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Representation numbers of certain quaternary quadratic forms in a genus consisting of a single class

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An explicit formula is given for the representation number of each of the 75 reduced, positive-definite, integral, primitive, quaternary quadratic forms $ax^2+bxy+cy^2+dz^2+ezt+ft^2$, which belong to a genus with discriminant ≤ 1732 containing one and only one form class.

Keywords: Representation number; sum of two binary quadratic forms; genus of quaternary quadratic forms; theta functions; double Gauss sums; local density; Siegel's mass formula; modular forms; Dirichlet L-series.

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1. Introduction

Let a, b, c, d, e and f be integers satisfying

$$a > 0, \quad c > 0, \quad b^2 - 4ac < 0, \quad d > 0, \quad f > 0, \quad e^2 - 4df < 0,$$

so that $ax^2 + bxy + cy^2$ and $dz^2 + ezt + ft^2$ are positive-definite, integral, binary quadratic forms. We are interested in the positive-definite, integral, quaternary quadratic form, which is the sum of these two binary quadratic forms, namely, the form

$$ax^2 + bxy + cy^2 + dz^2 + ezt + ft^2. \tag{1.1}$$

The discriminant of this quaternary form is

$$16\begin{vmatrix} a & b/2 & 0 & 0 \\ b/2 & c & 0 & 0 \\ 0 & 0 & d & e/2 \\ 0 & 0 & e/2 & f \end{vmatrix} = (b^2-4ac)(e^2-4df)(>0),$$

which is the product of the discriminants of $ax^2+bxy+cy^2$ and $dz^2+ezt+ft^2$. Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Q}$ and $\mathbb{C}$ denote the sets of all integers, positive integers, nonnegative integers, rational numbers and complex numbers, respectively. For $n \in \mathbb{N}_0$ the representation number of the form (1.1) is

$$N(a,b,c,d,e,f;n) \\ = \text{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = ax^2+bxy+cy^2+dz^2+ezt+ft^2\}.$$

If $n \in \mathbb{Q}\backslash\mathbb{N}_0$ we set $N(a,b,c,d,e,f;n)=0$. Clearly we have

$$N(a,b,c,d,e,f;0)=1, \quad N(d,e,f,a,b,c;n)=N(a,b,c,d,e,f;n),$$

and

$$N(a,b,c,d,e,f;n)=N(a/m,b/m,c/m,d/m,e/m,f/m;n/m),$$

where m denotes the greatest common divisor of a,b,c,d,e and f, written $m=(a,b,c,d,e,f)$.

Our objective in this paper is to determine an explicit formula for the representation number $N(a,b,c,d,e,f;n)$, valid for all $n \in \mathbb{N}$, for each of the 75 reduced, positive-definite, quaternary, quadratic forms of the type (1.1) listed in Nipp's table [18], which belong to a genus of discriminant less than or equal to 1732 containing one and only one form (class). Of these 75 forms, 27 are diagonal forms $((b,e)=(0,0))$ and 48 are non-diagonal forms $((b,e)\neq(0,0))$. We do not need to consider the 27 diagonal forms as $N(a,b,c,d,e,f;n)$ has been determined for each of these. Table 1 provides references.

Of the 48 non-diagonal forms, 13 of them have been treated in the literature using theta functions. These are listed with references in Table 2.

Of the remaining 35 non-diagonal forms, 11 of them have been be treated by the method introduced by Alaca, Pehlivan and Williams in [13] (see Table 3).

Of the remaining 24 non-diagonal forms, 2 of them can be treated by the method in [13], see Table 4 and Theorems 1.1 and 1.2. The proofs of these two theorems are given in Sec. 2.

Of the remaining 22 non-diagonal forms, 5 of them are treated here by the (p,k)-parametrization method introduced by Alaca, Alaca and Williams [8], see Table 5 and Theorems 1.3–1.7. The proofs of these theorems are given in Sec. 3.

The final 17 forms are treated using Siegel's method of local densities [19], see Table 6 and Theorems 1.8–1.24. In order to do this we require the evaluation of several double Gauss sums. These evaluations were carried out by the authors in [11] in anticipation of this paper. The proofs of Theorems 1.8–1.24 are given in Sec. 4.

Table 1. Twenty-seven diagonal forms.

Form	Disc.	Ref.
$x^2+y^2+z^2+t^2$	16	[3, Theorem 1.6]
$x^2+y^2+z^2+2t^2$	32	[7, Theorem 5.1]
$x^2+y^2+z^2+3t^2$	48	[5, Theorem 4.1]
$x^2+y^2+z^2+4t^2$	64	[3, Theorem 1.7]
$x^2+y^2+2z^2+2t^2$	64	[3, Theorem 1.8]
$x^2+y^2+z^2+5t^2$	80	[10, Theorem 5.1]
$x^2+y^2+2z^2+3t^2$	96	[12, Theorem 4.1]
$x^2+y^2+z^2+8t^2$	128	[7, Theorem 5.10]
$x^2+y^2+2z^2+4t^2$	128	[7, Theorem 5.3]
$x^2+2y^2+2z^2+2t^2$	128	[7, Theorem 5.2]
$x^2+y^2+3z^2+3t^2$	144	[3, Theorem 1.9]
$x^2+y^2+2z^2+6t^2$	192	[5, Theorem 5.1]
$x^2+2y^2+2z^2+3t^2$	192	[5, Theorem 6.1]
$x^2+y^2+4z^2+4t^2$	256	[3, Theorem 1.11]
$x^2+2y^2+2z^2+4t^2$	256	[3, Theorem 1.14]
$x^2+2y^2+2z^2+6t^2$	384	[12, Theorem 4.1]
$x^2+y^2+3z^2+9t^2$	432	[1, Theorem 1.3]
$x^2+3y^2+3z^2+3t^2$	432	[5, Theorem 8.1]
$x^2+y^2+4z^2+8t^2$	512	[7, Theorem 5.5]
$x^2+2y^2+2z^2+8t^2$	512	[7, Theorem 5.7]
$x^2+2y^2+4z^2+4t^2$	512	[7, Theorem 5.4]
$x^2+2y^2+4z^2+6t^2$	768	[5, Theorem 7.1]
$x^2+3y^2+3z^2+6t^2$	864	[12, Theorem 4.1]
$x^2+4y^2+4z^2+4t^2$	1024	[3, Theorem 1.18]
$x^2+3y^2+3z^2+9t^2$	1296	[1, Theorem 1.1]
$x^2+3y^2+6z^2+6t^2$	1728	[5, Theorem 9.1]
$2x^2+3y^2+3z^2+6t^2$	1728	[5, Theorem 10.1]

Table 2. Thirteen forms.

Form	Disc.	Ref.
$x^2+xy+y^2+z^2+zt+t^2$	9	[8, Theorem 12]
$x^2+xy+y^2+z^2+t^2$	12	[5, Theorem 11.1]
$x^2+xy+y^2+2z^2+2zt+2t^2$	36	[8, Theorem 13]
$x^2+y^2+2z^2+2zt+2t^2$	48	[5, Theorem 12.1]
$x^2+xy+2y^2+z^2+zt+2t^2$	49	[23, Theorem 1.1]
$x^2+xy+y^2+3z^2+3zt+3t^2$	81	[8, Theorem 14]
$x^2+xy+y^2+3z^2+3t^2$	108	[5, Theorem 14.1]
$x^2+xy+y^2+4z^2+4zt+4t^2$	144	[8, Theorem 15]
$x^2+y^2+4z^2+4zt+4t^2$	192	[5, Theorem 13.1]
$x^2+xy+y^2+6z^2+6zt+6t^2$	324	[8, Theorem 16]
$2x^2+2xy+2y^2+3z^2+3zt+3t^2$	324	[8, Theorem 17]
$2x^2+2xy+2y^2+3z^2+3t^2$	432	[5, Theorem 15.1]
$3x^2+3y^2+4z^2+4zt+4t^2$	1728	[5, Theorem 16.1]

Table 3. Eleven forms.

Form	Disc.	Ref.
$x^2 + xy + y^2 + z^2 + 3t^2$	36	[13, Theorem 1.2(iii)]
$x^2 + xy + y^2 + 2z^2 + 2t^2$	48	[13, Theorem 1.4]
$x^2 + y^2 + 3z^2 + 2zt + 3t^2$	128	[13, Theorem 1.3]
$x^2 + 3y^2 + 2z^2 + 2zt + 2t^2$	144	[13, Theorem 1.2(iii)]
$x^2 + 2y^2 + 3z^2 + 2zt + 3t^2$	256	[13, Theorem 1.2(i)]
$x^2 + xy + y^2 + 6z^2 + 6t^2$	432	[13, Theorem 1.4]
$x^2 + 2y^2 + 4z^2 + 4zt + 5t^2$	512	[13, Theorem 1.3]
$x^2 + 4y^2 + 3z^2 + 2zt + 3t^2$	512	[13, Theorem 1.3]
$x^2 + 3y^2 + 4z^2 + 4zt + 4t^2$	576	[13, Theorem 1.2(iii)]
$x^2 + 4y^2 + 4z^2 + 4zt + 4t^2$	768	[13, Theorem 1.4]
$3x^2 + 2xy + 3y^2 + 3z^2 + 2zt + 3t^2$	1024	[13, Theorem 1.2(i)]

Table 4. Two forms.

Form	Disc.	Theorem
$2x^2 + 2y^2 + 3z^2 + 2zt + 3t^2$	512	1.1
$x^2 + 2y^2 + 6z^2 + 4zt + 6t^2$	1024	1.2

Table 5. Five forms.

Form	Disc.	Theorem
$x^2 + 4y^2 + 2z^2 + 2zt + 2t^2$	192	1.3
$x^2 + y^2 + 8z^2 + 8zt + 8t^2$	768	1.4
$x^2 + 3y^2 + 6z^2 + 6zt + 6t^2$	1296	1.5
$2x^2 + 2xy + 2y^2 + 3z^2 + 9t^2$	1296	1.6
$2x^2 + 2xy + 2y^2 + 3z^2 + 12t^2$	1728	1.7

Table 6. Seventeen forms.

Form	Disc.	Theorem
$x^2 + xy + y^2 + z^2 + zt + 2t^2$	21	1.8
$x^2 + xy + y^2 + z^2 + zt + 4t^2$	45	1.9
$x^2 + xy + y^2 + 2z^2 + zt + 2t^2$	45	1.10
$x^2 + y^2 + 2z^2 + 2zt + 5t^2$	144	1.11
$x^2 + xy + y^2 + 3z^2 + 3zt + 6t^2$	189	1.12
$x^2 + 3y^2 + 2z^2 + 2zt + 3t^2$	240	1.13
$x^2 + y^2 + 5z^2 + 2zt + 5t^2$	384	1.14
$x^2 + 5y^2 + 2z^2 + 2zt + 3t^2$	400	1.15
$x^2 + xy + y^2 + 3z^2 + 3zt + 12t^2$	405	1.16
$2x^2 + xy + 2y^2 + 3z^2 + 3zt + 3t^2$	405	1.17
$x^2 + 3y^2 + 2z^2 + 2zt + 5t^2$	432	1.18
$x^2 + 2y^2 + 3z^2 + 2zt + 5t^2$	448	1.19
$x^2 + xy + 3y^2 + 2z^2 + 2zt + 6t^2$	484	1.20
$x^2 + xy + 2y^2 + 7z^2 + 7zt + 7t^2$	1029	1.21
$x^2 + xy + 4y^2 + 5z^2 + 5zt + 5t^2$	1125	1.22
$2x^2 + xy + 2y^2 + 5z^2 + 5zt + 5t^2$	1125	1.23
$2x^2 + 2xy + 5y^2 + 3z^2 + 3t^2$	1296	1.24

We now state Theorems 1.1–1.24. As usual $\sigma(n)$ denotes the sum of the positive divisors of the positive integer n, that is

$$\sigma(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d = \prod_{p^\gamma \| n} \frac{p^{\gamma+1}-1}{p-1},$$

where p runs through the primes dividing n and p^γ denotes the exact power of p dividing n. For an odd prime p and an integer m, $(\frac{m}{p})$ is the Legendre symbol, that is

$$\left(\frac{m}{p}\right) = \begin{cases} 0 & \text{if } m \equiv 0 \pmod{p}, \\ 1 & \text{if } m \not\equiv 0 \pmod{p} \text{ and } x^2 \equiv m \pmod{p} \text{ solvable}, \\ -1 & \text{if } m \not\equiv 0 \pmod{p} \text{ and } x^2 \equiv m \pmod{p} \text{ insolvable}. \end{cases}$$

For an integer m with $m \equiv 0, 1 \pmod 4$, $(\frac{m}{2})$ is the Kronecker symbol, that is

$$\left(\frac{m}{2}\right) = \begin{cases} 0 & \text{if } m \equiv 0 \pmod{4}, \\ 1 & \text{if } m \equiv 1 \pmod{8}, \\ -1 & \text{if } m \equiv 5 \pmod{8}. \end{cases}$$

For integers m and $N \neq 0$ (with $m \equiv 0, 1 \pmod 4$ if $N \equiv 0 \pmod 2$), we define

$$F_m(N) := \prod_{p^\gamma \| N} \frac{p^{\gamma+1} - \left(\frac{m}{p}\right)^{\gamma+1}}{p - \left(\frac{m}{p}\right)}.$$

Theorem 1.1. *Let $n \in \mathbb{N}$. Let $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 2) = 1$. Then*

$$N(2,0,2,3,2,3;n) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod{8}, \\ 2F_8(N) & \text{if } n \equiv 3 \pmod{4}, \\ 4F_8(N) & \text{if } n \equiv 5 \pmod{8}, \\ 4F_8(N) & \text{if } n \equiv 2 \pmod{4}, \\ \left(2^{\alpha+1} - 2\left(\frac{8}{N}\right)\right) F_8(N) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

Theorem 1.2. *Let $n \in \mathbb{N}$. Let $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 2) = 1$. Then*

$$N(1,0,2,6,4,6;n) = \begin{cases} 2\sigma(N) & \text{if } n \equiv 1 \pmod{8}, \\ 0 & \text{if } n \equiv 5 \pmod{8}, \\ \sigma(N) & \text{if } n \equiv 3 \pmod{4}, \\ 2\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 2\sigma(N) & \text{if } n \equiv 4 \pmod{16}, \\ 6\sigma(N) & \text{if } n \equiv 12 \pmod{16}, \\ 12\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 8\sigma(N) & \text{if } n \equiv 16 \pmod{32}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{32}. \end{cases}$$

Theorem 1.3. *Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N,6) = 1$. Then*

$$N(1,0,4,2,2,2;n) = \begin{cases} (2^{\alpha-1} - (-1)^{\alpha+\beta+(N-1)/2}) \\ \qquad \times\left(3^{\beta+1} + (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right)F_{12}(N) & \text{if } n \equiv 0 \pmod 4, \\ \frac{1}{2}\left(3^{\beta+1} + (-1)^{\beta}\left(\frac{N}{3}\right)\right)F_{12}(N) & \text{if } n \equiv 1 \pmod 4, \\ 3\left(3^{\beta+1} - (-1)^{\beta}\left(\frac{N}{3}\right)\right)F_{12}(N) & \text{if } n \equiv 2 \pmod 4, \\ \frac{3}{2}\left(3^{\beta+1} + (-1)^{\beta}\left(\frac{N}{3}\right)\right)F_{12}(N) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Theorem 1.4. *Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N,6) = 1$. Then*

$$N(1,0,1,8,8,8;n) = \begin{cases} (2^{\alpha-1} - (-1)^{\alpha+\beta+(N-1)/2}) \\ \qquad \times\left(3^{\beta+1} + (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right)F_{12}(N) & \text{if } n \equiv 0,2,4 \pmod 8, \\ \left(3^{\beta+1} + (-1)^{\beta}\left(\frac{N}{3}\right)\right)F_{12}(N) & \text{if } n \equiv 1,5 \pmod 8, \\ 0 & \text{if } n \equiv 3,6,7 \pmod 8. \end{cases}$$

Theorem 1.5. *Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N,6) = 1$. Then*

$$N(1,0,3,6,6,6;n) = \begin{cases} 2\sigma(N) & \text{if } n \equiv 1 \pmod 6, \\ 6\sigma(N) & \text{if } n \equiv 4 \pmod 6, \\ 0 & \text{if } n \equiv 2,5 \pmod 6, \\ 2(3^\beta - 2)\sigma(N) & \text{if } n \equiv 3 \pmod 6, \\ 6(3^\beta - 2)\sigma(N) & \text{if } n \equiv 0 \pmod 6. \end{cases}$$

Theorem 1.6. *Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N,6) = 1$. Then*

$$N(2,2,2,3,0,9;n) = \begin{cases} 0 & \text{if } n \equiv 1,4 \pmod 6, \\ 2\sigma(N) & \text{if } n \equiv 5 \pmod 6, \\ 6\sigma(N) & \text{if } n \equiv 2 \pmod 6, \\ 2(3^\beta - 2)\sigma(N) & \text{if } n \equiv 3 \pmod 6, \\ 6(3^\beta - 2)\sigma(N) & \text{if } n \equiv 0 \pmod 6. \end{cases}$$

Theorem 1.7. *Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 6) = 1$. Then*

$$N(2,2,2,3,0,12;n) = \begin{cases} (2^{\alpha-1} + (-1)^{\alpha+\beta+(N-1)/2}) \\ \qquad \times \left(3^\beta - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) F_{12}(N) & \text{if } n \equiv 0 \pmod 4, \\ \frac{3}{2}\left(3^\beta - (-1)^\beta \left(\frac{N}{3}\right)\right) F_{12}(N) & \text{if } n \equiv 1 \pmod 4, \\ 3\left(3^\beta + (-1)^\beta \left(\frac{N}{3}\right)\right) F_{12}(N) & \text{if } n \equiv 2 \pmod 4, \\ \frac{1}{2}\left(3^\beta - (-1)^\beta \left(\frac{N}{3}\right)\right) F_{12}(N) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Theorem 1.8. *Let $n \in \mathbb{N}$. Let $n = 3^\alpha 7^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 21) = 1$. Then*

$$N(1,1,1,1,1,2;n) = \frac{1}{2}\left(3^{\alpha+1} - (-1)^\alpha \left(\frac{N}{3}\right)\right)\left(7^{\beta+1} + (-1)^\alpha \left(\frac{N}{7}\right)\right) F_{21}(N).$$

Theorem 1.9. *Let $n \in \mathbb{N}$. Let $n = 3^\alpha 5^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 15) = 1$. Then*

$$N(1,1,1,1,1,4;n) = \frac{1}{4}(3^{\alpha+1} + (-1)^\alpha 5)\left(5^{\beta+1} - (-1)^\alpha \left(\frac{N}{5}\right)\right) F_5(N).$$

Theorem 1.10. *Let $n \in \mathbb{N}$. Let $n = 3^\alpha 5^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 15) = 1$. Then*

$$N(1,1,1,2,1,2;n) = \frac{1}{4}(3^{\alpha+2} - (-1)^\alpha 5)\left(5^{\beta+1} + (-1)^\alpha \left(\frac{N}{5}\right)\right) F_5(N).$$

Theorem 1.11. *Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 6) = 1$. Then*

$$N(1,0,1,2,2,5;n) = \begin{cases} 12(3^\beta - 1)\sigma(N) & \text{if } n \equiv 0 \pmod 6, \\ 4\sigma(N) & \text{if } n \equiv 1 \pmod 6, \\ 6\sigma(N) & \text{if } n \equiv 2 \pmod 6, \\ 4(3^\beta - 1)\sigma(N) & \text{if } n \equiv 3 \pmod 6, \\ 12\sigma(N) & \text{if } n \equiv 4 \pmod 6, \\ 2\sigma(N) & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

Theorem 1.12. *Let $n \in \mathbb{N}$. Let $n = 3^\alpha 7^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 21) = 1$. Then*

$$N(1,1,1,3,3,6;n) = \frac{1}{2}\left(3^\alpha + (-1)^\alpha \left(\frac{N}{3}\right)\right)\left(7^{\beta+1} - (-1)^\alpha \left(\frac{N}{7}\right)\right) F_{21}(N).$$

We remark that when $n \equiv 2 \pmod 3$ we have $\alpha = 0$ and $N \equiv 2 \pmod 3$ so that $3^\alpha + (-1)^\alpha \left(\frac{N}{3}\right) = 0$ and thus

$$N(1,1,1,3,3,6;n) = 0 \quad \text{if } n \equiv 2 \pmod 3.$$

Theorem 1.13. *Let* $n \in \mathbb{N}$. *Let* $n = 2^\alpha 3^\beta 5^\gamma N$, *where* $\alpha, \beta, \gamma \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $(N, 30) = 1$. *Then*

$$N(1,0,3,2,2,3;n) = \frac{1}{12}(2^{\alpha+1} - (-1)^{\beta + \frac{N-1}{2}}) \left(3^{\beta+1} + (-1)^{\alpha+\gamma} \left(\frac{N}{3}\right)\right) \times \left(5^{\gamma+1} + (-1)^{\alpha+\beta+\gamma} \left(\frac{N}{5}\right)\right) F_{15}(N).$$

Theorem 1.14. *Let* $n \in \mathbb{N}$. *Let* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $(N, 6) = 1$. *Then*

$$N(1,0,1,5,2,5;n) = \begin{cases} \frac{1}{6}\left(4 + 2(-1)^\beta \left(\frac{-1}{N}\right) + (-1)^\beta \left(\frac{2}{N}\right) - \left(\frac{-2}{N}\right)\right) \\ \qquad \times \left(3^{\beta+1} + \left(\frac{N}{3}\right)\right) F_6(N) & \text{if } n \equiv 1 \pmod 2, \\ 2\left(3^{\beta+1} - \left(\frac{N}{3}\right)\right) F_6(N) & \text{if } n \equiv 2 \pmod 4, \\ \frac{1}{3}\left(2^\alpha - (-1)^\alpha \left(\frac{-2}{N}\right)\right) \\ \qquad \times \left(3^{\beta+1} + (-1)^\alpha \left(\frac{N}{3}\right)\right) F_6(N) & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

We remark that when $n \equiv 3 \pmod 8$ we have $\alpha = 0$ and either $\beta \equiv 0 \pmod 2$, $N \equiv 3 \pmod 8$ or $\beta \equiv 1 \pmod 2, N \equiv 1 \pmod 8$ so that

$$4 + 2(-1)^\beta \left(\frac{-1}{N}\right) + (-1)^\beta \left(\frac{2}{N}\right) - \left(\frac{-2}{N}\right) = 0$$

and thus

$$N(1,0,1,5,2,5;n) = 0 \quad \text{for } n \equiv 3 \pmod 8.$$

Theorem 1.15. *Let* $n \in \mathbb{N}$. *Let* $n = 2^\alpha 5^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $(N, 10) = 1$. *Then*

$$N(1,0,5,2,2,3;n) = \begin{cases} 2\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 2(2^{\alpha+1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Theorem 1.16. *Let* $n \in \mathbb{N}$. *Let* $n = 3^{\alpha}5^{\beta}N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $(N, 15) = 1$. *Then*

$$N(1,1,1,3,3,12;n) = \begin{cases} \left(5^{\beta+1} + \left(\frac{N}{5}\right)\right) F_5(N) & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3, \\ \frac{1}{4}(3^{\alpha} - (-1)^{\alpha}5)\left(5^{\beta+1} + (-1)^{\alpha}\left(\frac{N}{5}\right)\right) F_5(N) & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

Theorem 1.17. *Let* $n \in \mathbb{N}$. *Let* $n = 2^{\alpha}3^{\beta}5^{\gamma}N$, *where* $\alpha, \beta, \gamma \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $(N, 30) = 1$. *Then*

$$N(2,1,2,3,3,3;n) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 3, \\ \frac{1}{3}(2^{\alpha+1} + (-1)^{\alpha}) \\ \quad \times \left(5^{\gamma+1} + (-1)^{\alpha}\left(\frac{N}{5}\right)\right) F_5(N) & \text{if } n \equiv 2 \pmod 3, \\ \frac{1}{12}(2^{\alpha+1} + (-1)^{\alpha})(3^{\beta} - (-1)^{\beta}5) \\ \quad \times \left(5^{\gamma+1} + (-1)^{\alpha+\beta}\left(\frac{N}{5}\right)\right) F_5(N) & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

Theorem 1.18. *Let* $n \in \mathbb{N}$. *Let* $n = 2^{\alpha}3^{\beta}N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $(N, 6) = 1$. *Then*

$$N(1,0,3,2,2,5;n) = \begin{cases} \frac{2}{3}(2^{\alpha+1} + (-1)^{\alpha+\frac{N-1}{2}})F_3(N) & \text{if } n \not\equiv 0 \pmod 3, \\ (2^{\alpha+1} + (-1)^{\alpha+\beta+\frac{N-1}{2}}) \\ \quad \times \left(5 \cdot 3^{\beta-1} - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) F_3(N) & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

Theorem 1.19. *Let* $n \in \mathbb{N}$. *Let* $n = 2^{\alpha}7^{\beta}N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $(N, 14) = 1$. *Then*

$$N(1,0,2,3,2,5;n) = \begin{cases} \frac{1}{4}\left(7^{\beta+1} + (-1)^{\beta}\left(\frac{N}{7}\right)\right) F_7(N) & \text{if } n \equiv 1 \pmod 2, \\ \frac{1}{4}(2^{\alpha} - (-1)^{\beta+\frac{N-1}{2}})\left(7^{\beta+1} + (-1)^{\beta}\left(\frac{N}{7}\right)\right) F_7(N) \\ & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Theorem 1.20. *Let $n \in \mathbb{N}$. Let $n = 2^\alpha 11^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 22) = 1$. Then*

$$N(1,1,3,2,2,6;n) = \frac{2}{5}(11^{\beta+1} - 6)\sigma(N).$$

Theorem 1.21. *Let $n \in \mathbb{N}$. Set $n = 3^\alpha 7^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 21) = 1$. Then*

$$N(1,1,2,7,7,7;n) = \frac{1}{2}\left(3^{\alpha+1} - (-1)^\alpha \left(\frac{N}{3}\right)\right)\left(7^\beta + (-1)^\alpha \left(\frac{N}{7}\right)\right) F_{21}(N).$$

We note that if $n \equiv 3, 5$ or $6 \pmod 7$ then $\beta = 0$ and $3^\alpha N \equiv 3, 5, 6 \pmod 7$ so

$$7^\beta + (-1)^\alpha \left(\frac{N}{7}\right) = 1 + \left(\frac{3^\alpha N}{7}\right) = 1 - 1 = 0$$

and thus

$$N(1,1,2,7,7,7;n) = 0 \quad \text{for } n \equiv 3,5,6 \pmod 7.$$

Theorem 1.22. *Let $n \in \mathbb{N}$. Let $n = 3^\alpha 5^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 15) = 1$. Then*

$$N(1,1,4,5,5,5;n) = \frac{1}{4}(3^{\alpha+2} - (-1)^\alpha 5)\left(5^\beta + (-1)^\alpha \left(\frac{N}{5}\right)\right) F_5(N).$$

We note that if $n \equiv 2$ or $3 \pmod 5$ then $\beta = 0$ and $(-1)^\alpha(\frac{N}{5}) = -1$ so that

$$N(1,1,4,5,5,5;n) = 0 \quad \text{for } n \equiv 2 \text{ or } 3 \pmod 5.$$

Theorem 1.23. *Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta 5^\gamma N$, where $\alpha, \beta, \gamma \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 30) = 1$. Then*

$$\begin{aligned} N(2,1,2,5,5,5;n) &= \frac{1}{12}(2^{\alpha+1} + (-1)^\alpha)(3^{\beta+1} + (-1)^\beta 5) \\ &\quad \times \left(5^\gamma - (-1)^{\alpha+\beta}\left(\frac{N}{5}\right)\right) F_5(N). \end{aligned}$$

We note that if $n \equiv 1$ or $4 \pmod 5$ then $\gamma = 0$ and $(-1)^{\alpha+\beta}(\frac{N}{5}) = 1$ so that

$$N(2,1,2,5,5,5;n) = 0 \quad \text{for } n \equiv 1,4 \pmod 5.$$

Theorem 1.24. *Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 6) = 1$. Then*

$$N(2,2,5,3,0,3;n) = \begin{cases} 4(2^{\alpha+1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod 6, \\ 0 & \text{if } n \equiv 1,4 \pmod 6, \\ 2(2^{\alpha+1} - 3)\sigma(N) & \text{if } n \equiv 2 \pmod 6, \\ 4\sigma(N) & \text{if } n \equiv 3 \pmod 6, \\ 2\sigma(N) & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

In Sec. 5 we determine the representation number of the form $3x^2 + 3xy + 6y^2 + 7z^2 + 7zt + 7t^2$ using modular forms. The discriminant of this form lies outside the range of Nipp's table [18]. We relate this representation number to those of the forms in Theorems 1.8, 1.12 and 1.21. In Sec. 6 we show that the values of certain Dirichlet L-series follow from the theorems in Table 6.

2. Proofs of Theorems 1.1 and 1.2

We use a recent theorem of Alaca, Pehlivan and Williams [13] to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By [13, Theorem 1.1(i)] we have

$$N(2,0,2,3,2,3;n) = N(1,2,2,2;n) - N(1,2,2,8;n) - N(1,1,1,2;n/2) + 2N(1,1,2,4;n/2). \tag{2.1}$$

The evaluations of the representation numbers $N(1,1,1,2;n)$, $N(1,1,2,4;n)$, $N(1,2,2,2;n)$ and $N(1,2,2,8;n)$ in terms of

$$\sum_{d|N} \left(\frac{8}{d}\right) \frac{N}{d} = \prod_{p^\gamma \| N} \frac{p^{\gamma+1} - \left(\frac{8}{p}\right)^{\gamma+1}}{p - \left(\frac{8}{p}\right)} = F_8(N)$$

are given in Theorems 5.1, 5.3, 5.2 and 5.7 of [7] respectively. Making use of these in (2.1), we obtain the formulae of Theorem 1.1. □

Proof of Theorem 1.2. By [13, Theorem 1.1(i)] we have

$$N(1,0,2,6,4,6;n) = N(1,2,2,4;n) - N(1,2,2,16;n) - N(1,2,4,8;n) + 2N(1,2,8,16;n). \tag{2.2}$$

The evaluations of the representation numbers $N(1,2,2,4;n)$, $N(1,2,2,16;n)$, $N(1,2,4,8;n)$ and $N(1,2,8,16;n)$ are given in Proposition 4.2, Theorems 4.9, 4.1 and 4.13 of [4] respectively. Making use of these in (2.2), we obtain the formulae of Theorem 1.2. □

3. Proofs of Theorems 1.3–1.7

We make use of the (p,k)-parametrization method introduced by Alaca, Alaca and Williams in [8] to prove Theorems 1.3–1.7. As extensive use of this method has been made recently by a number of authors to determine the representation numbers of a variety of quadratic forms, see for example [1, 2, 14, 24], we just use this method to prove Theorem 1.3 as Theorems 1.4–1.7 can be proved in a similar manner.

Proof of Theorem 1.3. We begin by recalling Ramanujan's one-dimensional theta function

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q \in \mathbb{C}, \ |q| < 1, \tag{3.1}$$

and the Borweins' two-dimensional theta function

$$a(q) := \sum_{(x,y)\in\mathbb{Z}^2} q^{x^2+xy+y^2}, \quad q \in \mathbb{C}, \ |q| < 1. \tag{3.2}$$

From (3.1) and (3.2) we deduce

$$\sum_{n=0}^{\infty} N(1,0,4,2,2,2;n)q^n = \varphi(q)\varphi(q^4)a(q^2). \tag{3.3}$$

As in [8, p. 178; 9, pp. 32, 33], we define the parameters p and k by

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \quad \text{and} \quad k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{3.4}$$

The parametrizations in terms of p and k that we require are given in [5, Theorem 2.4, pp. 222–223; 8, Theorem 2, p. 178], namely,

$$\varphi(q) = (1+2p)^{3/4}k^{1/2}, \tag{3.5}$$

$$\varphi(q^3) = (1+2p)^{1/4}k^{1/2}, \tag{3.6}$$

$$\varphi(q^4) = \frac{1}{2}((1+2p)^{3/4} + (1-p)^{3/4}(1+p)^{1/4})k^{1/2}, \tag{3.7}$$

$$\varphi(-q) = (1-p)^{3/4}(1+p)^{1/4}k^{1/2}, \tag{3.8}$$

$$\varphi(-q^3) = (1-p)^{1/4}(1+p)^{3/4}k^{1/2}, \tag{3.9}$$

$$a(q^2) = (1+p+p^2)k. \tag{3.10}$$

The parametrizations (3.5)–(3.10) allow us to verify the identity

$$\begin{aligned}\varphi(q)\varphi(q^4)a(q^2) &= \frac{3}{4}\varphi(q)\varphi^3(q^3) + \frac{3}{4}\varphi(-q)\varphi^3(q^3) \\ &\quad + \frac{1}{4}\varphi^3(q)\varphi(q^3) + \frac{1}{4}\varphi^2(q)\varphi(-q)\varphi(q^3) \\ &\quad - \frac{1}{2}\varphi^2(q)\varphi(-q)\varphi(-q^3) - \frac{1}{2}\varphi(q)\varphi^2(-q)\varphi(-q^3). \end{aligned} \tag{3.11}$$

For $n \in \mathbb{N}$ we define (as in [5, Definition 3.1, p. 225; 6, Definition 2.2, p. 544])

$$A(n) := \sum_{d\mid n} \left(\frac{12}{n/d}\right) d = \sum_{d\mid n} \left(\frac{12}{d}\right)\frac{n}{d}, \tag{3.12}$$

$$B(n) := \sum_{d\mid n} \left(\frac{-3}{d}\right)\left(\frac{-4}{n/d}\right) d = \sum_{d\mid n} \left(\frac{-3}{n/d}\right)\left(\frac{-4}{d}\right)\frac{n}{d}, \tag{3.13}$$

$$C(n) := \sum_{d|n} \left(\frac{-3}{n/d}\right)\left(\frac{-4}{d}\right) d = \sum_{d|n} \left(\frac{-3}{d}\right)\left(\frac{-4}{n/d}\right) \frac{n}{d}, \tag{3.14}$$

$$D(n) := \sum_{d|n} \left(\frac{12}{d}\right) d = \sum_{d|n} \left(\frac{12}{n/d}\right) \frac{n}{d}, \tag{3.15}$$

$$E(n) := \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ i,j \text{ odd} \\ 4n = i^2+3j^2}} (-1)^{(i-1)/2} i. \tag{3.16}$$

It follows from [5, Theorem 3.3, p. 227; 6, Theorem 4.1 and Theorem 6.1, pp. 555, 565–566] that

$$\varphi(q)\varphi^3(q^3) = 1 + \sum_{n=1}^{\infty} (2A(n) + 2B(n) - C(n) - D(n)) q^n,$$

$$\varphi(-q)\varphi^3(q^3) = 1 - \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2} (A(n) + B(n) + C(n) + D(n)) q^n - 2 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} E(n) q^n,$$

$$\varphi^3(q)\varphi(q^3) = 1 + \sum_{n=1}^{\infty} (6A(n) - 2B(n) + 3C(n) - D(n)) q^n,$$

$$\varphi^2(q)\varphi(-q)\varphi(q^3) = 1 + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2} (3A(n) - B(n) + 3C(n) - D(n)) q^n + 2 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} E(n) q^n,$$

$$\varphi^2(q)\varphi(-q)\varphi(-q^3) = 1 + \sum_{n=1}^{\infty} (3C(n) - D(n)) q^n,$$

$$\varphi(q)\varphi^2(-q)\varphi(-q^3) = 1 + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2} (3A(n) - B(n) + 3C(n) - D(n)) q^n - 2 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} E(n) q^n.$$

Using these in (3.11), and appealing to (3.3), we obtain

$$\sum_{n=0}^{\infty} N(1,0,4,2,2,2;n)q^n = 1 + \sum_{n=1}^{\infty}(3A(n) + B(n) - \frac{3}{2}C(n) - \frac{1}{2}D(n)q^n$$
$$- \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} (-1)^{n/2}\left(\frac{3}{2}A(n) + \frac{1}{2}B(n) + \frac{3}{2}C(n) + \frac{1}{2}D(n)\right)q^n.$$

Equating the coefficients of q^n $(n \in \mathbb{N})$, we obtain

$$N(1,0,4,2,2,2;n) = \begin{cases} \frac{3}{2}A(n) + \frac{1}{2}B(n) - 3C(n) - D(n) & \text{if } n \equiv 0 \pmod 4, \\ 3A(n) + B(n) - \frac{3}{2}C(n) - \frac{1}{2}D(n) & \text{if } n \equiv 1 \pmod 2, \\ \frac{9}{2}A(n) + \frac{3}{2}B(n) & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Set $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N,6) = 1$. From [5, Theorem 3.1, pp. 225–226] we have

$$A(n) = 2^\alpha 3^\beta A(N),$$
$$B(n) = (-1)^{\alpha+\beta} 2^\alpha \left(\frac{N}{3}\right) A(N),$$
$$C(n) = (-1)^{\alpha+\beta+(N-1)/2} 3^\beta A(N),$$
$$D(n) = (-1)^{(N-1)/2} \left(\frac{N}{3}\right) A(N),$$

and

$$A(N) = \sum_{d \mid N} \left(\frac{12}{N/d}\right) d = \prod_{p^\gamma \| N} \frac{p^{\gamma+1} - \left(\frac{12}{p}\right)^{\gamma+1}}{p - \left(\frac{12}{p}\right)} = F_{12}(N).$$

The formulae for $N(1,0,4,2,2,2;n)$ in terms of $F_{12}(N)$ now follow. □

4. Proofs of Theorems 1.8–1.24

For positive integers k and n, and a prime p, we define

$N_{p^k}(a,b,c,d,e,f;n) :=$ number of solutions of the congruence

$$ax^2 + bxy + cy^2 + dz^2 + ezt + ft^2 \equiv n \pmod{p^k}, \quad (4.1)$$

where the quaternary quadratic form $ax^2 + bxy + cy^2 + dz^2 + ezt + ft^2$ was specified in (1.1). The discriminant of this form is $D := (b^2 - 4ac)(e^2 - 4df)$. We assume that the class of this form belongs to a genus (of discriminant D) containing one and only one form class. Then Siegel's mass formula [16, 19] asserts that

$$N(a,b,c,d,e,f;n) = \frac{4\pi^2 n}{\sqrt{D}} \prod_p d_p(a,b,c,d,e,f;n), \tag{4.2}$$

where the product runs over all primes p and for each prime p the local density $d_p(a,b,c,d,e,f;n)$ is given by

$$d_p(a,b,c,d,e,f;n) := \lim_{k\to\infty} \frac{N_{p^k}(a,b,c,d,e,f;n)}{p^{3k}}. \tag{4.3}$$

We let ℓ denote the least positive integer represented by the form $ax^2+bxy+cy^2+dz^2+ezt+ft^2$ so that

$$N(a,b,c,d,e,f;\ell) > 0. \tag{4.4}$$

By (4.2) with $n=\ell$ we have

$$N(a,b,c,d,e,f;\ell) = \frac{4\pi^2 \ell}{\sqrt{D}} \prod_p d_p(a,b,c,d,e,f;\ell). \tag{4.5}$$

Dividing (4.2) by (4.5), we obtain in view of (4.4) the formula

$$N(a,b,c,d,e,f;n) = N(a,b,c,d,e,f;\ell)\frac{n}{\ell} \prod_p \frac{d_p(a,b,c,d,e,f;n)}{d_p(a,b,c,d,e,f;\ell)}, \tag{4.6}$$

which is valid for all $n \in \mathbb{N}$. In our paper [11] we gave a method of determining $N_{p^k}(a,b,c,d,e,f;n)$ using double Gauss sums. Then, using (4.3), we can determine the local density $d_p(a,b,c,d,e,f;n)$ for each prime p. Finally, from (4.6), we are able to determine $N(a,b,c,d,e,f;n)$ for each $n \in \mathbb{N}$. We carry this out for the seventeen forms in Table 6.

Proof of Theorem 1.8. Let $f := x^2+xy+y^2+z^2+zt+2t^2$. Let p denote a prime, K a positive integer not divisible by p, θ a nonnegative integer, and k a positive integer with $k \geq \theta+1$. Using the method of double Gauss sums as demonstrated by the authors in [11], we find that

$$N_{p^k}(f;p^\theta K) = \begin{cases} 3^{3k} - (-1)^\theta 3^{3k-\theta-1}\left(\dfrac{K}{3}\right) & \text{if } p=3, \\ 7^{3k} + 7^{3k-\theta-1}\left(\dfrac{K}{7}\right) & \text{if } p=7, \\ \dfrac{p^{3k-\theta-2}\left(p^2-\left(\dfrac{21}{p}\right)\right)\left(p^{\theta+1}-\left(\dfrac{21}{p}\right)^{\theta+1}\right)}{p-\left(\dfrac{21}{p}\right)} & \text{if } p \neq 3,7. \end{cases} \tag{4.7}$$

Appealing to (4.7) the local density

$$d_p(f;p^\theta K) = \lim_{k\to\infty} \frac{N_{p^k}(f;p^\theta K)}{p^{3k}}$$

is given by

$$d_p(f;p^\theta K)=\begin{cases}1-\dfrac{(-1)^\theta}{3^{\theta+1}}\left(\dfrac{K}{3}\right) & \text{if } p=3,\\ 1+\dfrac{1}{7^{\theta+1}}\left(\dfrac{K}{7}\right) & \text{if } p=7,\\ \dfrac{\left(p^2-\left(\dfrac{21}{p}\right)\right)\left(p^{\theta+1}-\left(\dfrac{21}{p}\right)^{\theta+1}\right)}{p^{\theta+2}\left(p-\left(\dfrac{21}{p}\right)\right)} & \text{if } p\neq 3,7.\end{cases} \tag{4.8}$$

Taking $\theta=0$ and $K=1$ in (4.8), we deduce

$$d_p(f;1)=\begin{cases}\dfrac{2}{3} & \text{if } p=3,\\ \dfrac{8}{7} & \text{if } p=7,\\ \dfrac{p^2-\left(\dfrac{21}{p}\right)}{p^2} & \text{if } p\neq 3,7.\end{cases} \tag{4.9}$$

Thus, from (4.8) and (4.9), we obtain

$$p^\theta\frac{d_p(f;p^\theta K)}{d_p(f;1)}=\begin{cases}\dfrac{1}{2}\left(3^{\theta+1}-(-1)^\theta\left(\dfrac{K}{3}\right)\right) & \text{if } p=3,\\ \dfrac{1}{8}\left(7^{\theta+1}+\left(\dfrac{K}{7}\right)\right) & \text{if } p=7,\\ \dfrac{p^{\theta+1}-\left(\dfrac{21}{p}\right)^{\theta+1}}{p-\left(\dfrac{21}{p}\right)} & \text{if } p\neq 3,7.\end{cases} \tag{4.10}$$

Let $n\in\mathbb{N}$. Let $n=3^\alpha 7^\beta N$, where $\alpha,\beta\in\mathbb{N}_0$, $N\in\mathbb{N}$ and $(N,21)=1$. Then, by (4.10), we have

$$3^\alpha\frac{d_3(f;n)}{d_3(f;1)}=\frac{1}{2}\left(3^{\alpha+1}-(-1)^\alpha\left(\frac{N}{3}\right)\right), \tag{4.11}$$

$$7^\beta\frac{d_7(f;n)}{d_7(f;1)}=\frac{1}{8}\left(7^{\beta+1}+(-1)^\alpha\left(\frac{N}{7}\right)\right), \tag{4.12}$$

$$p^\gamma\frac{d_p(f;n)}{d_p(f;1)}=\frac{p^{\gamma+1}-\left(\dfrac{21}{p}\right)^{\gamma+1}}{p-\left(\dfrac{21}{p}\right)} \text{ if } p\neq 3,7 \text{ and } p^\gamma\| N. \tag{4.13}$$

Since $N(f;1) = 8$ the least positive integer represented by the form f is $\ell = 1$. Thus, by (4.6), we have

$$N(f;n) = 8n \prod_p \frac{d_p(f;n)}{d_p(f;1)}. \tag{4.14}$$

Appealing to (4.11)–(4.14) we deduce

$$N(f;n) = \frac{1}{2}\left(3^{\alpha+1} - (-1)^\alpha \left(\frac{N}{3}\right)\right)\left(7^{\beta+1} + (-1)^\alpha \left(\frac{N}{7}\right)\right) F_{21}(N),$$

as asserted. □

As the proofs of Theorems 1.9–1.24 are essentially the same as that of Theorem 1.8 we just give the evaluations of the required quantities. Throughout the proofs p denotes a prime, K a positive integer not divisible by p, θ a nonnegative integer, and k a positive integer with $k \geq \theta + 1$.

Proof of Theorem 1.9. Let $f := x^2 + xy + y^2 + z^2 + zt + 4t^2$. Using the method in [11], we find that

$$N_{p^k}(f;p^\theta K) = \begin{cases} \frac{1}{2}(3^{3k} + 5(-1)^\theta 3^{3k-\theta-1}) & \text{if } p = 3, \\ 5^{3k} - \left(\frac{K}{5}\right) 5^{3k-\theta-1} & \text{if } p = 5, \\ \dfrac{p^{3k-\theta-2}\left(p^2 - \left(\frac{5}{p}\right)\right)\left(p^{\theta+1} - \left(\frac{5}{p}\right)^{\theta+1}\right)}{p - \left(\frac{5}{p}\right)} & \text{if } p \neq 3, 5. \end{cases}$$

Hence

$$d_p(f;p^\theta K) = \lim_{k\to\infty} \frac{N_{p^k}(f;p^\theta K)}{p^{3k}} = \begin{cases} \frac{1}{2}\left(1 + \frac{(-1)^\theta 5}{3^{\theta+1}}\right) & \text{if } p = 3, \\ 1 - \left(\frac{K}{5}\right)\frac{1}{5^{\theta+1}} & \text{if } p = 5, \\ \dfrac{\left(p^2 - \left(\frac{5}{p}\right)\right)\left(p^{\theta+1} - \left(\frac{5}{p}\right)^{\theta+1}\right)}{p^{\theta+2}\left(p - \left(\frac{5}{p}\right)\right)} & \text{if } p \neq 3, 5; \end{cases}$$

$$d_p(f;1) = \begin{cases} \dfrac{4}{3} & \text{if } p = 3, \\ \dfrac{4}{5} & \text{if } p = 5, \\ \dfrac{p^2 - \left(\dfrac{5}{p}\right)}{p^2} & \text{if } p \neq 3, 5; \end{cases}$$

$$p^\theta \frac{d_p(f;p^\theta K)}{d_p(f;1)} = \begin{cases} \dfrac{1}{8}(3^{\theta+1} + (-1)^\theta 5) & \text{if } p = 3, \\ \dfrac{1}{4}\left(5^{\theta+1} - \left(\dfrac{K}{5}\right)\right) & \text{if } p = 5, \\ \dfrac{p^{\theta+1} - \left(\dfrac{5}{p}\right)^{\theta+1}}{p - \left(\dfrac{5}{p}\right)} & \text{if } p \neq 3, 5. \end{cases}$$

Let $n \in \mathbb{N}$. Let $n = 3^\alpha 5^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 15) = 1$. Then as $N(f;1) = 8$ the least positive integer represented by the form f is $\ell = 1$ and

$$N(f;n) = 8n \prod_p \frac{d_p(f;n)}{d_p(f;1)} = \frac{1}{4}(3^{\alpha+1} + (-1)^\alpha 5)\left(5^{\beta+1} - (-1)^\alpha \left(\frac{N}{5}\right)\right) F_5(N),$$

as asserted. □

Proof of Theorem 1.10. Let $f := x^2 + xy + y^2 + 2z^2 + zt + 2t^2$. Using the method in [11], we find that

$$N_{p^k}(f;p^\theta K) = \begin{cases} \dfrac{1}{2}(3^{3k+1} - (-1)^\theta 3^{3k-\theta-1} 5) & \text{if } p = 3, \\ 5^{3k} + \left(\dfrac{K}{5}\right) 5^{3k-\theta-1} & \text{if } p = 5, \\ \dfrac{p^{3k-\theta-2}\left(p^2 - \left(\dfrac{5}{p}\right)\right)\left(p^{\theta+1} - \left(\dfrac{5}{p}\right)^{\theta+1}\right)}{p - \left(\dfrac{5}{p}\right)} & \text{if } p \neq 3, 5. \end{cases}$$

Hence

$$d_p(f;p^\theta K) = \begin{cases} \dfrac{1}{2}\left(3 - \dfrac{(-1)^\theta 5}{3^{\theta+1}}\right) & \text{if } p = 3, \\ 1 + \left(\dfrac{K}{5}\right)\dfrac{1}{5^{\theta+1}} & \text{if } p = 5, \\ \dfrac{\left(p^2 - \left(\dfrac{5}{p}\right)\right)\left(p^{\theta+1} - \left(\dfrac{5}{p}\right)^{\theta+1}\right)}{p^{\theta+2}\left(p - \left(\dfrac{5}{p}\right)\right)} & \text{if } p \neq 3, 5; \end{cases}$$

$$d_p(f;1) = \begin{cases} \dfrac{2}{3} & \text{if } p=3, \\ \dfrac{6}{5} & \text{if } p=5, \\ \dfrac{p^2-\left(\dfrac{5}{p}\right)}{p^2} & \text{if } p\neq 3,5; \end{cases}$$

$$p^\theta \frac{d_p(f;p^\theta K)}{d_p(f;1)} = \begin{cases} \dfrac{1}{4}\left(3^{\theta+2}-(-1)^\theta 5\right) & \text{if } p=3, \\ \dfrac{1}{6}\left(5^{\theta+1}+\left(\dfrac{K}{5}\right)\right) & \text{if } p=5, \\ \dfrac{p^{\theta+1}-\left(\dfrac{5}{p}\right)^{\theta+1}}{p-\left(\dfrac{5}{p}\right)} & \text{if } p\neq 3,5. \end{cases}$$

Let $n \in \mathbb{N}$. Let $n = 3^\alpha 5^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N,15)=1$. Then, as $N(f;1)=6$, the least positive integer represented by the form f is $\ell = 1$ and by (4.6) we have

$$N(f;n) = 6n\prod_p \frac{d_p(f;n)}{d_p(f;1)} = \frac{1}{4}(3^{\alpha+2}-(-1)^\alpha 5)\left(5^{\beta+1}+(-1)^\alpha\left(\frac{N}{5}\right)\right)F_5(N),$$

as asserted. □

Proof of Theorem 1.11. Let $f := x^2+y^2+2z^2+2zt+5t^2$. The value of $N_{p^k}(f;p^\theta K)$ is given in [11, Theorem 1.4]. Using this value we find that

$$d_p(f;p^\theta K) = \begin{cases} 1 & \text{if } p=2,\ \theta=0, \\ \dfrac{3}{2^\theta} & \text{if } p=2,\ \theta\geq 1, \\ 1+\left(\dfrac{K}{3}\right)\dfrac{1}{3} & \text{if } p=3,\ \theta=0, \\ \dfrac{4}{3}\left(1-\dfrac{1}{3^\theta}\right) & \text{if } p=3,\ \theta\geq 1, \\ (p+1)\left(\dfrac{1}{p}-\dfrac{1}{p^{\theta+2}}\right) & \text{if } p\neq 2,3; \end{cases}$$

$$d_p(f;1) = \begin{cases} 1 & \text{if } p=2, \\ \dfrac{4}{3} & \text{if } p=3, \\ (p+1)\left(\dfrac{1}{p}-\dfrac{1}{p^2}\right) & \text{if } p\neq 2,3; \end{cases}$$

$$p^{\theta}\frac{d_p(f;p^{\theta}K)}{d_p(f;1)}=\begin{cases}1 & \text{if } p=2,\ \theta=0,\\ 3 & \text{if } p=2,\ \theta\geq 1,\\ \frac{1}{4}\left(3+\left(\frac{K}{3}\right)\right) & \text{if } p=3,\ \theta=0,\\ 3^{\theta}-1 & \text{if } p=3,\ \theta\geq 1,\\ \frac{p^{\theta+1}-1}{p-1} & \text{if } p\neq 2,3.\end{cases}$$

Let $n\in\mathbb{N}$. Let $n=2^{\alpha}3^{\beta}N$, where $\alpha,\beta\in\mathbb{N}_0$, $N\in\mathbb{N}$ and $(N,6)=1$. Then, as $N(f;1)=4$, the least positive integer represented by the form f is $\ell=1$ and by (4.6) we have

$$\begin{aligned}N(f;n)&=4n\prod_p\frac{d_p(f;n)}{d_p(f;1)}\\ &=4\begin{Bmatrix}1 & \text{if } \alpha=0\\ 3 & \text{if } \alpha\geq 1\end{Bmatrix}\begin{Bmatrix}\frac{1}{4}\left(3+\left(\frac{2^{\alpha}N}{3}\right)\right) & \text{if } \beta=0\\ 3^{\beta}-1 & \text{if } \beta\geq 1\end{Bmatrix}\prod_{p^{\gamma}\| N}\frac{p^{\gamma+1}-1}{p-1}\\ &=\begin{cases}12(3^{\beta}-1)\sigma(N) & \text{if } n\equiv 0 \pmod 6,\\ 4\sigma(N) & \text{if } n\equiv 1 \pmod 6,\\ 6\sigma(N) & \text{if } n\equiv 2 \pmod 6,\\ 4(3^{\beta}-1)\sigma(N) & \text{if } n\equiv 3 \pmod 6,\\ 12\sigma(N) & \text{if } n\equiv 4 \pmod 6,\\ 2\sigma(N) & \text{if } n\equiv 5 \pmod 6,\end{cases}\end{aligned}$$

as asserted. □

Proof of Theorem 1.12. Let $f:=x^2+xy+y^2+3z^2+3zt+6t^2$. Using the method in [11], we find that

$$N_{p^k}(f;p^{\theta}K)=\begin{cases}3^{3k}+(-1)^{\theta}3^{3k-\theta}\left(\frac{K}{3}\right) & \text{if } p=3,\\ 7^{3k}-\left(\frac{K}{7}\right)7^{3k-\theta-1} & \text{if } p=7,\\ \dfrac{p^{3k-\theta-2}\left(p^2-\left(\frac{21}{p}\right)\right)\left(p^{\theta+1}-\left(\frac{21}{p}\right)^{\theta+1}\right)}{p-\left(\frac{21}{p}\right)} & \text{if } p\neq 3,7.\end{cases}$$

Hence

$$d_p(f;p^\theta K) = \lim_{k\to\infty} \frac{N_{p^k}(f;p^\theta K)}{p^{3k}}$$
$$= \begin{cases} 1 + \left(\frac{K}{3}\right)\frac{(-1)^\theta}{3^\theta} & \text{if } p=3,\\ 1 - \left(\frac{K}{7}\right)\frac{1}{7^{\theta+1}} & \text{if } p=7,\\ \dfrac{\left(p^2 - \left(\frac{21}{p}\right)\right)\left(p^{\theta+1} - \left(\frac{21}{p}\right)^{\theta+1}\right)}{p^{\theta+2}\left(p - \left(\frac{21}{p}\right)\right)} & \text{if } p \neq 3,7; \end{cases}$$

$$d_p(f;1) = \begin{cases} 2 & \text{if } p=3,\\ \frac{6}{7} & \text{if } p=7,\\ \dfrac{p^2 - \left(\frac{21}{p}\right)}{p^2} & \text{if } p\neq 3,7; \end{cases}$$

$$p^\theta \frac{d_p(f;p^\theta K)}{d_p(f;1)} = \begin{cases} \frac{1}{2}\left(3^\theta + (-1)^\theta\left(\frac{K}{3}\right)\right) & \text{if } p=3,\\ \frac{1}{6}\left(7^{\theta+1} - \left(\frac{K}{7}\right)\right) & \text{if } p=7,\\ \dfrac{p^{\theta+1} - \left(\frac{21}{p}\right)^{\theta+1}}{p - \left(\frac{21}{p}\right)} & \text{if } p\neq 3,7. \end{cases}$$

Let $n \in \mathbb{N}$. Let $n = 3^\alpha 7^\beta N$, where $\alpha,\beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N,21)=1$. Then, as $N(f;1)=6$, the least positive integer represented by the form f is $\ell = 1$ and by (4.6) we have

$$N(f;n) = 6n\prod_p \frac{d_p(f;n)}{d_p(f;1)} = \frac{1}{2}\left(3^\alpha + (-1)^\alpha\left(\frac{N}{3}\right)\right)\left(7^{\beta+1} - (-1)^\alpha\left(\frac{N}{7}\right)\right)F_{21}(N),$$

as asserted. □

Proof of Theorem 1.13. Let $f := x^2+3y^2+2z^2+2zt+3t^2$. Using the value of

$N_{p^k}(f;p^\theta K)$ given in [11, Theorem 1.5], we find that

$$d_p(f;p^\theta K) = \begin{cases} 1 - \dfrac{(-1)^{(K-1)/2}}{2^{\theta+1}} & \text{if } p = 2, \\ 1 + \left(\dfrac{K}{3}\right)\dfrac{1}{3^{\theta+1}} & \text{if } p = 3, \\ 1 + (-1)^\theta \left(\dfrac{K}{5}\right)\dfrac{1}{5^{\theta+1}} & \text{if } p = 5, \\ \dfrac{\left(p^2 - \left(\dfrac{15}{p}\right)\right)\left(p^{\theta+1} - \left(\dfrac{15}{p}\right)^{\theta+1}\right)}{p^{\theta+2}\left(p - \left(\dfrac{15}{p}\right)\right)} & \text{if } p \neq 2,3,5; \end{cases}$$

$$d_p(f;1) = \begin{cases} \dfrac{1}{2} & \text{if } p = 2, \\ \dfrac{4}{3} & \text{if } p = 3, \\ \dfrac{6}{5} & \text{if } p = 5, \\ \dfrac{p^2 - \left(\dfrac{15}{p}\right)}{p^2} & \text{if } p \neq 2,3,5; \end{cases}$$

$$p^\theta \frac{d_p(f;p^\theta K)}{d_p(f;1)} = \begin{cases} 2^{\theta+1} - (-1)^{(K-1)/2} & \text{if } p = 2, \\ \dfrac{1}{4}\left(3^{\theta+1} + \left(\dfrac{K}{3}\right)\right) & \text{if } p = 3, \\ \dfrac{1}{6}\left(5^{\theta+1} + (-1)^\theta\left(\dfrac{K}{5}\right)\right) & \text{if } p = 5, \\ \dfrac{p^{\theta+1} - \left(\dfrac{15}{p}\right)^{\theta+1}}{p - \left(\dfrac{15}{p}\right)} & \text{if } p \neq 2,3,5. \end{cases}$$

Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta 5^\gamma N$, where $\alpha, \beta, \gamma \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 30) = 1$. Then, as $N(f;1) = 2$, the least positive integer represented by the form f is $\ell = 1$ and by (4.6) we have

$$N(f;n) = 2n\prod_p \frac{d_p(f;n)}{d_p(f;1)} = \frac{1}{12}(2^{\alpha+1} - (-1)^{\beta+(N-1)/2})\left(3^{\beta+1} + (-1)^{\alpha+\gamma}\left(\frac{N}{3}\right)\right) \times \left(5^{\gamma+1} + (-1)^{\alpha+\beta+\gamma}\left(\frac{N}{5}\right)\right)F_{15}(N),$$

as asserted. □

Proof of Theorem 1.14. Let $f := x^2 + y^2 + 5z^2 + 2zt + 5t^2$. Using the value of $N_{p^k}(f; p^\theta K)$ given in [11, Theorem 1.6], we find that

$$d_p(f; p^\theta K) = \begin{cases} \dfrac{1}{4}\left(4 + 2\left(\dfrac{-1}{K}\right) + \left(\dfrac{2}{K}\right) - \left(\dfrac{-2}{K}\right)\right) & \text{if } p = 2,\ \theta = 0, \\ \dfrac{3}{2} & \text{if } p = 2,\ \theta = 1, \\ \dfrac{1}{2}\left(1 - (-1)^\theta \left(\dfrac{-2}{K}\right)\dfrac{1}{2^\theta}\right) & \text{if } p = 2,\ \theta \geq 2, \\ 1 + \left(\dfrac{K}{3}\right)\dfrac{1}{3^{\theta+1}} & \text{if } p = 3, \\ \dfrac{\left(p^2 - \left(\dfrac{6}{p}\right)\right)\left(p^{\theta+1} - \left(\dfrac{6}{p}\right)^{\theta+1}\right)}{p^{\theta+2}\left(p - \left(\dfrac{6}{p}\right)\right)} & \text{if } p \neq 2, 3; \end{cases}$$

$$d_p(f; 1) = \begin{cases} \dfrac{3}{2} & \text{if } p = 2, \\ \dfrac{4}{3} & \text{if } p = 3, \\ \dfrac{p^2 - \left(\dfrac{6}{p}\right)}{p^2} & \text{if } p \neq 2, 3; \end{cases}$$

$$p^\theta \frac{d_p(f; p^\theta K)}{d_p(f; 1)} = \begin{cases} \dfrac{1}{6}\left(4 + 2\left(\dfrac{-1}{K}\right) + \left(\dfrac{2}{K}\right) - \left(\dfrac{-2}{K}\right)\right) & \text{if } p = 2,\ \theta = 0, \\ 2 & \text{if } p = 2,\ \theta = 1, \\ \dfrac{1}{3}\left(2^\theta - (-1)^\theta \left(\dfrac{-2}{K}\right)\right) & \text{if } p = 2,\ \theta \geq 2, \\ \dfrac{1}{4}\left(3^{\theta+1} + \left(\dfrac{K}{3}\right)\right) & \text{if } p = 3, \\ \dfrac{p^{\theta+1} - \left(\dfrac{6}{p}\right)^{\theta+1}}{p - \left(\dfrac{6}{p}\right)} & \text{if } p \neq 2, 3. \end{cases}$$

Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 6) = 1$. Then, as $N(f; 1) = 4$, the least positive integer represented by the form f is $\ell = 1$ and by (4.6) we have

$$N(f; n) = 4n \prod_p \frac{d_p(f; n)}{d_p(f; 1)}.$$

If $n \equiv 1 \pmod 2$ we have

$$N(f;n) = 4 \cdot \frac{1}{6}\left(4 + 2\left(\frac{-1}{3^\beta N}\right) + \left(\frac{2}{3^\beta N}\right) - \left(\frac{-2}{3^\beta N}\right)\right)\frac{1}{4}\left(3^{\beta+1} + \left(\frac{N}{3}\right)\right)F_6(N)$$
$$= \frac{1}{6}\left(4 + 2(-1)^{\beta+(N-1)/2} + (-1)^\beta\left(\frac{2}{N}\right) - (-1)^{(N-1)/2}\left(\frac{2}{N}\right)\right)$$
$$\times\left(3^{\beta+1} + \left(\frac{N}{3}\right)\right)F_6(N).$$

If $n \equiv 2 \pmod 4$ we have

$$N(f;n) = 4 \cdot 2 \cdot \frac{1}{4}\left(3^{\beta+1} + \left(\frac{2N}{3}\right)\right)F_6(N) = 2\left(3^{\beta+1} - \left(\frac{N}{3}\right)\right)F_6(N).$$

If $n \equiv 0 \pmod 4$ we have

$$N(f;n) = 4 \cdot \frac{1}{3}\left(2^\alpha - (-1)^\alpha\left(\frac{-2}{3^\beta N}\right)\right)\frac{1}{4}\left(3^{\beta+1} + \left(\frac{2^\alpha N}{3}\right)\right)F_6(N)$$
$$= \frac{1}{3}\left(2^\alpha - (-1)^{\alpha+(N-1)/2}\left(\frac{2}{N}\right)\right)\left(3^{\beta+1} + (-1)^\alpha\left(\frac{N}{3}\right)\right)F_6(N).$$

This completes the proof of Theorem 1.14. □

Proof of Theorem 1.15. Let $f := x^2 + 5y^2 + 2z^2 + 2zt + 3t^2$. Using the value of $N_{p^k}(f;p^\theta K)$ given in [11, Theorem 1.7], we find that

$$d_p(f;p^\theta K) = \begin{cases} 1 & \text{if } p=2,\ \theta=0, \\ 2 - \dfrac{3}{2^\theta} & \text{if } p=2,\ \theta \ge 1, \\ \dfrac{6}{5^{\theta+1}} & \text{if } p=5, \\ (p+1)\left(\dfrac{1}{p} - \dfrac{1}{p^{\theta+2}}\right) & \text{if } p \ne 2,5; \end{cases}$$

$$d_p(f;1) = \begin{cases} 1 & \text{if } p=2, \\ \dfrac{6}{5} & \text{if } p=5, \\ (p+1)\left(\dfrac{1}{p} - \dfrac{1}{p^2}\right) & \text{if } p \ne 2,5; \end{cases}$$

$$p^\theta \frac{d_p(f;p^\theta K)}{d_p(f;1)} = \begin{cases} 1 & \text{if } p=2,\ \theta=0, \\ 2^{\theta+1} - 3 & \text{if } p=2,\ \theta \ge 1, \\ 1 & \text{if } p=5, \\ \dfrac{p^{\theta+1}-1}{p-1} & \text{if } p \ne 2,5. \end{cases}$$

Let $n \in \mathbb{N}$. Let $n = 2^\alpha 5^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N,10)=1$. Then, as $N(f;1) = 2$, the least positive integer represented by the form f is $\ell = 1$ and by (4.6) we have

$$N(f;n) = 2n \prod_p \frac{d_p(f;n)}{d_p(f;1)}.$$

If $n \equiv 1 \pmod 2$ we have

$$N(f;n) = 2 \prod_{p^\gamma \| N} \frac{p^{\gamma+1}-1}{p-1} = 2\sigma(N).$$

If $n \equiv 0 \pmod 2$ we have

$$N(f;n) = 2(2^{\alpha+1}-3) \prod_{p^\gamma \| N} \frac{p^{\gamma+1}-1}{p-1} = 2(2^{\alpha+1}-3)\sigma(N).$$

This completes the proof of Theorem 1.15. □

Proof of Theorem 1.16. Let $f := x^2+xy+y^2+3z^2+3zt+12t^2$. Using the value of $N_{p^k}(f;p^\theta K)$ given in [11, Theorem 1.8], we find that

$$d_p(f;p^\theta K) = \begin{cases} 1+\left(\dfrac{K}{3}\right) & \text{if } p=3,\ \ \theta=0, \\ \dfrac{1}{2}\left(1-(-1)^\theta\dfrac{5}{3^\theta}\right) & \text{if } p=3,\ \ \theta\geq 1, \\ 1+\left(\dfrac{K}{5}\right)\dfrac{1}{5^{\theta+1}} & \text{if } p=5, \\ \dfrac{\left(p^2-\left(\dfrac{5}{p}\right)\right)\left(p^{\theta+1}-\left(\dfrac{5}{p}\right)^{\theta+1}\right)}{p^{\theta+2}\left(p-\left(\dfrac{5}{p}\right)\right)} & \text{if } p\neq 3,5; \end{cases}$$

$$d_p(f;1) = \begin{cases} 2 & \text{if } p=3, \\ \dfrac{6}{5} & \text{if } p=5, \\ \dfrac{p^2-\left(\dfrac{5}{p}\right)}{p^2} & \text{if } p\neq 3,5; \end{cases}$$

$$p^\theta\frac{d_p(f;p^\theta K)}{d_p(f;1)} = \begin{cases} \dfrac{1}{2}\left(1+\left(\dfrac{K}{3}\right)\right) & \text{if } p=3,\ \ \theta=0, \\ \dfrac{1}{4}(3^\theta-(-1)^\theta 5) & \text{if } p=3,\ \ \theta\geq 1, \\ \dfrac{1}{6}\left(5^{\theta+1}+\left(\dfrac{K}{5}\right)\right) & \text{if } p=5, \\ \dfrac{p^{\theta+1}-\left(\dfrac{5}{p}\right)^{\theta+1}}{p-\left(\dfrac{5}{p}\right)} & \text{if } p\neq 3,5. \end{cases}$$

Let $n \in \mathbb{N}$. Let $n = 3^\alpha 5^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 15) = 1$. Then, as $N(f;1) = 6$, the least positive integer represented by the form f is $\ell = 1$ and by (4.6) we have

$$N(f;n) = 6n \prod_p \frac{d_p(f;n)}{d_p(f;1)}.$$

If $n \equiv 0 \pmod 3$ we have $\alpha \geq 1$ and

$$\begin{aligned} N(f;n) &= 6 \cdot \frac{1}{4}(3^\alpha - (-1)^\alpha 5)\frac{1}{6}\left(5^{\beta+1} + \left(\frac{3^\alpha N}{5}\right)\right) \prod_{p^\gamma \| N} \frac{p^{\gamma+1} - \left(\frac{5}{p}\right)^{\gamma+1}}{p - \left(\frac{5}{p}\right)} \\ &= \frac{1}{4}(3^\alpha - (-1)^\alpha 5)\left(5^{\beta+1} + (-1)^\alpha \left(\frac{N}{5}\right)\right) F_5(N). \end{aligned}$$

If $n \equiv 1 \pmod 3$ we have $\alpha = 0$ and $5^\beta N \equiv 1 \pmod 3$ so $(\frac{5^\beta N}{3}) = 1$, and thus

$$\begin{aligned} N(f;n) &= 6 \cdot \frac{1}{2}\left(1 + \left(\frac{5^\beta N}{3}\right)\right) \frac{1}{6}\left(5^{\beta+1} + \left(\frac{N}{5}\right)\right) \prod_{p^\gamma \| N} \frac{p^{\gamma+1} - \left(\frac{5}{p}\right)^{\gamma+1}}{p - \left(\frac{5}{p}\right)} \\ &= \left(5^{\beta+1} + \left(\frac{N}{5}\right)\right) F_5(N). \end{aligned}$$

If $n \equiv 2 \pmod 3$ we have $\alpha = 0$ and $5^\beta N \equiv 2 \pmod 3$ so $(\frac{5^\beta N}{3}) = -1$, and thus

$$N(f;n) = 6 \cdot \frac{1}{2}\left(1 + \left(\frac{5^\beta N}{3}\right)\right) \frac{1}{6}\left(5^{\beta+1} + \left(\frac{N}{5}\right)\right) \prod_{p^\gamma \| N} \frac{p^{\gamma+1} - \left(\frac{5}{p}\right)^{\gamma+1}}{p - \left(\frac{5}{p}\right)} = 0.$$

This completes the proof of Theorem 1.16. □

Proof of Theorem 1.17. Let $f := 2x^2 + xy + 2y^2 + 3z^2 + 3zt + 3t^2$. Using the value of $N_{p^k}(f;p^\theta K)$ given in [11, Theorem 1.9], we find that

$$d_p(f;p^\theta K) = \begin{cases} 1 - \left(\frac{K}{3}\right) & \text{if } p = 3, \ \ \theta = 0, \\ \frac{1}{2}\left(1 - (-1)^\theta \frac{5}{3^\theta}\right) & \text{if } p = 3, \ \ \theta \geq 1, \\ 1 + \left(\frac{K}{5}\right) \frac{1}{5^{\theta+1}} & \text{if } p = 5, \\ \frac{\left(p^2 - \left(\frac{5}{p}\right)\right)\left(p^{\theta+1} - \left(\frac{5}{p}\right)^{\theta+1}\right)}{p^{\theta+2}\left(p - \left(\frac{5}{p}\right)\right)} & \text{if } p \neq 3, 5; \end{cases}$$

$$d_p(f;2) = \begin{cases} \dfrac{5}{8} & \text{if } p = 2, \\ 2 & \text{if } p = 3, \\ \dfrac{4}{5} & \text{if } p = 5, \\ \dfrac{p^2 - \left(\dfrac{5}{p}\right)}{p^2} & \text{if } p \neq 2, 3, 5; \end{cases}$$

$$p^\theta \frac{d_p(f; p^\theta K)}{d_p(f;2)} = \begin{cases} \dfrac{2}{3}(2^{\theta+1} + (-1)^\theta) & \text{if } p = 2, \\ \dfrac{1}{2}\left(1 - \left(\dfrac{K}{3}\right)\right) & \text{if } p = 3, \ \ \theta = 0, \\ \dfrac{1}{4}(3^\theta - (-1)^\theta 5) & \text{if } p = 3, \ \ \theta \geq 1, \\ \dfrac{1}{4}\left(5^{\theta+1} + \left(\dfrac{K}{5}\right)\right) & \text{if } p = 5, \\ \dfrac{p^{\theta+1} - \left(\dfrac{5}{p}\right)^{\theta+1}}{p - \left(\dfrac{5}{p}\right)} & \text{if } p \neq 2, 3, 5. \end{cases}$$

Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta 5^\gamma N$, where $\alpha, \beta, \gamma \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 30) = 1$. Then, as $N(f;1) = 0$ and $N(f;2) = 4$, the least positive integer represented by the form f is $\ell = 2$ and by (4.6) we have

$$N(f;n) = 2n \prod_p \frac{d_p(f;n)}{d_p(f;2)}.$$

If $n \equiv 0 \pmod 3$ then $\beta \geq 1$ and

$$\begin{aligned} N(f;n) &= 2 \cdot \frac{2}{3}(2^{\alpha+1} + (-1)^\alpha)\frac{1}{4}(3^\beta - (-1)^\beta 5) \\ &\quad \times \frac{1}{4}\left(5^{\gamma+1} + \left(\frac{2^\alpha 3^\beta N}{5}\right)\right) \prod_{p^\delta \| N} \frac{p^{\delta+1} - \left(\dfrac{5}{p}\right)^{\delta+1}}{p - \left(\dfrac{5}{p}\right)} \\ &= \frac{1}{12}(2^{\alpha+1} + (-1)^\alpha)(3^\beta - (-1)^\beta 5)\left(5^{\gamma+1} + (-1)^{\alpha+\beta}\left(\frac{N}{5}\right)\right) F_5(N). \end{aligned}$$

If $n \equiv 1 \pmod 3$ then $\beta = 0$ and $2^\alpha 5^\gamma N \equiv 1 \pmod 3$ so

$$3^\beta \frac{d_3(f; 2^\alpha 3^\beta 5^\gamma N)}{d_3(f;2)} = \frac{d_3(f; 2^\alpha 5^\gamma N)}{d_3(f;2)} = \frac{1}{2}\left(1 - \left(\frac{2^\alpha 5^\gamma N}{3}\right)\right) = 0$$

and thus

$$N(f;n) = 0.$$

If $n \equiv 2 \pmod 3$ then $\beta = 0$ and $2^\alpha 5^\gamma N \equiv 2 \pmod 3$ so

$$N(f;n) = 2 \cdot \frac{2}{3}(2^{\alpha+1} + (-1)^\alpha)\frac{1}{2}\left(1 - \left(\frac{2^\alpha 5^\gamma N}{3}\right)\right)$$

$$\times \frac{1}{4}\left(5^{\gamma+1} + \left(\frac{2^\alpha N}{5}\right)\right) \prod_{p^\delta \| N} \frac{p^{\delta+1} - \left(\frac{5}{p}\right)^{\delta+1}}{p - \left(\frac{5}{p}\right)}$$

$$= \frac{1}{3}(2^{\alpha+1} + (-1)^\alpha)\left(5^{\gamma+1} + (-1)^\alpha \left(\frac{N}{5}\right)\right) F_5(N).$$

This completes the proof of Theorem 1.17. □

Proof of Theorem 1.18. Let $f := x^2 + 3y^2 + 2z^2 + 2zt + 5t^2$. Using the value of $N_{p^k}(f;p^\theta K)$ given in [11, Theorem 1.10], we obtain

$$d_p(f;p^\theta K) = \begin{cases} \dfrac{2^{\theta+1} + (-1)^{\theta+(K-1)/2}}{2^{\theta+1}} & \text{if } p = 2, \\ \dfrac{2}{3} & \text{if } p = 3,\ \theta = 0, \\ \dfrac{5}{3} - \dfrac{(-1)^\theta}{3^\theta}\left(\dfrac{K}{3}\right) & \text{if } p = 3,\ \theta \geq 1, \\ \dfrac{\left(p^2 - \left(\frac{3}{p}\right)\right)\left(p^{\theta+1} - \left(\frac{3}{p}\right)^{\theta+1}\right)}{p^{\theta+2}\left(p - \left(\frac{3}{p}\right)\right)} & \text{if } p \neq 2, 3; \end{cases}$$

$$d_p(f;1) = \begin{cases} \dfrac{3}{2} & \text{if } p = 2, \\ \dfrac{2}{3} & \text{if } p = 3, \\ \dfrac{p^2 - \left(\frac{3}{p}\right)}{p^2} & \text{if } p \neq 2, 3; \end{cases}$$

$$p^\theta \frac{d_p(f;p^\theta K)}{d_p(f;1)} = \begin{cases} \dfrac{1}{3}(2^{\theta+1} + (-1)^{\theta+(K-1)/2}) & \text{if } p = 2, \\ 1 & \text{if } p = 3,\ \theta = 0, \\ \dfrac{1}{2}\left(5 \cdot 3^\theta - (-1)^\theta \left(\dfrac{K}{3}\right) 3\right) & \text{if } p = 3,\ \theta \geq 1, \\ \dfrac{p^{\theta+1} - \left(\frac{3}{p}\right)^{\theta+1}}{p - \left(\frac{3}{p}\right)} & \text{if } p \neq 2, 3. \end{cases}$$

Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 6) = 1$. Then, as $N(f; 1) = 2$, the least positive integer represented by the form f is $\ell = 1$ and by (4.6) we have

$$N(f;n) = 2n \prod_p \frac{d_p(f;n)}{d_p(f;1)}.$$

If $n \not\equiv 0 \pmod 3$ we have $\beta = 0$ and

$$\begin{aligned} N(f;n) &= 2 \cdot \frac{1}{3}(2^{\alpha+1} + (-1)^{\alpha+(N-1)/2}) \prod_{p^\gamma \| N} \frac{p^{\gamma+1} - \left(\dfrac{3}{p}\right)^{\gamma+1}}{p - \left(\dfrac{3}{p}\right)} \\ &= \frac{2}{3}(2^{\alpha+1} + (-1)^{\alpha+(N-1)/2}) F_3(N). \end{aligned}$$

If $n \equiv 0 \pmod 3$ we have $\beta \geq 1$ and by (4.6) we have

$$\begin{aligned} N(f;n) &= \frac{2}{3}(2^{\alpha+1} + (-1)^{\alpha+(3^\beta N-1)/2}) \\ &\quad \times \frac{1}{2}\left(5 \cdot 3^\beta - (-1)^\beta \left(\frac{2^\alpha N}{3}\right) 3\right) \prod_{p^\gamma \| N} \frac{p^{\gamma+1} - \left(\dfrac{3}{p}\right)^{\gamma+1}}{p - \left(\dfrac{3}{p}\right)} \\ &= \frac{1}{3}(2^{\alpha+1} + (-1)^{\alpha+\beta+(N-1)/2}) \left(5 \cdot 3^\beta - (-1)^{\alpha+\beta} \left(\frac{N}{3}\right) 3\right) F_3(N). \end{aligned}$$

This completes the proof of Theorem 1.18. □

Proof of Theorem 1.19. Let $f := x^2 + 2y^2 + 3z^2 + 2zt + 5t^2$. Using the value of $N_{p^k}(f; p^\theta K)$ given in [11, Theorem 1.11], we obtain

$$d_p(f; p^\theta K) = \begin{cases} 1 & \text{if } p = 2,\ \theta = 0, \\ \dfrac{2^\theta - (-1)^{(K-1)/2}}{2^\theta} & \text{if } p = 2,\ \theta \geq 1, \\ \dfrac{7^{\theta+1} + (-1)^\theta \left(\dfrac{K}{7}\right)}{7^{\theta+1}} & \text{if } p = 7, \\ \dfrac{\left(p^2 - \left(\dfrac{7}{p}\right)\right)\left(p^{\theta+1} - \left(\dfrac{7}{p}\right)^{\theta+1}\right)}{p^{\theta+2}\left(p - \left(\dfrac{7}{p}\right)\right)} & \text{if } p \neq 2, 7; \end{cases}$$

$$d_p(f;1)=\begin{cases}1 & \text{if } p=2,\\ \dfrac{8}{7} & \text{if } p=7,\\ \dfrac{p^2-\left(\dfrac{7}{p}\right)}{p^2} & \text{if } p\neq 2,7;\end{cases}$$

$$p^\theta\frac{d_p(f;p^\theta K)}{d_p(f;1)}=\begin{cases}1 & \text{if } p=2,\ \ \theta=0,\\ 2^\theta-(-1)^{(K-1)/2} & \text{if } p=2,\ \ \theta\geq 1,\\ \dfrac{1}{8}\left(7^{\theta+1}+(-1)^\theta\left(\dfrac{K}{7}\right)\right) & \text{if } p=7,\\ \dfrac{p^{\theta+1}-\left(\dfrac{7}{p}\right)^{\theta+1}}{p-\left(\dfrac{7}{p}\right)} & \text{if } p\neq 2,7.\end{cases}$$

Let $n\in\mathbb{N}$. Let $n=2^\alpha 7^\beta N$, where $\alpha,\beta\in\mathbb{N}_0$, $N\in\mathbb{N}$ and $(N,14)=1$. Then, as $N(f;1)=2$, the least positive integer represented by the form f is $\ell=1$ and by (4.6) we have

$$N(f;n)=2n\prod_p\frac{d_p(f;n)}{d_p(f;1)}.$$

If $n\equiv 1 \pmod 2$ we have $\alpha=0$, $n=7^\beta N$ and

$$N(f;n)=\frac{1}{4}\left(7^{\beta+1}+(-1)^\beta\left(\frac{N}{7}\right)\right)F_7(N).$$

If $n\equiv 0 \pmod 2$ we have $\alpha\geq 1$ and

$$\begin{aligned}N(f;n)&=2(2^\alpha-(-1)^{(7^\beta N-1)/2})\frac{1}{8}\left(7^{\beta+1}+(-1)^\beta\left(\frac{2^\alpha N}{7}\right)\right)F_7(N)\\&=\frac{1}{4}(2^\alpha-(-1)^{\beta+(N-1)/2})\left(7^{\beta+1}+(-1)^\beta\left(\frac{N}{7}\right)\right)F_7(N).\end{aligned}$$

This completes the proof of Theorem 1.19. □

Proof of Theorem 1.20. Let $f:=x^2+xy+3y^2+2z^2+2zt+6t^2$. Using the method in [11], we find that

$$N_{p^k}(f;p^\theta K)=\begin{cases}3\cdot 2^{3k-\theta-1} & \text{if } p=2,\\ 2(11^{3k}-6\cdot 11^{3k-\theta-1}) & \text{if } p=11,\\ (p+1)(p^{3k-1}-p^{3k-\theta-2}) & \text{if } p\neq 2,11.\end{cases}$$

Hence

$$d_p(f;p^\theta K) = \lim_{k\to\infty} \frac{N_{p^k}(f;p^\theta K)}{p^{3k}} = \begin{cases} \dfrac{3}{2^{\theta+1}} & \text{if } p=2,\\ \dfrac{2(11^{\theta+1}-6)}{11^{\theta+1}} & \text{if } p=11,\\ \dfrac{(p+1)(p^{\theta+1}-1)}{p^{\theta+2}} & \text{if } p\neq 2,11; \end{cases}$$

$$d_p(f;1) = \begin{cases} \dfrac{3}{2} & \text{if } p=2,\\ \dfrac{10}{11} & \text{if } p=11,\\ \dfrac{(p+1)(p-1)}{p^2} & \text{if } p\neq 2,11; \end{cases}$$

$$p^\theta \frac{d_p(f;p^\theta K)}{d_p(f;1)} = \begin{cases} 1 & \text{if } p=2,\\ \dfrac{1}{5}(11^{\theta+1}-6) & \text{if } p=11,\\ \dfrac{p^{\theta+1}-1}{p-1} & \text{if } p\neq 2,11. \end{cases}$$

Let $n\in\mathbb{N}$. Let $n = 2^\alpha 11^\beta N$, where $\alpha,\beta\in\mathbb{N}_0$, $N\in\mathbb{N}$ and $(N,22)=1$. Then, as $N(f;1)=2$, the least positive integer represented by the form f is $\ell=1$ and by (4.6) we have

$$N(f;n) = 2n\prod_p \frac{d_p(f;n)}{d_p(f;1)} = \frac{2}{5}(11^{\beta+1}-6)\prod_{p^\gamma\| N}\frac{p^{\gamma+1}-1}{p-1} = \frac{2}{5}(11^{\beta+1}-6)\sigma(N),$$

as asserted. □

Proof of Theorem 1.21. Let $f := x^2+xy+2y^2+7z^2+7zt+7t^2$. Using the method in [11], we find that

$$N_{p^k}(f;p^\theta K) = \begin{cases} 3^{3k} - (-1)^\theta \left(\dfrac{K}{3}\right) 3^{3k-\theta-1} & \text{if } p=3,\\ 7^{3k} + \left(\dfrac{K}{7}\right) 7^{3k-\theta} & \text{if } p=7,\\ \dfrac{p^{3k-\theta-2}\left(p^2-\left(\dfrac{21}{p}\right)\right)\left(p^{\theta+1}-\left(\dfrac{21}{p}\right)^{\theta+1}\right)}{p-\left(\dfrac{21}{p}\right)} & \text{if } p\neq 3,7. \end{cases}$$

Hence

$$d_p(f;p^\theta K) = \lim_{k\to\infty} \frac{N_{p^k}(f;p^\theta K)}{p^{3k}}$$

$$= \begin{cases} \dfrac{3^{\theta+1} - (-1)^\theta \left(\dfrac{K}{3}\right)}{3^{\theta+1}} & \text{if } p = 3, \\ \dfrac{7^\theta + \left(\dfrac{K}{7}\right)}{7^\theta} & \text{if } p = 7, \\ \dfrac{\left(p^2 - \left(\dfrac{21}{p}\right)\right)\left(p^{\theta+1} - \left(\dfrac{21}{p}\right)^{\theta+1}\right)}{p^{\theta+2}\left(p - \left(\dfrac{21}{p}\right)\right)} & \text{if } p \neq 3, 7; \end{cases}$$

$$d_p(f;1) = \begin{cases} \dfrac{2}{3} & \text{if } p = 3, \\ 2 & \text{if } p = 7, \\ \dfrac{p^2 - \left(\dfrac{21}{p}\right)}{p^2} & \text{if } p \neq 3, 7; \end{cases}$$

$$p^\theta \frac{d_p(f;p^\theta K)}{d_p(f;1)} = \begin{cases} \dfrac{1}{2}\left(3^{\theta+1} - (-1)^\theta \left(\dfrac{K}{3}\right)\right) & \text{if } p = 3, \\ \dfrac{1}{2}\left(7^\theta + \left(\dfrac{K}{7}\right)\right) & \text{if } p = 7, \\ \dfrac{p^{\theta+1} - \left(\dfrac{21}{p}\right)^{\theta+1}}{p - \left(\dfrac{21}{p}\right)} & \text{if } p \neq 3, 7. \end{cases}$$

Let $n \in \mathbb{N}$. Let $n = 3^\alpha 7^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 21) = 1$. Then, as $N(f;1) = 2$, the least positive integer represented by the form f is $\ell = 1$ and by (4.6) we have

$$N(f;n) = 2n \prod_p \frac{d_p(f;n)}{d_p(f;1)}$$

$$= \left(3^{\alpha+1} - (-1)^\alpha \left(\frac{7^\beta N}{3}\right)\right) \frac{1}{2}\left(7^\beta + \left(\frac{3^\alpha N}{7}\right)\right) \prod_{p^\gamma \| N} \frac{p^{\gamma+1} - \left(\dfrac{21}{p}\right)^{\gamma+1}}{p - \left(\dfrac{21}{p}\right)}$$

$$= \frac{1}{2}\left(3^{\alpha+1} - (-1)^\alpha \left(\frac{N}{3}\right)\right)\left(7^\beta + (-1)^\alpha \left(\frac{N}{7}\right)\right) F_{21}(N),$$

completing the proof of Theorem 1.21. □

Proof of Theorem 1.22. Let $f := x^2+xy+4y^2+5z^2+5zt+5t^2$. Using the value of $N_{p^k}(f;p^\theta K)$ given in [11, Theorem 1.13], we find that

$$d_p(f;p^\theta K) = \lim_{k\to\infty} \frac{N_{p^k}(f;p^\theta K)}{p^{3k}} = \begin{cases} \dfrac{3^{\theta+2}-(-1)^\theta 5}{2\cdot 3^{\theta+1}} & \text{if } p=3, \\ \dfrac{5^\theta + \left(\dfrac{K}{5}\right)}{5^\theta} & \text{if } p=5, \\ \dfrac{\left(p^2-\left(\dfrac{5}{p}\right)\right)\left(p^{\theta+1}-\left(\dfrac{5}{p}\right)^{\theta+1}\right)}{p^{\theta+2}\left(p-\left(\dfrac{5}{p}\right)\right)} & \text{if } p\neq 3,5; \end{cases}$$

$$d_p(f;1) = \begin{cases} \dfrac{2}{3} & \text{if } p=3, \\ 2 & \text{if } p=5, \\ \dfrac{p^2-\left(\dfrac{5}{p}\right)}{p^2} & \text{if } p\neq 3,5; \end{cases}$$

$$p^\theta \frac{d_p(f;p^\theta K)}{d_p(f;1)} = \begin{cases} \dfrac{3^{\theta+2}-(-1)^\theta 5}{4} & \text{if } p=3, \\ \dfrac{5^\theta + \left(\dfrac{K}{5}\right)}{2} & \text{if } p=5, \\ \dfrac{p^{\theta+1}-\left(\dfrac{5}{p}\right)^{\theta+1}}{p-\left(\dfrac{5}{p}\right)} & \text{if } p\neq 3,5. \end{cases}$$

Let $n\in\mathbb{N}$. Let $n=3^\alpha 5^\beta N$, where $\alpha,\beta\in\mathbb{N}_0$, $N\in\mathbb{N}$ and $(N,15)=1$. Then, as $N(f;1)=2$, the least positive integer represented by the form f is $\ell=1$ and by (4.6) we have

$$\begin{aligned} N(f;n) = 2n\prod_p \frac{d_p(f;n)}{d_p(f;1)} &= \frac{1}{2}(3^{\alpha+2}-(-1)^\alpha 5) \\ &\quad \times \frac{1}{2}\left(5^\beta + \left(\frac{3^\alpha N}{5}\right)\right) \prod_{p^\gamma \| N} \frac{p^{\gamma+1}-\left(\dfrac{5}{p}\right)^{\gamma+1}}{p-\left(\dfrac{5}{p}\right)} \\ &= \frac{1}{4}(3^{\alpha+2}-(-1)^\alpha 5)\left(5^\beta + (-1)^\alpha\left(\frac{N}{5}\right)\right) F_5(N), \end{aligned}$$

completing the proof of Theorem 1.22. □

Proof of Theorem 1.23. Let $f := 2x^2 + xy + 2y^2 + 5z^2 + 5zt + 5t^2$. Using the value of $N_{p^k}(f; p^\theta K)$ given in [11, Theorem 1.14], we find that

$$d_p(f; p^\theta K) = \lim_{k\to\infty} \frac{N_{p^k}(f; p^\theta K)}{p^{3k}} = \begin{cases} \dfrac{3^{\theta+1} + (-1)^\theta 5}{2\cdot 3^{\theta+1}} & \text{if } p = 3, \\ \dfrac{5^\theta - \left(\dfrac{K}{5}\right)}{5^\theta} & \text{if } p = 5, \\ \dfrac{\left(p^2 - \left(\dfrac{5}{p}\right)\right)\left(p^{\theta+1} - \left(\dfrac{5}{p}\right)^{\theta+1}\right)}{p^{\theta+2}\left(p - \left(\dfrac{5}{p}\right)\right)} & \text{if } p \neq 3, 5; \end{cases}$$

$$d_p(f; 2) = \begin{cases} \dfrac{5}{8} & \text{if } p = 2, \\ \dfrac{4}{3} & \text{if } p = 3, \\ 2 & \text{if } p = 5, \\ \dfrac{p^2 - \left(\dfrac{5}{p}\right)}{p^2} & \text{if } p \neq 2, 3, 5; \end{cases}$$

$$p^\theta \frac{d_p(f; p^\theta K)}{d_p(f; 2)} = \begin{cases} \dfrac{2}{3}(2^{\theta+1} + (-1)^\theta) & \text{if } p = 2, \\ \dfrac{1}{8}(3^{\theta+1} + (-1)^\theta 5) & \text{if } p = 3, \\ \dfrac{1}{2}\left(5^\theta - \left(\dfrac{K}{5}\right)\right) & \text{if } p = 5, \\ \dfrac{p^{\theta+1} - \left(\dfrac{5}{p}\right)^{\theta+1}}{p - \left(\dfrac{5}{p}\right)} & \text{if } p \neq 2, 3, 5. \end{cases}$$

Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta 5^\gamma N$, where $\alpha, \beta, \gamma \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 30) = 1$. Then, as $N(f; 1) = 0$ and $N(f; 2) = 4$, the least positive integer represented by the form f is $\ell = 2$ and by (4.6) we have

$$\begin{aligned} N(f; n) &= 2n \prod_p \frac{d_p(f; n)}{d_p(f; 2)} \\ &= 2 \cdot \frac{2}{3}(2^{\alpha+1} + (-1)^\alpha)\frac{1}{8}(3^{\beta+1} + (-1)^\beta 5)\frac{1}{2}\left(5^\gamma - \left(\frac{2^\alpha 3^\beta N}{5}\right)\right) \end{aligned}$$

$$\times \prod_{p^\delta \| N} \frac{p^{\delta+1} - \left(\frac{5}{p}\right)^{\delta+1}}{p - \left(\frac{5}{p}\right)}$$

$$= \frac{1}{12}(2^{\alpha+1} + (-1)^\alpha)(3^{\beta+1} + (-1)^\beta 5)\left(5^\gamma - (-1)^{\alpha+\beta}\left(\frac{N}{5}\right)\right) F_5(N),$$

completing the proof of Theorem 1.23. □

Proof of Theorem 1.24. Let $f := 2x^2 + 2xy + 5y^2 + 3z^2 + 3t^2$. Using the value of $N_{p^k}(f; p^\theta K)$ given in [11, Theorem 1.15], we find that

$$d_p(f; p^\theta K) = \lim_{k\to\infty} \frac{N_{p^k}(f; p^\theta K)}{p^{3k}} = \begin{cases} 1 & \text{if } p = 2, \theta = 0, \\ \dfrac{2^{\theta+1} - 3}{2^\theta} & \text{if } p = 2, \theta \geq 1, \\ 1 - \left(\dfrac{K}{3}\right) & \text{if } p = 3, \theta = 0, \\ \dfrac{4}{3^\theta} & \text{if } p = 3, \theta \geq 1, \\ \dfrac{(p+1)(p^{\theta+1} - 1)}{p^{\theta+2}} & \text{if } p \neq 2, 3; \end{cases}$$

$$d_p(f; 2) = \begin{cases} \dfrac{1}{2} & \text{if } p = 2, \\ 2 & \text{if } p = 3, \\ \dfrac{(p+1)(p-1)}{p^2} & \text{if } p \neq 2, 3; \end{cases}$$

$$p^\theta \frac{d_p(f; p^\theta K)}{d_p(f; 2)} = \begin{cases} 2 & \text{if } p = 2, \theta = 0, \\ 2(2^{\theta+1} - 3) & \text{if } p = 2, \theta \geq 1, \\ \dfrac{1 - \left(\frac{K}{3}\right)}{2} & \text{if } p = 3, \theta = 0, \\ 2 & \text{if } p = 3, \theta \geq 1, \\ \dfrac{p^{\theta+1} - 1}{p - 1} & \text{if } p \neq 2, 3. \end{cases}$$

Let $n \in \mathbb{N}$. Let $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 6) = 1$. Then, as $N(f; 1) = 0$ and $N(f; 2) = 2$, the least positive integer represented by the form f is $\ell = 2$ and by (4.6) we have

$$N(f; n) = n \prod_p \frac{d_p(f; n)}{d_p(f; 2)}.$$

If $n \equiv 1$ or $5 \pmod 6$ we have $\alpha = \beta = 0$, $n = N$ and

$$N(f;n) = \left(1 - \left(\frac{N}{3}\right)\right) \prod_{p^\gamma \| N} \frac{p^{\gamma+1}-1}{p-1} = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 6, \\ 2\sigma(N) & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

If $n \equiv 2$ or $4 \pmod 6$ we have $\alpha \geq 1, \beta = 0$, $n = 2^\alpha N$ and

$$\begin{aligned} N(f;n) &= 2(2^{\alpha+1} - 3)\frac{1}{2}\left(1 - \left(\frac{2^\alpha N}{3}\right)\right) \prod_{p^\gamma \| N} \frac{p^{\gamma+1}-1}{p-1} \\ &= (2^{\alpha+1} - 3)\left(1 - \left(\frac{n}{3}\right)\right)\sigma(N) \\ &= \begin{cases} 0 & \text{if } n \equiv 4 \pmod 6, \\ 2(2^{\alpha+1}-3)\sigma(N) & \text{if } n \equiv 2 \pmod 6. \end{cases} \end{aligned}$$

If $n \equiv 3 \pmod 6$ we have $\alpha = 0$, $\beta \geq 1$, $n = 3^\beta N$ and

$$N(f;n) = 4 \prod_{p^\gamma \| N} \frac{p^{\gamma+1}-1}{p-1} = 4\sigma(N).$$

If $n \equiv 0 \pmod 6$ we have $\alpha \geq 1$, $\beta \geq 1$ and

$$N(f;n) = 2(2^{\alpha+1} - 3)2 \prod_{p^\gamma \| N} \frac{p^{\gamma+1}-1}{p-1} = 4(2^{\alpha+1} - 3)\sigma(N).$$

This completes the proof of Theorem 1.24. □

5. The Form $3x^2 + 3xy + 6y^2 + 7z^2 + 7zt + 7t^2$

In this section we determine a formula for the representation number $N(3, 3, 6, 7, 7, 7; n)$ of the quaternary quadratic form

$$3x^2 + 3xy + 6y^2 + 7z^2 + 7zt + 7t^2 \tag{5.1}$$

valid for all $n \in \mathbb{N}$, see Theorem 5.1. The discriminant of the form (5.1) is 9261 so that it lies outside the range of Nipp's table and thus we do not know if it belongs to a genus containing a single form class or not. Without knowing this we cannot use Siegel's mass formula to evaluate $N(3, 3, 6, 7, 7, 7; n)$. Thus we proceed differently. We use modular forms to prove Theorem 5.1.

Later in this section we show that

$$N(3, 3, 6, 7, 7, 7; n) = \frac{1}{2}A(n) - \frac{1}{2}B(n) + \frac{1}{2}C(n) - \frac{1}{2}D(n), \quad n \in \mathbb{N}, \tag{5.2}$$

where

$$A(n) := \sum_{d|n} \left(\frac{21}{n/d}\right) d, \tag{5.3}$$

$$B(n) := \sum_{d|n} \left(\frac{-3}{n/d}\right)\left(\frac{-7}{d}\right) d, \tag{5.4}$$

$$C(n) := \sum_{d|n} \left(\frac{-3}{d}\right)\left(\frac{-7}{n/d}\right) d, \tag{5.5}$$

$$D(n) := \sum_{d|n} \left(\frac{21}{d}\right) d. \tag{5.6}$$

(We note that this is a different usage of $A(n)$, $B(n)$, $C(n)$ and $D(n)$ from that in (3.12)–(3.15) but this will not cause confusion.) Letting $n = 3^{\alpha}7^{\beta}N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $(N, 21) = 1$, it is easy to show that

$$A(n) = 3^{\alpha}7^{\beta}F_{21}(N), \tag{5.7}$$

$$B(n) = (-1)^{\alpha}\left(\frac{N}{7}\right)3^{\alpha}F_{21}(N), \tag{5.8}$$

$$C(n) = (-1)^{\alpha}\left(\frac{N}{3}\right)7^{\beta}F_{21}(N), \tag{5.9}$$

$$D(n) = \left(\frac{N}{21}\right)F_{21}(N). \tag{5.10}$$

Using (5.7)–(5.10) in (5.2), we obtain the following theorem.

Theorem 5.1. *Let* $n \in \mathbb{N}$. *Let* $n = 3^{\alpha}7^{\beta}N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $(N, 21) = 1$. *Then*

$$N(3,3,6,7,7,7;n) = \frac{1}{2}\left(3^{\alpha} + (-1)^{\alpha}\left(\frac{N}{3}\right)\right)\left(7^{\beta} - (-1)^{\alpha}\left(\frac{N}{7}\right)\right)F_{21}(N).$$

Next we exhibit the interrelationship between the representation numbers $N(1,1,1,1,1,2;n)$, $N(1,1,1,3,3,6;n)$, $N(1,1,2,7,7,7;n)$ and $N(3,3,6,7,7,7;n)$. To do this, we use (5.7)–(5.10) in Theorems 1.8, 1.12 and 1.21 to obtain

$$N(1,1,1,1,1,2;n) = \frac{21}{2}A(n) + \frac{3}{2}B(n) - \frac{7}{2}C(n) - \frac{1}{2}D(n), \tag{5.11}$$

$$N(1,1,1,3,3,6;n) = \frac{7}{2}A(n) - \frac{1}{2}B(n) + \frac{7}{2}C(n) - \frac{1}{2}D(n), \tag{5.12}$$

$$N(1,1,2,7,7,7;n) = \frac{3}{2}A(n) + \frac{3}{2}B(n) - \frac{1}{2}C(n) - \frac{1}{2}D(n). \tag{5.13}$$

The formulae (5.2), (5.11), (5.12) and (5.13) give the required relationship.

Further, we express each of

$$\sum_{n=1}^{\infty} A(n)q^n, \quad \sum_{n=1}^{\infty} B(n)q^n, \quad \sum_{n=1}^{\infty} C(n)q^n, \quad \sum_{n=1}^{\infty} D(n)q^n,$$

in terms of theta functions. Solving (5.2), (5.11), (5.12) and (5.13) for $A(n)$, $B(n)$, $C(n)$ and $D(n)$, we obtain

$$\begin{aligned} A(n) &= \frac{1}{12}N(1,1,1,1,1,2;n) + \frac{1}{12}N(1,1,1,3,3,6;n) \\ &\quad - \frac{1}{12}N(1,1,2,7,7,7;n) - \frac{1}{12}N(3,3,6,7,7,7;n), \end{aligned} \tag{5.14}$$

$$\begin{aligned} B(n) &= -\frac{1}{12}N(1,1,1,1,1,2;n) + \frac{1}{12}N(1,1,1,3,3,6;n) \\ &\quad + \frac{7}{12}N(1,1,2,7,7,7;n) - \frac{7}{12}N(3,3,6,7,7,7;n), \end{aligned} \tag{5.15}$$

$$\begin{aligned} C(n) &= -\frac{1}{12}N(1,1,1,1,1,2;n) + \frac{1}{4}N(1,1,1,3,3,6;n) \\ &\quad + \frac{1}{12}N(1,1,2,7,7,7;n) - \frac{1}{4}N(3,3,6,7,7,7;n), \end{aligned} \tag{5.16}$$

$$\begin{aligned} D(n) &= \frac{1}{12}N(1,1,1,1,1,2;n) + \frac{1}{4}N(1,1,1,3,3,6;n) \\ &\quad - \frac{7}{12}N(1,1,2,7,7,7;n) - \frac{7}{4}N(3,3,6,7,7,7;n). \end{aligned} \tag{5.17}$$

For $q \in \mathbb{C}$ with $|q| < 1$, we recall the Borweins' theta function $a(q)$ from (3.2) and define the theta function $b(q)$ by

$$a(q) := \sum_{(x,y)\in\mathbb{Z}^2} q^{x^2+xy+y^2}, \quad b(q) := \sum_{(x,y)\in\mathbb{Z}^2} q^{x^2+xy+2y^2}, \tag{5.18}$$

so that

$$\sum_{n=0}^{\infty} N(1,1,1,1,1,2;n)q^n = a(q)b(q), \tag{5.19}$$

$$\sum_{n=0}^{\infty} N(1,1,1,3,3,6;n)q^n = a(q)b(q^3), \tag{5.20}$$

$$\sum_{n=0}^{\infty} N(1,1,2,7,7,7;n)q^n = a(q^7)b(q), \tag{5.21}$$

$$\sum_{n=0}^{\infty} N(3,3,6,7,7,7;n)q^n = a(q^7)b(q^3). \tag{5.22}$$

Summing (5.14)–(5.17) over $n \in \mathbb{N}_0$, we obtain the following result.

Theorem 5.2.

$$\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-3}{n/d}\right)\left(\frac{-7}{n/d}\right)d\right)q^n$$
$$=\frac{1}{12}a(q)b(q)+\frac{1}{12}a(q)b(q^3)-\frac{1}{12}a(q^7)b(q)-\frac{1}{12}a(q^7)b(q^3),$$

$$\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-3}{n/d}\right)\left(\frac{-7}{d}\right)d\right)q^n$$
$$=-\frac{1}{12}a(q)b(q)+\frac{1}{12}a(q)b(q^3)+\frac{7}{12}a(q^7)b(q)-\frac{7}{12}a(q^7)b(q^3),$$

$$\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-3}{d}\right)\left(\frac{-7}{n/d}\right)d\right)q^n$$
$$=-\frac{1}{12}a(q)b(q)+\frac{1}{4}a(q)b(q^3)+\frac{1}{12}a(q^7)b(q)-\frac{1}{4}a(q^7)b(q^3),$$

$$-2+\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-3}{d}\right)\left(\frac{-7}{d}\right)d\right)q^n$$
$$=\frac{1}{12}a(q)b(q)+\frac{1}{4}a(q)b(q^3)-\frac{7}{12}a(q^7)b(q)-\frac{7}{4}a(q^7)b(q^3).$$

We now give the proof of Theorem 5.1. It suffices to prove (5.2).

Proof of Theorem 5.1. The discriminant of the binary quadratic form $3x^2+3xy+6y^2$ is $\Delta=-63$ and the matrix A of this form is given by

$$3x^2+3xy+6y^2=\frac{1}{2}(x \quad y)A\begin{pmatrix}x\\ y\end{pmatrix},$$

that is

$$A:=\begin{pmatrix}6 & 3\\ 3 & 12\end{pmatrix}.$$

The level of the form $3x^2+3xy+6y^2$ is the least positive integer L such that LA^{-1} has integral entries with even diagonal entries. As

$$21A^{-1}=\begin{pmatrix}4 & -1\\ -1 & 2\end{pmatrix}$$

the level of the form $3x^2+3xy+6y^2$ is $L=21$. The character χ of the form $3x^2+3xy+6y^2$ is the unique Dirichlet character modulo the level 21 satisfying

$$\chi(p)=\left(\frac{\Delta}{p}\right)=\left(\frac{-63}{p}\right)$$

for every odd prime p that does not divide the level $L = 21$. Hence, for all primes $p \neq 2, 3, 7$, we have

$$\chi(p) = \left(\frac{-7}{p}\right) = \begin{cases} +1 & \text{if } p \equiv 1, 2, 4 \pmod 7, \\ -1 & \text{if } p \equiv 3, 5, 6 \pmod 7. \end{cases}$$

For all $n \in \mathbb{N}$, the character χ is given by

$$\chi(n) = \begin{cases} +1 & \text{if } n \equiv 1, 2, 4, 8, 11, 16 \pmod{21}, \\ -1 & \text{if } n \equiv 5, 10, 13, 17, 19, 20 \pmod{21}, \\ 0 & \text{if } n \equiv 0, 3, 6, 7, 9, 12, 14, 15, 18 \pmod{21}. \end{cases}$$

Thus the theta series corresponding to the form $3x^2 + 3xy + 6y^2$, namely,

$$\sum_{(x,y)\in\mathbb{Z}^2} q^{3x^2+3xy+6y^2}, \quad q = e^{2\pi iz}, \quad \mathrm{Im}(z) > 0,$$

is a modular form on $\Gamma_0(21)$ of weight 1 and character χ, see for example [15, Theorem 2.2; 20, Theorem 3.7]. Similarly, the theta series of the form $7z^2 + 7zt + 7t^2$, namely,

$$\sum_{(z,t)\in\mathbb{Z}^2} q^{7z^2+7zt+7t^2}, \quad q = e^{2\pi iz}, \quad \mathrm{Im}(z) > 0,$$

is a modular form on $\Gamma_0(21)$ of weight 1 and character ξ given by

$$\xi(n) = \begin{cases} +1 & \text{if } n \equiv 1, 4, 10, 13, 16, 19 \pmod{21}, \\ -1 & \text{if } n \equiv 2, 5, 8, 11, 17, 20 \pmod{21}, \\ 0 & \text{if } n \equiv 0, 3, 6, 7, 9, 12, 14, 15, 18 \pmod{21}. \end{cases}$$

Hence the theta series of the form (5.1), namely,

$$\sum_{(x,y,z,t)\in\mathbb{Z}^4} q^{3x^2+3xy+6y^2+7z^2+7zt+7t^2} = \sum_{(x,y)\in\mathbb{Z}^2} q^{3x^2+3xy+6y^2} \sum_{(z,t)\in\mathbb{Z}^2} q^{7z^2+7zt+7t^2}$$

is a modular form on $\Gamma_0(21)$ of weight 2 and character $\chi\xi$ given by

$$\chi\xi(n) = \begin{cases} +1 & \text{if } n \equiv 1, 4, 5, 16, 17, 20 \pmod{21}, \\ -1 & \text{if } n \equiv 2, 8, 10, 11, 13, 19 \pmod{21}, \\ 0 & \text{if } n \equiv 0, 3, 6, 7, 9, 12, 14, 15, 18 \pmod{21}. \end{cases}$$

Clearly we have

$$\chi\xi(n) = \left(\frac{21}{n}\right).$$

Let $M_2(\Gamma_0(21), (\frac{21}{*}))$ denote the complex space of modular forms of weight 2 for $\Gamma_0(21)$ with multiplier system $(\frac{21}{*})$ so that

$$\sum_{n=0}^{\infty} N(3,3,6,7,7,7;n)q^n = \sum_{(x,y,z,t)\in\mathbb{Z}^4} q^{3x^2+3xy+6y^2+7z^2+7zt+7t^2} \in M_2\left(\Gamma_0(21), \left(\frac{21}{*}\right)\right).$$

By the formula in [21, Sec. 6.3, p. 100] we have

$$\dim M_2\left(\Gamma_0(21), \left(\frac{21}{*}\right)\right) = 4.$$

Let $S_2\left(\Gamma_0(21), (\frac{21}{*})\right)$ denote the subspace of $M_2\left(\Gamma_0(21), (\frac{21}{*})\right)$ consisting of cusp forms. Using the formula given in [21, Sec. 6.3, p. 98], we deduce that

$$\dim S_2\left(\Gamma_0(21), \left(\frac{21}{*}\right)\right) = 0.$$

Now let $E_2\left(\Gamma_0(21), (\frac{21}{*})\right)$ denote the subspace of $M_2\left(\Gamma_0(21), (\frac{21}{*})\right)$ consisting of Eisenstein forms. By [21, p. 83] we have

$$M_2\left(\Gamma_0(21), \left(\frac{21}{*}\right)\right) = E_2\left(\Gamma_0(21), \left(\frac{21}{*}\right)\right) \oplus S_2\left(\Gamma_0(21), \left(\frac{21}{*}\right)\right).$$

Hence

$$M_2\left(\Gamma_0(21), \left(\frac{21}{*}\right)\right) = E_2\left(\Gamma_0(21), \left(\frac{21}{*}\right)\right).$$

Let χ_1 denote the trivial character and $\chi_{-3}, \chi_{-7}, \chi_{21}$ the primitive Dirichlet characters of conductor $3, 7, 21$ respectively given by

$$\chi_{-3}(n) = \left(\frac{-3}{n}\right), \quad \chi_{-7}(n) = \left(\frac{-7}{n}\right), \quad \chi_{21}(n) = \left(\frac{21}{n}\right), \ n \in \mathbb{N}.$$

Clearly

$$\chi_{-3}\chi_{-7} = \chi_{21}.$$

It follows from [17, Chap. 7], see also [21, p. 88], that the four Eisenstein series

$$E_{2,\chi_{21},\chi_1}(q) := \sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{21}{n/d}\right)d\right)q^n = q + q^2 + 3q^3 + \cdots,$$

$$E_{2,\chi_{-3},\chi_{-7}}(q) := \sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-3}{n/d}\right)\left(\frac{-7}{d}\right)d\right)q^n = q + q^2 - 3q^3 + \cdots,$$

$$E_{2,\chi_{-7},\chi_{-3}}(q) := \sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-7}{n/d}\right)\left(\frac{-3}{d}\right)d\right)q^n = q - q^2 - q^3 + \cdots,$$

$$E_{2,\chi_1,\chi_{21}}(q) := -2 + \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{21}{d} \right) d \right) q^n = -2 + q - q^2 + q^3 + \cdots,$$

form a basis for $E_2(\Gamma_0(21), (\frac{21}{*}))$ and thus for $M_2(\Gamma_0(21), (\frac{21}{*}))$. Hence there exist rational numbers a, b, c, d such that

$$\sum_{n=0}^{\infty} N(3,3,6,7,7,7;n)q^n$$
$$= aE_{2,\chi_{21},\chi_1}(q) + bE_{2,\chi_{-3},\chi_{-7}}(q) + cE_{2,\chi_{-7},\chi_{-3}}(q) + dE_{2,\chi_1,\chi_{21}}(q),$$

that is

$$1 + 2q^3 + \cdots = a(q + q^2 + 3q^3 + \cdots) + b(q + q^2 - 3q^3 + \cdots)$$
$$+ c(q - q^2 - q^3 + \cdots) + d(-2 + q - q^2 + q^3 + \cdots).$$

Equating coefficients of $1, q, q^2$ and q^3, we obtain

$$-2d = 1, \quad a + b + c + d = 0, \quad a + b - c - d = 0, \quad 3a - 3b - c + d = 2,$$

so

$$a = \frac{1}{2}, \quad b = -\frac{1}{2}, \quad c = \frac{1}{2}, \quad d = -\frac{1}{2}.$$

Hence

$$\sum_{n=0}^{\infty} N(3,3,6,7,7,7;n)q^n$$
$$= \frac{1}{2}E_{2,\chi_{21},\chi_1}(q) - \frac{1}{2}E_{2,\chi_{-3},\chi_{-7}}(q) + \frac{1}{2}E_{2,\chi_{-7},\chi_{-3}}(q) - \frac{1}{2}E_{2,\chi_1,\chi_{21}}(q),$$

and thus equating coefficients of q^n $(n \in \mathbb{N})$ we obtain

$$N(3,3,6,7,7,7;n) = \frac{1}{2}A(n) - \frac{1}{2}B(n) + \frac{1}{2}C(n) - \frac{1}{2}D(n), \quad n \in \mathbb{N},$$

which is (5.2). This completes the proof of Theorem 5.1. □

These results show that the representation numbers of the four quaternary quadratic forms

$$x^2 + xy + y^2 + z^2 + zt + 2t^2, \quad x^2 + xy + y^2 + 3z^2 + 3zt + 6t^2,$$
$$x^2 + xy + 2y^2 + 7z^2 + 7zt + 7t^2, \quad 3x^2 + 3xy + 6y^2 + 7z^2 + 7zt + 7t^2,$$

have a special relationship one to another. Other quadruples of quaternary quadratic forms have a similar special relationship, for example

$$x^2 + 4y^2 + 2z^2 + 2zt + 2t^2, \quad x^2 + 4y^2 + 4z^2 + 4zt + 4t^2,$$
$$3x^2 + 12y^2 + 2z^2 + 2zt + 2t^2, \quad 3x^2 + 12y^2 + 4z^2 + 4zt + 4t^2,$$

but we will not pursue this here.

6. Evaluation of Some Dirichlet Series

In proving Theorems 1.8–1.24 we used the quotient of the formulae (4.2) and (4.5) to determine the number of representations $N(a,b,c,d,e,f;n)$ for the 17 forms $ax^2+bxy+cy^2+dz^2+ezt+ft^2$ in Table 6 using Siegel's mass formula. In doing this we determined the local density $d_p(a,b,c,d,e,f;\ell)$ explicitly for all primes p, where ℓ is the least positive integer represented by $ax^2+bxy+cy^2+dz^2+ezt+ft^2$. Armed with this information, we can now evaluate the infinite product $\prod_p d_p(a,b,c,d,e,f;\ell)$ by means of (4.5), namely,

$$\prod_p d_p(a,b,c,d,e,f;\ell) = \frac{\sqrt{D}}{4\pi^2\ell} N(a,b,c,d,e,f;\ell). \tag{6.1}$$

The left-hand side of (6.1) turns out to be a multiple of an Euler product and this enables us to determine the sum of some Dirichlet L-series. The Dirichlet L-series we are interested in are

$$L\left(2,\left(\frac{E}{*}\right)\right) := \sum_{n=1}^{\infty}\left(\frac{E}{n}\right)\frac{1}{n^2} = \prod_p\left(1-\left(\frac{E}{p}\right)\frac{1}{p^2}\right)^{-1},$$

where E is a positive nonsquare integer with $E\equiv 0$ or $1 \pmod 4$. We prove the following theorem.

Theorem 6.1.

$$L\left(2,\left(\frac{5}{*}\right)\right) = \frac{4\pi^2\sqrt{5}}{125}, \tag{6.2}$$

$$L\left(2,\left(\frac{12}{*}\right)\right) = \frac{\pi^2\sqrt{3}}{18}, \tag{6.3}$$

$$L\left(2,\left(\frac{21}{*}\right)\right) = \frac{8\pi^2\sqrt{21}}{441}, \tag{6.4}$$

$$L\left(2,\left(\frac{24}{*}\right)\right) = \frac{\pi^2\sqrt{6}}{24}, \tag{6.5}$$

$$L\left(2,\left(\frac{28}{*}\right)\right) = \frac{2\pi^2\sqrt{7}}{49}. \tag{6.6}$$

Proof. We just prove (6.6) as the others can be proved similarly using other forms from Table 6. We choose $(a,b,c,d,e,f)=(1,0,2,3,2,5)$. The representation number of the form $x^2+2y^2+3z^2+2zt+5t^2$ was determined in Theorem 1.19. Here $D=448=2^6\cdot 7$, $\sqrt{D}=8\sqrt{7}$, $\ell=1$, $N(1,0,2,3,2,5;1)=2$ and

$$d_p(1,0,2,3,2,5;1) = \begin{cases} 1 & \text{if } p=2,\\ \dfrac{8}{7} & \text{if } p=7,\\ \dfrac{p^2-\left(\frac{7}{p}\right)}{p^2} & \text{if } p\neq 2,7. \end{cases}$$

Then, by (6.1), we obtain

$$\frac{8}{7}\prod_{p\neq 2,7}\left(1-\left(\frac{7}{p}\right)\frac{1}{p^2}\right)=\frac{8\sqrt{7}}{4\pi^2}\cdot 2=\frac{4\sqrt{7}}{\pi^2}$$

so that

$$\prod_{p}\left(1-\left(\frac{28}{p}\right)\frac{1}{p^2}\right)=\prod_{p\neq 2,7}\left(1-\left(\frac{7}{p}\right)\frac{1}{p^2}\right)=\frac{7\sqrt{7}}{2\pi^2}=\frac{49}{2\pi^2\sqrt{7}}.$$

Thus

$$L\left(2,\left(\frac{28}{*}\right)\right)=\sum_{n=1}^{\infty}\left(\frac{28}{n}\right)\frac{1}{n^2}=\prod_{p}\left(1-\left(\frac{28}{p}\right)\frac{1}{p^2}\right)^{-1}=\frac{2\pi^2\sqrt{7}}{49},$$

as asserted. □

Formulae (6.2)–(6.4) are given in [22]. The authors have not found (6.5) and (6.6) in the literature.

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