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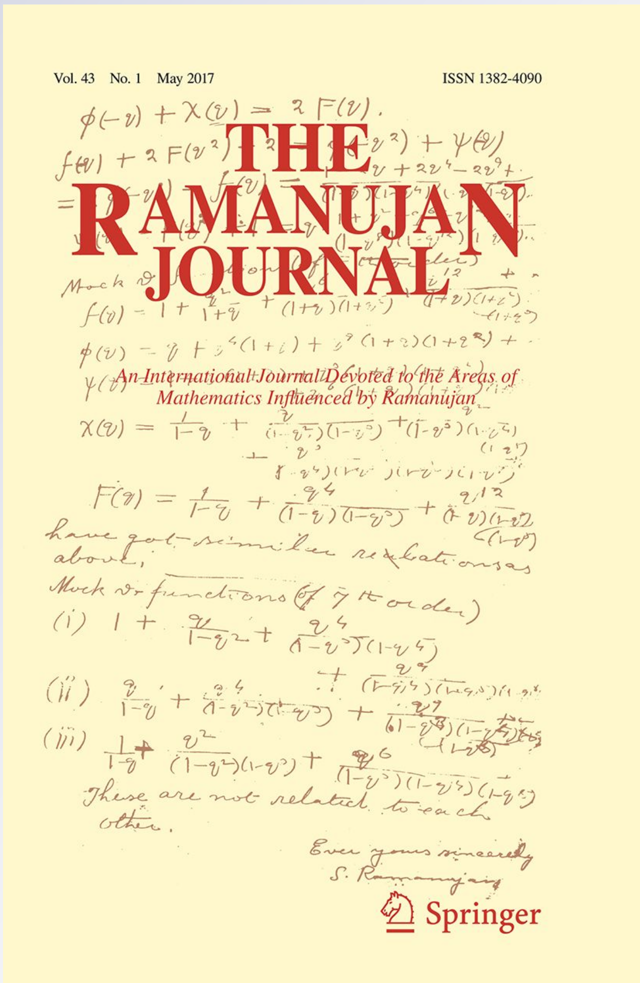

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Abstract Let n be a positive integer. Let $\delta_3(n)$ denote the difference between the number of (positive) divisors of n congruent to 1 modulo 3 and the number of those congruent to 2 modulo 3. In 2004, Farkas proved that the arithmetic convolution sum

$$D_3(n) := \sum_{j=1}^{n-1} \delta_3(j)\delta_3(n-j)$$

satisfies the relation

$$3D_3(n) + \delta_3(n) = \sum_{\substack{d \mid n \\ 3 \nmid d}} d.$$

In this paper, we use a result about binary quadratic forms to prove a general arithmetic convolution identity which contains Farkas' formula and two other similar known formulas as special cases. From our identity, we deduce a number of analogous new convolution formulas.

Keywords Divisor functions · Arithmetic convolution identities · Representations by binary quadratic forms · Sums of two binary quadratic forms

Mathematics Subject Classification 11A25 · 11E25

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1 Introduction

As usual let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ denote the sets of positive integers, integers and rational numbers, respectively. In 2004, Farkas [7, Theorem 1] showed using the theory of theta functions that the arithmetic function

$$\delta_3(n) := \sum_{\substack{d\in\mathbb{N} \\ d\mid n \\ d\equiv 1 \pmod 3}} 1 - \sum_{\substack{d\in\mathbb{N} \\ d\mid n \\ d\equiv 2 \pmod 3}} 1 \tag{1.1}$$

satisfies the relation

$$3\sum_{j=1}^{n-1} \delta_3(j)\delta_3(n-j) + \delta_3(n) = \sigma(n) - 3\sigma(n/3) \tag{1.2}$$

for all positive integers n, where

$$\sigma(m) = \begin{cases} \sum_{\substack{d\in\mathbb{N} \\ d\mid m}} d & \text{if } m \in \mathbb{N}, \\ 0 & \text{if } m \in \mathbb{Q}\setminus\mathbb{N}. \end{cases} \tag{1.3}$$

(If $f : \mathbb{N} \to \mathbb{Z}$, we understand throughout this paper that $f(x) = 0$ for all $x \in \mathbb{Q}\setminus\mathbb{N}$.)

In 2005, Farkas [8] derived the arithmetic identity

$$2\sum_{j=1}^{n-1} \delta_4(j)\delta_4(n-j) + \delta_4(n) = \sigma(n) - 4\sigma(n/4) \tag{1.4}$$

for all positive integers n, where

$$\delta_4(n) := \sum_{\substack{d\in\mathbb{N} \\ d\mid n \\ d\equiv 1 \pmod 4}} 1 - \sum_{\substack{d\in\mathbb{N} \\ d\mid n \\ d\equiv 3 \pmod 4}} 1. \tag{1.5}$$

This identity was reproved by Raji [14] in 2008 using eta products and their logarithmic derivatives.

In 2009, Guerzhoy and Raji [9] showed that

$$\sum_{j=1}^{n-1} \delta_7(j)\delta_7(n-j) + \delta_7(n) = \sigma(n) - 7\sigma(n/7) \tag{1.6}$$

for all positive integers n, where

$$\delta_7(n) := \sum_{\substack{d\in\mathbb{N}\\ d\mid n\\ d\equiv 1,2,4 \pmod 7}} 1 - \sum_{\substack{d\in\mathbb{N}\\ d\mid n\\ d\equiv 3,5,6 \pmod 7}} 1. \tag{1.7}$$

In Sect. 2, we prove a general arithmetic convolution identity, see Theorem 2.1, from which the identities (1.2), (1.4) and (1.6) follow. Theorem 2.1 is proved using a result from the theory of binary quadratic forms, specifically an extension of a theorem of Dirichlet due to Kaplan and Williams [11] giving the number of representations of a positive integer n by a representative system of inequivalent, primitive, positive-definite, integral, binary quadratic forms with fundamental discriminant. In order to apply Theorem 2.1, one requires the number of representations of n by certain quaternary quadratic forms of the type $a_1x^2 + b_1xy + c_1y^2 + a_2z^2 + b_2zt + c_2t^2$ of which there are many examples in the literature. Then, in Sect. 3, we deduce from Theorem 2.1 twelve new arithmetic convolution identities similar to (1.2), (1.4) and (1.6), see Theorem 3.1. Further examples illustrating Theorem 2.1 are given in Sects. 4 and 5.

2 A general arithmetic convolution identity

We begin with some definitions.

Definition 2.1 An integer D is called a discriminant if D is not a perfect square and $D \equiv 0 \text{ or } 1 \pmod 4$.

Definition 2.2 A discriminant D is called a fundamental discriminant if there is no integer $g > 1$ such that $g^2 \mid D$ and $D/g^2 \equiv 0 \text{ or } 1 \pmod 4$.

Definition 2.3 Let D be a negative discriminant. Let $\left(\frac{D}{*}\right)$ be the Legendre–Jacobi–Kronecker symbol. For $n \in \mathbb{N}$, define

$$\delta_{|D|}(n) := \sum_{\substack{d\in\mathbb{N}\\ d\mid n}} \left(\frac{D}{d}\right). \tag{2.1}$$

For $d \in \mathbb{N}$, we have

$$\left(\frac{-3}{d}\right) = \begin{cases} 1 & \text{if} \quad d \equiv 1 \pmod 3,\\ -1 & \text{if} \quad d \equiv 2 \pmod 3,\\ 0 & \text{if} \quad d \equiv 0 \pmod 3,\end{cases}$$

$$\left(\frac{-4}{d}\right) = \begin{cases} 1 & \text{if} \quad d \equiv 1 \pmod 4,\\ -1 & \text{if} \quad d \equiv 3 \pmod 4,\\ 0 & \text{if} \quad d \equiv 0 \pmod 2,\end{cases}$$

$$\left(\frac{-7}{d}\right) = \begin{cases} 1 & \text{if} \quad d \equiv 1, 2, 4 \pmod 7, \\ -1 & \text{if} \quad d \equiv 3, 5, 6 \pmod 7, \\ 0 & \text{if} \quad d \equiv 0 \pmod 7, \end{cases}$$

so that (2.1) agrees with (1.1), (1.5) and (1.7) when $D = -3, -4$ and -7, respectively.

We now recall a result from the theory of binary quadratic forms.

Proposition 2.1 *Let D be a negative fundamental discriminant. Let*

$$\{f_i(x, y) = a_i x^2 + b_i xy + c_i y^2 \mid i = 1, 2, \dots, h\}$$

be a representative set of inequivalent, primitive, positive-definite, integral, binary quadratic forms of discriminant D. (The positive integer $h = h(D)$ is called the class number of discriminant D.) Let n be a positive integer. Let $N(n, d)$ denote the number of representations of n by the forms $f_i(x, y) (i = 1, 2, \dots, h)$, that is

$$N(n, D) = \sum_{i=1}^{h} \text{card}\{(x, y) \in \mathbb{Z}^2 \mid f_i(x, y) = n\}.$$

Then

$$N(n, D) = w(D)\delta_{|D|}(n),$$

where

$$w(D) = \begin{cases} 6 & \text{if} \quad D = -3, \\ 4 & \text{if} \quad D = -4, \\ 2 & \text{if} \quad D < -4. \end{cases}$$

Proof This is the special case of an extension of a formula due to Dirichlet proved by Kaplan and Williams [11] when the discriminant D is restricted to be negative and fundamental. □

We now state our main result.

Theorem 2.1 *Let k and ℓ be positive integers. Let D and E be negative fundamental discriminants. Let*

$$\{f_i(x, y) = a_i x^2 + b_i xy + c_i y^2 \mid i = 1, 2, \dots, h(D)\}$$

and

$$\{g_j(z, t) = A_j z^2 + B_j zt + C_j t^2 \mid j = 1, 2, \dots, h(E)\}$$

be representative sets of inequivalent, primitive, positive-definite, integral, binary quadratic forms of discriminants D and E, respectively. For $n \in \mathbb{N}$, $i \in \{1, 2, \dots, h(D)\}$ and $j \in \{1, 2, \dots, h(E)\}$, let

$$N(kf_i + \ell g_j; n) := \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = kf_i(x, y) + \ell g_j(z, t)\}.$$

Then

$$w(D)w(E) \sum_{\substack{(r,s)\in\mathbb{N}^2 \\ kr+\ell s=n}} \delta_{|D|}(r)\delta_{|E|}(s) + w(D)h(E)\delta_{|D|}(n/k) + w(E)h(D)\delta_{|E|}(n/\ell)$$
$$= \sum_{i=1}^{h(D)} \sum_{j=1}^{H(E)} N(kf_i + \ell g_j; n).$$

Proof Let $n \in \mathbb{N}$. Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We have

$$\sum_{i=1}^{h(D)} \sum_{j=1}^{H(E)} N(kf_i + \ell g_j; n) = \sum_{i=1}^{h(D)} \sum_{j=1}^{H(E)} \sum_{\substack{(r,s)\in\mathbb{N}_0^2 \\ kr+\ell s=n}} \sum_{\substack{(x,y,z,t)\in\mathbb{Z}^4 \\ f_i(x,y)=r \\ g_j(z,t)=s}} 1$$
$$= \sum_{\substack{(r,s)\in\mathbb{N}_0^2 \\ kr+\ell s=n}} \left(\sum_{i=1}^{h(D)} \sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ f_i(x,y)=r}} 1 \right) \left(\sum_{j=1}^{h(E)} \sum_{\substack{(z,t)\in\mathbb{Z}^2 \\ g_j(z,t)=s}} 1 \right).$$

In the sum over $(r, s) \in \mathbb{N}_0^2$ satisfying $kr + \ell s = n$, we cannot have $(r, s) = (0, 0)$ as $n \in \mathbb{N}$. In the same sum, the terms with $s = 0$ contribute

$$\sum_{\substack{r\in\mathbb{N} \\ r=n/k}} \left(\sum_{i=1}^{h(D)} \sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ f_i(x,y)=r}} 1 \right) \left(\sum_{j=1}^{h(E)} \sum_{\substack{(z,t)\in\mathbb{Z}^2 \\ g_j(z,t)=0}} 1 \right)$$
$$= N(n/k, D) \sum_{j=1}^{h(E)} 1 = w(D)\delta_{|D|}(n/k)h(E)$$

by Proposition 2.1, where $N(n/k, D) = \delta_{|D|}(n/k) = 0$ if $k \nmid n$. Similarly, the terms in the sum with $r = 0$ contribute

$$w(E)\delta_{|E|}(n/\ell)h(D).$$

Finally the terms with $r \neq 0$ and $s \neq 0$ contribute

$$\sum_{\substack{(r,s)\in\mathbb{N}^2 \\ kr+\ell s=n}} N(r, D)N(s, E) = w(D)w(E) \sum_{\substack{(r,s)\in\mathbb{N}^2 \\ kr+\ell s=n}} \delta_{|D|}(r)\delta_{|E|}(s)$$

by Proposition 2.1. Putting the contributions together, we obtain the asserted identity. □

We conclude this section by deducing identities (1.2), (1.4) and (1.6) from Theorem 2.1.

To prove Farkas' identity (1.2), we take $D = E = -3$ and $k = \ell = 1$ in Theorem 2.1. With this choice, we have $h(D) = h(E) = h(-3) = 1$, $w(D) = w(E) = w(-3) = 6$, $f_1(x, y) = x^2 + xy + y^2$ and $g_1(z, t) = z^2 + zt + t^2$. Now for $n \in \mathbb{N}$ we have

$$\text{card}\,\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = x^2 + xy + y^2 + z^2 + zt + t^2\} = 12\sigma(n) - 36\sigma(n/3). \quad (2.2)$$

This formula goes back to Liouville [12]. An elementary proof of Liouville's formula was given in 2002 by Huard, Ou Spearman and Williams [10, Theorem 13] and in 2008 by Chapman [6]. Then Theorem 2.1 gives

$$36\sum_{j=1}^{n-1}\delta_3(j)\delta_3(n-j) + 12\delta_3(n) = N(f_1 + g_1; n) = 12\sigma(n) - 36\sigma(n/3)$$

from which (1.2) follows on dividing by 12.

To prove Farkas' identity (1.4), we take $D = E = -4$ and $k = \ell = 1$ in Theorem 2.1. With this choice, we have $h(D) = h(E) = h(-4) = 1$, $w(D) = w(E) = w(-4) = 4$, $f_1(x, y) = x^2 + y^2$ and $g_1(z, t) = z^2 + t^2$. Now for $n \in \mathbb{N}$, we have Jacobi's well-known formula

$$\text{card}\,\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + t^2\} = 8\sigma(n) - 32\sigma(n/4). \quad (2.3)$$

A simple arithmetic proof of Jacobi's formula was given in 2000 by Spearman and Williams [15]. Theorem 2.1 gives

$$16\sum_{j=1}^{n-1}\delta_4(j)\delta_4(n-j) + 8\delta_4(n) = N(f_1 + g_1; n) = 8\sigma(n) - 32\sigma(n/4)$$

from which (1.4) follows on dividing by 8.

To prove Guerzhoy and Raji's identity (1.6), we take $D = E = -7$ and $k = \ell = 1$ in Theorem 2.1. With this choice, we have $h(D) = h(E) = h(-7) = 1$, $w(D) = w(E) = w(-7) = 2$, $f_1(x, y) = x^2 + xy + 2y^2$ and $g_1(z, t) = z^2 + zt + 2t^2$. Now for $n \in \mathbb{N}$, we have

$$\text{card}\,\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = x^2 + xy + 2y^2 + z^2 + zt + 2t^2\} = 4\sigma(n) - 28\sigma(n/7),$$

which was proved arithmetically by the author [16] in 2006. Then Theorem 2.1 gives

$$4\sum_{j=1}^{n-1}\delta_7(j)\delta_7(n-j) + 4\delta_7(n) = N(f_1 + g_1; n) = 4\sigma(n) - 28\sigma(n/7)$$

from which Guerzhoy and Raji's identity (1.6) follows on dividing by 4.

3 Twelve new arithmetic convolution identities

In this section, we derive from Theorem 2.1 twelve new arithmetic convolution identities which are analogous to (1.2), (1.4) and (1.6). We put the twelve identities together as Theorem 3.1. In what follows $\mathbb{C}$ denotes the field of complex numbers and $[x]$ denotes the greatest integer less than or equal to the real number x.

Part (ix) of Theorem 3.1 involves the function $c(n)$ $(n \in \mathbb{N})$ defined by

$$q\prod_{n=1}^{\infty}(1-q^{2n})^2(1-q^{10n})^2 = \sum_{n=1}^{\infty} c(n)q^n, \quad q \in \mathbb{C}, \quad |q| < 1. \tag{3.1}$$

Clearly $c(n) = 0$ for $n \equiv 0 \pmod 2$. It is known that $c(n)$ is a multiplicative function of n [13, Table 1, p. 4853].

Theorem 3.1 *Let $n \in \mathbb{N}$. The following identities hold:*

(i) $6\sum_{1\le m<n/2} \delta_3(m)\delta_3(n-2m) + \delta_3(n) + \delta_3(n/2)$
$= \sigma(n) - 2\sigma(n/2) + 3\sigma(n/3) - 6\sigma(n/6).$

(ii) $6\sum_{1\le m<n/3} \delta_3(m)\delta_3(n-3m) + \delta_3(n) + \delta_3(n/3)$
$= (2 + 3[n/3] - n)\sigma(n) - 6\sigma(n/3).$

(iii) $6\sum_{1\le m<n/4} \delta_3(m)\delta_3(n-4m) + \delta_3(n) + \delta_3(n/4)$
$= \sigma(n) - 3\sigma(n/2) - 3\sigma(n/3) + 4\sigma(n/4) + 9\sigma(n/6) - 12\sigma(n/12).$

(iv) $6\sum_{1\le m<n/6} \delta_3(m)\delta_3(n-6m) + \delta_3(n) + \delta_3(n/6)$
$= A\sigma(n) - 2A\sigma(n/2) + 5\sigma(n/3) - 10\sigma(n/6),$
where

$$A = \begin{cases} -1 & \text{if} \quad n \equiv 0 \pmod 3, \\ 1 & \text{if} \quad n \equiv 1 \pmod 3, \\ 0 & \text{if} \quad n \equiv 2 \pmod 3. \end{cases}$$

(v) $6\sum_{\substack{r,s\ge 1\\ 2r+3s=n}} \delta_3(r)\delta_3(s) + \delta_3(n/2) + \delta_3(n/3)$
$= B\sigma(n) - 2B\sigma(n/2) + 5\sigma(n/3) - 10\sigma(n/6),$
where

$$B = \begin{cases} -1 & \text{if} \quad n \equiv 0 \pmod 3, \\ 0 & \text{if} \quad n \equiv 1 \pmod 3, \\ 1 & \text{if} \quad n \equiv 2 \pmod 3. \end{cases}$$

(vi) $4 \sum_{1 \le m < n/2} \delta_4(m)\delta_4(n-2m) + \delta_4(n) + \delta_4(n/2)$
$= \sigma(n) - \sigma(n/2) + 2\sigma(n/4) - 8\sigma(n/8).$

(vii) $4 \sum_{1 \le m < n/3} \delta_4(m)\delta_4(n-3m) + \delta_4(n) + \delta_4(n/3)$
$= \sigma(n) - 2\sigma(n/2) - 3\sigma(n/3) + 4\sigma(n/4) + 6\sigma(n/6) - 12\sigma(n/12).$

(viii) $8 \sum_{1 \le m < n/4} \delta_4(m)\delta_4(n-4m) + 2\delta_4(n) + 2\delta_4(n/4)$
$= \left(1 + \left(\frac{-4}{n}\right)\right)\sigma(n) - \sigma(n/2) + 4\sigma(n/8) - 16\sigma(n/16).$

(ix) $12 \sum_{1 \le m < n/5} \delta_4(m)\delta_4(n-5m) + 3\delta_4(n) + 3\delta_4(n/5)$
$= \sigma(n) - 4\sigma(n/4) + 5\sigma(n/5) - 20\sigma(n/20) + 2c(n).$

(x) $\sum_{m=1}^{n-1} \delta_8(m)\delta_8(n-m) + \delta_8(n)$
$= \sigma(n) - \sigma(n/2) + 2\sigma(n/4) - 8\sigma(n/8).$

(xi) $2 \sum_{1 \le m < n/2} \delta_8(m)\delta_8(n-2m) + \delta_8(n) + \delta_8(n/2)$
$= \sigma(n) - \sigma(n/2) + 4\sigma(n/8) - 16\sigma(n/16).$

(xii) $2 \sum_{1 \le m < n/2} \delta_{11}(m)\delta_{11}(n-2m) + \delta_{11}(n) + \delta_{11}(n/2)$
$= \sigma(n) - 2\sigma(n/2) + 11\sigma(n/11) - 22\sigma(n/22).$

Proof (i) We choose $D = E = -3$, $k = 1$ and $\ell = 2$ in Theorem 2.1. With this choice, we have $h(D) = h(E) = h(-3) = 1$, $w(D) = w(E) = w(-3) = 6$, $f_1(x, y) = x^2 + xy + y^2$ and $g_1(z, t) = z^2 + zt + t^2$. By [1, Theorem 13, p. 180], we have

$$\begin{aligned} N(f_1 + 2g_1; n) &= \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid x^2 + xy + y^2 + 2z^2 + 2zt + 2t^2 = n\} \\ &= 6\sigma(n) - 12\sigma(n/2) + 18\sigma(n/3) - 36\sigma(n/6). \end{aligned}$$

Appealing to Theorem 2.1, we obtain the asserted formula after dividing through by 6.

(ii) The proof is similar to that of (i). We choose $D = E = -3$, $k = 1$ and $\ell = 3$ in Theorem 2.1 and use the result

$$\begin{aligned} N(f_1 + 3g_1; n) &= \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid x^2 + xy + y^2 + 3z^2 + 3zt + 3t^2 = n\} \\ &= 6(2 + 3[n/3] - n)\sigma(n) - 36\sigma(n/3), \end{aligned}$$

see [1, Theorem 14, p. 180].

(iii) The proof proceeds as in (i) and (ii). We choose $D = E = -3$, $k = 1$ and $\ell = 4$ in Theorem 2.1 and use the result

$$\begin{aligned} N(f_1 + 4g_1; n) &= \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid x^2 + xy + y^2 + 4z^2 + 4zt + 4t^2 = n\} \\ &= 6\sigma(n) - 18\sigma(n/2) - 18\sigma(n/3) + 24\sigma(n/4) + 54\sigma(n/6) \\ &\quad - 72\sigma(n/12), \end{aligned}$$

which follows from [1, Theorem 15, p. 181] using the elementary identities $\sigma(n) = 3\sigma(n/2) - 2\sigma(n/4)$ and $\sigma(n/3) = 3\sigma(n/6) - 2\sigma(n/12)$ when n is even.

(iv) The proof is the same as those in (i), (ii) and (iii) except that we use the result

$$N(f_1 + 6g_1; n) = 6A\sigma(n) - 12A\sigma(n/2) + 30\sigma(n/3) - 60\sigma(n/6),$$

see [1, Theorem 16, p. 181].

(v) The proof is similar to those in (i)–(iv). We choose $D = E = -3$, $k = 2$ and $\ell = 3$ in Theorem 2.1 and use the result

$$N(2f_1 + 3g_1; n) = 6B\sigma(n) - 12B\sigma(n/2) + 30\sigma(n/3) - 60\sigma(n/6),$$

see [1, Theorem 17, p. 181].

(vi) We choose $D = E = -4$, $k = 1$ and $\ell = 2$. Here $h(D) = h(E) = h(-4) = 1$, $w(D) = w(E) = w(-4) = 4$, $f_1(x, y) = x^2 + y^2$ and $g_1(z, t) = z^2 + t^2$. By [3, p. 297], we have

$$\begin{aligned} N(f_1 + 2g_1; n) &= \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid x^2 + y^2 + 2z^2 + 2t^2 = n\} \\ &= 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8). \end{aligned}$$

Appealing to Theorem 2.1, we obtain (vi) after dividing by 4.

(vii) The proof is similar to (vi). We choose $D = E = -4$, $k = 1$ and $\ell = 3$ in Theorem 2.1 and use the result

$$\begin{aligned} N(f_1 + 3g_1; n) &= \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid x^2 + y^2 + 3z^2 + 3t^2 = n\} \\ &= 4\sigma(n) - 8\sigma(n/2) - 12\sigma(n/3) + 16\sigma(n/4) \\ &\quad + 24\sigma(n/6) - 48\sigma(n/12), \end{aligned}$$

see [3, p. 297].

(viii) The proof is as in (vi) and (vii). We choose $D = E = -4$, $k = 1$ and $\ell = 4$ in Theorem 2.1 and use the result

$$N(f_1 + 4g_1; n) = \left(2 + 2\left(\frac{-4}{n}\right)\right)\sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16),$$

see [3, p. 298].

(ix) The proof proceeds as in (vi), (vii) and (viii). We choose $D = E = -4$, $k = 1$ and $\ell = 5$ in Theorem 2.1 and use the result

$$N(f_1 + 5g_1; n) = \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma(n/4) + \frac{20}{3}\sigma(n/5) - \frac{80}{3}\sigma(n/20) + \frac{8}{3}c(n),$$

where $c(n)$ ($n \in \mathbb{N}$) is defined in (3.1), see [2, Eq. (7.3), p. 49].

(x) We choose $D = E = -8$ and $k = \ell = 1$ in Theorem 2.1. Here $h(D) = h(E) = h(-8) = 1$, $w(D) = w(E) = w(-8) = 2$, $f_1(x, y) = x^2 + 2y^2$ and $g_1(z, t) = z^2 + 2t^2$. By [3, Theorem 1.8, p. 297], we have

$$\begin{aligned} N(f_1+g_1;n) &= \text{card}\{(x,y,z,t)\in\mathbb{Z}^4 \mid x^2+2y^2+z^2+2t^2=n\} \\ &= 4\sigma(n)-4\sigma(n/2)+8\sigma(n/4)-32\sigma(n/8). \end{aligned}$$

Appealing to Theorem 2.1, we obtain (x) after dividing by 4.

(xi) The proof proceeds as in (x) except that we choose $\ell=2$ and use the formula [3, Theorem 1.14, p. 300]

$$\begin{aligned} N(f_1+2g_1;n) &= \text{card}\{(x,y,z,t)\in\mathbb{Z}^4 \mid x^2+2y^2+2z^2+4t^2=n\} \\ &= 2\sigma(n)-2\sigma(n/2)+8\sigma(n/8)-32\sigma(n/16). \end{aligned}$$

(xii) We choose $D=E=-11$, $k=1$ and $\ell=2$. Here $h(D)=h(E)=h(-11)=1$, $w(D)=w(E)=w(-11)=2$, $f_1(x,y)=x^2+xy+3y^2$ and $g_1(z,t)=z^2+zt+3t^2$. By [5, Theorem 1.20], we have

$$\begin{aligned} N(f_1+2g_1;n) &= \text{card}\{(x,y,z,t)\in\mathbb{Z}^4 \mid x^2+xy+3y^2+2z^2+2zt+6t^2=n\} \\ &= 2\sigma(n)-4\sigma(n/2)+22\sigma(n/11)-44\sigma(n/22). \end{aligned}$$

Appealing to Theorem 2.1, we obtain (xii) after dividing by 2. □

In the proofs of all twelve parts of Theorem 3.1, we choose the fundamental discriminants D and E such that $D=E$ and $h(D)=h(E)=1$. In Sect. 4 we illustrate Theorem 2.1 with an example where we choose D and E such that $D=E$ and $h(D)=h(E)>1$ and in Sect. 5 an example with $D\neq E$.

4 An example with the class number greater than 1

A representative set of inequivalent, primitive, positive-definite, binary quadratic forms of discriminant -20 is $\{x^2+5y^2, 2x^2+2xy+3y^2\}$, so that $h(-20)=2$. We define for $n\in\mathbb{N}$

$$\begin{aligned} N(1,5,1,5;n) &:= \text{card}\{(x,y,z,t)\in\mathbb{Z}^4 \mid x^2+5y^2+z^2+5t^2=n\}, \\ N(1,5,2,2,3;n) &:= \text{card}\{(x,y,z,t)\in\mathbb{Z}^4 \mid x^2+5y^2+2z^2+2zt+3t^2=n\} \\ &\text{and} \\ N(2,2,3,2,2,3;n) &:= \text{card}\{(x,y,z,t)\in\mathbb{Z}^4 \mid 2x^2+2xy \\ &\qquad +3y^2+2z^2+2zt+3t^2=n\}. \end{aligned}$$

We recall from the proof of Theorem 3.1(ix) that

$$N(1,5,1,5;n)=\frac{4}{3}\sigma(n)-\frac{16}{3}\sigma(n/4)+\frac{20}{3}\sigma(n/5)-\frac{80}{3}\sigma(n/20)+\frac{8}{3}c(n), \tag{4.1}$$

where $c(n)$ is defined in (3.1). Recently a formula for $N(1,5,2,2,3;n)$ has been determined by Alaca, Alaca and Williams [5, Theorem 1.15], namely

$$N(1,5,2,2,3;n) = 2\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 10\sigma(n/5) + 20\sigma(n/10) - 40\sigma(n/20). \tag{4.2}$$

As far as the author is aware, the number of representations of n by $2x^2+2xy+3y^2+2z^2+2zt+3t^2$ has not been determined, and as we require this number, we give a formula for it in the next theorem.

Theorem 4.1 *Let $n \in \mathbb{N}$. Then*

$$N(2,2,3,2,2,3;n) = \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma(n/4) + \frac{20}{3}\sigma(n/5) - \frac{80}{3}\sigma(n/20) - \frac{4}{3}c(n).$$

Proof For $q \in \mathbb{C}$ with $|q| < 1$, we define

$$\varphi(q) := \sum_{x\in\mathbb{Z}} q^{x^2}$$

and

$$h(q) := \sum_{(x,y)\in\mathbb{Z}^2} q^{2x^2+2xy+3y^2}.$$

The function $\varphi(q)$ is known as Ramanujan's theta function. We have

$$h(q^2) = \sum_{(x,y)\in\mathbb{Z}^2} q^{4x^2+4xy+6y^2} = \sum_{(x,y)\in\mathbb{Z}^2} q^{(2x+y)^2+5y^2} = \sum_{\substack{(x,y)\in\mathbb{Z}^2\\ x\equiv y \pmod 2}} q^{x^2+5y^2}.$$

On the other hand, we have

$$\varphi(q)\varphi(q^5) + \varphi(-q)\varphi(-q^5) = \sum_{(x,y)\in\mathbb{Z}^2} q^{x^2+5y^2} + \sum_{(x,y)\in\mathbb{Z}^2} (-1)^{x-y} q^{x^2+5y^2},$$

as $(-1)^{x^2+5y^2} = (-1)^{x-y}$. Thus

$$\varphi(q)\varphi(q^5) + \varphi(-q)\varphi(-q^5) = 2\sum_{\substack{(x,y)\in\mathbb{Z}^2\\ x\equiv y \pmod 2}} q^{x^2+5y^2}.$$

Hence

$$\varphi(q)\varphi(q^5) + \varphi(-q)\varphi(-q^5) = 2h(q^2).$$

Squaring this equation, we obtain (as $\varphi(q)\varphi(-q) = \varphi^2(-q^2)$)

$$\varphi^2(q)\varphi^2(q^5) + \varphi^2(-q)\varphi^2(-q^5) + 2\varphi^2(-q^2)\varphi^2(-q^{10}) = 4h^2(q^2).$$

Now

$$\varphi^2(q)\varphi^2(q^5) = \sum_{n=0}^{\infty} N(1,5,1,5;n)q^n$$

and

$$\varphi^2(-q)\varphi^2(-q^5) = \sum_{n=0}^{\infty} N(1,5,1,5;n)(-q)^n.$$

Therefore

$$\begin{aligned}\varphi^2(q)\varphi^2(q^5) + \varphi^2(-q)\varphi^2(-q^5) &= 2 \sum_{\substack{n=0 \\ n\equiv 0 \pmod 2}}^{\infty} N(1,5,1,5;n)q^n \\ &= 2\sum_{n=0}^{\infty} N(1,5,1,5;2n)q^{2n}.\end{aligned}$$

Also

$$\varphi^2(-q^2)\varphi^2(-q^{10}) = \sum_{n=0}^{\infty} N(1,5,1,5;n)(-q^2)^n = \sum_{n=0}^{\infty} N(1,5,1,5;n)(-1)^n q^{2n}.$$

Thus

$$\begin{aligned}4h^2(q^2) &= \varphi^2(q)\varphi^2(q^5) + \varphi^2(-q)\varphi^2(-q^5) + 2\varphi^2(-q^2)\varphi^2(-q^{10}) \\ &= \sum_{n=0}^{\infty}(2N(1,5,1,5;2n) + 2(-1)^n N(1,5,1,5;n))q^{2n}.\end{aligned}$$

Now

$$h^2(q) = \sum_{n=0}^{\infty} N(2,2,3,2,2,3;n)q^n.$$

Thus

$$h^2(q^2) = \sum_{n=0}^{\infty} N(2,2,3,2,2,3;n)q^{2n}.$$

Hence

$$\sum_{n=0}^{\infty}(2N(1,5,1,5;2n)+2(-1)^n N(1,5,1,5;n))q^{2n} = 4\sum_{n=0}^{\infty} N(2,2,3,2,2,3;n)q^{2n}.$$

Equating coefficients of q^{2n} ($n \in \mathbb{N}$), we obtain

$$N(2,2,3,2,2,3;n) = \frac{1}{2}N(1,5,1,5;2n) + \frac{(-1)^n}{2}N(1,5,1,5;n).$$

Replacing n by $2n$ in (4.1), we deduce

$$N(1,5,1,5;2n) = \frac{4}{3}\sigma(2n) - \frac{16}{3}\sigma(n/2) + \frac{20}{3}\sigma(2n/5) - \frac{80}{3}\sigma(n/10)$$

as $c(2n) = 0$. Now

$$\sigma(2n) = 3\sigma(n) - 2\sigma(n/2), \quad \sigma(2n/5) = 3\sigma(n/5) - 2\sigma(n/10),$$

so that

$$N(1,5,1,5;2n) = 4\sigma(n) - 8\sigma(n/2) + 20\sigma(n/5) - 40\sigma(n/10).$$

We now treat two cases according as n is odd or even.

If n is odd, we have

$$N(1,5,1,5;2n) = 4\sigma(n) + 20\sigma(n/5).$$

Thus

$$\begin{aligned} N(2,2,3,2,2,3;n) &= \frac{1}{2}N(1,5,1,5;2n) - \frac{1}{2}N(1,5,1,5;n) \\ &= \frac{1}{2}(4\sigma(n) + 20\sigma(n/5)) - \frac{1}{2}\left(\frac{4}{3}\sigma(n) + \frac{20}{3}\sigma(n/5) + \frac{8}{3}c(n)\right) \\ &= \frac{4}{3}\sigma(n) + \frac{20}{3}\sigma(n/5) - \frac{4}{3}c(n) \\ &= \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma(n/4) + \frac{20}{3}\sigma(n/5) - \frac{80}{3}\sigma(n/20) - \frac{4}{3}c(n), \end{aligned}$$

as asserted.

If n is even, as

$$\sigma(n/2) = \frac{1}{3}\sigma(n) + \frac{2}{3}\sigma(n/4), \quad \sigma(n/10) = \frac{1}{3}\sigma(n/5) + \frac{2}{3}\sigma(n/20), \quad c(n) = 0,$$

we have by (4.1)

$$\begin{aligned} N(1,5,1,5;2n) &= \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma(n/4) + \frac{20}{3}\sigma(n/5) - \frac{80}{3}\sigma(n/20) \\ &= N(1,5,1,5;n). \end{aligned}$$

Thus

$$\begin{aligned} N(2,2,3,2,2,3;n) &= \frac{1}{2}N(1,5,1,5;2n) + \frac{1}{2}N(1,5,1,5;n) \\ &= N(1,5,1,5;n) \\ &= \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma(n/4) + \frac{20}{3}\sigma(n/5) - \frac{80}{3}\sigma(n/20) - \frac{4}{3}c(n), \end{aligned}$$

as asserted. □.

Theorem 4.2 *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} 3\sum_{m=1}^{n-1} \delta_{20}(m)\delta_{20}(n-m) + 6\delta_{20}(n) &= 5\sigma(n) - 6\sigma(n/2) + 4\sigma(n/4) - 5\sigma(n/5) \\ &\quad + 30\sigma(n/10) - 100\sigma(n/20) + c(n). \end{aligned}$$

Proof We choose $D = E = -20$ and $k = \ell = 1$ in Theorem 2.1. Here $h(D) = h(E) = h(-20) = 2$, $w(D) = w(E) = w(-20) = 2$, $f_1(x,y) = x^2 + 5y^2$, $f_2(x,y) = 2x^2 + 2xy + 3y^2$, $g_1(z,t) = z^2 + 5t^2$ and $g_2(z,t) = 2z^2 + 2xy + 3y^2$. Also, appealing to (4.1), (4.2) and Theorem 4.1, we have

$$\begin{aligned} N(f_1+g_1;n) &= N(1,5,1,5;n) \\ &= \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma(n/4) + \frac{20}{3}\sigma(n/5) - \frac{80}{3}\sigma(n/20) + \frac{8}{3}c(n), \\ N(f_1+g_2;n) &= N(f_2+g_1;n) = N(1,5,2,2,3;n) \\ &= 2\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 10\sigma(n/5) \\ &\quad + 20\sigma(n/10) - 40\sigma(n/20), \\ N(f_2+g_2;n) &= N(2,2,3,2,2,3;n) \\ &= \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma(n/4) + \frac{20}{3}\sigma(n/5) - \frac{80}{3}\sigma(n/20) - \frac{4}{3}c(n). \end{aligned}$$

Thus

$$\begin{aligned} \sum_{i=1}^{2}\sum_{j=1}^{2} N(f_i+g_j;n) &= \frac{20}{3}\sigma(n) - 8\sigma(n/2) + \frac{16}{3}\sigma(n/4) - \frac{20}{3}\sigma(n/5) \\ &\quad + 40\sigma(n/10) - \frac{400}{3}\sigma(n/20) + \frac{4}{3}c(n). \end{aligned}$$

Appealing to Theorem 2.1, we obtain the asserted result on multiplying the resulting formula by 3/4. □

5 An example with unequal discriminants

We treat the special case of Theorem 2.1 when $D = -3$ and $E = -4$. In this case, the evaluation of the right-hand side of Theorem 2.1 requires the function

$$\sigma(r, s, 12; n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n \\ d \equiv r \pmod{12} \\ n/d \equiv s \pmod{12}}} d \tag{5.1}$$

rather than $\sigma(n)$. We note that $\sigma(r, s, 12; n) = 0$ if $n \not\equiv rs \pmod{12}$.

Theorem 5.1 *Let $n \in \mathbb{N}$. For $r, s \in \mathbb{Z}$, we define*

$$c(r, s) := \left(3\left(\frac{-3}{s}\right) - \left(\frac{-3}{r}\right)\right)\left(4\left(\frac{-4}{s}\right) + \left(\frac{-4}{r}\right)\right). \tag{5.2}$$

Then

$$24\sum_{m=1}^{n-1} \delta_3(m)\delta_4(n-m) + 6\delta_3(n) + 4\delta_4(n) = \sum_{\substack{r,s=0 \\ rs \equiv n \pmod{12}}}^{11} c(r, s)\sigma(r, s, 12; n).$$

Proof In [4, p. 225], the arithmetic functions $A(n)$, $B(n)$, $C(n)$ and $D(n)$ are defined for $n \in \mathbb{N}$ by

$$A(n) := \sum_{d \mid n} d\left(\frac{12}{n/d}\right),$$

$$B(n) := \sum_{d \mid n} d\left(\frac{-3}{d}\right)\left(\frac{-4}{n/d}\right),$$

$$C(n) := \sum_{d \mid n} d\left(\frac{-3}{n/d}\right)\left(\frac{-4}{d}\right),$$

$$D(n) := \sum_{d \mid n} d\left(\frac{12}{d}\right),$$

and their basic properties given. Appealing to (5.2), we see that

$$A(n) = \sum_{r,s=0}^{11} \left(\frac{12}{s}\right)\sigma(r, s, 12; n),$$

$$B(n)=\sum_{r,s=0}^{11}\left(\frac{-3}{r}\right)\left(\frac{-4}{s}\right)\sigma(r,s,12;n),$$

$$C(n)=\sum_{r,s=0}^{11}\left(\frac{-4}{r}\right)\left(\frac{-3}{s}\right)\sigma(r,s,12;n),$$

$$D(n)=\sum_{r,s=0}^{11}\left(\frac{12}{r}\right)\sigma(r,s,12;n).$$

By [4, p. 233], we have

$$\begin{aligned}
&\operatorname{card}\{(x,y,z,t)\in\mathbb{Z}^4 \mid x^2+xy+y^2+z^2+t^2=n\}\\
&\quad=12A(n)-4B(n)+3C(n)-D(n)\\
&\quad=\sum_{r,s=0}^{11}\left(3\left(\frac{-3}{s}\right)-\left(\frac{-3}{r}\right)\right)\left(4\left(\frac{-4}{s}\right)+\left(\frac{-4}{r}\right)\right)\sigma(r,s,12;n),
\end{aligned}$$

that is

$$\begin{aligned}
&\operatorname{card}\{(x,y,z,t)\in\mathbb{Z}^4 \mid x^2+xy+y^2+z^2+t^2=n\}\\
&\quad=\sum_{r,s=0}^{11}c(r,s)\sigma(r,s,12;n),
\end{aligned}\tag{5.3}$$

by (5.1). Choosing $D=-3$, $E=-4$ and $k=\ell=1$ in Theorem 2.1, as $h(D)=h(-3)=1$, $h(E)=h(-4)=1$, $w(D)=w(-3)=6$, $w(E)=w(-4)=4$, $f_1(x,y)=x^2+xy+y^2$ and $g_1(z,t)=z^2+t^2$, we obtain

$$\begin{aligned}
&24\sum_{m=1}^{n-1}\delta_3(m)\delta_4(n-m)+6\delta_3(n)+4\delta_4(n)=N(f_1+g_1;n)\\
&\quad=\operatorname{card}\{(x,y,z,t)\in\mathbb{Z}^4 \mid x^2+xy+y^2+z^2+t^2=n\}\\
&\quad=\sum_{r,s=0}^{11}c(r,s)\sigma(r,s,12;n)
\end{aligned}$$

by (5.2). This is the asserted identity. □

As we have already mentioned, elementary arithmetic proofs of the formulas in (2.2) and (2.3) are known. As far as the author is aware, no such proof is known for the formula (5.3). It would be interesting to give such a proof.

The referee has pointed out that the identities derived in this paper might be of interest from the point of view of modular forms by considering the connection between sums of classes of theta series associated to positive-definite quadratic forms and Eisenstein series. The author thanks the referee for his/her helpful review of his paper.

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