1.

$$\begin{aligned} \text{(a)} \quad C_{1,12}(q) &= \frac{1}{16}\varphi^4(q) - \frac{1}{4}\varphi^3(q)\varphi(q^4) - \frac{1}{4}\varphi^3(q)\varphi(q^9) + \frac{1}{4}\varphi^3(q)\varphi(q^{36}) \\ &+ \frac{3}{4}\varphi^2(q)\varphi(q^4)\varphi(q^9) + \frac{3}{8}\varphi^2(q)\varphi^2(q^9) - \frac{3}{4}\varphi^2(q)\varphi(q^9)\varphi(q^{36}) \\ &+ \frac{1}{4}\varphi(q)\varphi^3(q^4) - \frac{3}{4}\varphi(q)\varphi^2(q^4)\varphi(q^{36}) - \frac{3}{4}\varphi(q)\varphi(q^4)\varphi^2(q^9) \\ &+ \frac{3}{4}\varphi(q)\varphi(q^4)\varphi^2(q^{36}) - \frac{3}{4}\varphi(q)\varphi^3(q^9) + \frac{9}{4}\varphi(q)\varphi^2(q^9)\varphi(q^{36}) \\ &- \frac{3}{4}\varphi(q)\varphi^3(q^{36}) - \frac{1}{4}\varphi^3(q^4)\varphi(q^9) + \frac{3}{4}\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) + \frac{3}{4}\varphi(q^4)\varphi^3(q^9) \\ &- \frac{9}{4}\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}) + \frac{9}{16}\varphi^4(q^9) - \frac{9}{4}\varphi^3(q^9)\varphi(q^{36}) + \frac{9}{4}\varphi(q^9)\varphi^3(q^{36}), \\ \text{(b)} \quad C_{7,12}(q) &= -\frac{1}{8}\varphi^4(q) + \frac{1}{4}\varphi^3(q)\varphi(q^4) + \frac{1}{2}\varphi^3(q)\varphi(q^9) - \frac{1}{4}\varphi^3(q)\varphi(q^{36}) \\ &- \frac{3}{4}\varphi^2(q)\varphi(q^4)\varphi(q^9) - \frac{3}{4}\varphi^2(q)\varphi^2(q^9) + \frac{3}{4}\varphi^2(q)\varphi(q^9)\varphi(q^{36}) - \frac{1}{4}\varphi(q)\varphi^3(q^4) \\ &+ \frac{3}{4}\varphi(q)\varphi^2(q^4)\varphi(q^{36}) + \frac{3}{4}\varphi(q)\varphi(q^4)\varphi^2(q^9) - \frac{3}{4}\varphi(q)\varphi(q^4)\varphi^2(q^{36}) \\ &+ \frac{3}{2}\varphi(q)\varphi^3(q^9) - \frac{9}{4}\varphi(q)\varphi^2(q^9)\varphi(q^{36}) + \frac{3}{4}\varphi(q)\varphi^3(q^{36}) + \frac{1}{4}\varphi^3(q^4)\varphi(q^9) \\ &- \frac{3}{4}\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) - \frac{3}{4}\varphi(q^4)\varphi^3(q^9) + \frac{9}{4}\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}) \\ &- \frac{9}{8}\varphi^4(q^9) + \frac{9}{4}\varphi^3(q^9)\varphi(q^{36}) - \frac{9}{4}\varphi(q^9)\varphi^3(q^{36}). \end{aligned}$$

Proof. Recall that $c(n) = 0$ for $n \not\equiv 1 \pmod{6}$. We have

$$\begin{aligned} C_{1,12}(q) - C_{7,12}(q) &= \sum_{\substack{n=1 \\ n \equiv 1 \pmod{6} \\ n \equiv 1 \pmod{4}}}^{\infty} c(n)q^n - \sum_{\substack{n=1 \\ n \equiv 1 \pmod{6} \\ n \equiv 3 \pmod{4}}}^{\infty} c(n)q^n \\ &= \sum_{\substack{n=1 \\ n \equiv 1 \pmod{6}}}^{\infty} c(n)q^n \left(\frac{i^n - (-i)^n}{2i} \right) = \frac{1}{2i}C_{1,6}(iq) - \frac{1}{2i}C_{1,6}(-iq). \end{aligned}$$

Appealing to (4.9) and (7.1), we obtain

$$\begin{aligned}
 C_{1,12}(q) - C_{7,12}(q) &= \frac{1}{4}\varphi^3(q)\varphi(q^4) - \frac{1}{4}\varphi^3(q)\varphi(q^{36}) - \frac{3}{4}\varphi^2(q)\varphi^2(q^4) \\
 &\quad - \frac{3}{4}\varphi^2(q)\varphi(q^4)\varphi(q^9) + \frac{3}{2}\varphi^2(q)\varphi(q^4)\varphi(q^{36}) + \frac{3}{4}\varphi^2(q)\varphi(q^9)\varphi(q^{36}) \\
 &\quad - \frac{3}{4}\varphi^2(q)\varphi^2(q^{36}) + \frac{1}{2}\varphi(q)\varphi^3(q^4) + \frac{3}{2}\varphi(q)\varphi^2(q^4)\varphi(q^9) \\
 &\quad - \frac{3}{2}\varphi(q)\varphi^2(q^4)\varphi(q^{36}) + \frac{3}{4}\varphi(q)\varphi(q^4)\varphi^2(q^9) - 3\varphi(q)\varphi(q^4)\varphi(q^9)\varphi(q^{36}) \\
 &\quad + \frac{3}{2}\varphi(q)\varphi(q^4)\varphi^2(q^{36}) - \frac{9}{4}\varphi(q)\varphi^2(q^9)\varphi(q^{36}) + \frac{9}{2}\varphi(q)\varphi(q^9)\varphi^2(q^{36}) \\
 &\quad - \frac{3}{2}\varphi(q)\varphi^3(q^{36}) - \frac{1}{2}\varphi^3(q^4)\varphi(q^9) - \frac{3}{4}\varphi^2(q^4)\varphi^2(q^9) + \frac{3}{2}\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) \\
 &\quad - \frac{3}{4}\varphi(q^4)\varphi^3(q^9) + \frac{9}{2}\varphi(q^4)\varphi^2(q^9)\varphi(q^{36}) - \frac{9}{2}\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}) \\
 &\quad + \frac{9}{4}\varphi^3(q^9)\varphi(q^{36}) - \frac{27}{4}\varphi^2(q^9)\varphi^2(q^{36}) + \frac{9}{2}\varphi(q^9)\varphi^3(q^{36}), \\
 C_{1,12}(q) + C_{7,12}(q) &= \frac{3}{4}\varphi(q)\varphi^3(q^9) - \frac{3}{8}\varphi^2(q)\varphi^2(q^9) + \frac{1}{4}\varphi^3(q)\varphi(q^9) \\
 &\quad - \frac{1}{16}\varphi^4(q) - \frac{9}{16}\varphi^4(q^9).
 \end{aligned}$$

The assertions follow by adding and subtracting the above two equations and using identity (5.2). \square

Theorem 7.2.

$$C_{1,6}(q^2) = \frac{1}{4}\varphi^2(q)\varphi^2(q^{36}) + \frac{1}{4}\varphi^2(q^4)\varphi^2(q^9) - \frac{1}{2}\varphi(q)\varphi(q^4)\varphi(q^9)\varphi(q^{36}).$$

Proof. Sun [11] has shown that for $a, b \in \mathbb{N}$ and $q \in \mathbb{R}$ with $|q| < 1$

$$\begin{aligned}
 \prod_{n=1}^{\infty} (1 - q^{an})(1 - q^{bn}) &= 1 + \sum_{n=1}^{\infty} \frac{1}{2} [R(a + b, 12(a - b), 36(a + b); 24n + a + b) \\
 &\quad - R(4(a + b), 12(a - b), 9(a + b); 24n + a + b)] q^n,
 \end{aligned}$$

where $R(a, b, c; n) = \text{card}\{(x, y) \in \mathbb{Z}^2 \mid n = ax^2 + bxy + cy^2\}$. Take $a = b = 12$. Then

$$\begin{aligned}
 \prod_{n=1}^{\infty} (1 - q^{12n})^2 &= 1 + \sum_{n=1}^{\infty} \frac{1}{2} [R(24, 0, 864; 24n + 24) - R(96, 0, 216; 24n + 24)] q^n \\
 &= 1 + \frac{1}{2} \sum_{n=1}^{\infty} [R(1, 0, 36; n + 1) - R(4, 0, 9; n + 1)] q^n.
 \end{aligned}$$

Multiplying both sides by $2q$, we obtain

$$2q \prod_{n=1}^{\infty} (1 - q^{12n})^2 = 2q + \sum_{n=2}^{\infty} R(1, 0, 36; n)q^n - \sum_{n=2}^{\infty} R(4, 0, 9; n)q^n$$

$$= \varphi(q)\varphi(q^{36}) - \varphi(q^4)\varphi(q^9),$$

as $R(1, 0, 36; 0) = 1$, $R(1, 0, 36; 1) = 2$, $R(4, 0, 9; 0) = 1$, $R(4, 0, 9; 1) = 0$. Squaring both sides, we obtain the asserted identity by (1.5). \square

8. Evaluation of $L_{i,j}(q)$

Theorem 8.1.

$$L_{1,2}(q) = \frac{1}{2} \varphi(q^4) [\varphi^3(q) - 3\varphi^2(q)\varphi(q^4) + 4\varphi(q)\varphi^2(q^4) - 2\varphi^3(q^4)].$$

Proof. From [3, Theorem 2.4(i)] we have

$$L_{1,2}(q) = \frac{1}{4} (\varphi(q) - \varphi(-q)) \varphi^2(q^2) \varphi(q^4).$$

The asserted result now follows using (4.4) and (4.5). \square

Lemma 8.1. $L_{2,4}(q) = 3L_{1,2}(q^2)$.

Proof. Recall that for all $n \in \mathbb{N}$, we have

$$\sigma(2n) = 3\sigma(n) - 2\sigma(n/2). \tag{8.1}$$

Hence, appealing to (8.1), we deduce

$$L_{2,4}(q) = \sum_{\substack{n=1 \\ n \equiv 2 \pmod{4}}}^{\infty} \sigma(n)q^n = \sum_{\substack{n=1 \\ n \equiv 1 \pmod{2}}}^{\infty} \sigma(2n)q^{2n}$$

$$= 3 \sum_{\substack{n=1 \\ n \equiv 1 \pmod{2}}}^{\infty} \sigma(n)q^{2n} = 3L_{1,2}(q^2). \quad \square$$

Theorem 8.2.

- (a) $L_{1,4}(q) = \frac{1}{2} \varphi^3(q^4) (\varphi(q) - \varphi(q^4)),$
- (b) $L_{2,4}(q) = \frac{3}{4} \varphi^2(q^4) (\varphi(q) - \varphi(q^4))^2,$
- (c) $L_{3,4}(q) = \frac{1}{2} \varphi(q^4) (\varphi(q) - \varphi(q^4))^3.$

Proof. (a) From [3, Theorem 2.4(ii)] we have by (4.4),

$$L_{1,4}(q) = \frac{1}{4}[\varphi(q) - \varphi(-q)]\varphi^3(q^4) = \frac{1}{2}(\varphi(q) - \varphi(q^4))\varphi^3(q^4).$$

(b) Replacing q by q^2 in (4.5), we have

$$\varphi^2(q^2) - \varphi^2(q^4) = 2(\varphi(q^2) - \varphi(q^8))\varphi(q^8). \tag{8.2}$$

Substituting (4.5) into (8.2), we obtain

$$(\varphi(q) - \varphi(q^4))^2 = 2\varphi(q^8)(\varphi(q^2) - \varphi(q^8)). \tag{8.3}$$

Appealing to Lemma 8.1, Theorem 8.1, (4.5) and (8.3) we obtain

$$\begin{aligned} L_{2,4}(q) &= \frac{3}{2}\varphi(q^8)[\varphi^3(q^2) - 3\varphi^2(q^2)\varphi(q^8) + 4\varphi(q^2)\varphi^2(q^8) - 2\varphi^3(q^8)] \\ &= \frac{3}{2}\varphi(q^8)[\varphi(q^2) - \varphi(q^8)][\varphi^2(q^2) - 2\varphi(q^2)\varphi(q^8) + 2\varphi^2(q^8)] \\ &= \frac{3}{4}\varphi^2(q^4)[\varphi(q) - \varphi(q^4)]^2. \end{aligned}$$

(c) From [3, Theorem 2.4(iii)] we have

$$L_{3,4}(q) = \frac{1}{4}[\varphi(q) - \varphi(-q)][\varphi(q^2) - \varphi(-q^2)]\varphi(q^4)\varphi(q^8).$$

Then, by (4.4) and (8.3), we obtain

$$\begin{aligned} L_{3,4}(q) &= \varphi(q^4)[\varphi(q) - \varphi(q^4)]\varphi(q^8)[\varphi(q^2) - \varphi(q^8)] \\ &= \varphi(q^4)[\varphi(q) - \varphi(q^4)]\frac{1}{2}[\varphi(q) - \varphi(q^4)]^2 = \frac{1}{2}\varphi(q^4)[\varphi(q) - \varphi(q^4)]^3. \quad \square \end{aligned}$$

Theorem 8.3.

- (a) $L_{1,4}(q^9) = \frac{1}{2}\varphi^3(q^{36})(\varphi(q^9) - \varphi(q^{36})),$
- (b) $L_{2,4}(q^9) = \frac{3}{4}\varphi^2(q^{36})(\varphi(q^9) - \varphi(q^{36}))^2,$
- (c) $L_{3,4}(q^9) = \frac{1}{2}\varphi(q^{36})(\varphi(q^9) - \varphi(q^{36}))^3.$

Proof. Replace q by q^9 in Theorem 8.2. \square

Theorem 8.4.

- (a) $\alpha(q) = \frac{1}{8}(-1 + \varphi^4(q)),$
- (b) $\alpha(q^{12}) = \frac{1}{8}(-1 + 3\varphi(q^4)\varphi^3(q^{36}) - 3\varphi^2(q^4)\varphi^2(q^{36}) + \varphi^3(q^4)\varphi(q^{36})),$
- (c) $\alpha(q^{36}) = \frac{1}{8}(-1 + \varphi^4(q^{36})).$

Proof. (a) The number of representations of $n \in \mathbb{N}$ as the sum of four squares is $8\sigma(n) - 32\sigma(n/4)$ so

$$\varphi^4(q) - 1 = \sum_{n=1}^{\infty} (8\sigma(n) - 32\sigma(n/4))q^n = 8L_{0,1}(q) - 32L_{0,1}(q^4) = 8\alpha(q).$$

(b) Replacing q by q^4 in (4.7), we obtain

$$\varphi^4(q^{12}) = 3\varphi(q^4)\varphi^3(q^{36}) - 3\varphi^2(q^4)\varphi^2(q^{36}) + \varphi^3(q^4)\varphi(q^{36}). \tag{8.4}$$

The assertion follows from (a) on replacing q by q^{12} and then using (8.4).

(c) Replace q by q^{36} in part (a). \square

Theorem 8.5. Let $\omega = e^{2\pi i/3}$. Then, for $s = 1, 2, \dots, 11,$

$$L_{s,12}(q) = \frac{1}{3}L_{s,4}(q) + \frac{1}{3}\omega^{2s}L_{s,4}(\omega q) + \frac{1}{3}\omega^sL_{s,4}(\omega^2q).$$

Proof. We have

$$\begin{aligned} L_{s,12}(q) &= \sum_{\substack{n=1 \\ n \equiv s \pmod{3} \\ n \equiv s \pmod{4}}}^{\infty} \sigma(n)q^n \\ &= \frac{1}{3} \sum_{\substack{n=1 \\ n \equiv s \pmod{4}}}^{\infty} \sigma(n)q^n (1 + \omega^{n-s} + \omega^{2(n-s)}) \\ &= \frac{1}{3}L_{s,4}(q) + \frac{1}{3}\omega^{2s}L_{s,4}(\omega q) + \frac{1}{3}\omega^sL_{s,4}(\omega^2q). \quad \square \end{aligned}$$

Theorem 8.6. For $s \in \{1, 2, 3, 5, 6, 7, 9, 10, 11\},$ we have

$$L_{s,12}(q) = \sum_{\substack{i,j,k,l=0 \\ i+j+k+l=4}}^4 c_s(i, j, k, l)\varphi^i(q)\varphi^j(q^4)\varphi^k(q^9)\varphi^l(q^{36}).$$

The coefficients $c_s(i, j, k, l)$ are listed in Table 8.1. We note that $L_{s,12}(q)$ is shortened as $L_{s,12}.$

Table 8.1

$c_s(i, j, k, l)$	$s = 1$ $L_{1,12}$	$s = 2$ $L_{2,12}$	$s = 3$ $L_{3,12}$	$s = 5$ $L_{5,12}$	$s = 6$ $L_{6,12}$	$s = 7$ $L_{7,12}$	$s = 9$ $L_{9,12}$	$s = 10$ $L_{10,12}$	$s = 11$ $L_{11,12}$
$c_5(4, 0, 0, 0)$	0	0	0	0	0	0	0	0	0
$c_5(3, 1, 0, 0)$	0	0	0	0	0	1/2	0	0	0
$c_5(3, 0, 1, 0)$	0	0	0	0	0	0	0	0	0
$c_5(3, 0, 0, 1)$	0	0	1/2	0	0	-1/2	0	0	0
$c_5(2, 2, 0, 0)$	0	0	0	0	0	-3/2	0	3/4	0
$c_5(2, 1, 1, 0)$	0	0	3/2	0	0	-3/2	0	0	0
$c_5(2, 1, 0, 1)$	0	0	-3	0	3/2	3	0	-3/2	0
$c_5(2, 0, 2, 0)$	0	0	0	0	0	0	0	0	0
$c_5(2, 0, 1, 1)$	0	0	-3	0	0	3/2	0	0	3/2
$c_5(2, 0, 0, 2)$	0	3/4	3	0	-3/2	-3/2	0	3/4	-3/2
$c_5(1, 3, 0, 0)$	1/2	0	0	0	0	3/2	0	-3/2	0
$c_5(1, 2, 1, 0)$	0	0	-3	0	3/2	3	0	-3/2	0
$c_5(1, 2, 0, 1)$	-3/2	0	9/2	0	-9/2	-9/2	3/2	9/2	0
$c_5(1, 1, 2, 0)$	0	0	-3	0	0	3/2	0	0	3/2
$c_5(1, 1, 1, 1)$	0	3	12	0	-6	-6	0	3	-6
$c_5(1, 1, 0, 2)$	3/2	-9/2	-9	3/2	9	9/2	-3	-9/2	9/2
$c_5(1, 0, 3, 0)$	0	0	0	0	0	0	0	0	0
$c_5(1, 0, 2, 1)$	0	0	9/2	0	0	0	0	0	-9/2
$c_5(1, 0, 1, 2)$	0	-9/2	-9	0	9/2	0	0	0	9
$c_5(1, 0, 0, 3)$	0	9/2	9/2	-3/2	-9/2	0	3/2	0	-9/2
$c_5(0, 4, 0, 0)$	-1/2	0	0	0	0	-1/2	0	3/4	0
$c_5(0, 3, 1, 0)$	-1/2	0	3/2	0	-3/2	-3/2	1/2	3/2	0
$c_5(0, 3, 0, 1)$	2	0	-2	0	3	2	-2	-3	0
$c_5(0, 2, 2, 0)$	0	3/4	3	0	-3/2	-3/2	0	3/4	-3/2
$c_5(0, 2, 1, 1)$	3/2	-9/2	-9	3/2	9	9/2	-3	-9/2	9/2
$c_5(0, 2, 0, 2)$	-3	9/2	6	-3	-9	-3	6	9/2	-3
$c_5(0, 1, 3, 0)$	0	0	3/2	0	0	0	0	0	-3/2
$c_5(0, 1, 2, 1)$	0	-9/2	-9	0	9/2	0	0	0	9
$c_5(0, 1, 1, 2)$	0	27/2	27/2	-9/2	-27/2	0	9/2	0	-27/2
$c_5(0, 1, 0, 3)$	0	-9	-6	6	9	0	-6	0	6
$c_5(0, 0, 4, 0)$	0	0	0	0	0	0	0	0	0
$c_5(0, 0, 3, 1)$	0	0	-3/2	0	0	-3/2	0	0	3
$c_5(0, 0, 2, 2)$	0	9/2	9/2	0	-9/4	9/2	0	-9/4	-9
$c_5(0, 0, 1, 3)$	-3/2	-9	-9/2	3	9/2	-9/2	-3/2	9/2	9
$c_5(0, 0, 0, 4)$	3/2	9/2	3/2	-3	-9/4	3/2	3/2	-9/4	-3

Proof. We do the proof only for $L_{1,12}(q)$ as the other ones can be done similarly. By Theorem 8.5, we have

$$L_{1,12}(q) = \frac{1}{3}L_{1,4}(q) + \frac{1}{3}\omega^2 L_{1,4}(\omega q) + \frac{1}{3}\omega L_{1,4}(\omega^2 q). \tag{8.5}$$

Appealing to Theorem 8.2(a), (6.2) and (6.3), we obtain

$$\begin{aligned} L_{1,4}(\omega q) &= \frac{3}{2}\varphi(q)\varphi^2(q^4)\varphi(q^{36}) - \frac{9}{2}\varphi(q)\varphi(q^4)\varphi^2(q^{36}) + 3\varphi(q)\varphi^3(q^{36}) \\ &+ \frac{1}{2}\varphi^3(q^4)\varphi(q^9) - 2\varphi^3(q^4)\varphi(q^{36}) - \frac{9}{2}\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) \\ &+ 9\varphi^2(q^4)\varphi^2(q^{36}) + \frac{9}{2}\varphi^4(q^{36}) - 12\varphi(q^4)\varphi^3(q^{36}) - \frac{9}{2}\varphi(q^9)\varphi^3(q^{36}) \\ &+ 9\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}) + \omega A(q), \end{aligned}$$

$$\begin{aligned}
 L_{1,4}(\omega^2 q) &= -\frac{1}{2}\varphi(q)\varphi^3(q^4) + 3\varphi(q)\varphi^2(q^4)\varphi(q^{36}) - \frac{9}{2}\varphi(q)\varphi(q^4)\varphi^2(q^{36}) \\
 &\quad + \frac{3}{2}\varphi(q)\varphi^3(q^{36}) + \frac{1}{2}\varphi^4(q^4) + \varphi^3(q^4)\varphi(q^9) - 4\varphi^3(q^4)\varphi(q^{36}) - 6\varphi(q^4)\varphi^3(q^{36}) \\
 &\quad - \frac{9}{2}\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) + 9\varphi^2(q^4)\varphi^2(q^{36}) + \frac{9}{2}\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}) - \omega A(q),
 \end{aligned}$$

where

$$\begin{aligned}
 A(q) &= \frac{1}{2}\varphi(q)\varphi^3(q^4) - \frac{3}{2}\varphi(q)\varphi^2(q^4)\varphi(q^{36}) + \frac{3}{2}\varphi(q)\varphi^3(q^{36}) \\
 &\quad - \frac{1}{2}\varphi^4(q^4) - \frac{1}{2}\varphi^3(q^4)\varphi(q^9) + 2\varphi^3(q^4)\varphi(q^{36}) + \frac{9}{2}\varphi^4(q^{36}) \\
 &\quad - 6\varphi(q^4)\varphi^3(q^{36}) - \frac{9}{2}\varphi(q^9)\varphi^3(q^{36}) + \frac{9}{2}\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}).
 \end{aligned}$$

Substituting $L_{1,4}(q)$, $L_{1,4}(\omega q)$, $L_{1,4}(\omega^2 q)$ into (8.5) completes the proof. \square

We note that $L_{4,12}(q)$ and $L_{8,12}(q)$ can be obtained in a similar manner. They are excluded as they are not needed in this paper.

9. Proofs of theorems of Section 3

Theorem 3.10 follows from Theorems 4.5, 4.6, 8.3(a), 8.4(c), Tables 6.1 and 8.1 immediately. We just prove Theorem 3.7. The rest can be proved similarly.

By Theorems 4.5, 4.6, 7.1(a), 8.3(a), 8.4(b), Table 6.1, Table 8.1 and the identity (5.3), the right-hand side of Theorem 3.7 becomes

$$\begin{aligned}
 &\frac{1}{24}[\varphi^4(q) - 4\varphi^3(q)\varphi(q^4) - 4\varphi^3(q)\varphi(q^9) + 4\varphi^3(q)\varphi(q^{36}) + 12\varphi^2(q)\varphi(q^4)\varphi(q^9) \\
 &\quad + 6\varphi^2(q)\varphi^2(q^9) - 12\varphi^2(q)\varphi(q^9)\varphi(q^{36}) + 8\varphi(q)\varphi^3(q^4) - 12\varphi(q)\varphi^3(q^9) \\
 &\quad - 24\varphi(q)\varphi^2(q^4)\varphi(q^{36}) - 12\varphi(q)\varphi(q^4)\varphi^2(q^9) + 24\varphi(q)\varphi(q^4)\varphi^2(q^{36}) \\
 &\quad + 36\varphi(q)\varphi^2(q^9)\varphi(q^{36}) - 24\varphi(q)\varphi^3(q^{36}) - 4\varphi^4(q^4) - 8\varphi^3(q^4)\varphi(q^9) \\
 &\quad + 16\varphi^3(q^4)\varphi(q^{36}) + 24\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) - 24\varphi^2(q^4)\varphi^2(q^{36}) + 9\varphi^4(q^9) \\
 &\quad - 72\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}) + 48\varphi(q^4)\varphi^3(q^{36}) + 12\varphi(q^4)\varphi^3(q^9) - 36\varphi^4(q^{36}) \\
 &\quad + 72\varphi(q^9)\varphi^3(q^{36}) - 36\varphi^3(q^9)\varphi(q^{36})] + \varphi(q)\varphi^2(q^4)\varphi(q^{36}) \\
 &= \varphi(q)\varphi^2(q^4)\varphi(q^{36}).
 \end{aligned}$$

10. Proofs of theorems of Section 2

We deduce Theorem 2.7 from Theorem 3.7. The rest can be proved similarly. By Theorem 3.7 and (1.1), (1.5), (1.7), (1.8), we have

$$\begin{aligned}
 & \sum_{n=0}^{\infty} N(1, 4, 4, 36; n)q^n \\
 &= 1 + 8 \sum_{n=1}^{\infty} \sigma(n)q^{12n} - 32 \sum_{n=1}^{\infty} \sigma(n)q^{48n} + 2 \sum_{\substack{n=1 \\ n \equiv 1 \pmod{3}}}^{\infty} \sigma(n)q^{4n} \\
 &\quad - 8 \sum_{\substack{n=1 \\ n \equiv 1 \pmod{3}}}^{\infty} \sigma(n)q^{16n} + 4 \sum_{\substack{n=1 \\ n \equiv 2 \pmod{3}}}^{\infty} \sigma(n)q^{4n} - 16 \sum_{\substack{n=1 \\ n \equiv 2 \pmod{3}}}^{\infty} \sigma(n)q^{16n} \\
 &\quad + \sum_{\substack{n=1 \\ n \equiv 1 \pmod{4}}}^{\infty} \sigma(n)q^{9n} + \frac{5}{9} \sum_{\substack{n=1 \\ n \equiv 1 \pmod{12}}}^{\infty} \sigma(n)q^n + \frac{8}{9} \sum_{\substack{n=1 \\ n \equiv 5 \pmod{12}}}^{\infty} \sigma(n)q^n \\
 &\quad + \frac{5}{9} \sum_{\substack{n=1 \\ n \equiv 9 \pmod{12}}}^{\infty} \sigma(n)q^n + \frac{2}{3} \sum_{\substack{n=1 \\ n \equiv 1 \pmod{12}}}^{\infty} c(n)q^n + 4 \sum_{\substack{n=1 \\ n \equiv 1 \pmod{6}}}^{\infty} c(n)q^{4n} \\
 &\quad + \frac{1}{9} \left(\sum_{\substack{n=1 \\ n \equiv 1 \pmod{12}}}^{\infty} t(n)q^n + \sum_{\substack{n=1 \\ n \equiv 5 \pmod{12}}}^{\infty} t(n)q^n + \sum_{\substack{n=1 \\ n \equiv 9 \pmod{12}}}^{\infty} t(n)q^n \right) \\
 &= 1 + 8 \sum_{n=1}^{\infty} \sigma(n/12)q^n - 32 \sum_{n=1}^{\infty} \sigma(n/48)q^n + 2 \sum_{\substack{n=1 \\ n \equiv 1 \pmod{3}}}^{\infty} \sigma(n/4)q^n \\
 &\quad - 8 \sum_{\substack{n=1 \\ n \equiv 1 \pmod{3}}}^{\infty} \sigma(n/16)q^n + 4 \sum_{\substack{n=1 \\ n \equiv 2 \pmod{3}}}^{\infty} \sigma(n/4)q^n - 16 \sum_{\substack{n=1 \\ n \equiv 2 \pmod{3}}}^{\infty} \sigma(n/16)q^n \\
 &\quad + \sum_{\substack{n=1 \\ n \equiv 1 \pmod{4}}}^{\infty} \sigma(n/9)q^n + \frac{5}{9} \sum_{\substack{n=1 \\ n \equiv 1 \pmod{12}}}^{\infty} \sigma(n)q^n + \frac{8}{9} \sum_{\substack{n=1 \\ n \equiv 5 \pmod{12}}}^{\infty} \sigma(n)q^n \\
 &\quad + \frac{5}{9} \sum_{\substack{n=1 \\ n \equiv 9 \pmod{12}}}^{\infty} \sigma(n)q^n + \frac{2}{3} \sum_{\substack{n=1 \\ n \equiv 1 \pmod{12}}}^{\infty} c(n)q^n + 4 \sum_{\substack{n=1 \\ n \equiv 4 \pmod{6}}}^{\infty} c(n/4)q^n \\
 &\quad + \frac{1}{9} \left(\sum_{\substack{n=1 \\ n \equiv 1 \pmod{12}}}^{\infty} t(n)q^n + \sum_{\substack{n=1 \\ n \equiv 5 \pmod{12}}}^{\infty} t(n)q^n + \sum_{\substack{n=1 \\ n \equiv 9 \pmod{12}}}^{\infty} t(n)q^n \right).
 \end{aligned}$$

Equating the coefficients of q^n completes the proof.

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