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On the number of representations of a positive integer as a sum of two binary quadratic forms

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Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{Z}$ the set of all integers. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $a_1x^2 + b_1xy + c_1y^2$ and $a_2z^2 + b_2zt + c_2t^2$ be two positive-definite, integral, binary quadratic forms. The number of representations of $n \in \mathbb{N}_0$ as a sum of these two binary quadratic forms is

$$N(a_1, b_1, c_1, a_2, b_2, c_2; n) := \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = a_1x^2 + b_1xy + c_1y^2 + a_2z^2 + b_2zt + c_2t^2\}.$$

When $(b_1, b_2) \neq (0, 0)$ we prove under certain conditions on a_1, b_1, c_1, a_2, b_2 and c_2 that $N(a_1, b_1, c_1, a_2, b_2, c_2; n)$ can be expressed as a finite linear combination of quantities of the type $N(a, 0, b, c, 0, d; n)$ with a, b, c and d positive integers. Thus, when the quantities $N(a, 0, b, c, 0, d; n)$ are known, we can determine $N(a_1, b_1, c_1, a_2, b_2, c_2; n)$. This determination is carried out explicitly for a number of quaternary quadratic forms $a_1x^2 + b_1xy + c_1y^2 + a_2z^2 + b_2zt + c_2t^2$. For example, in Theorem 1.2 we show

for $n \in \mathbb{N}$ that

$$N(3,2,3,3,2,3;n) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 4, \\ 2\sigma(N) & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{if } n \equiv 2 \pmod 8, \\ 4\sigma(N) & \text{if } n \equiv 4, 6 \pmod 8, \\ 8\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{16}, \end{cases}$$

where N is the largest odd integer dividing n and

$$\sigma(N) = \sum_{\substack{d \in \mathbb{N} \\ d \mid N}} d.$$

Keywords: Sum of two binary quadratic forms; number of representations.

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1. Introduction

Let $\mathbb{Z}$ denote the domain of all integers and $\mathbb{N}$ the set of positive integers. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Q}$ and $\mathbb{C}$ denote the fields of rational and complex numbers, respectively.

If $a_1x^2 + b_1xy + c_1y^2$ and $a_2z^2 + b_2zt + c_2t^2$ are two integral, positive-definite, binary quadratic forms, the number of representations of $n \in \mathbb{N}_0$ by the quaternary quadratic form $a_1x^2 + b_1xy + c_1y^2 + a_2z^2 + b_2zt + c_2t^2$ is given by

$$\begin{aligned} &N(a_1, b_1, c_1, a_2, b_2, c_2; n) \\ &\quad := \operatorname{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = a_1x^2 + b_1xy + c_1y^2 + a_2z^2 + b_2zt + c_2t^2\}. \end{aligned}$$

Clearly $N(a_1, b_1, c_1, a_2, b_2, c_2; 0) = 1$. The number $N(a_1, b_1, c_1, a_2, b_2, c_2; n)$ is called the representation number of the form $a_1x^2 + b_1xy + c_1y^2 + a_2z^2 + b_2zt + c_2t^2$ and has the following simple properties:

$$N(a_1, b_1, c_1, a_2, b_2, c_2; n) = N(a_2, b_2, c_2, a_1, b_1, c_1; n),$$

$$\begin{aligned} N(a_1, b_1, c_1, a_2, b_2, c_2; n) &= N(a_1, b_1, c_1, c_2, b_2, a_2; n) = N(c_1, b_1, a_1, a_2, b_2, c_2; n) \\ &= N(c_1, b_1, a_1, c_2, b_2, a_2; n), \end{aligned}$$

and

$$\begin{aligned} N(a_1, b_1, c_1, a_2, b_2, c_2; n) &= N(a_1, -b_1, c_1, a_2, b_2, c_2; n) \\ &= N(a_1, b_1, c_1, a_2, -b_2, c_2; n) \\ &= N(a_1, -b_1, c_1, a_2, -b_2, c_2; n). \end{aligned}$$

If $m \in \mathbb{Q} \backslash \mathbb{N}_0$ we define $N(a_1, b_1, c_1, a_2, b_2, c_2; m) = 0$. Thus, for $k \in \mathbb{N}$, we have

$$N(ka_1, kb_1, kc_1, ka_2, kb_2, kc_2; n) = N(a_1, b_1, c_1, a_2, b_2, c_2; n/k).$$

We recall that two integral binary quadratic forms $A_1x^2 + B_1xy + C_1y^2$ and $A_2x^2 + B_2xy + C_2y^2$ are said to be equivalent, written $A_1x^2 + B_1xy + C_1y^2 \sim A_2x^2 + B_2xy + C_2y^2$, if there exist $p, q, r, s \in \mathbb{Z}$ with $ps - qr = 1$ such that

$$A_1(px + qy)^2 + B_1(px + qy)(rx + sy) + C_1(rx + sy)^2 = A_2x^2 + B_2xy + C_2y^2.$$

For example, if a and c are positive integers with $a < 4c$ then $ax^2 + axy + cy^2$ and $cx^2 + (2c - a)xy + cy^2$ are positive-definite, integral, binary quadratic forms, which are equivalent as

$$a(-y)^2 + a(-y)(x + y) + c(x + y)^2 = cx^2 + (2c - a)xy + cy^2.$$

If $a_1x^2 + b_1xy + c_1y^2 \sim A_1x^2 + B_1xy + C_1y^2$ and $a_2x^2 + b_2xy + c_2y^2 \sim A_2x^2 + B_2xy + C_2y^2$ then

$$N(a_1, b_1, c_1, a_2, b_2, c_2; n) = N(A_1, B_1, C_1, A_2, B_2, C_2; n).$$

Thus, for example, we have

$$N(a_1, b_1, c_1, a, a, c; n) = N(a_1, b_1, c_1, c, 2c - a, c; n).$$

Hence in particular we have

$$N(a_1, b_1, c_1, 4, 4, 5; n) = N(a_1, b_1, c_1, 5, 6, 5; n).$$

If $b_1 = b_2 = 0$ we define

$$N(a_1, c_1, a_2, c_2; n) := N(a_1, 0, c_1, a_2, 0, c_2; n).$$

In Sec. 2 we prove Theorem 1.1, which shows that in the case $(b_1, b_2) \neq (0, 0)$, under certain conditions on a_1, b_1, c_1, a_2, b_2 and c_2, the representation number $N(a_1, b_1, c_1, a_2, b_2, c_2; n)$ can be expressed as a finite linear combination with integral coefficients of certain representation numbers

$$N(A_1, B_1, C_1, D_1; n), \ldots, N(A_m, B_m, C_m, D_m; n).$$

Theorem 1.1. (i) *Let $a_1x^2 + c_1y^2$ and $a_2z^2 + b_2zt + c_2t^2$ be positive-definite, integral, binary quadratic forms with $b_2 \neq 0$. Suppose that*

$$a_2z^2 + b_2zt + c_2t^2 \sim M_2z^2 + 2N_2zt + M_2t^2,$$

where M_2 and N_2 are positive integers such that $M_2 > N_2$ and $M_2 \equiv N_2 \pmod 2$. Then, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} N(a_1, 0, c_1, a_2, b_2, c_2; n) = {} & N\left(a_1, c_1, \frac{M_2 - N_2}{2}, \frac{M_2 + N_2}{2}; n\right) \\ & - N\left(a_1, c_1, \frac{M_2 - N_2}{2}, 2(M_2 + N_2); n\right) \\ & - N\left(a_1, c_1, \frac{M_2 + N_2}{2}, 2(M_2 - N_2); n\right) \\ & + 2N(a_1, c_1, 2(M_2 - N_2), 2(M_2 + N_2); n). \end{aligned}$$

(ii) *Let $a_1x^2+b_1xy+c_1y^2$ and $a_2z^2+b_2zt+c_2t^2$ be positive-definite, integral, binary quadratic forms with $b_1 \neq 0$ and $b_2 \neq 0$. Suppose that*

$$a_1x^2 + b_1xy + c_1y^2 \sim M_1x^2 + 2N_1xy + M_1y^2$$

and

$$a_2z^2 + b_2zt + c_2t^2 \sim M_2z^2 + 2N_2zt + M_2t^2,$$

where M_1, M_2, N_1 and N_2 are positive integers such that $M_1 > N_1, M_1 \equiv N_1 \pmod 2), M_2 > N_2$ and $M_2 \equiv N_2 \pmod 2$. Then, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&N(a_1, b_1, c_1, a_2, b_2, c_2; n)\\
&\quad = N\left(\frac{M_1-N_1}{2}, \frac{M_1+N_1}{2}, \frac{M_2-N_2}{2}, \frac{M_2+N_2}{2}; n\right)\\
&\quad\quad - N\left(\frac{M_1-N_1}{2}, \frac{M_1+N_1}{2}, \frac{M_2+N_2}{2}, 2(M_2-N_2); n\right)\\
&\quad\quad - N\left(\frac{M_1-N_1}{2}, \frac{M_1+N_1}{2}, \frac{M_2-N_2}{2}, 2(M_2+N_2); n\right)\\
&\quad\quad + 2N\left(\frac{M_1-N_1}{2}, \frac{M_1+N_1}{2}, 2(M_2-N_2), 2(M_2+N_2); n\right)\\
&\quad\quad - N\left(\frac{M_1+N_1}{2}, 2(M_1-N_1), \frac{M_2-N_2}{2}, \frac{M_2+N_2}{2}; n\right)\\
&\quad\quad + N\left(\frac{M_1+N_1}{2}, 2(M_1-N_1), \frac{M_2+N_2}{2}, 2(M_2-N_2); n\right)\\
&\quad\quad + N\left(\frac{M_1+N_1}{2}, 2(M_1-N_1), \frac{M_2-N_2}{2}, 2(M_2+N_2); n\right)\\
&\quad\quad - 2N\left(\frac{M_1+N_1}{2}, 2(M_1-N_1), 2(M_2-N_2), 2(M_2+N_2); n\right)\\
&\quad\quad - N\left(\frac{M_1-N_1}{2}, 2(M_1+N_1), \frac{M_2-N_2}{2}, \frac{M_2+N_2}{2}; n\right)\\
&\quad\quad + N\left(\frac{M_1-N_1}{2}, 2(M_1+N_1), \frac{M_2+N_2}{2}, 2(M_2-N_2); n\right)\\
&\quad\quad + N\left(\frac{M_1-N_1}{2}, 2(M_1+N_1), \frac{M_2-N_2}{2}, 2(M_2+N_2); n\right)\\
&\quad\quad - 2N\left(\frac{M_1-N_1}{2}, 2(M_1+N_1), 2(M_2-N_2), 2(M_2+N_2); n\right)\\
&\quad\quad + 2N\left(2(M_1-N_1), 2(M_1+N_1), \frac{M_2-N_2}{2}, \frac{M_2+N_2}{2}; n\right)
\end{aligned}$$

$$- 2N\left(2(M_1 - N_1), 2(M_1 + N_1), \frac{M_2 + N_2}{2}, 2(M_2 - N_2); n\right)$$
$$- 2N\left(2(M_1 - N_1), 2(M_1 + N_1), \frac{M_2 - N_2}{2}, 2(M_2 + N_2); n\right)$$
$$+ 4N(2(M_1 - N_1), 2(M_1 + N_1), 2(M_2 - N_2), 2(M_2 + N_2); n).$$

(iii) *Let* $r, s \in \mathbb{N}$. *Then, for all* $n \in \mathbb{N}$, *we have*

$$N(r, 0, s, 1, 1, 1; n) = N(1, 3, 4r, 4s; 4n).$$

(iv) *Let* $r, s \in \mathbb{N}$. *Then, for all* $n \in \mathbb{N}$, *we have*

$$N(2r, 0, 2s, 1, 1, 1; n) = (2 - (-1)^n)N(1, 3, 2r, 2s; n).$$

(v) *Let* $r, s \in \mathbb{N}$. *Then, for all* $n \in \mathbb{N}$, *we have*

$$N(r, 0, s, 2, 2, 2; n) = N(1, 3, 2r, 2s; 2n).$$

In Secs. 3–8 we apply Theorem 1.1 to determine explicit formulae for the representation numbers $N(a_1, b_1, c_1, a_2, b_2, c_2; n)$ of 51 quaternary quadratic forms $a_1x^2 + b_1xy + c_1y^2 + a_2z^2 + b_2zt + c_2t^2$, see Theorems 1.2–1.7. Forty of these evaluations are new.

In Theorem 1.2 we give 10 forms whose representation numbers can be given in terms of the sum of divisors function $\sigma(n)$, which is defined for $n \in \mathbb{N}$ by

$$\sigma(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d. \tag{1.1}$$

Theorem 1.2. (i) *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0, N \in \mathbb{N}$ *and* $\gcd(N, 2) = 1$. *Then*

$$N(1, 0, 2, 3, 2, 3; n) = \begin{cases} 2\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 2\sigma(N) & \text{if } n \equiv 2 \pmod 8, \\ 6\sigma(N) & \text{if } n \equiv 6 \pmod 8, \\ 12\sigma(N) & \text{if } n \equiv 4 \pmod 8, \\ 8\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{16}; \end{cases}$$

$$N(4, 0, 8, 3, 2, 3; n) = \begin{cases} 0 & \text{if } n \equiv 1, 2 \pmod 4, \\ \sigma(N) & \text{if } n \equiv 3 \pmod 4, \\ 4\sigma(N) & \text{if } n \equiv 4 \pmod 8, \\ 8\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{16}; \end{cases}$$

$$N(8,0,16,3,2,3;n) = \begin{cases} 0 & \text{if } n \equiv 1,5,7 \pmod{8}, \\ \sigma(N) & \text{if } n \equiv 3 \pmod{8}, \\ 0 & \text{if } n \equiv 2 \pmod{4}, \\ 2\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 4\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 8\sigma(N) & \text{if } n \equiv 16 \pmod{32}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{32}; \end{cases}$$

$$N(3,2,3,3,2,3;n) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod{4}, \\ 2\sigma(N) & \text{if } n \equiv 3 \pmod{4}, \\ 0 & \text{if } n \equiv 2 \pmod{8}, \\ 4\sigma(N) & \text{if } n \equiv 4,6 \pmod{8}, \\ 8\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{16}. \end{cases}$$

(ii) *Let* $n \in \mathbb{N}$. *Set* $n = 3^\beta N$, *where* $\beta \in \mathbb{N}_0, N \in \mathbb{N}$ *and* $\gcd(N,3) = 1$. *Then*

$$N(1,1,1,1,1,1;n) = 12\sigma(N).$$

(iii) *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0, N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,0,3,1,1,1;n) = \begin{cases} 8\sigma(N) & \text{if } n \equiv 1 \pmod{2}, \\ 12(2^\alpha - 1)\sigma(N) & \text{if } n \equiv 0 \pmod{2}; \end{cases}$$

$$N(1,0,3,2,2,2;n) = \begin{cases} 2(3^{\beta+1} - 2)\sigma(N) & \text{if } n \equiv 1 \pmod{2}, \\ 6(3^{\beta+1} - 2)\sigma(N) & \text{if } n \equiv 0 \pmod{2}; \end{cases}$$

$$N(1,0,3,4,4,4;n) = \begin{cases} 2\sigma(N) & \text{if } n \equiv 1 \pmod{2}, \\ 0 & \text{if } n \equiv 2 \pmod{4}, \\ 12(2^{\alpha-1} - 1)\sigma(N) & \text{if } n \equiv 0 \pmod{4}; \end{cases}$$

$$N(1,0,12,4,4,4;n) = \begin{cases} 2\sigma(N) & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 2,3 \pmod{4}, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 12(2^{\alpha-2} - 1)\sigma(N) & \text{if } n \equiv 0 \pmod{8}; \end{cases}$$

$$N(3,0,4,4,4,4;n) = \begin{cases} 0 & \text{if } n \equiv 1,2 \pmod{4}, \\ 2\sigma(N) & \text{if } n \equiv 3 \pmod{4}, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 12(2^{\alpha-2} - 1)\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

The formula for $N(1,1,1,1,1,1;n)$ was conjectured by Liouville [10] and proved by a number of authors including Chapman [6], Huard, Ou, Spearman and Williams [7] and Walfisz [13]. The formulae for $N(1,0,3,2,2,2;n)$ and $N(1,0,3,4,4,4;n)$ were proved by Walfisz [13]. The remaining seven formulae stated in Theorem 1.2 are new.

In Theorem 1.3 we give nine forms whose representation numbers can be given in terms of the function $S(n)$, which is defined for $n \in \mathbb{N}$ by

$$S(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \frac{n}{d} \left(\frac{8}{d}\right) \quad \text{(see [2, p. 151; 5, p. 27])}, \tag{1.2}$$

where $(\frac{8}{d})$ is the Legendre–Jacobi–Kronecker symbol for discriminant 8, that is

$$\left(\frac{8}{d}\right) = \begin{cases} 1 & \text{if } d \equiv 1,7 \pmod 8, \\ -1 & \text{if } d \equiv 3,5 \pmod 8, \\ 0 & \text{if } d \equiv 0 \pmod 2. \end{cases}$$

Theorem 1.3. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0, N \in \mathbb{N}$ and* $\gcd(N,2)=1$. *Then*

$$N(1,0,1,3,2,3;n) = \begin{cases} \left(6 - 2\left(\frac{8}{N}\right)\right) S(N) & \text{if } n \equiv 1 \pmod 4, \\ 2S(N) & \text{if } n \equiv 3 \pmod 4, \\ 4S(N) & \text{if } n \equiv 2 \pmod 4, \\ 2\left(3 \cdot 2^\alpha - \left(\frac{8}{N}\right)\right) S(N) & \text{if } n \equiv 0 \pmod 4; \end{cases}$$

$$N(1,0,4,3,2,3;n) = \begin{cases} 2S(N) & \text{if } n \equiv 1 \pmod 2, \\ 0 & \text{if } n \equiv 2 \pmod 4, \\ 2\left(2^{\alpha+1} - \left(\frac{8}{N}\right)\right) S(N) & \text{if } n \equiv 0 \pmod 4; \end{cases}$$

$$N(2,0,8,3,2,3;n) = \begin{cases} 0 & \text{if } n \equiv 1,7 \pmod 8, \\ 2S(N) & \text{if } n \equiv 3,5 \pmod 8, \\ 2S(N) & \text{if } n \equiv 2 \pmod 4, \\ 2\left(2^{\alpha-1} - \left(\frac{8}{N}\right)\right) S(N) & \text{if } n \equiv 0 \pmod 4; \end{cases}$$

$$N(4,0,4,3,2,3;n) = \begin{cases} 0 & \text{if } n \equiv 1,2 \pmod 4, \\ 2S(N) & \text{if } n \equiv 3 \pmod 4, \\ 2\left(2^{\alpha} - \left(\frac{8}{N}\right)\right) S(N) & \text{if } n \equiv 0 \pmod 4; \end{cases}$$

$$N(8,0,8,3,2,3;n)=\begin{cases}0 & \textit{if } n\equiv 1,5,7 \pmod 8,\\ 2S(N) & \textit{if } n\equiv 3 \pmod 8,\\ 0 & \textit{if } n\equiv 2 \pmod 4,\\ 2\left(2^{\alpha-1}-\left(\frac{8}{N}\right)\right)S(N) & \textit{if } n\equiv 0 \pmod 4;\end{cases}$$

$$N(1,0,2,4,4,5;n)=\begin{cases}2S(N) & \textit{if } n\equiv 1 \pmod 2,\\ 2S(N) & \textit{if } n\equiv 2 \pmod 8,\\ 8S(N) & \textit{if } n\equiv 6 \pmod{16},\\ 4S(N) & \textit{if } n\equiv 14 \pmod{16},\\ 4S(N) & \textit{if } n\equiv 4 \pmod 8,\\ \left(3\cdot 2^{\alpha}-2\left(\frac{8}{N}\right)\right)S(N) & \textit{if } n\equiv 0 \pmod 8;\end{cases}$$

$$N(1,0,8,4,4,5;n)=\begin{cases}2S(N) & \textit{if } n\equiv 1 \pmod 4,\\ 0 & \textit{if } n\equiv 3 \pmod 4,\\ 0 & \textit{if } n\equiv 2 \pmod 8,\\ 4S(N) & \textit{if } n\equiv 6 \pmod 8,\\ 4S(N) & \textit{if } n\equiv 4 \pmod 8,\\ \left(2^{\alpha}-2\left(\frac{8}{N}\right)\right)S(N) & \textit{if } n\equiv 0 \pmod 8;\end{cases}$$

$$N(2,0,16,4,4,5;n)=\begin{cases}0 & \textit{if } n\equiv 1,3 \pmod 8,\\ S(N) & \textit{if } n\equiv 5,7 \pmod 8,\\ 2S(N) & \textit{if } n\equiv 2,6 \pmod{16},\\ 0 & \textit{if } n\equiv 10,14 \pmod{16},\\ 2S(N) & \textit{if } n\equiv 4 \pmod 8,\\ 2\left(2^{\alpha-2}-\left(\frac{8}{N}\right)\right)S(N) & \textit{if } n\equiv 0 \pmod 8;\end{cases}$$

$$N(8,0,16,4,4,5;n)=\begin{cases}0 & \textit{if } n\equiv 1,3,7 \pmod 8,\\ S(N) & \textit{if } n\equiv 5 \pmod 8,\\ 0 & \textit{if } n\equiv 2 \pmod 4,\\ 2S(N) & \textit{if } n\equiv 4 \pmod 8,\\ \left(2^{\alpha-1}-2\left(\frac{8}{N}\right)\right)S(N) & \textit{if } n\equiv 0 \pmod 8.\end{cases}$$

All nine evaluations in Theorem 1.3 are new.

In Theorem 1.4 we give 10 forms all of whose representation numbers can be given in terms of the function $A(n)$ defined for $n \in \mathbb{N}$ by

$$A(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \frac{n}{d} \left(\frac{12}{d} \right) \quad \text{(see [3, p. 225])}, \tag{1.3}$$

where $(\frac{12}{d})$ is the Legendre–Jacobi–Kronecker symbol for discriminant 12, that is

$$\left(\frac{12}{d} \right) = \begin{cases} 1 & \text{if } d \equiv 1, 11 \pmod{12}, \\ -1 & \text{if } d \equiv 5, 7 \pmod{12}, \\ 0 & \text{if } d \equiv 0 \pmod{2} \text{ or } d \equiv 0 \pmod{3}. \end{cases}$$

Theorem 1.4. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0, N \in \mathbb{N}$ *and* $\gcd(N, 6) = 1$. *Then*

$$N(1,0,1,1,1,1;n) = (2^{\alpha+2} + (-1)^{\alpha+\beta+(N-1)/2}) \left(3^{\beta+1} - (-1)^{\alpha+\beta} \left(\frac{N}{3} \right) \right) A(N);$$

$$N(2,0,2,1,1,1;n) = \begin{cases} 3 \left(3^{\beta+1} - (-1)^{\beta} \left(\frac{N}{3} \right) \right) A(N) & \text{if } n \equiv 1 \pmod{2}, \\ (2^{\alpha} + (-1)^{\alpha+\beta+(N-1)/2}) \left(3^{\beta+1} - (-1)^{\alpha+\beta} \left(\frac{N}{3} \right) \right) A(N) & \text{if } n \equiv 0 \pmod{2}; \end{cases}$$

$$N(3,0,3,1,1,1;n) = (2^{\alpha+2} - (-1)^{\alpha+\beta+(N-1)/2}) \left(3^{\beta} + (-1)^{\alpha+\beta} \left(\frac{N}{3} \right) \right) A(N);$$

$$N(6,0,6,1,1,1;n) = \begin{cases} 3 \left(3^{\beta} + (-1)^{\beta} \left(\frac{N}{3} \right) \right) A(N) & \text{if } n \equiv 1 \pmod{2}, \\ (2^{\alpha} - (-1)^{\alpha+\beta+(N-1)/2}) \left(3^{\beta} + (-1)^{\alpha+\beta} \left(\frac{N}{3} \right) \right) A(N) & \text{if } n \equiv 0 \pmod{2}; \end{cases}$$

$$N(1,0,1,2,2,2;n) = (2^{\alpha+1} - (-1)^{\alpha+\beta+(N-1)/2}) \left(3^{\beta+1} + (-1)^{\alpha+\beta} \left(\frac{N}{3} \right) \right) A(N);$$

$$N(3,0,3,2,2,2;n) = (2^{\alpha+1} + (-1)^{\alpha+\beta+(N-1)/2}) \left(3^{\beta} - (-1)^{\alpha+\beta} \left(\frac{N}{3} \right) \right) A(N);$$

$$N(1,0,1,4,4,4;n) = (2^\alpha + (-1)^{\alpha+\beta+(N-1)/2}) \left(3^{\beta+1} - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) A(N);$$

$$N(1,0,4,4,4,4;n) = \begin{cases} \frac{1}{2}(1 + (-1)^{\beta+(N-1)/2}) \left(3^{\beta+1} - (-1)^{\beta}\left(\frac{N}{3}\right)\right) A(N) & \textit{if } n \equiv 1 \pmod 2, \\ 0 & \textit{if } n \equiv 2 \pmod 4, \\ (2^\alpha + (-1)^{\alpha+\beta+(N-1)/2}) \left(3^{\beta+1} - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) A(N) & \textit{if } n \equiv 0 \pmod 4; \end{cases}$$

$$N(3,0,3,4,4,4;n) = (2^\alpha - (-1)^{\alpha+\beta+(N-1)/2}) \left(3^{\beta} + (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) A(N);$$

$$N(3,0,12,4,4,4;n) = \begin{cases} \frac{1}{2}(1 - (-1)^{\beta+(N-1)/2}) \left(3^{\beta} + (-1)^{\beta}\left(\frac{N}{3}\right)\right) A(N) & \textit{if } n \equiv 1 \pmod 2, \\ 0 & \textit{if } n \equiv 2 \pmod 4, \\ (2^\alpha - (-1)^{\alpha+\beta+(N-1)/2}) \left(3^{\beta} + (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) A(N) & \textit{if } n \equiv 0 \pmod 4; \end{cases}$$

where

$$\left(\frac{N}{3}\right) = \begin{cases} 1 & \textit{if } N \equiv 1 \pmod 3, \\ -1 & \textit{if } N \equiv 2 \pmod 3. \end{cases}$$

The formulae for $N(1,0,1,1,1,1;n)$, $N(3,0,3,1,1,1;n)$, $N(1,0,1,2,2,2;n)$ and $N(3,0,3,2,2,2;n)$ were conjectured by Liouville in [9], [12], [8] and [11], respectively, and proved by Alaca, Alaca, Lemire and Williams [3] and Walfisz [14]. The formulae for $N(2,0,2,1,1,1;n)$ and $N(6,0,6,1,1,1;n)$ are due to Walfisz [14]. The formulae for $N(1,0,1,4,4,4;n)$ and $N(3,0,3,4,4,4;n)$ are also due to Walfisz [14] and were also proved by Alaca, Alaca, Lemire and Williams [3]. The remaining two formulae are new.

Theorem 1.5 gives the representation numbers of six forms which can be given in terms of the functions $S(n)$ (see (1.2)) and $K_1(n)$, where $K_1(n)$ is defined for $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$ by

$$K_1(n) := \sum_{\substack{(r,s)\in\mathbb{N}\times\mathbb{Z} \\ r\equiv 1 \pmod 2 \\ n=r^2+2s^2}} (-1)^{(r-1)/2} r \quad \text{(see [2, p. 146])}. \tag{1.4}$$

It is observed in [2, p. 146] that $K_1(n) = 0$ for $n \equiv 5, 7 \pmod 8$.

Theorem 1.5. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0, N \in \mathbb{N}$ *and* $\gcd(N, 2) = 1$. *Then*

$$N(1,0,16,3,2,3;n) = \begin{cases} S(N) + K_1(N) & \text{if } n \equiv 1,3 \pmod 8, \\ S(N) & \text{if } n \equiv 5,7 \pmod 8, \\ 0 & \text{if } n \equiv 2 \pmod 4, \\ 12S(N) & \text{if } n \equiv 4 \pmod 8, \\ 2\left(2^{\alpha-1} - \left(\frac{8}{N}\right)\right) S(N) & \text{if } n \equiv 0 \pmod 8; \end{cases}$$

$$N(16,0,16,3,2,3;n) = \begin{cases} 0 & \text{if } n \equiv 1,5,7 \pmod 8, \\ S(N) + K_1(N) & \text{if } n \equiv 3 \pmod 8, \\ 0 & \text{if } n \equiv 2 \pmod 4, \\ 2S(N) & \text{if } n \equiv 4 \pmod 8, \\ 2\left(2^{\alpha-2} - \left(\frac{8}{N}\right)\right) S(N) & \text{if } n \equiv 0 \pmod 8; \end{cases}$$

$$N(2,0,4,4,4,5;n) = \begin{cases} S(N) - K_1(N) & \text{if } n \equiv 1,3 \pmod 8, \\ S(N) & \text{if } n \equiv 5,7 \pmod 8, \\ 2S(N) & \text{if } n \equiv 2 \pmod 8, \\ 4S(N) & \text{if } n \equiv 6 \pmod{16}, \\ 0 & \text{if } n \equiv 14 \pmod{16}, \\ 4S(N) & \text{if } n \equiv 4 \pmod 8, \\ 2\left(2^{\alpha-1} - \left(\frac{8}{N}\right)\right) S(N) & \text{if } n \equiv 0 \pmod 8; \end{cases}$$

$$N(4,0,8,4,4,5;n) = \begin{cases} S(N) - K_1(N) & \text{if } n \equiv 1 \pmod 8, \\ S(N) & \text{if } n \equiv 5 \pmod 8, \\ 0 & \text{if } n \equiv 2,3 \pmod 4, \\ 4S(N) & \text{if } n \equiv 4 \pmod 8, \\ 2\left(2^{\alpha-1} - \left(\frac{8}{N}\right)\right) S(N) & \text{if } n \equiv 0 \pmod 8; \end{cases}$$

$$N(3,2,3,4,4,5;n) = \begin{cases} S(N) - \left(\frac{8}{N}\right) K_1(N) & \text{if } n \equiv 1,3 \pmod 8, \\ S(N) & \text{if } n \equiv 5,7 \pmod 8, \\ 0 & \text{if } n \equiv 2 \pmod 4, \\ 4S(N) & \text{if } n \equiv 4 \pmod 8, \\ 2\left(3 \cdot 2^{\alpha-1} - \left(\frac{8}{N}\right)\right) S(N) & \text{if } n \equiv 0 \pmod 8; \end{cases}$$

$$N(4,4,5,6,4,6;n) = \begin{cases} 0 & \text{if } n \equiv 1,7 \pmod{8}, \\ S(N) - K_1(N) & \text{if } n \equiv 3 \pmod{8}, \\ S(N) & \text{if } n \equiv 5 \pmod{8}, \\ 0 & \text{if } n \equiv 2,14 \pmod{16}, \\ 2S(N) & \text{if } n \equiv 6,10 \pmod{16}, \\ 2S(N) & \text{if } n \equiv 4 \pmod{8}, \\ 2\left(2^{\alpha-2} - \left(\frac{8}{N}\right)\right) S(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

All six formulae of Theorem 1.5 are new.

In Theorem 1.6 we give 14 forms all of whose representation numbers can be expressed in terms of the functions $\sigma(n)$ (see (1.1)) and $K_2(n)$, where $K_2(n)$ is defined for $n \in \mathbb{N}$ with $n \equiv 1 \pmod{2}$ by

$$K_2(n) := \sum_{\substack{(r,s)\in\mathbb{N}\times\mathbb{Z} \\ r\equiv 1 \pmod{2} \\ n=r^2+4s^2}} (-1)^{(r-1)/2} r \quad \text{(see [2, p. 146])}. \tag{1.5}$$

It is noted in [2, p. 147] that $K_2(n) = 0$ if $n \equiv 3 \pmod{4}$.

Theorem 1.6. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0, N \in \mathbb{N}$ *and* $\gcd(N,2) = 1$. *Then*

$$N(1,0,8,3,2,3;n) = \begin{cases} \sigma(N) + \left(\frac{8}{N}\right) K_2(N) & \text{if } n \equiv 1 \pmod{4}, \\ \sigma(N) & \text{if } n \equiv 3 \pmod{4}, \\ 0 & \text{if } n \equiv 2 \pmod{4}, \\ 12\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 8\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{16}; \end{cases}$$

$$N(2,0,4,3,2,3;n) = \begin{cases} \sigma(N) - \left(\frac{8}{N}\right) K_2(N) & \text{if } n \equiv 1 \pmod{4}, \\ \sigma(N) & \text{if } n \equiv 3 \pmod{4}, \\ 2\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 4\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 8\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{16}; \end{cases}$$

$$N(2,0,16,3,2,3;n)=\begin{cases}0 & \text{if } n\equiv 1,7 \pmod 8,\\ \sigma(N) & \text{if } n\equiv 3 \pmod 8,\\ \sigma(N)+K_2(N) & \text{if } n\equiv 5 \pmod 8,\\ \sigma(N)+\left(\dfrac{8}{N}\right)K_2(N) & \text{if } n\equiv 2 \pmod 8,\\ \sigma(N) & \text{if } n\equiv 6 \pmod 8,\\ 2\sigma(N) & \text{if } n\equiv 4 \pmod 8,\\ 4\sigma(N) & \text{if } n\equiv 8 \pmod{16},\\ 8\sigma(N) & \text{if } n\equiv 16 \pmod{32},\\ 24\sigma(N) & \text{if } n\equiv 0 \pmod{32};\end{cases}$$

$$N(1,0,1,4,4,5;n)=\begin{cases}3\sigma(N)+K_2(N) & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 3 \pmod 8,\\ 2\sigma(N) & \text{if } n\equiv 7 \pmod 8,\\ 6\sigma(N)-2\left(\dfrac{8}{N}\right)K_2(N) & \text{if } n\equiv 2 \pmod 8,\\ 6\sigma(N) & \text{if } n\equiv 6 \pmod 8,\\ 6\sigma(N) & \text{if } n\equiv 4 \pmod{16},\\ 2\sigma(N) & \text{if } n\equiv 12 \pmod{16},\\ 12\sigma(N) & \text{if } n\equiv 8 \pmod{16},\\ 8\sigma(N) & \text{if } n\equiv 16 \pmod{32},\\ 24\sigma(N) & \text{if } n\equiv 0 \pmod{32};\end{cases}$$

$$N(1,0,4,4,4,5;n)=\begin{cases}2\sigma(N) & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 3 \pmod 4,\\ 2\sigma(N)-2\left(\dfrac{8}{N}\right)K_2(N) & \text{if } n\equiv 2 \pmod 8,\\ 2\sigma(N) & \text{if } n\equiv 6 \pmod 8,\\ 6\sigma(N) & \text{if } n\equiv 4 \pmod{16},\\ 2\sigma(N) & \text{if } n\equiv 12 \pmod{16},\\ 12\sigma(N) & \text{if } n\equiv 8 \pmod{16},\\ 8\sigma(N) & \text{if } n\equiv 16 \pmod{32},\\ 24\sigma(N) & \text{if } n\equiv 0 \pmod{32};\end{cases}$$

$$N(1,0,16,4,4,5;n) = \begin{cases} \sigma(N) + K_2(N) & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 3 \pmod{4}, \\ 0 & \text{if } n \equiv 2 \pmod{8}, \\ 2\sigma(N) & \text{if } n \equiv 6 \pmod{8}, \\ 4\sigma(N) & \text{if } n \equiv 4 \pmod{16}, \\ 0 & \text{if } n \equiv 12 \pmod{16}, \\ 4\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 8\sigma(N) & \text{if } n \equiv 16 \pmod{32}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{32}; \end{cases}$$

$$N(2,0,2,4,4,5;n) = \begin{cases} \sigma(N) - K_2(N) & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 3 \pmod{8}, \\ 2\sigma(N) & \text{if } n \equiv 7 \pmod{8}, \\ 2\sigma(N) + 2\left(\frac{8}{N}\right)K_2(N) & \text{if } n \equiv 2 \pmod{8}, \\ 2\sigma(N) & \text{if } n \equiv 6 \pmod{8}, \\ 6\sigma(N) & \text{if } n \equiv 4 \pmod{16}, \\ 2\sigma(N) & \text{if } n \equiv 12 \pmod{16}, \\ 12\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 8\sigma(N) & \text{if } n \equiv 16 \pmod{32}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{32}; \end{cases}$$

$$N(2,0,8,4,4,5;n) = \begin{cases} 0 & \text{if } n \equiv 1,3 \pmod{8}, \\ \sigma(N) - K_2(N) & \text{if } n \equiv 5 \pmod{8}, \\ \sigma(N) & \text{if } n \equiv 7 \pmod{8}, \\ \sigma(N) + \left(\frac{8}{N}\right)K_2(N) & \text{if } n \equiv 2 \pmod{8}, \\ \sigma(N) & \text{if } n \equiv 6 \pmod{8}, \\ 2\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 4\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 8\sigma(N) & \text{if } n \equiv 16 \pmod{32}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{32}; \end{cases}$$

$$N(4,0,4,4,4,5;n)=\begin{cases}\sigma(N)-K_2(N) & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 2,3 \pmod 4,\\ 6\sigma(N) & \text{if } n\equiv 4 \pmod{16},\\ 2\sigma(N) & \text{if } n\equiv 12 \pmod{16},\\ 12\sigma(N) & \text{if } n\equiv 8 \pmod{16},\\ 8\sigma(N) & \text{if } n\equiv 16 \pmod{32},\\ 24\sigma(N) & \text{if } n\equiv 0 \pmod{32};\end{cases}$$

$$N(4,0,16,4,4,5;n)=\begin{cases}\frac{1}{2}\sigma(N)-\frac{1}{2}\left(\frac{8}{N}\right)K_2(N) & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 2,3 \pmod 4,\\ 4\sigma(N) & \text{if } n\equiv 4,8 \pmod{16},\\ 0 & \text{if } n\equiv 12 \pmod{16},\\ 8\sigma(N) & \text{if } n\equiv 16 \pmod{32},\\ 24\sigma(N) & \text{if } n\equiv 0 \pmod{32};\end{cases}$$

$$N(8,0,8,4,4,5;n)=\begin{cases}0 & \text{if } n\equiv 1,3,7 \pmod 8,\\ \sigma(N)-K_2(N) & \text{if } n\equiv 5 \pmod 8,\\ 0 & \text{if } n\equiv 2 \pmod 4,\\ 2\sigma(N) & \text{if } n\equiv 4 \pmod 8,\\ 4\sigma(N) & \text{if } n\equiv 8 \pmod{16},\\ 8\sigma(N) & \text{if } n\equiv 16 \pmod{32},\\ 24\sigma(N) & \text{if } n\equiv 0 \pmod{32};\end{cases}$$

$$N(16,0,16,4,4,5;n)=\begin{cases}0 & \text{if } n\equiv 1,3,7 \pmod 8,\\ \frac{1}{2}\sigma(N)+\frac{1}{2}K_2(N) & \text{if } n\equiv 5 \pmod 8,\\ 0 & \text{if } n\equiv 2 \pmod 4,\\ 2\sigma(N) & \text{if } n\equiv 4 \pmod{16},\\ 0 & \text{if } n\equiv 8,12 \pmod{16},\\ 8\sigma(N) & \text{if } n\equiv 16 \pmod{32},\\ 24\sigma(N) & \text{if } n\equiv 0 \pmod{32};\end{cases}$$

$$N(3,2,3,6,4,6;n) = \begin{cases} \sigma(N) - K_2(N) & \text{if } n \equiv 1 \pmod{8}, \\ \sigma(N) & \text{if } n \equiv 3 \pmod{8}, \\ 0 & \text{if } n \equiv 5,7 \pmod{8}, \\ \sigma(N) - \left(\frac{8}{N}\right) K_2(N) & \text{if } n \equiv 2 \pmod{8}, \\ \sigma(N) & \text{if } n \equiv 6 \pmod{8}, \\ 2\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 4\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 8\sigma(N) & \text{if } n \equiv 16 \pmod{32}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{32}; \end{cases}$$

$$N(4,4,5,4,4,5;n) = \begin{cases} \sigma(N) - \left(\frac{8}{N}\right) K_2(N) & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 3 \pmod{4}, \\ 2\sigma(N) - 2\left(\frac{8}{N}\right) K_2(N) & \text{if } n \equiv 2 \pmod{8}, \\ 0 & \text{if } n \equiv 6 \pmod{8}, \\ 4\sigma(N) & \text{if } n \equiv 4 \pmod{16}, \\ 0 & \text{if } n \equiv 12 \pmod{16}, \\ 4\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 8\sigma(N) & \text{if } n \equiv 16 \pmod{32}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{32}. \end{cases}$$

All 14 formulae of Theorem 1.6 are new.

In Theorem 1.7 we give two forms whose representation numbers can be expressed in terms of the functions $\sigma(n)$ and $a(n)$, where $a(n)$ is defined for $n \in \mathbb{N}_0$ by

$$\sum_{n=0}^{\infty} a(n) q^n = q \prod_{n=1}^{\infty} (1-q^{2n})(1-q^{4n})(1-q^{6n})(1-q^{12n}) \quad \text{(see [1, p. 278])}, \tag{1.6}$$

where $q \in \mathbb{C}$, $|q| < 1$. It is noted in [1, p. 278] that $a(n) = 0$ for $n \equiv 0 \pmod{2}$.

Theorem 1.7. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0, N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(2,0,6,1,1,1;n) = \begin{cases} 3(3^{\beta+1}-2)\sigma(N) + 3a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2(3^{\beta+1}-2)\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 6(3^{\beta+1}-2)\sigma(N) & \text{if } n \equiv 0 \pmod{4}; \end{cases}$$

$$N(4,0,12,1,1,1;n) = \begin{cases} 3\sigma(N) + 3(-1)^{(n-1)/2}a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 0 & \text{if } n \equiv 2 \pmod{4}, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 12(2^{\alpha-2}-1)\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

Both formulae of Theorem 1.7 are new.

2. Proof of Theorem 1.1

We begin with a result on theta functions that we will use in the proof of Theorem 1.1.

Theorem 2.1. *Let M and N be positive integers such that*

$$M > N, \quad M \equiv N \pmod{2}. \tag{2.1}$$

Then, for $q \in \mathbb{C}$ with $|q| < 1$, we have

$$\begin{aligned} \sum_{(x,y)\in\mathbb{Z}^2} & q^{Mx^2+2Nxy+My^2} \\ &= \varphi(q^{\frac{M-N}{2}})\varphi(q^{\frac{M+N}{2}}) - \varphi(q^{\frac{M-N}{2}})\varphi(q^{2(M+N)}) \\ &\quad - \varphi(q^{\frac{M+N}{2}})\varphi(q^{2(M-N)}) + 2\varphi(q^{2(M-N)})\varphi(q^{2(M+N)}), \end{aligned} \tag{2.2}$$

where Ramanujan's theta function φ is defined by

$$\varphi(q) := \sum_{x\in\mathbb{Z}} q^{x^2}. \tag{2.3}$$

Proof. As $M, N \in \mathbb{N}$, $M > N$ and $M \equiv N \pmod{2}$, we see that

$$A := \frac{M+N}{2} \in \mathbb{N}, \quad B := \frac{M-N}{2} \in \mathbb{N}.$$

Thus

$$M = A + B, \quad N = A - B.$$

Now

$$\begin{aligned} \sum_{(x,y)\in\mathbb{Z}^2} q^{Mx^2+2Nxy+My^2} &= \sum_{(x,y)\in\mathbb{Z}^2} q^{M(y-x)^2+2(M+N)(y-x)x+2(M+N)x^2} \\ &= \sum_{(x,z)\in\mathbb{Z}^2} q^{Mz^2+2(M+N)zx+2(M+N)x^2} \\ &= \sum_{(x,z)\in\mathbb{Z}^2} q^{(A+B)z^2+4Azx+4Ax^2} \end{aligned}$$

$$\begin{aligned}
&= \sum_{(x,z)\in\mathbb{Z}^2} q^{A(2x+z)^2+Bz^2} = \sum_{\substack{(t,z)\in\mathbb{Z}^2\\ t\equiv z \pmod 2}} q^{At^2+Bz^2}\\
&= \sum_{(t,z)\in\mathbb{Z}^2} q^{At^2+Bz^2} - \sum_{\substack{(t,z)\in\mathbb{Z}^2\\ t\equiv 0 \pmod 2}} q^{At^2+Bz^2}\\
&\quad - \sum_{\substack{(t,z)\in\mathbb{Z}^2\\ z\equiv 0 \pmod 2}} q^{At^2+Bz^2} + 2 \sum_{\substack{(t,z)\in\mathbb{Z}^2\\ t\equiv z\equiv 0 \pmod 2}} q^{At^2+Bz^2}\\
&= \varphi(q^A)\varphi(q^B) - \varphi(q^{4A})\varphi(q^B)\\
&\quad - \varphi(q^A)\varphi(q^{4B}) + 2\varphi(q^{4A})\varphi(q^{4B})\\
&= \varphi(q^{\frac{M-N}{2}})\varphi(q^{\frac{M+N}{2}}) - \varphi(q^{\frac{M-N}{2}})\varphi(q^{2(M+N)})\\
&\quad - \varphi(q^{\frac{M+N}{2}})\varphi(q^{2(M-N)}) + 2\varphi(q^{2(M-N)})\varphi(q^{2(M+N)}),
\end{aligned}$$

as required. □

We are now ready to prove Theorem 1.1.

Proof of part (i) of Theorem 1.1. For $q \in \mathbb{C}$ with $|q| < 1$, appealing to Theorem 2.1, we obtain

$$\begin{aligned}
&\sum_{n\in\mathbb{N}_0} N(a_1, 0, c_1, a_2, b_2, c_2; n)q^n\\
&\quad = \sum_{(x,y,z,t)\in\mathbb{Z}^4} q^{a_1x^2+c_1y^2+a_2z^2+b_2zt+c_2t^2}\\
&\quad = \sum_{(x,y)\in\mathbb{Z}^2} q^{a_1x^2+c_1y^2} \sum_{(z,t)\in\mathbb{Z}^2} q^{a_2z^2+b_2zt+c_2t^2}\\
&\quad = \varphi(q^{a_1})\varphi(q^{c_1}) \sum_{(z,t)\in\mathbb{Z}^2} q^{M_2z^2+2N_2zt+M_2t^2}\\
&\quad = \varphi(q^{a_1})\varphi(q^{c_1})(\varphi(q^{\frac{M_2-N_2}{2}})\varphi(q^{\frac{M_2+N_2}{2}}) - \varphi(q^{\frac{M_2-N_2}{2}})\varphi(q^{2(M_2+N_2)})\\
&\quad\quad - \varphi(q^{\frac{M_2+N_2}{2}})\varphi(q^{2(M_2-N_2)}) + 2\varphi(q^{2(M_2-N_2)})\varphi(q^{2(M_2+N_2)}))\\
&\quad = \varphi(q^{a_1})\varphi(q^{c_1})\varphi(q^{\frac{M_2-N_2}{2}})\varphi(q^{\frac{M_2+N_2}{2}}) - \varphi(q^{a_1})\varphi(q^{c_1})\varphi(q^{\frac{M_2-N_2}{2}})\varphi(q^{2(M_2+N_2)})\\
&\quad\quad - \varphi(q^{a_1})\varphi(q^{c_1})\varphi(q^{\frac{M_2+N_2}{2}})\varphi(q^{2(M_2-N_2)})\\
&\quad\quad + 2\varphi(q^{a_1})\varphi(q^{c_1})\varphi(q^{2(M_2-N_2)})\varphi(q^{2(M_2+N_2)})
\end{aligned}$$

$$= \sum_{n\in\mathbb{N}_0} N\left(a_1, c_1, \frac{M_2-N_2}{2}, \frac{M_2+N_2}{2}; n\right) q^n$$
$$- \sum_{n\in\mathbb{N}_0} N\left(a_1, c_1, \frac{M_2-N_2}{2}, 2(M_2+N_2); n\right) q^n$$
$$- \sum_{n\in\mathbb{N}_0} N\left(a_1, c_1, \frac{M_2+N_2}{2}, 2(M_2-N_2); n\right) q^n$$
$$+ 2\sum_{n\in\mathbb{N}_0} N(a_1, c_1, 2(M_2-N_2), 2(M_2+N_2); n) q^n.$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain the asserted formula. □

Proof of part (ii) of Theorem 1.1. For $q \in \mathbb{C}$ with $|q| < 1$, appealing to Theorem 2.1, we obtain

$$\sum_{n\in\mathbb{N}_0} N(a_1, b_1, c_1, a_2, b_2, c_2; n) q^n$$
$$= \sum_{(x,y,z,t)\in\mathbb{Z}^4} q^{a_1x^2+b_1xy+c_1y^2+a_2z^2+b_2zt+c_2t^2}$$
$$= \sum_{(x,y)\in\mathbb{Z}^2} q^{a_1x^2+b_1xy+c_1y^2} \sum_{(z,t)\in\mathbb{Z}^2} q^{a_2z^2+b_2zt+c_2t^2}$$
$$= \sum_{(x,y)\in\mathbb{Z}^2} q^{M_1x^2+2N_1xy+M_1y^2} \sum_{(z,t)\in\mathbb{Z}^2} q^{M_2z^2+2N_2zt+M_2t^2}$$
$$= (\varphi(q^{\frac{M_1-N_1}{2}})\varphi(q^{\frac{M_1+N_1}{2}}) - \varphi(q^{\frac{M_1-N_1}{2}})\varphi(q^{2(M_1+N_1)})$$
$$- \varphi(q^{\frac{M_1+N_1}{2}})\varphi(q^{2(M_1-N_1)}) + 2\varphi(q^{2(M_1-N_1)})\varphi(q^{2(M_1+N_1)}))$$
$$\times (\varphi(q^{\frac{M_2-N_2}{2}})\varphi(q^{\frac{M_2+N_2}{2}}) - \varphi(q^{\frac{M_2-N_2}{2}})\varphi(q^{2(M_2+N_2)})$$
$$- \varphi(q^{\frac{M_2+N_2}{2}})\varphi(q^{2(M_2-N_2)}) + 2\varphi(q^{2(M_2-N_2)})\varphi(q^{2(M_2+N_2)})).$$

Multiplying this out and using $\varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d) = \sum_{n\in\mathbb{N}_0} N(a,b,c,d;n)q^n$ on the resulting 16 products to evaluate coefficients of q^n $(n \in \mathbb{N})$, we obtain the asserted formula. □

Proof of part (iii) of Theorem 1.1. Appealing to part (i) of this theorem, we have

$$N(r, 0, s, 1, 1, 1; n) = N(4r, 0, 4s, 4, 4, 4; 4n)$$
$$= N(4r, 4s, 1, 3; 4n) - N(4r, 4s, 1, 12; 4n)$$
$$- N(4r, 4s, 3, 4; 4n) + 2N(4r, 4s, 4, 12; 4n).$$

Now

$$N(4r,4s,1,12;4n) = N(4r,4s,4,12;4n) = N(1,3,r,s;n),$$
$$N(4r,4s,3,4;4n) = N(4r,4s,12,4;4n) = N(1,3,r,s;n),$$
$$N(4r,4s,4,12;4n) = N(1,3,r,s;n),$$

so that

$$N(r,0,s,1,1,1;n) = N(4r,4s,1,3;4n) = N(1,3,4r,4s;4n),$$

as claimed. □

Proof of part (iv) of Theorem 1.1. For $q \in \mathbb{C}$ with $|q| < 1$ we define

$$a(q) := \sum_{(z,t)\in\mathbb{Z}^2} q^{z^2+zt+t^2}.$$

It was shown in [3, Theorem 2.7(a)] that

$$a(q) = 2\varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3).$$

Hence

$$\begin{aligned}\sum_{n=0}^{\infty} N(2r,0,2s,1,1,1;n)q^n &= \varphi(q^{2r})\varphi(q^{2s})a(q)\\ &= \varphi(q^{2r})\varphi(q^{2s})(2\varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3))\\ &= 2\varphi(q)\varphi(q^3)\varphi(q^{2r})\varphi(q^{2s})\\ &\quad - \varphi(-q)\varphi((-q)^3)\varphi((-q)^{2r})\varphi((-q)^{2s})\\ &= 2\sum_{n=0}^{\infty} N(1,3,2r,2s;n)q^n\\ &\quad - \sum_{n=0}^{\infty} N(1,3,2r,2s;n)(-q)^n.\end{aligned}$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we deduce

$$N(2r,0,2s,1,1,1;n) = 2N(1,3,2r,2s;n) - (-1)^n N(1,3,2r,2s;n)$$

from which the asserted formula follows. □

Proof of part (v) of Theorem 1.1. Appealing to part (i) of this theorem, we have

$$\begin{aligned}N(r,0,s,2,2,2;n) &= N(2r,0,2s,4,4,4;2n)\\ &= N(2r,2s,1,3;2n) - N(2r,2s,1,12;2n)\\ &\quad - N(2r,2s,3,4;2n) + 2N(2r,2s,4,12;2n).\end{aligned}$$

Now

$$N(2r, 2s, 1, 12; 2n) = N(2r, 2s, 4, 12; 2n) = N(2, 6, r, s; n),$$
$$N(2r, 2s, 3, 4; 2n) = N(2r, 2s, 12, 4; 2n) = N(2, 6, r, s; n),$$
$$N(2r, 2s, 4, 12; 2n) = N(2, 6, r, s; n),$$

so that

$$N(r, 0, s, 2, 2, 2; n) = N(2r, 2s, 1, 3; 2n) = N(1, 3, 2r, 2s; 2n),$$

as claimed. □

3. Proof of Theorem 1.2

We just give the details for $N(1, 0, 2, 3, 2, 3; n)$ and $N(1, 0, 3, 1, 1, 1; n)$ as the remaining eight representation numbers can be evaluated in a similar manner.

To determine $N(1, 0, 2, 3, 2, 3; n)$ we take $a_1 = 1$, $c_1 = 2$, $a_2 = 3$, $b_2 = 2$ and $c_2 = 3$ in Theorem 1.1(i). Thus $M_2 = 3$ and $N_2 = 1$. Then, for all $n \in \mathbb{N}$, we have

$$N(1, 0, 2, 3, 2, 3; n) = N(1, 1, 2, 2; n) - N(1, 1, 2, 8; n) - N(1, 2, 2, 4; n) + 2N(1, 2, 4, 8; n). \quad (3.1)$$

Consulting Table A.1 in Appendix we see that $N(1, 1, 2, 2; n)$, $N(1, 1, 2, 8; n)$, $N(1, 2, 2, 4; n)$ and $N(1, 2, 4, 8; n)$ are evaluated in [1, Theorem 1.8], [2, Theorem 4.3], [1, Theorem 1.14] and [2, Theorem 4.1], respectively. The evaluations are given in terms of σ and K_2. Using these evaluations in (3.1), we obtain the formula for $N(1, 0, 2, 3, 2, 3; n)$ given in Theorem 1.2(i).

To determine $N(1, 0, 3, 1, 1, 1; n)$ we appeal to Theorem 1.1(iii). Taking $r = 1$ and $s = 3$ we obtain

$$N(1, 0, 3, 1, 1, 1; n) = N(1, 3, 4, 12; 4n). \quad (3.2)$$

Table A.1 tells us that $N(1, 3, 4, 12; n)$ is evaluated in [1, Theorem 1.17]. Using this evaluation, we obtain the formula for $N(1, 0, 3, 1, 1, 1; n)$ given in Theorem 1.2(iii).

4. Proof of Theorem 1.3

We just give the proof of the formula for $N(2, 0, 16, 4, 4, 5; n)$. The remaining eight representation numbers can be determined in a similar manner.

From Sec. 1 we have

$$N(2, 0, 16, 4, 4, 5; n) = N(2, 0, 16, 5, 6, 5; n).$$

Taking $a_1 = 2$, $c_1 = 16$, $a_2 = 5$, $b_2 = 6$ and $c_2 = 5$ in Theorem 1.1(i), we have $M_2 = 5$ and $N_2 = 3$, so that

$$N(2,0,16,4,4,5;n) = N(2,16,1,4;n) - N(2,16,1,16;n) - N(2,16,4,4;n) + 2N(2,16,4,16;n).$$

Hence

$$N(2,0,16,4,4,5;n) = N(1,2,4,16;n) - N(1,2,16,16;n) - N(1,2,2,8;n/2) + 2N(1,2,8,8;n/2). \quad (4.1)$$

From Table A.1 we see that formulae for $N(1,2,4,16;n)$, $N(1,2,16,16;n)$, $N(1,2,2,8;n)$ and $N(1,2,8,8;n)$ are given in [2, Theorem 4.17], [2, Theorem 4.18], [5, Theorem 5.7] and [5, Theorem 5.8], respectively. Using these formulae in (4.1), we obtain the formula for $N(2,0,16,4,4,5;n)$ stated in Theorem 1.3.

5. Proof of Theorem 1.4

We determine the formula for $N(2,0,2,1,1,1;n)$. The remaining nine representation numbers can be obtained in a similar manner.

Taking $r = s = 1$ in Theorem 1.1(iv) we obtain

$$N(2,0,2,1,1,1;n) = (2-(-1)^n)N(1,2,2,3;n). \quad (5.1)$$

Appealing to Table A.1 we see that a formula for $N(1,2,2,3;n)$ is given in [3, Theorem 6.1], namely,

$$N(1,2,2,3;n) = \left(2^\alpha + \frac{1}{2}(1+(-1)^n)(-1)^{\alpha+\beta+(N-1)/2}\right) \times \left(3^{\beta+1} - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) A(N).$$

If n is odd then $\alpha = 0$ and we have

$$N(2,0,2,1,1,1;n) = 3N(1,2,2,3;n) = 3\left(3^{\beta+1} - (-1)^{\beta}\left(\frac{N}{3}\right)\right) A(N).$$

If n is even we have

$$N(2,0,2,1,1,1;n) = N(1,2,2,3;n) = (2^\alpha + (-1)^{\alpha+\beta+(N-1)/2})\left(3^{\beta+1} - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right)\right) A(N).$$

This completes the evaluation of $N(2,0,2,1,1,1;n)$.

6. Proof of Theorem 1.5

We just give the details for the proof of the formula for $N(3,2,3,4,4,5;n)$ as the remaining five representation numbers can be treated similarly.

As noted in Sec. 1 we have

$$N(3,2,3,4,4,5;n) = N(3,2,3,5,6,5;n). \tag{6.1}$$

Thus we can take $M_1 = 3$, $N_1 = 1$, $M_2 = 5$ and $N_2 = 3$ in Theorem 1.1(ii). We obtain

$$\begin{aligned} N(3,2,3,4,4,5;n) = &-2N(1,1,1,2;n/4) + N(1,1,2,4;n) \\ &+ 4N(1,1,2,4;n/4) - N(1,1,2,16;n) \\ &- N(1,1,4,8;n) + N(1,1,8,16;n) \\ &+ N(1,2,2,2;n/2) - 2N(1,2,2,8;n/2) \\ &- 2N(1,2,4,4;n) + 3N(1,2,4,16;n) \\ &+ 3N(1,4,4,8;n) - 4N(1,4,8,16;n) \end{aligned} \tag{6.2}$$

as

$$\begin{aligned} N(2,4,4,4;n) &= N(1,2,2,2;n/2), \\ N(2,4,4,16;n) &= N(1,2,2,8;n/2), \\ N(4,8,4,4;n) &= N(1,1,1,2;n/4), \\ N(4,8,4,16;n) &= N(1,1,2,4;n/4). \end{aligned}$$

Appealing to Table A.1 we see that explicit formulae for $N(1,1,1,2;n)$, $N(1,1,2,4;n)$, $N(1,1,4,8;n)$, $N(1,2,2,2;n)$, $N(1,2,2,8;n)$, $N(1,2,4,4;n)$ and $N(1,4,4,8;n)$ are given in [5] and for $N(1,1,2,16;n)$, $N(1,1,8,16;n)$, $N(1,2,4,16;n)$ and $N(1,4,8,16;n)$ in [2]. Using these formulae in (6.2), and then appealing to (6.1), we obtain the formula for $N(3,2,3,4,4,5;n)$ stated in Theorem 1.5.

7. Proof of Theorem 1.6

We just prove the formula for $N(3,2,3,6,4,6;n)$. The remaining 13 representation numbers can be treated similarly.

We take $M_1 = 3$, $N_1 = 1$, $M_2 = 6$ and $N_2 = 2$ in Theorem 1.1(ii). We obtain

$$\begin{aligned} N(3,2,3,6,4,6;n) = &-N(1,1,2,2;n/2) - 2N(1,1,2,2;n/4) \\ &+ 3N(1,2,2,4;n/2) + N(1,1,2,8;n/2) \\ &- 4N(1,2,4,8;n/2) + 4N(1,2,2,4;n/4) \\ &+ N(1,2,2,4;n) - 2N(1,2,4,8;n) \\ &- N(1,2,2,16;n) + 3N(1,2,8,16;n) \\ &+ N(1,4,8,8;n) - 2N(1,8,8,16;n). \end{aligned} \tag{7.1}$$

Appealing to Table A.1 we see that explicit formulae for $N(1,1,2,2;n)$ and $N(1,2,2,4;n)$ are given in [1] and for $N(1,1,2,8;n)$, $N(1,2,4,8;n)$, $N(1,2,2,16;n)$, $N(1,2,8,16;n)$, $N(1,4,8,8;n)$ and $N(1,8,8,16;n)$ in [2]. Putting these formulae into (7.1), we obtain the formula for $N(3,2,3,6,4,6;n)$ stated in Theorem 1.6.

8. Proof of Theorem 1.7

We evaluate $N(2,0,6,1,1,1;n)$. The representation number $N(4,0,12,1,1,1;n)$ can be done similarly.

We choose $r=1$ and $s=3$ in Theorem 1.1(iv). We obtain

$$N(2,0,6,1,1,1;n) = (2-(-1)^n)N(1,2,3,6;n). \tag{8.1}$$

From Table A.1 we see that a formula for $N(1,2,3,6;n)$ is given in [1, Theorem 1.15], namely

$$N(1,2,3,6;n) = \begin{cases} (3^{\beta+1}-2)\sigma(N)+a(n) & \text{if } n \equiv 1 \pmod 2, \\ 2(3^{\beta+1}-2)\sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 6(3^{\beta+1}-2)\sigma(N) & \text{if } n \equiv 0 \pmod 4. \end{cases} \tag{8.2}$$

The formula for $N(2,0,6,1,1,1;n)$ stated in Theorem 1.7 now follows from (8.1) and (8.2).

9. Final Remarks

The authors have checked all the formulae of Theorems 1.2–1.7 numerically for all positive integers less than or equal to 400.

Other explicit formulae for the number of representations of a positive integer as a sum of two positive-definite, integral, binary quadratic forms can be deduced from Theorem 1.1. For example, using Theorem 1.1(iv), $N(4,0,4,1,1,1;n)$ can be determined in terms of the function $A(n)$ and the function $E(n)$ given by

$$E(n) := \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ i,j \text{ odd} \\ 4n=i^2+3j^2}} (-1)^{(i-1)/2} i$$

and $N(12,0,12,1,1,1;n)$ can be determined in terms of the function $A(n)$ and the function $F(n)$ given by

$$F(n) := \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ i,j \text{ odd} \\ 4n=i^2+3j^2}} (-1)^{(j-1)/2} j.$$

The functions $E(n)$ and $F(n)$ are related, see [4] for details. However, there are some representation numbers which are known explicitly, which we are unable to deduce from Theorem 1.1. This occurs, for example, for the representation number $N(1,0,4,1,1,1;n)$ [15] since to apply Theorem 1.1 (parts (i) or (iii)) in this case we need an explicit formula for $N(1,3,4,16;n)$ and this is not known.

Appendix

Table A.1. Location of evaluation of $N(a, b, c, d;\ n)$.

$a\ b\ c\ d$	Evaluation of $N(a, b, c, d; n)$	$a\ b\ c\ d$	Evaluation of $N(a, b, c, d; n)$
1 1 1 1	[1, Theorem 1.6]	1 2 8 8	[5, Theorem 5.8]
1 1 1 2	[5, Theorem 5.1]	1 2 8 16	[2, Theorem 4.13]
1 1 1 3	[3, Theorem 4.1]	1 2 16 16	[2, Theorem 4.18
1 1 1 4	[1, Theorem 1.7]	1 3 3 3	[3, Theorem 8.1
1 1 1 8	[5, Theorem 5.10]	1 3 3 4	[1, Theorem 1.16
1 1 1 12	[4, Corollary 7.1(a)]	1 3 3 12	[4, Corollary 7.1(e)]
1 1 1 16	[2, Theorem 4.10]	1 3 4 4	[4, Corollary 7.1(f)]
1 1 2 2	[1, Theorem 1.8]	1 3 4 12	[1, Theorem 1.17]
1 1 2 4	[5, Theorem 5.3]	1 3 6 6	[3, Theorem 9.1]
1 1 2 6	[3, Theorem 5.1]	1 3 12 12	[4, Corollary 7.1(g)]
1 1 2 8	[2, Theorem 4.3]	1 4 4 4	[1, Theorem 1.18]
1 1 2 16	[2, Theorem 4.15]	1 4 4 8	[5, Theorem 5.6]
1 1 3 3	[1, Theorem 1.9]	1 4 4 12	[4, Corollary 7.1(h)]
1 1 3 4	[4, Corollary 7.1(b)]	1 4 4 16	[2, Theorem 4.7]
1 1 3 12	[1, Theorem 1.10]	1 4 8 8	[2, Theorem 4.4]
1 1 4 4	[1, Theorem 1.11]	1 4 8 16	[2, Theorem 4.14]
1 1 4 8	[5, Theorem 5.5]	1 4 12 12	[1, Theorem 1.20]
1 1 4 12	[4, Corollary 7.1(c)]	1 4 16 16	[2, Theorem 4.5]
1 1 4 16	[2, Theorem 4.8]	1 6 6 12	[4, Corollary 7.1(i)]
1 1 8 8	[2, Theorem 4.2]	1 8 8 8	[5, Theorem 5.9]
1 1 8 16	[2, Theorem 4.16]	1 8 8 16	[2, Theorem 4.11]
1 1 12 12	[1, Theorem 1.13]	1 8 16 16	[2, Theorem 4.19]
1 1 16 16	[2, Theorem 4.6]	1 12 12 12	[4, Corollary 7.1(j)]
1 2 2 2	[5, Theorem 5.2]	1 16 16 16	[2, Theorem 4.12]
1 2 2 3	[3, Theorem 6.1]	2 2 3 4	[4, Corollary 7.1(k)]
1 2 2 4	[1, Theorem 1.14]	3 3 3 4	[4, Corollary 7.1(l)]
1 2 2 8	[5, Theorem 5.7]	3 3 4 4	[1, Theorem 1.23]
1 2 2 12	[4, Corollary 7.1(d)]	3 3 4 12	[4, Corollary 7.1(m)]
1 2 2 16	[2, Theorem 4.9]	3 4 4 4	[4, Corollary 7.1(n)]
1 2 3 6	[1, Theorem 1.15]	3 4 4 12	[1, Theorem 1.24]
1 2 4 4	[5, Theorem 5.4]	3 4 6 6	[4, Corollary 7.1(o)]
1 2 4 8	[2, Theorem 4.1]	3 4 12 12	[4, Corollary 7.1(p)]
1 2 4 16	[2, Theorem 4.17]		

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