Ramanujan J (2014) 33:23–53
DOI 10.1007/s11139-013-9503-1

The power series expansion of certain infinite products $q^r \prod_{n=1}^{\infty}(1-q^n)^{a_1}(1-q^{2n})^{a_2}\cdots(1-q^{mn})^{a_m}$

Lerna Pehlivan · Kenneth S. Williams

Received: 12 June 2012 / Accepted: 30 May 2013 / Published online: 21 August 2013

Abstract Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, rational numbers, and complex numbers, respectively. If $f(q)$ is a complex-valued function with

$$f(q)=\sum_{n=0}^{\infty} f_n q^n \quad \bigl(q\in\mathbb{C}, |q|<1\bigr)$$

we define

$$\bigl[f(q)\bigr]_n := f_n \quad (n\in\mathbb{N}_0).$$

For $k\in\mathbb{N}$ we define

$$E_k := \prod_{n=1}^{\infty}\bigl(1-q^{kn}\bigr) \quad \bigl(q\in\mathbb{C}, |q|<1\bigr).$$

We show how modular equations of a special form can be used in conjunction with the representation numbers of certain quadratic forms to determine

$$\bigl[q^r E_1^{a_1}\cdots E_m^{a_m}\bigr]_n \quad (r\in\mathbb{N}_0, m\in\mathbb{N}, a_1,\ldots,a_m\in\mathbb{Z})$$

L. Pehlivan
Department of Mathematics and Statistics, York University, 4700 Keele Street, North York M3J 1P3, Ontario, Canada
e-mail: pehlivan@mathstat.yorku.ca

K.S. Williams (✉)
Centre for Research in Algebra and Number Theory, School of Mathematics and Statistics, Carleton University, Ottawa K1S 5B6, Ontario, Canada
e-mail: kwilliam@connect.carleton.ca

for certain products $q^r E_1^{a_1} \cdots E_m^{a_m}$. For example, we show that

$$\left[q^2 \frac{E_1^4 E_{16}^4}{E_2^2 E_8^2}\right]_n = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 4, \\ -\sigma(N) & \text{if } n \equiv 3 \pmod 4, \\ \sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 4\sigma(N) & \text{if } n \equiv 4 \pmod 8, \\ 0 & \text{if } n \equiv 0 \pmod 8, \end{cases}$$

where N denotes the odd part of the positive integer n and

$$\sigma(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d.$$

Keywords Infinite products · Quadratic forms · Representations · Theta functions · Modular equations

Mathematics Subject Classification 11E25 · 11F27 · 11B65 · 05A19

1 Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, rational numbers, and complex numbers, respectively. Throughout this paper, q denotes a complex variable such that $|q| < 1$. If the complex-valued function $f(q)$ has a power series expansion $\sum_{n=0}^{\infty} f_n q^n$ valid for $|q| < 1$, we use $[f(q)]_n$ to denote the coefficient f_n $(n \in \mathbb{N}_0)$.

For $k \in \mathbb{N}$ we define

$$E_k = E_k(q) := \prod_{n=1}^{\infty} (1 - q^{kn}). \tag{1.1}$$

We are interested in determining the coefficient of q^n $(n \in \mathbb{N}_0)$ in the expansion of a product $q^r E_1^{a_1} \cdots E_m^{a_m}$ $(r \in \mathbb{N}_0, m \in \mathbb{N}, a_1, \ldots, a_m \in \mathbb{Z}, (a_1, \ldots, a_m) \neq (0, \ldots, 0))$ in powers of q. Clearly, the coefficient of q^n is 0 for $n < r$ and is 1 for $n = r$. We describe a simple method by which it is possible to do this for certain products $q^r E_1^{a_1} \cdots E_m^{a_m}$ using modular equations of a special type (Sect. 2) in conjunction with known formulae for the number of representations of a positive integer by certain quadratic forms (Sect. 3). Applying this method in Sects. 4–9, we obtain the following results, where $(\frac{D}{d})$ $(d \in \mathbb{N})$ is the Legendre–Jacobi–Kronecker symbol for discriminant D. Section 10 gives an application of our results.

Theorem 1.1 *Let* $n \in \mathbb{N}$. *Then*

(i) $$\left[\frac{E_1^4}{E_2^2}\right]_n = -4\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right) + 8\sum_{\substack{d\in\mathbb{N}\\ d\mid n/2}}\left(\frac{-4}{d}\right);$$

(ii) $$\left[q\frac{E_8^4}{E_4^2}\right]_n = \sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right) - \sum_{\substack{d\in\mathbb{N}\\ d\mid n/2}}\left(\frac{-4}{d}\right);$$

(iii) $$\left[\frac{E_1^2E_2E_6^3}{E_3^2E_4E_{12}}\right]_n = -2\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right) + 6\sum_{\substack{d\in\mathbb{N}\\ d\mid n/3}}\left(\frac{-4}{d}\right);$$

(iv) $$\left[q\frac{E_2^3E_6E_{12}^2}{E_1E_3E_4^2}\right]_n = \sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right) - \sum_{\substack{d\in\mathbb{N}\\ d\mid n/3}}\left(\frac{-4}{d}\right);$$

(v) $$\left[\frac{E_1E_4E_{10}^2}{E_5E_{20}}\right]_n = -\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right) + 5\sum_{\substack{d\in\mathbb{N}\\ d\mid n/5}}\left(\frac{-4}{d}\right);$$

(vi) $$\left[q\frac{E_2^2E_5E_{20}}{E_1E_4}\right]_n = \sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right) - \sum_{\substack{d\in\mathbb{N}\\ d\mid n/5}}\left(\frac{-4}{d}\right);$$

(vii) $$\left[q\frac{E_2E_3E_4E_{24}}{E_1E_8}\right]_n = \sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-8}{d}\right) - \sum_{\substack{d\in\mathbb{N}\\ d\mid n/3}}\left(\frac{-8}{d}\right);$$

(viii) $$\left[\frac{E_1^6E_6}{E_2^3E_3^2}\right]_n = -6\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-3}{d}\right) + 12\sum_{\substack{d\in\mathbb{N}\\ d\mid n/2}}\left(\frac{-3}{d}\right);$$

(ix) $$\left[\frac{E_2^6E_3}{E_1^3E_6^2}\right]_n = 3\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-3}{d}\right) + 3\sum_{\substack{d\in\mathbb{N}\\ d\mid n/2}}\left(\frac{-3}{d}\right);$$

(x) $$\left[\frac{E_2E_3^6}{E_1^2E_6^3}\right]_n = 2\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-3}{d}\right) + 4\sum_{\substack{d\in\mathbb{N}\\ d\mid n/2}}\left(\frac{-3}{d}\right);$$

(xi) $$\left[\frac{E_1^3}{E_3}\right]_n = -3\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-3}{d}\right) + 9\sum_{\substack{d\in\mathbb{N}\\ d\mid n/3}}\left(\frac{-3}{d}\right);$$

(xii) $$\left[q\frac{E_1E_6^6}{E_2^2E_3^3}\right]_n = \sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-3}{d}\right) - \sum_{\substack{d\in\mathbb{N}\\ d\mid n/2}}\left(\frac{-3}{d}\right).$$

For our next theorem we need the arithmetic function

$$K_2(n) := \sum_{\substack{(r,s)\in\mathbb{N}\times\mathbb{Z}\\ r\equiv 1 \pmod 2\\ n=r^2+4s^2}} (-1)^{(r-1)/2} r, \quad n\in\mathbb{N}, \tag{1.2}$$

which was defined in [3, p. 146] and was originally used by Liouville in his work on quadratic forms [20, p. 413].

Theorem 1.2 *Let* $n \in \mathbb{N}$. *Set* $n = 2^{\alpha} N$, *where* $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N, 2) = 1$. *Then*

(i)
$$\left[\frac{E_1^8}{E_2^4}\right]_n = \begin{cases} -8\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 2; \end{cases}$$

(ii)
$$\left[\frac{E_1^4 E_2^2}{E_4^2}\right]_n = \begin{cases} 4(-1)^{(n+1)/2}\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 0 & \text{if } n \equiv 2 \pmod 4, \\ -8\sigma(N) & \text{if } n \equiv 4 \pmod 8, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 8; \end{cases}$$

(iii)
$$\left[\frac{E_2^{14}}{E_1^4 E_4^6}\right]_n = \begin{cases} 4(\frac{-4}{N})\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 0 & \text{if } n \equiv 2 \pmod 4, \\ -8\sigma(N) & \text{if } n \equiv 4 \pmod 8, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 8; \end{cases}$$

(iv)
$$\left[\frac{E_2^{10}}{E_1^4 E_8^2}\right]_n = \begin{cases} 4(\frac{8}{N})K_2(N) & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4, \\ 4(\frac{-4}{N})\sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{if } n \equiv 4 \pmod 8, \\ -8\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{16}; \end{cases}$$

(v)
$$\left[\frac{E_1^2 E_2 E_4^3}{E_8^2}\right]_n = -2 \sum_{\substack{d \in \mathbb{N} \\ d \mid N}} d\left(\frac{8}{d}\right);$$

(vi)
$$\left[\frac{E_1^4 E_4^{10}}{E_2^6 E_8^4}\right]_n = \begin{cases} -4\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 8\sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 4; \end{cases}$$

(vii)
$$\left[\frac{E_1^4 E_4^4}{E_2^2 E_8^2}\right]_n = \begin{cases} -4(\frac{8}{N})K_2(N) & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4, \\ 4(\frac{-4}{N})\sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{if } n \equiv 4 \pmod 8, \\ -8\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{16}; \end{cases}$$

(viii)
$$\left[\frac{E_2^7 E_4}{E_1^2 E_8^2}\right]_n = 2(-1)^{n+1} \sum_{\substack{d \in \mathbb{N} \\ d \mid N}} d\left(\frac{8}{d}\right);$$

(ix) $$\left[\frac{E_2^3E_4^7}{E_1^2E_8^4}\right]_n=\begin{cases}2(-1)^{(n-1)/2}\sum_{\substack{d\in\mathbb{N}\\d\mid N}}d(\frac{8}{d}) & \textit{if } n\equiv 1 \pmod 2,\\ 2(-1)^{(n+2)/2}\sum_{\substack{d\in\mathbb{N}\\d\mid N}}d(\frac{8}{d}) & \textit{if } n\equiv 0 \pmod 2;\end{cases}$$

(x) $$\left[\frac{E_1^4E_8^{10}}{E_2^2E_4^4E_{16}^4}\right]_n=\begin{cases}-4\sigma(N) & \textit{if } n\equiv 1 \pmod 4,\\ 0 & \textit{if } n\equiv 3 \pmod 4,\\ 4\sigma(N) & \textit{if } n\equiv 2 \pmod 4,\\ 8\sigma(N) & \textit{if } n\equiv 4 \pmod 8,\\ 24\sigma(N) & \textit{if } n\equiv 0 \pmod 8;\end{cases}$$

(xi) $$\left[\frac{E_1^4E_{16}^{10}}{E_2^2E_8^4E_{32}^4}\right]_n=\begin{cases}-2\sigma(N)-2(\frac{8}{N})K_2(N) & \textit{if } n\equiv 1 \pmod 4,\\ 0 & \textit{if } n\equiv 3 \pmod 4,\\ 4\sigma(N) & \textit{if } n\equiv 2 \pmod 8,\\ 0 & \textit{if } n\equiv 6 \pmod 8,\\ 4\sigma(N) & \textit{if } n\equiv 4 \pmod 8,\\ 8\sigma(N) & \textit{if } n\equiv 8 \pmod{16},\\ 24\sigma(N) & \textit{if } n\equiv 0 \pmod{16};\end{cases}$$

(xii) $$\left[\frac{E_1^4E_4^3E_8^3}{E_2^4E_{16}^2}\right]_n=\begin{cases}-4\sum_{\substack{d\in\mathbb{N}\\d\mid N}}\frac{N}{d}(\frac{8}{d}) & \textit{if } n\equiv 1 \pmod 2,\\ (2^{\alpha+2}-2(\frac{8}{N}))\sum_{\substack{d\in\mathbb{N}\\d\mid N}}\frac{N}{d}(\frac{8}{d}) & \textit{if } n\equiv 0 \pmod 2;\end{cases}$$

(xiii) $$\left[\frac{E_1^4E_8^3E_{16}^3}{E_2^2E_4^2E_{32}^2}\right]_n=\begin{cases}-4\sum_{\substack{d\in\mathbb{N}\\d\mid N}}\frac{N}{d}(\frac{8}{d}) & \textit{if } n\equiv 1 \pmod 4,\\ 0 & \textit{if } n\equiv 3 \pmod 4,\\ 4\sum_{\substack{d\in\mathbb{N}\\d\mid N}}\frac{N}{d}(\frac{8}{d}) & \textit{if } n\equiv 2 \pmod 4,\\ (2^{\alpha+1}-2(\frac{8}{N}))\sum_{\substack{d\in\mathbb{N}\\d\mid N}}\frac{N}{d}(\frac{8}{d}) & \textit{if } n\equiv 0 \pmod 4;\end{cases}$$

(xiv) $$\left[q\frac{E_4^8}{E_2^4}\right]_n=\begin{cases}\sigma(N) & \textit{if } n\equiv 1 \pmod 2,\\ 0 & \textit{if } n\equiv 0 \pmod 2;\end{cases}$$

(xv) $$\left[qE_4^2E_8^2\right]_n=\begin{cases}(\frac{8}{N})K_2(N) & \textit{if } n\equiv 1 \pmod 4,\\ 0 & \textit{if } n\not\equiv 1 \pmod 4;\end{cases}$$

(xvi) $$\left[q\frac{E_8^{14}}{E_4^6E_{16}^4}\right]_n=\begin{cases}\sigma(N) & \textit{if } n\equiv 1 \pmod 4,\\ 0 & \textit{if } n\not\equiv 1 \pmod 4;\end{cases}$$

(xvii) $$\left[q\frac{E_2^4E_8^4}{E_4^4}\right]_n=\begin{cases}(\frac{-4}{N})\sigma(N) & \textit{if } n\equiv 1 \pmod 2,\\ 0 & \textit{if } n\equiv 0 \pmod 2;\end{cases}$$

(xviii) $$\left[q\frac{E_{16}^{10}}{E_4^2E_{32}^4}\right]_n=\begin{cases}\frac{1}{2}\sigma(N)+\frac{1}{2}(\frac{8}{N})K_2(N) & \textit{if } n\equiv 1 \pmod 4,\\ 0 & \textit{if } n\not\equiv 1 \pmod 4;\end{cases}$$

(xix) $$\left[q\frac{E_2^3E_4E_8^2}{E_1^2}\right]_n=\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\frac{n}{d}\left(\frac{8}{d}\right);$$

(xx) $$\left[q\frac{E_1^4E_8^4}{E_2^2E_4^2}\right]_n=\begin{cases}\sigma(N) & \text{if } n\equiv 1 \pmod 2,\\ -4\sigma(N) & \text{if } n\equiv 2 \pmod 4,\\ 0 & \text{if } n\equiv 0 \pmod 4;\end{cases}$$

(xxi) $$\left[q\frac{E_2^{10}E_8^4}{E_1^4E_4^6}\right]_n=\begin{cases}\sigma(N) & \text{if } n\equiv 1 \pmod 2,\\ 4\sigma(N) & \text{if } n\equiv 2 \pmod 4,\\ 0 & \text{if } n\equiv 0 \pmod 4;\end{cases}$$

(xxii) $$\left[q\frac{E_4E_8^7}{E_2^2E_{16}^2}\right]_n=\begin{cases}\sum_{\substack{d\in\mathbb{N}\\ d\mid N}}\frac{N}{d}(\frac{8}{d}) & \text{if } n\equiv 1 \pmod 2,\\ 0 & \text{if } n\equiv 0 \pmod 2;\end{cases}$$

(xxiii) $$\left[q\frac{E_8^7E_{16}^3}{E_4^4E_{32}^2}\right]_n=\begin{cases}\sum_{\substack{d\in\mathbb{N}\\ d\mid N}}\frac{N}{d}(\frac{8}{d}) & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\not\equiv 1 \pmod 4;\end{cases}$$

(xxiv) $$\left[q^2\frac{E_1^4E_{16}^4}{E_2^2E_8^2}\right]_n=\begin{cases}0 & \text{if } n\equiv 1 \pmod 4,\\ -\sigma(N) & \text{if } n\equiv 3 \pmod 4,\\ \sigma(N) & \text{if } n\equiv 2 \pmod 4,\\ 4\sigma(N) & \text{if } n\equiv 4 \pmod 8,\\ 0 & \text{if } n\equiv 0 \pmod 8;\end{cases}$$

(xxv) $$\left[q^2\frac{E_2^{10}E_{16}^4}{E_1^4E_4^4E_8^2}\right]_n=\begin{cases}0 & \text{if } n\equiv 1 \pmod 4,\\ \sigma(N) & \text{if } n\equiv 2,3 \pmod 4,\\ 4\sigma(N) & \text{if } n\equiv 4 \pmod 8,\\ 0 & \text{if } n\equiv 0 \pmod 8;\end{cases}$$

(xxvi) $$\left[q^2\frac{E_2^3E_4^3E_{16}^4}{E_1^2E_8^4}\right]_n=\begin{cases}0 & \text{if } n\equiv 1,7 \pmod 8,\\ \sum_{\substack{d\in\mathbb{N}\\ d\mid N}}\frac{N}{d}(\frac{8}{d}) & \text{if } n\equiv 3,5 \pmod 8,\\ \frac{1}{2}\sum_{\substack{d\in\mathbb{N}\\ d\mid N}}\frac{n}{d}(\frac{8}{d}) & \text{if } n\equiv 0 \pmod 2;\end{cases}$$

(xxvii) $$\left[q^3\frac{E_8^2E_{16}^4}{E_4^2}\right]_n=\begin{cases}\frac{1}{4}\sigma(N) & \text{if } n\equiv 3 \pmod 4,\\ 0 & \text{if } n\not\equiv 3 \pmod 4;\end{cases}$$

(xxviii) $$\left[q^4\frac{E_1^4E_{32}^4}{E_2^2E_{16}^2}\right]_n=\begin{cases}-\frac{1}{2}\sigma(N)+\frac{1}{2}(\frac{8}{N})K_2(N) & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 2,3,7 \pmod 8,\\ \sigma(N) & \text{if } n\equiv 4,6 \pmod 8,\\ 4\sigma(N) & \text{if } n\equiv 8 \pmod{16},\\ 0 & \text{if } n\equiv 0 \pmod{16};\end{cases}$$

$$\text{(xxix)}\quad \left[q^4\frac{E_2^{10}E_{32}^4}{E_1^4E_4^4E_{16}^2}\right]_n = \begin{cases} \frac{1}{2}\sigma(N) - \frac{1}{2}(\frac{8}{N})K_2(N) & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 2,3,7 \pmod 8, \\ \sigma(N) & \text{if } n \equiv 4,6 \pmod 8, \\ 4\sigma(N) & \text{if } n \equiv 8 \pmod{16}, \\ 0 & \text{if } n \equiv 0 \pmod{16}; \end{cases}$$

$$\text{(xxx)}\quad \left[q^4\frac{E_2^3E_4^3E_{32}^4}{E_1^2E_8^2E_{16}^2}\right]_n = \begin{cases} 0 & \text{if } n \equiv 1,3 \pmod 8, \\ \frac{1}{2}\sum_{\substack{d\in\mathbb{N}\\ d\mid N}} \frac{N}{d}(\frac{8}{d}) & \text{if } n \equiv 5,7 \pmod 8, \\ \frac{1}{2}(1-(\frac{8}{N}))\sum_{\substack{d\in\mathbb{N}\\ d\mid N}} \frac{N}{d}(\frac{8}{d}) & \text{if } n \equiv 2 \pmod 4, \\ 2^{\alpha-2}\sum_{\substack{d\in\mathbb{N}\\ d\mid N}} \frac{N}{d}(\frac{8}{d}) & \text{if } n \equiv 0 \pmod 4; \end{cases}$$

$$\text{(xxxi)}\quad \left[q^5\frac{E_8^4E_{32}^4}{E_4^2E_{16}^2}\right]_n = \begin{cases} \frac{1}{8}\sigma(N) - \frac{1}{8}(\frac{8}{N})K_2(N) & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \not\equiv 1 \pmod 4. \end{cases}$$

Theorem 1.3 *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$\text{(i)}\quad \left[\frac{E_1^6}{E_3^2}\right]_n = \begin{cases} 12(2^{\alpha+1}-1)\sigma(N) & \text{if } n \equiv 0 \pmod 3, \\ -6(2^{\alpha+1}-1)\sigma(N) & \text{if } n \equiv 1 \pmod 3, \\ 3(2^{\alpha+1}-1)\sigma(N) & \text{if } n \equiv 2 \pmod 3; \end{cases}$$

$$\text{(ii)}\quad \left[\frac{E_1^3E_2^3}{E_3E_6}\right]_n = 3\big(3^{\beta+1}-4\big)\sigma(N);$$

$$\text{(iii)}\quad \left[\frac{E_1^4E_3^4}{E_2^2E_6^2}\right]_n = 4\big(2^{\alpha+1}-3\big)\sigma(N);$$

$$\text{(iv)}\quad \left[\frac{E_2^7E_3^7}{E_1^5E_6^5}\right]_n = \big(2^{\alpha+3}+3^{\beta+2}-12\big)\sigma(N);$$

$$\text{(v)}\quad \left[\frac{E_1^{12}E_6^2}{E_2^6E_3^4}\right]_n = 12\big(3\cdot 2^{\alpha+1}-2\cdot 3^{\beta+1}-1\big)\sigma(N);$$

$$\text{(vi)}\quad \left[\frac{E_2^{12}E_3^2}{E_1^6E_6^4}\right]_n = 3\big(3\cdot 2^{\alpha}+3^{\beta+1}-4\big)\sigma(N);$$

$$\text{(vii)}\quad \left[\frac{E_2^2E_3^{12}}{E_1^4E_6^6}\right]_n = 4\big(2^{\alpha+1}+2\cdot 3^{\beta}-3\big)\sigma(N);$$

$$\text{(viii)}\quad \left[\frac{E_1^9E_6}{E_2^3E_3^3}\right]_n = \begin{cases} -12(2^{\alpha+1}+1)\sigma(N) & \text{if } n \equiv 0 \pmod 3, \\ -3(2^{\alpha+1}+1)\sigma(N) & \text{if } n \equiv 1 \pmod 3, \\ 6(2^{\alpha+1}+1)\sigma(N) & \text{if } n \equiv 2 \pmod 3; \end{cases}$$

(ix) $$\left[\frac{E_1E_2E_3^5}{E_6^3}\right]_n = \begin{cases} 4(2^{\alpha+1}-3)\sigma(N) & \text{if } n\equiv 0 \pmod 3,\\ (2^{\alpha+1}-3)\sigma(N) & \text{if } n\equiv 1 \pmod 3,\\ -2(2^{\alpha+1}-3)\sigma(N) & \text{if } n\equiv 2 \pmod 3;\end{cases}$$

(x) $$\left[\frac{E_1^4E_2^2E_6^6}{E_3^4E_4^2E_{12}^2}\right]_n = \begin{cases} 4(3^{\beta+1}-4)\sigma(N) & \text{if } n\equiv 1 \pmod 2,\\ 12(3^{\beta+1}-2^{\alpha}-1)\sigma(N) & \text{if } n\equiv 0 \pmod 2;\end{cases}$$

(xi) $$\left[\frac{E_2^{11}E_6^3}{E_1^2E_3^2E_4^5E_{12}}\right]_n = \begin{cases} 2(4-3^{\beta+1})\sigma(N) & \text{if } n\equiv 1 \pmod 2,\\ 6(2^{\alpha+1}-3^{\beta+1}-2)\sigma(N) & \text{if } n\equiv 0 \pmod 2;\end{cases}$$

(xii) $$\left[q\frac{E_1^4E_6^6}{E_2^3E_3^4}\right]_n = \begin{cases} 2^{\alpha}\sigma(N) & \text{if } n\equiv 1 \pmod 3,\\ -2^{\alpha+1}\sigma(N) & \text{if } n\equiv 2 \pmod 3,\\ 2^{\alpha+2}\sigma(N) & \text{if } n\equiv 0 \pmod 3;\end{cases}$$

(xiii) $$\left[q\frac{E_3^3E_6^3}{E_1E_2}\right]_n = 3^{\beta}\sigma(N);$$

(xiv) $$\left[q\frac{E_2^4E_6^4}{E_1^2E_3^2}\right]_n = 2^{\alpha}\sigma(N);$$

(xv) $$\left[q\frac{E_1^7E_6^7}{E_2^5E_3^5}\right]_n = \left(3^{\beta+2}-2^{\alpha+3}\right)\sigma(N);$$

(xvi) $$\left[q\frac{E_1E_2^4E_6^4E_{12}}{E_3^3E_4^3}\right]_n = \begin{cases} (4-3^{\beta+1})\sigma(N) & \text{if } n\equiv 1 \pmod 2,\\ (2^{\alpha+2}-3^{\beta+2})\sigma(N) & \text{if } n\equiv 0 \pmod 2;\end{cases}$$

(xvii) $$\left[q\frac{E_2^{13}E_6E_{12}^2}{E_1^5E_3E_4^6}\right]_n = \begin{cases} (3^{\beta+1}-2)\sigma(N) & \text{if } n\equiv 1 \pmod 2,\\ (3^{\beta+2}-2^{\alpha+1})\sigma(N) & \text{if } n\equiv 0 \pmod 2;\end{cases}$$

(xviii) $$\left[q^2\frac{E_1^2E_6^{12}}{E_2^4E_3^6}\right]_n = \left(2^{\alpha}-3^{\beta}\right)\sigma(N);$$

(xix) $$\left[q^2\frac{E_2^6E_6^2E_{12}^4}{E_1^2E_3^2E_4^4}\right]_n = \begin{cases} (3^{\beta}-1)\sigma(N) & \text{if } n\equiv 1 \pmod 2,\\ (3^{\beta+1}-2^{\alpha})\sigma(N) & \text{if } n\equiv 0 \pmod 2.\end{cases}$$

Theorem 1.4 *Let $n\in\mathbb{N}$. Set $n=2^{\alpha}N$, where $\alpha\in\mathbb{N}_0$, $N\in\mathbb{N}$ and* $\gcd(N,2)=1$. *Let*

$$N=\prod_{p\mid N} p^{\alpha_p}$$

be the prime factorization of N. Set

$$F(N) := N^2\prod_{p\mid N}\frac{1-(\frac{-4}{p})^{\alpha_p+1}p^{-2\alpha_p-2}}{1-(\frac{-4}{p})p^{-2}}. \tag{1.3}$$

Then

(i) $$\left[\frac{E_1^{12}}{E_2^6}\right]_n = (-1)^n \left(2^{2\alpha+4} - 4\left(\frac{-4}{N}\right)\right) F(N);$$

(ii) $$\left[q\frac{E_1^8 E_8^4}{E_2^4 E_4^2}\right]_n = \frac{1}{2}(-1)^{n+1}\left(2^{2\alpha+2} - \left(1-(-1)^n\right)\left(\frac{-4}{N}\right)\right) F(N);$$

(iii) $$\left[q^2\frac{E_1^4 E_8^8}{E_2^2 E_4^4}\right]_n = \frac{1}{8}(-1)^n\left(2^{2\alpha+1} - \left(1-(-1)^n\right)\left(\frac{-4}{N}\right)\right) F(N);$$

(iv) $$\left[q^3\frac{E_8^{12}}{E_4^6}\right]_n = \frac{1}{32}\left(1-\left(\frac{-4}{N}\right)\right)\left(1-(-1)^n\right) F(N).$$

Theorem 1.5 *Let $n \in \mathbb{N}$. Then*

(i) $$\left[\frac{E_2^5 E_{10}^7}{E_1 E_4 E_5^3 E_{20}^3}\right]_n = \sum_{\substack{d\in\mathbb{N}\\ d\mid n}} (-1)^{n+d} d \left(\frac{5}{d}\right);$$

(ii) $$\left[q\frac{E_2^7 E_{10}^5}{E_1^3 E_4^3 E_5 E_{20}}\right]_n = \sum_{\substack{d\in\mathbb{N}\\ d\mid n}} (-1)^{n+d} d \left(\frac{5}{n/d}\right).$$

For our final theorem of this introduction, we require the multiplicative arithmetic function

$$a(n) := [qE_2E_4E_6E_{12}]_n \quad (n \in \mathbb{N}) \tag{1.4}$$

which was defined and used extensively in [2]. Clearly, $a(n) = 0$ if n is even.

Theorem 1.6 *Let $n \in \mathbb{N}$. Set $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and* $\gcd(N, 6) = 1$. *Then*

$$\left[q\frac{E_2^4 E_3 E_4^4 E_{24}}{E_1^3 E_8^3}\right]_n = \begin{cases} \frac{1}{2}(3^{\beta+1}\sigma(N) - a(n)) & \text{if } n \equiv 1 \pmod 2,\\ 3^{\beta+1}\sigma(N) & \text{if } n \equiv 2 \pmod 4,\\ 3^{\beta+2}\sigma(N) & \text{if } n \equiv 0 \pmod 4.\end{cases}$$

We remark that equivalent forms of some of the identities of Theorems 1.2 and 1.3 can be found in [32, Table 1, p. 999].

Theorems 1.1–1.6 give the evaluation of $[q^r E_1^{a_1} \cdots E_m^{a_m}]_n$ for 69 products $q^r E_1^{a_1} \cdots E_m^{a_m}$. However, many other such products can be treated by our method. For example, equation (1.11) in [1] (with q replaced by $-q$) and equation (2.13) in [11], which are not used in this paper, provide other starting points for our method.

2 Some modular equations

Ramanujan's theta function φ is defined by

$$\varphi(q)=\sum_{n=-\infty}^{\infty} q^{n^2}=\frac{E_2^5}{E_1^2E_4^2}, \tag{2.1}$$

see, for example, [10, p. 6]. We are interested in modular equations of the form

$$q^rE_1^{a_1}\cdots E_m^{a_m}=b_1\varphi\left(q^{c_1}\right)\varphi\left(q^{c_2}\right)+b_2\varphi\left(q^{c_3}\right)\varphi\left(q^{c_4}\right), \tag{2.2}$$

where $r\in\mathbb{N}_0$, $m\in\mathbb{N}$, $a_1,\ldots,a_m\in\mathbb{Z}$, $(a_1,\ldots,a_m)\neq(0,\ldots,0)$, $b_1,b_2\in\mathbb{Q}$, $c_1,c_2,c_3,c_4\in\mathbb{N}$, $b_1b_2\neq 0$ and $(c_1,c_2)\neq(c_3,c_4)$ or (c_4,c_3). Many examples of identities of the type (2.2) occur in the literature. We list in the next theorem those that we make use of.

Theorem 2.1

(i) $$\frac{E_1^4}{E_2^2}=-\varphi^2(q)+2\varphi^2\left(q^2\right).$$

(ii) $$q\frac{E_8^4}{E_4^2}=\frac{1}{4}\left(\varphi^2(q)-\varphi^2\left(q^2\right)\right).$$

(iii) $$\frac{E_1^2E_2E_6^3}{E_3^2E_4E_{12}}=\frac{1}{2}\left(-\varphi^2(q)+3\varphi^2\left(q^3\right)\right).$$

(iv) $$q\frac{E_2^3E_6E_{12}^2}{E_1E_3E_4^2}=\frac{1}{4}\left(\varphi^2(q)-\varphi^2\left(q^3\right)\right).$$

(v) $$\frac{E_1E_4E_{10}^2}{E_5E_{20}}=\frac{1}{4}\left(-\varphi^2(q)+5\varphi^2\left(q^5\right)\right).$$

(vi) $$q\frac{E_2^2E_5E_{20}}{E_1E_4}=\frac{1}{4}\left(\varphi^2(q)-\varphi^2\left(q^5\right)\right).$$

(vii) $$q\frac{E_2E_3E_4E_{24}}{E_1E_8}=\frac{1}{2}\left(\varphi(q)\varphi\left(q^2\right)-\varphi\left(q^3\right)\varphi\left(q^6\right)\right).$$

Proof First, we prove (i) and (ii). We begin with the following two well-known identities

$$\varphi^2(q)+\varphi^2(-q)=2\varphi^2\left(q^2\right) \tag{2.3}$$

and

$$\varphi^2(q)-\varphi^2(-q)=8q\psi^2\left(q^4\right), \tag{2.4}$$

see, for example, [10, pp. 71, 72], where

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{E_2^2}{E_1}. \tag{2.5}$$

By (2.3)–(2.5), we have

$$\varphi^2(q) - \varphi^2(q^2) = \frac{1}{2}\big(\varphi^2(q) - \varphi^2(-q)\big) = 4q\psi^2(q^4) = 4q\frac{E_8^4}{E_4^2},$$

which gives (ii), and, as $\varphi(-q) = E_1^2/E_2$,

$$\varphi^2(q) - 2\varphi^2(q^2) = -\varphi^2(-q) = -\frac{E_1^4}{E_2^2},$$

which gives (i).

Next we prove (iii) and (iv). Becksmith, Brillhart, and Gerst [12, Theorem 3] (see also [14, Theorem 3]) have shown that

$$\varphi(q) + \varphi(q^3) = 2\prod_{n=0}^{\infty} \frac{(1-q^{12n+2})(1-q^{12n+6})(1-q^{12n+10})(1-q^{12n+12})}{(1-q^{12n+1})(1-q^{12n+3})(1-q^{12n+9})(1-q^{12n+11})}$$

and

$$\varphi(q) - \varphi(q^3) = 2q\prod_{n=0}^{\infty} \frac{(1-q^{12n+2})(1-q^{12n+6})(1-q^{12n+10})(1-q^{12n+12})}{(1-q^{12n+3})(1-q^{12n+5})(1-q^{12n+7})(1-q^{12n+9})}.$$

Multiplying these identities together, we obtain

$$\varphi^2(q) - \varphi^2(q^3) = 4qE_{12}^2\prod_{n=0}^{\infty} \frac{(1-q^{4n+2})^2}{(1-q^{2n+1})(1-q^{12n+3})(1-q^{12n+9})}.$$

As

$$\prod_{n=0}^{\infty}(1-q^{2n+1}) = \frac{E_1}{E_2}, \qquad \prod_{n=0}^{\infty}(1-q^{4n+2}) = \frac{E_2}{E_4},$$

$$\prod_{n=0}^{\infty}(1-q^{12n+3})(1-q^{12n+9}) = \frac{E_3}{E_6},$$

we deduce

$$\varphi^2(q) - \varphi^2(q^3) = 4q\frac{E_2^3E_6E_{12}^2}{E_1E_3E_4^2},$$

which gives (iv).

From [9, p. 232], we have

$$\frac{\varphi^3(q)}{\varphi(q^3)} + 2\frac{\varphi^3(-q^2)}{\varphi(-q^6)} = 3\varphi(q)\varphi\bigl(q^3\bigr).$$

Hence

$$\begin{aligned} 3\varphi^2\bigl(q^3\bigr) - \varphi^2(q) &= \left(3\varphi(q)\varphi\bigl(q^3\bigr) - \frac{\varphi^3(q)}{\varphi(q^3)}\right)\frac{\varphi(q^3)}{\varphi(q)} = 2\frac{\varphi^3(-q^2)\varphi(q^3)}{\varphi(-q^6)\varphi(q)} \\ &= 2\frac{\varphi^3(-q^2)\varphi(-q^6)}{\varphi(q)\varphi(-q^3)}, \end{aligned}$$

as $\varphi(q^3)\varphi(-q^3) = \varphi^2(-q^6)$. Then, using $\varphi(q)\varphi(-q) = \varphi^2(-q^2)$, we have

$$\varphi^2(q) - 3\varphi^2\bigl(q^3\bigr) = -2\frac{\varphi(-q)\varphi(-q^2)\varphi(-q^6)}{\varphi(-q^3)}.$$

As

$$\varphi(-q) = \frac{E_1^2}{E_2}, \qquad \varphi\bigl(-q^2\bigr) = \frac{E_2^2}{E_4}, \qquad \varphi\bigl(-q^3\bigr) = \frac{E_3^2}{E_6}, \qquad \varphi\bigl(-q^6\bigr) = \frac{E_6^2}{E_{12}},$$

we have

$$\varphi^2(q) - 3\varphi^2\bigl(q^3\bigr) = -2\frac{E_1^2 E_2 E_6^3}{E_3^2 E_4 E_{12}},$$

which gives (iii).

Next we prove (v) and (vi). From [6, Theorem 3.3, p. 41] we have

$$\varphi^2(q) - \varphi^2\bigl(q^5\bigr) = 4q\frac{E_2^2 E_5 E_{20}}{E_1 E_4},$$

which gives (vi), and from [6, Theorem 3.4, p. 42]

$$\varphi^2(q) - 5\varphi^2\bigl(q^5\bigr) = -4\frac{E_1 E_4 E_{10}^2}{E_5 E_{20}},$$

which gives (v).

Finally, we prove (vii). Ramanujan's theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad a,b \in \mathbb{C},\ |ab| < 1, \tag{2.6}$$

see, for example, [10, p. 8]. Taking $a = q$ and $b = q^5$ in (2.6), we obtain

$$f\bigl(q,q^5\bigr) = \sum_{n=-\infty}^{\infty} q^{3n^2-2n}. \tag{2.7}$$

By Jacobi's triple product identity (see, for example, [15, Eq. (19.9.2), p. 283]), we have

$$\sum_{n=-\infty}^{\infty} q^{kn^2+\ell n} = \prod_{n=0}^{\infty} \big(1+q^{2kn+k-\ell}\big)\big(1+q^{2kn+k+\ell}\big)\big(1-q^{2kn+2k}\big). \tag{2.8}$$

Taking $k=3$ and $\ell=-2$ in (2.8), we deduce

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{3n^2-2n} &= \prod_{n=0}^{\infty} \big(1+q^{6n+5}\big)\big(1+q^{6n+1}\big)\big(1-q^{6n+6}\big)\\ &= E_6 \prod_{n=1}^{\infty} \frac{(1+q^n)(1+q^{6n})}{(1+q^{2n})(1+q^{3n})}\\ &= \frac{E_2^2 E_3 E_{12}}{E_1 E_4 E_6},\end{aligned}$$

as

$$\prod_{n=1}^{\infty} \big(1+q^{kn}\big) = \prod_{n=1}^{\infty} \frac{1-q^{2kn}}{1-q^{kn}} = \frac{E_{2k}}{E_k}.$$

Thus (2.7) gives

$$f\big(q,q^5\big) = \frac{E_2^2 E_3 E_{12}}{E_1 E_4 E_6}, \qquad f\big(q^2,q^{10}\big) = \frac{E_4^2 E_6 E_{24}}{E_2 E_8 E_{12}},$$

and so

$$f\big(q,q^5\big) f\big(q^2,q^{10}\big) = \frac{E_2 E_3 E_4 E_{24}}{E_1 E_8}. \tag{2.9}$$

The next identity is due to Chen and Huang [13, Corollary 3, p. 7]:

$$\varphi(q)\varphi\big(q^2\big) - \varphi\big(q^3\big)\varphi\big(q^6\big) = 2q f\big(q,q^5\big) f\big(q^2,q^{10}\big). \tag{2.10}$$

Appealing to (2.9), equation (2.10) becomes

$$\varphi(q)\varphi\big(q^2\big) - \varphi\big(q^3\big)\varphi\big(q^6\big) = 2q \frac{E_2 E_3 E_4 E_{24}}{E_1 E_8},$$

which gives (vii). □

The 2-dimensional theta function of the Borweins is

$$a(q) := \sum_{x,y=-\infty}^{\infty} q^{x^2+xy+y^2}, \tag{2.11}$$

see, for example, [5, p. 177]. The modular equations involving $a(q)$ that we need are given in the next theorem.

Theorem 2.2

(i) $$\frac{E_1^6 E_6}{E_2^3 E_3^2} = -a(q) + 2a\big(q^2\big).$$

(ii) $$\frac{E_2^6 E_3}{E_1^3 E_6^2} = \frac{1}{2}a(q) + \frac{1}{2}a\big(q^2\big).$$

(iii) $$\frac{E_2 E_3^6}{E_1^2 E_6^3} = \frac{1}{3}a(q) + \frac{2}{3}a\big(q^2\big).$$

(iv) $$\frac{E_1^3}{E_3} = -\frac{1}{2}a(q) + \frac{3}{2}a\big(q^3\big).$$

(v) $$q\frac{E_1 E_6^6}{E_2^2 E_3^3} = \frac{1}{6}a(q) - \frac{1}{6}a\big(q^2\big).$$

Proof We make use of the (p,k)-parametrization of theta functions given by Alaca, Alaca, and Williams, namely,

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, \qquad k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)},$$

see, for example, [5, p. 178]. By (2.1), we have

$$\varphi(q) = \frac{E_2^5}{E_1^2 E_4^2}, \qquad \varphi\big(q^3\big) = \frac{E_6^5}{E_3^2 E_{12}^2},$$

so that

$$k = \frac{E_1^2 E_4^2 E_6^{15}}{E_2^5 E_3^6 E_{12}^6}.$$

By Theorem 2.1(iv), we have

$$\varphi^2(q) - \varphi^2\big(q^3\big) = 4q\frac{E_2^3 E_6 E_{12}^2}{E_1 E_3 E_4^2},$$

so that

$$p = 2q\frac{E_2^3 E_3^3 E_{12}^6}{E_1 E_4^2 E_6^9}.$$

From [8, Eqs. (2.14)–(2.19), pp. 48–49] it follows that

$$1 - p = \frac{E_1^2 E_2 E_3^2 E_{12}^3}{E_4 E_6^7}, \qquad 1 + p = \frac{E_2^3 E_3^6 E_{12}^3}{E_1^2 E_4 E_6^9},$$

$$1 + 2p = \frac{E_2^{10} E_3^4 E_{12}^4}{E_1^4 E_4^4 E_6^{10}}, \qquad 2 + p = 2\frac{E_2 E_3^3 E_4^2 E_{12}^2}{E_1 E_6^7}.$$

Alaca, Alaca, and Williams [5, Theorems 1–3, p. 178] have shown that

$$a(q) = \left(1+4p+p^2\right)k, \qquad a\left(q^2\right) = \left(1+p+p^2\right)k,$$
$$a\left(q^3\right) = \frac{1}{3}\left(1+4p+p^2\right)k + \frac{1}{3}2^{2/3}\left((1-p)^4(1+2p)(2+p)\right)^{1/3}k.$$

Hence the Borweins' theta function (2.11) satisfies the following relations:

(i) $-a(q) + 2a\left(q^2\right) = (1-p)^2k = \dfrac{E_1^6E_6}{E_2^3E_3^2}$;

(ii) $a(q) + a\left(q^2\right) = (1+2p)(2+p)k = 2\dfrac{E_2^6E_3}{E_1^3E_6^2}$;

(iii) $a(q) + 2a\left(q^2\right) = 3(1+p)^2k = 3\dfrac{E_2E_3^6}{E_1^2E_6^3}$;

(iv) $-a(q) + 3a\left(q^3\right) = 2^{2/3}\left((1-p)^4(1+2p)(2+p)\right)^{1/3}k = 2\dfrac{E_1^3}{E_3}$;

(v) $a(q) - a\left(q^2\right) = 3pk = 6q\dfrac{E_1E_6^6}{E_2^2E_3^3}$.

This completes the proof of the theorem. □

3 Representation numbers for certain quadratic forms

For $a_1, a_2, \ldots, a_m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define

$$N(a_1, a_2, \ldots, a_m; n) = \text{card}\left\{(x_1, \ldots, x_m) \in \mathbb{Z}^m \mid n = a_1x_1^2 + \cdots + a_mx_m^2\right\}. \tag{3.1}$$

Taking $n = 0$ in (3.1), we see that

$$N(a_1, a_2, \ldots, a_m; 0) = 1. \tag{3.2}$$

From (2.1), (3.1), and (3.2), we see that

$$\varphi\left(q^{a_1}\right)\cdots\varphi\left(q^{a_m}\right) = 1 + \sum_{n=1}^{\infty} N(a_1, \ldots, a_m; n)q^n.$$

If $n \notin \mathbb{N}_0$ we understand $N(a_1, a_2, \ldots, a_m; n) = 0$. Also if $k = \gcd(a_1, \ldots, a_m)$ then

$$N(a_1, \ldots, a_m; n) = N(a_1/k, \ldots, a_m/k; n/k). \tag{3.3}$$

We use (3.3) without comment in what follows. For the remainder of this section, we take $n \in \mathbb{N}$.

When $m=2$, we make use of the classical formulae

$$N(1,1;n)=4\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right), \tag{3.4}$$

$$N(1,2;n)=2\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-8}{d}\right), \tag{3.5}$$

as well as

$$\operatorname{card}\{(x,y)\in\mathbb{Z}^2 \mid n=x^2+xy+y^2\}=6\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-3}{d}\right). \tag{3.6}$$

When $m=4$ we require the following formulae. Set $n=2^\alpha N$, where $\alpha\in\mathbb{N}_0$, $N\in\mathbb{N}$ and $\gcd(N,2)=1$. By Jacobi's four squares theorem (see, for example, [2, Theorem 1.6, p. 284]), we have

$$N(1,1,1,1;n)=\begin{cases}8\sigma(N) & \text{if } n\equiv 1 \pmod 2,\\ 24\sigma(N) & \text{if } n\equiv 0 \pmod 2.\end{cases} \tag{3.7}$$

By a theorem of Liouville [18] (see, for example, [2, Theorem 1.8, p. 284]), we have

$$N(1,1,2,2;n)=\begin{cases}4\sigma(N) & \text{if } n\equiv 1 \pmod 2,\\ 8\sigma(N) & \text{if } n\equiv 2 \pmod 4,\\ 24\sigma(N) & \text{if } n\equiv 0 \pmod 4.\end{cases} \tag{3.8}$$

From [19] and [2, Theorem 1.11, p. 285], we have

$$N(1,1,4,4;n)=\begin{cases}4\sigma(N) & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 3 \pmod 4,\\ 4\sigma(N) & \text{if } n\equiv 2 \pmod 4,\\ 8\sigma(N) & \text{if } n\equiv 4 \pmod 8,\\ 24\sigma(N) & \text{if } n\equiv 0 \pmod 8.\end{cases} \tag{3.9}$$

From [30] and [4, Theorem 5.1, p. 29], we have

$$N(1,1,1,2;n)=2\left(2^{\alpha+2}-\left(\frac{8}{N}\right)\right)\sum_{\substack{d\in\mathbb{N}\\ d\mid N}}\frac{N}{d}\left(\frac{8}{d}\right), \tag{3.10}$$

from [30] and [4, Theorem 5.2, p. 30],

$$N(1,2,2,2;n)=2\left(2^{\alpha+1}-\left(\frac{8}{N}\right)\right)\sum_{\substack{d\in\mathbb{N}\\ d\mid N}}\frac{N}{d}\left(\frac{8}{d}\right), \tag{3.11}$$

from [23] and [4, Theorem 5.3, p. 30],

$$N(1,1,2,4;n) = 2\left(2^{\alpha+2} - \left(1+(-1)^n\right)\left(\frac{8}{N}\right)\right)\sum_{\substack{d\in\mathbb{N}\\ d\mid N}} \frac{N}{d}\left(\frac{8}{d}\right), \tag{3.12}$$

from [21] and [4, Theorem 5.4, p. 31],

$$N(1,2,4,4;n) = 2\left(2^{\alpha+1} - \left(1+(-1)^n\right)\left(\frac{8}{N}\right)\right)\sum_{\substack{d\in\mathbb{N}\\ d\mid N}} \frac{N}{d}\left(\frac{8}{d}\right), \tag{3.13}$$

from [24] and [4, Theorem 5.5, p. 32],

$$N(1,1,4,8;n) = \begin{cases} 4\sum_{\substack{d\in\mathbb{N}\\ d\mid N}} \frac{N}{d}(\frac{8}{d}) & \text{if } n \equiv 1,2 \pmod 4,\\ 0 & \text{if } n \equiv 3 \pmod 4,\\ 2(2^\alpha - (\frac{8}{N}))\sum_{\substack{d\in\mathbb{N}\\ d\mid N}} \frac{N}{d}(\frac{8}{d}) & \text{if } n \equiv 0 \pmod 4, \end{cases} \tag{3.14}$$

and from [22] and [4, Theorem 5.8, p. 35],

$$N(1,2,8,8;n) = \begin{cases} 0 & \text{if } n \equiv 5,7 \pmod 8,\\ 2\sum_{\substack{d\in\mathbb{N}\\ d\mid N}} \frac{N}{d}(\frac{8}{d}) & \text{if } n \equiv 1,2,3,6 \pmod 8,\\ 2(2^{\alpha-1} - (\frac{8}{N}))\sum_{\substack{d\in\mathbb{N}\\ d\mid N}} \frac{N}{d}(\frac{8}{d}) & \text{if } n \equiv 0 \pmod 4. \end{cases} \tag{3.15}$$

From [25] and [3, Theorem 4.2, p. 163], we have

$$N(1,1,8,8;n) = \begin{cases} 2\sigma(N) + 2(\frac{8}{N})K_2(N) & \text{if } n \equiv 1 \pmod 4,\\ 0 & \text{if } n \equiv 3 \pmod 4,\\ 4\sigma(N) & \text{if } n \equiv 2 \pmod 8,\\ 0 & \text{if } n \equiv 6 \pmod 8,\\ 4\sigma(N) & \text{if } n \equiv 4 \pmod 8,\\ 8\sigma(N) & \text{if } n \equiv 8 \pmod{16},\\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{16}, \end{cases} \tag{3.16}$$

where $K_2(N)$ was defined in (1.2).

We also require $N(1,1,3,3;n)$ and $N(1,2,3,6;n)$. Let $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $\gcd(N,6) = 1$. Then

$$N(1,1,3,3;n) = \begin{cases} 4\sigma(N) & \text{if } n \equiv 1 \pmod 2,\\ 4(2^{\alpha+1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod 2, \end{cases} \tag{3.17}$$

$$N(1,2,3,6;n) = \begin{cases} (3^{\beta+1} - 2)\sigma(N) + a(n) & \text{if } n \equiv 1 \pmod 2,\\ 2(3^{\beta+1} - 2)\sigma(N) & \text{if } n \equiv 2 \pmod 4,\\ 6(3^{\beta+1} - 2)\sigma(N) & \text{if } n \equiv 0 \pmod 4, \end{cases} \tag{3.18}$$

where $a(n)$ was defined in (1.4). Formula (3.17) was stated without proof by Liouville [17] and later proved by other authors; see, for example, [2, Theorem 1.9, p. 284]. A brief history of (3.18), as well as a proof, was given in [2, Theorem 1.15, p. 286].

In addition, we need

$$N(1,1,1,5;n) = \sum_{\substack{d\in\mathbb{N}\\ d\mid n}} (-1)^{n+d}\left(\frac{5}{d}\right)d + 5\sum_{\substack{d\in\mathbb{N}\\ d\mid n}} (-1)^{n+d}\left(\frac{5}{n/d}\right)d \tag{3.19}$$

and

$$N(1,5,5,5;n) = \sum_{\substack{d\in\mathbb{N}\\ d\mid n}} (-1)^{n+d}\left(\frac{5}{d}\right)d + \sum_{\substack{d\in\mathbb{N}\\ d\mid n}} (-1)^{n+d}\left(\frac{5}{n/d}\right)d, \tag{3.20}$$

which are valid for all $n \in \mathbb{N}$, see [6, Theorems 5.1, 6.1].

When $m = 6$, we require the following formulae. Let $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $\gcd(N, 2) = 1$. Let

$$N = \prod_{p\mid N} p^{\alpha_p}$$

be the prime factorization of N. Define $F(N)$ as in (1.3). Then

$$N(1,1,1,1,1,1;n) = \left(2^{2\alpha+4} - 4\left(\frac{-4}{N}\right)\right)F(N), \tag{3.21}$$

$$N(1,1,1,1,2,2;n) = \left(2^{2\alpha+3} - 2\big(1+(-1)^n\big)\left(\frac{-4}{N}\right)\right)F(N), \tag{3.22}$$

$$N(1,1,2,2,2,2;n) = \left(2^{2\alpha+2} - 2\big(1+(-1)^n\big)\left(\frac{-4}{N}\right)\right)F(N), \tag{3.23}$$

see [7, Theorem 2.4, p. 553].

For $a, b \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define

$$M(a,b;n) := \operatorname{card}\big\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = a\big(x^2+xy+y^2\big) + b\big(z^2+zt+t^2\big)\big\}. \tag{3.24}$$

Taking $n = 0$ in (3.24), we see that

$$M(a,b;0) = 1. \tag{3.25}$$

Alaca, Alaca, and Williams [5, Theorems 12, 13, 14, 17] (see also [26–29]) have shown for $n \in \mathbb{N}$ that

$$M(1,1;n) = 12\sigma(n) - 36\sigma(n/3), \tag{3.26}$$

$$M(1,2;n) = 6\sigma(n) - 12\sigma(n/2) + 18\sigma(n/3) - 36\sigma(n/6), \tag{3.27}$$

$$M(1,3;n)=\begin{cases}12\sigma(n)-36\sigma(n/3) & \text{if } n\equiv 0 \pmod 3,\\ 6\sigma(n) & \text{if } n\equiv 1 \pmod 3,\\ 0 & \text{if } n\equiv 2 \pmod 3,\end{cases} \tag{3.28}$$

$$M(2,3;n)=\begin{cases}-6\sigma(n)+12\sigma(n/2)+30\sigma(n/3)-60\sigma(n/6) & \text{if } n\equiv 0 \pmod 3,\\ 0 & \text{if } n\equiv 1 \pmod 3,\\ 6\sigma(n)-12\sigma(n/2) & \text{if } n\equiv 2 \pmod 3.\end{cases} \tag{3.29}$$

4 Proof of Theorem 1.1

We just give the details for parts (iii) and (vi) as the rest can be proved similarly.

(iii) By Theorem 2.1(iii), we have

$$\begin{aligned}\frac{E_1^2E_2E_6^3}{E_3^2E_4E_{12}} &= \frac{1}{2}\left(-\varphi^2(q)+3\varphi^2(q^3)\right)\\ &= -\frac{1}{2}\sum_{n=0}^{\infty}N(1,1;n)q^n+\frac{3}{2}\sum_{n=0}^{\infty}N(1,1;n/3)q^n,\end{aligned}$$

where (with the notation of (3.1)–(3.3))

$$N(1,1;r)=\begin{cases}\operatorname{card}\{(x,y)\in\mathbb{Z}^2 \mid r=x^2+y^2\} & \text{if } r\in\mathbb{N}_0,\\ 0 & \text{if } r\in\mathbb{Q}\setminus\mathbb{N}_0.\end{cases} \tag{4.1}$$

As (see (3.4))

$$N(1,1;n)=\begin{cases}4\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right) & \text{if } n\in\mathbb{N},\\ 1 & \text{if } n=0,\end{cases} \tag{4.2}$$

we have

$$\frac{E_1^2E_2E_6^3}{E_3^2E_4E_{12}}=1+\sum_{n=1}^{\infty}\left(-2\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right)+6\sum_{\substack{d\in\mathbb{N}\\ d\mid n/3}}\left(\frac{-4}{d}\right)\right)q^n,$$

where the sum over $d \mid n/3$ is understood to be 0 if $3\nmid n$. Equating coefficients of q^n $(n\in\mathbb{N})$, we obtain

$$\left[\frac{E_1^2E_2E_6^3}{E_3^2E_4E_{12}}\right]_n=-2\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right)+6\sum_{\substack{d\in\mathbb{N}\\ d\mid n/3}}\left(\frac{-4}{d}\right)$$

as claimed.

(vi) By Theorem 2.1(vi) and (4.2), we have

$$\begin{aligned} q\frac{E_2^2E_5E_{20}}{E_1E_4} &= \frac{1}{4}\big(\varphi^2(q)-\varphi^2\big(q^5\big)\big) \\ &= \frac{1}{4}\sum_{n=0}^{\infty} N(1,1;n)q^n - \frac{1}{4}\sum_{n=0}^{\infty} N(1,1;n/5)q^n \\ &= \sum_{n=1}^{\infty}\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right)q^n - \sum_{n=1}^{\infty}\sum_{\substack{d\in\mathbb{N}\\ d\mid n/5}}\left(\frac{-4}{d}\right)q^n. \end{aligned}$$

Equating coefficients of q^n $(n\in\mathbb{N})$, we deduce

$$\left[q\frac{E_2^2E_5E_{20}}{E_1E_4}\right]_n = \sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right) - \sum_{\substack{d\in\mathbb{N}\\ d\mid n/5}}\left(\frac{-4}{d}\right)$$

as asserted.

Parts (i), (ii), (iv), and (v) can be proved similarly to parts (iii) and (vi). To prove part (vii), we use (3.5) in conjunction with part (vii) of Theorem 2.1. To prove parts (viii)–(xii), we use (3.6) in conjunction with Theorem 2.2.

5 Proof of Theorem 1.2

The 31 parts of Theorem 1.2 follow from the identities given below which are consequences of (2.1) and Theorem 2.1(i), (ii):

part (i) $$\frac{E_1^8}{E_2^4} = \big(-\varphi^2(q)+2\varphi^2\big(q^2\big)\big)^2,$$

part (ii) $$\frac{E_1^4E_2^2}{E_4^2} = \big(-\varphi^2(q)+2\varphi^2\big(q^2\big)\big)\big(-\varphi^2\big(q^2\big)+2\varphi^2\big(q^4\big)\big),$$

part (iii) $$\frac{E_2^{14}}{E_1^4E_4^6} = \varphi^2(q)\big(-\varphi^2\big(q^2\big)+2\varphi^2\big(q^4\big)\big),$$

part (iv) $$\frac{E_2^{10}}{E_1^4E_8^2} = \varphi^2(q)\big(-\varphi^2\big(q^4\big)+2\varphi^2\big(q^8\big)\big),$$

part (v) $$\frac{E_1^2E_2E_4^3}{E_8^2} = \varphi(q)\varphi\big(q^2\big)\big(-\varphi^2(q)+2\varphi^2\big(q^2\big)\big),$$

part (vi) $$\frac{E_1^4E_4^{10}}{E_2^6E_8^4} = \varphi^2\big(q^2\big)\big(-\varphi^2(q)+2\varphi^2\big(q^2\big)\big),$$

part (vii) $\dfrac{E_1^4E_4^4}{E_2^2E_8^2}=\big(-\varphi^2(q)+2\varphi^2(q^2)\big)\big(-\varphi^2(q^4)+2\varphi^2(q^8)\big),$

part (viii) $\dfrac{E_2^7E_4}{E_1^2E_8^2}=\varphi(q)\varphi(q^2)\big(-\varphi^2(q^2)+2\varphi^2(q^4)\big),$

part (ix) $\dfrac{E_2^3E_4^7}{E_1^2E_8^4}=\varphi(q)\varphi(q^2)\big(-\varphi^2(q^4)+2\varphi^2(q^8)\big),$

part (x) $\dfrac{E_1^4E_8^{10}}{E_2^2E_4^4E_{16}^4}=\varphi(q^4)\big(-\varphi^2(q)+2\varphi^2(q^2)\big),$

part (xi) $\dfrac{E_1^4E_{16}^{10}}{E_2^2E_8^4E_{32}^4}=\varphi^2(q^8)\big(-\varphi^2(q)+2\varphi^2(q^2)\big),$

part (xii) $\dfrac{E_1^4E_4^3E_8^3}{E_2^4E_{16}^2}=\varphi(q^2)\varphi(q^4)\big(-\varphi^2(q)+2\varphi^2(q^2)\big),$

part (xiii) $\dfrac{E_1^4E_8^3E_{16}^3}{E_2^2E_4^2E_{32}^2}=\varphi(q^4)\varphi(q^8)\big(-\varphi^2(q)+2\varphi^2(q^2)\big),$

part (xiv) $q\dfrac{E_4^8}{E_2^4}=\dfrac{1}{4}\varphi^2(q^2)\big(\varphi^2(q)-\varphi^2(q^2)\big),$

part (xv) $qE_4^2E_8^2=\dfrac{1}{4}\big(-\varphi^2(q^4)+2\varphi^2(q^8)\big)\big(\varphi^2(q)-\varphi^2(q^2)\big),$

part (xvi) $q\dfrac{E_8^{14}}{E_4^6E_{16}^4}=\dfrac{1}{4}\varphi^2(q^4)\big(\varphi^2(q)-\varphi^2(q^2)\big),$

part (xvii) $q\dfrac{E_2^4E_8^4}{E_4^4}=\dfrac{1}{4}\big(-\varphi^2(q^2)+2\varphi^2(q^4)\big)\big(\varphi^2(q)-\varphi^2(q^2)\big),$

part (xviii) $q\dfrac{E_{16}^{10}}{E_4^2E_{32}^4}=\dfrac{1}{4}\varphi^2(q^8)\big(\varphi^2(q)-\varphi^2(q^2)\big),$

part (xix) $q\dfrac{E_2^3E_4E_8^2}{E_1^2}=\dfrac{1}{4}\varphi(q)\varphi(q^2)\big(\varphi^2(q)-\varphi^2(q^2)\big),$

part (xx) $q\dfrac{E_1^4E_8^4}{E_2^2E_4^2}=\dfrac{1}{4}\big(-\varphi^2(q)+2\varphi^2(q^2)\big)\big(\varphi^2(q)-\varphi^2(q^2)\big),$

part (xxi) $q\dfrac{E_2^{10}E_8^4}{E_1^4E_4^6}=\dfrac{1}{4}\varphi^2(q)\big(\varphi^2(q)-\varphi^2(q^2)\big),$

part (xxii) $q\dfrac{E_4E_8^7}{E_2^2E_{16}^2}=\dfrac{1}{4}\varphi(q^2)\varphi(q^4)\big(\varphi^2(q)-\varphi^2(q^2)\big),$

part (xxiii) $q\dfrac{E_8^7E_{16}^3}{E_4^4E_{32}^2} = \dfrac{1}{4}\varphi(q^4)\varphi(q^8)\big(\varphi^2(q) - \varphi^2(q^2)\big),$

part (xxiv) $q^2\dfrac{E_1^4E_{16}^4}{E_2^2E_8^2} = \dfrac{1}{4}\big(-\varphi^2(q) + 2\varphi^2(q^2)\big)\big(\varphi^2(q^2) - \varphi^2(q^4)\big),$

part (xxv) $q^2\dfrac{E_2^{10}E_{16}^4}{E_1^4E_4^4E_8^2} = \dfrac{1}{4}\varphi^2(q)\big(\varphi^2(q^2) - \varphi^2(q^4)\big),$

part (xxvi) $q^2\dfrac{E_2^3E_4^3E_{16}^4}{E_1^2E_8^4} = \dfrac{1}{4}\varphi(q)\varphi(q^2)\big(\varphi^2(q^2) - \varphi^2(q^4)\big),$

part (xxvii) $q^3\dfrac{E_8^2E_{16}^4}{E_4^2} = \dfrac{1}{16}\big(\varphi^2(q) - \varphi^2(q^2)\big)\big(\varphi^2(q^2) - \varphi^2(q^4)\big),$

part (xxviii) $q^4\dfrac{E_1^4E_{32}^4}{E_2^2E_{16}^2} = \dfrac{1}{4}\big(-\varphi^2(q) + 2\varphi^2(q^2)\big)\big(\varphi^2(q^4) - \varphi^2(q^8)\big),$

part (xxix) $q^4\dfrac{E_2^{10}E_{32}^4}{E_1^4E_4^4E_{16}^2} = \dfrac{1}{4}\varphi^2(q)\big(\varphi^2(q^4) - \varphi^2(q^8)\big),$

part (xxx) $q^4\dfrac{E_2^3E_4^3E_{32}^4}{E_1^2E_8^2E_{16}^2} = \dfrac{1}{4}\varphi(q)\varphi(q^2)\big(\varphi^2(q^4) - \varphi^2(q^8)\big),$

part (xxxi) $q^5\dfrac{E_8^4E_{32}^4}{E_4^2E_{16}^2} = \dfrac{1}{16}\big(\varphi^2(q) - \varphi^2(q^2)\big)\big(\varphi^2(q^4) - \varphi^2(q^8)\big).$

The evaluations of the representation numbers that arise from these identities are given in (3.7)–(3.16). We give the details for parts (xxi) and (xxviii). The other parts can be proved in a similar manner.

(xxi) We have

$$\begin{aligned} q\frac{E_2^{10}E_8^4}{E_1^4E_4^6} &= \frac{1}{4}\big(\varphi^4(q) - \varphi^2(q)\varphi^2(q^2)\big) \\ &= \frac{1}{4}\left(\sum_{n=0}^{\infty} N(1,1,1,1;n)q^n - \sum_{n=0}^{\infty} N(1,1,2,2;n)q^n\right) \\ &= \frac{1}{4}\left(\sum_{n=1}^{\infty} N(1,1,1,1;n)q^n - \sum_{n=1}^{\infty} N(1,1,2,2;n)q^n\right) \end{aligned}$$

as $N(1,1,1,1;0) = N(1,1,2,2;0) = 1$ by (3.2). Equating coefficients of q^n $(n \in \mathbb{N})$, we deduce

$$\left[q\frac{E_2^{10}E_8^4}{E_1^4E_4^6}\right]_n = \frac{1}{4}\big(N(1,1,1,1;n) - N(1,1,2,2;n)\big).$$

The asserted formula now follows from (3.7) and (3.8).

(xxviii) We have

$$
\begin{aligned}
q^4\frac{E_1^4E_{32}^4}{E_2^2E_{16}^2} &= -\frac{1}{4}\varphi^2(q)\varphi^2\bigl(q^4\bigr)+\frac{1}{4}\varphi^2(q)\varphi^2\bigl(q^8\bigr)\\
&\quad+\frac{1}{2}\varphi^2\bigl(q^2\bigr)\varphi^2\bigl(q^4\bigr)-\frac{1}{2}\varphi^2\bigl(q^2\bigr)\varphi^2\bigl(q^8\bigr)\\
&= -\frac{1}{4}\sum_{n=0}^{\infty}N(1,1,4,4;n)q^n+\frac{1}{4}\sum_{n=0}^{\infty}N(1,1,8,8;n)q^n\\
&\quad+\frac{1}{2}\sum_{n=0}^{\infty}N(1,1,2,2;n/2)q^n-\frac{1}{2}\sum_{n=0}^{\infty}N(1,1,4,4;n/2)q^n\\
&= \sum_{n=1}^{\infty}\biggl(-\frac{1}{4}N(1,1,4,4;n)+\frac{1}{4}N(1,1,8,8;n)\\
&\quad+\frac{1}{2}N(1,1,2,2;n/2)-\frac{1}{2}N(1,1,4,4;n/2)\biggr)q^n
\end{aligned}
$$

as $N(1,1,2,2;0)=N(1,1,4,4;0)=N(1,1,8,8;0)=1$. Hence for $n\in\mathbb{N}$ we have

$$
\begin{aligned}
\left[q^4\frac{E_1^4E_{32}^4}{E_2^2E_{16}^2}\right]_n &= -\frac{1}{4}N(1,1,4,4;n)+\frac{1}{4}N(1,1,8,8;n)\\
&\quad+\frac{1}{2}N(1,1,2,2;n/2)-\frac{1}{2}N(1,1,4,4;n/2).
\end{aligned}
$$

Appealing to (3.8), (3.9), and (3.16), we obtain part (xxviii) of Theorem 1.2.

6 Proof of Theorem 1.3

The 19 parts of Theorem 1.3 follow from the identities listed below, which follow from (2.1), Theorem 2.1(iii), (iv), and Theorem 2.2:

part (i) $$\frac{E_1^6}{E_3^2}=\frac{1}{4}\bigl(-a(q)+3a\bigl(q^3\bigr)\bigr)^2,$$

part (ii) $$\frac{E_1^3E_2^3}{E_3E_6}=\frac{1}{2}\bigl(-a(q)+2a\bigl(q^2\bigr)\bigr)\bigl(a(q)+a\bigl(q^2\bigr)\bigr),$$

part (iii) $$\frac{E_1^4E_3^4}{E_2^2E_6^2}=\frac{1}{3}\bigl(-a(q)+2a\bigl(q^2\bigr)\bigr)\bigl(a(q)+2a\bigl(q^2\bigr)\bigr),$$

part (iv) $$\frac{E_2^7E_3^7}{E_1^5E_6^5}=\frac{1}{6}\bigl(a(q)+a\bigl(q^2\bigr)\bigr)\bigl(a(q)+2a\bigl(q^2\bigr)\bigr),$$

part (v) $$\frac{E_1^{12}E_6^2}{E_2^6E_3^4} = \left(-a(q) + 2a\left(q^2\right)\right)^2,$$

part (vi) $$\frac{E_2^{12}E_3^2}{E_1^6E_6^4} = \frac{1}{4}\left(a(q) + a\left(q^2\right)\right)^2,$$

part (vii) $$\frac{E_2^2E_3^{12}}{E_1^4E_6^6} = \frac{1}{9}\left(a(q) + 2a\left(q^2\right)\right)^2,$$

part (viii) $$\frac{E_1^9E_6}{E_2^3E_3^3} = \frac{1}{2}\left(-a(q) + 2a\left(q^2\right)\right)\left(-a(q) + 3a\left(q^3\right)\right),$$

part (ix) $$\frac{E_1E_2E_3^5}{E_6^3} = \frac{1}{6}\left(a(q) + 2a\left(q^2\right)\right)\left(-a(q) + 3a\left(q^3\right)\right),$$

part (x) $$\frac{E_1^4E_2^2E_6^6}{E_3^4E_4^2E_{12}^2} = \frac{1}{4}\left(-\varphi^2(q) + 3\varphi^2\left(q^3\right)\right)^2,$$

part (xi) $$\frac{E_2^{11}E_6^3}{E_1^2E_3^2E_4^5E_{12}} = \frac{1}{2}\varphi^2(q)\left(-\varphi^2(q) + 3\varphi^2\left(q^3\right)\right),$$

part (xii) $$q\frac{E_1^4E_6^6}{E_2^3E_3^4} = \frac{1}{12}\left(-a(q) + 3a\left(q^3\right)\right)\left(a(q) - a\left(q^2\right)\right),$$

part (xiii) $$q\frac{E_3^3E_6^3}{E_1E_2} = \frac{1}{18}\left(a(q) + 2a\left(q^2\right)\right)\left(a(q) - a\left(q^2\right)\right),$$

part (xiv) $$q\frac{E_2^4E_6^4}{E_1^2E_3^2} = \frac{1}{12}\left(a(q) + a\left(q^2\right)\right)\left(a(q) - a\left(q^2\right)\right),$$

part (xv) $$q\frac{E_1^7E_6^7}{E_2^5E_3^5} = \frac{1}{6}\left(-a(q) + 2a\left(q^2\right)\right)\left(a(q) - a\left(q^2\right)\right),$$

part (xvi) $$q\frac{E_1E_2^4E_6^4E_{12}}{E_3^3E_4^3} = \frac{1}{8}\left(-\varphi^2(q) + 3\varphi^2\left(q^3\right)\right)\left(\varphi^2(q) - \varphi^2\left(q^3\right)\right),$$

part (xvii) $$q\frac{E_2^{13}E_6E_{12}^2}{E_1^5E_3E_4^6} = \frac{1}{4}\varphi^2(q)\left(\varphi^2(q) - \varphi^2\left(q^3\right)\right),$$

part (xviii) $$q^2\frac{E_1^2E_6^{12}}{E_2^4E_3^6} = \frac{1}{36}\left(a(q) - a\left(q^2\right)\right)^2,$$

part (xix) $$q^2\frac{E_2^6E_6^2E_{12}^4}{E_1^2E_3^2E_4^4} = \frac{1}{16}\left(\varphi^2(q) - \varphi^2\left(q^3\right)\right)^2.$$

We just give the details for parts (v) and (x). The remaining parts can be treated similarly using (3.7), (3.17), and (3.26)–(3.29).

(v) We have

$$\begin{aligned}\frac{E_1^{12}E_6^2}{E_2^6E_3^4} &= a^2(q) - 4a(q)a\big(q^2\big) + 4a^2\big(q^2\big)\\ &= \sum_{n=0}^{\infty} M(1,1;n)q^n - 4\sum_{n=0}^{\infty} M(1,2;n)q^n + 4\sum_{n=0}^{\infty} M(1,1;n)q^{2n}\\ &= 1 + \sum_{n=1}^{\infty}\big(M(1,1;n) - 4M(1,2;n) + 4M(1,1;n/2)\big)q^n\end{aligned}$$

so that for $n \in \mathbb{N}$

$$\begin{aligned}\left[\frac{E_1^{12}E_6^2}{E_2^6E_3^4}\right]_n &= M(1,1;n) - 4M(1,2;n) + 4M(1,1;n/2)\\ &= -12\sigma(n) + 96\sigma(n/2) - 108\sigma(n/3)\end{aligned}$$

by (3.26) and (3.27). Setting $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $\gcd(N, 6) = 1$, we obtain

$$\begin{aligned}\left[\frac{E_1^{12}E_6^2}{E_2^6E_3^4}\right]_n &= -6\big(2^{\alpha+1} - 1\big)\big(3^{\beta+1} - 1\big)\sigma(N) + 48\big(2^\alpha - 1\big)\big(3^{\beta+1} - 1\big)\sigma(N)\\ &\quad - 54\big(2^{\alpha+1} - 1\big)\big(3^\beta - 1\big)\sigma(N)\\ &= 12\big(3.2^{\alpha+1} - 2.3^{\beta+1} - 1\big)\sigma(N)\end{aligned}$$

as asserted.

(x) We have

$$\frac{E_1^4E_2^2E_6^6}{E_3^4E_4^2E_{12}^2} = \frac{1}{4}\varphi^4(q) - \frac{3}{2}\varphi^2(q)\varphi^2\big(q^3\big) + \frac{9}{4}\varphi^4\big(q^3\big)$$

so that for $n \in \mathbb{N}$

$$\left[\frac{E_1^4E_2^2E_6^6}{E_3^4E_4^2E_{12}^2}\right]_n = \frac{1}{4}N(1,1,1,1;n) - \frac{3}{2}N(1,1,3,3;n) + \frac{9}{4}N(1,1,1,1;n/3).$$

Setting $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $\gcd(N, 6) = 1$, we obtain

$$\left[\frac{E_1^4E_2^2E_6^6}{E_3^4E_4^2E_{12}^2}\right]_n = \begin{cases} 4(3^{\beta+1} - 4)\sigma(N) & \text{if } n \equiv 1 \pmod 2,\\ 12(3^{\beta+1} - 2^\alpha - 1)\sigma(N) & \text{if } n \equiv 0 \pmod 2,\end{cases}$$

by (3.7) and (3.17).

7 Proof of Theorem 1.4

(i) We have by Theorem 2.1(i)

$$\begin{aligned}\frac{E_1^{12}}{E_2^6} = \left(\frac{E_1^4}{E_2^2}\right)^3 &= \left(-\varphi^2(q)+2\varphi^2\left(q^2\right)\right)^3\\ &= -\varphi^6(q)+6\varphi^4(q)\varphi^2\left(q^2\right)-12\varphi^2(q)\varphi^4\left(q^2\right)+8\varphi^6\left(q^2\right)\\ &= -\sum_{n=0}^{\infty}N(1,1,1,1,1,1;n)q^n+6\sum_{n=0}^{\infty}N(1,1,1,1,2,2;n)q^n\\ &\quad -12\sum_{n=0}^{\infty}N(1,1,2,2,2,2;n)q^n+8\sum_{n=0}^{\infty}N(2,2,2,2,2,2;n)q^n\\ &= 1+\sum_{n=1}^{\infty}\big(-N(1,1,1,1,1,1;n)+6N(1,1,1,1,2,2;n)\\ &\quad -12N(1,1,2,2,2,2;n)+8N(1,1,1,1,1,1;n/2)\big)q^n.\end{aligned}$$

For n even, appealing to (3.21), (3.22), and (3.23), we obtain

$$\begin{aligned}\left[\frac{E_1^{12}}{E_2^6}\right]_n &= -N(1,1,1,1,1,1;n)+6N(1,1,1,1,2,2;n)\\ &\quad -12N(1,1,2,2,2,2;n)+8N(1,1,1,1,1,1;n/2)\\ &= -\left(2^{2\alpha+4}-4\left(\frac{-4}{N}\right)\right)F(N)+6\left(2^{2\alpha+3}-4\left(\frac{-4}{N}\right)\right)F(N)\\ &\quad -12\left(2^{2\alpha+2}-4\left(\frac{-4}{N}\right)\right)F(N)+8\left(2^{2\alpha+2}-4\left(\frac{-4}{N}\right)\right)F(N)\\ &= \left(2^{2\alpha+4}-4\left(\frac{-4}{N}\right)\right)F(N).\end{aligned}$$

For n odd, appealing to (3.21), (3.22), and (3.23), we obtain

$$\begin{aligned}\left[\frac{E_1^{12}}{E_2^6}\right]_n &= -N(1,1,1,1,1,1;n)+6N(1,1,1,1,2,2;n)\\ &\quad -12N(1,1,2,2,2,2;n)\\ &= -\left(2^{2\alpha+4}-4\left(\frac{-4}{N}\right)\right)F(N)+6\cdot 2^{2\alpha+3}F(N)\\ &\quad -12\cdot 2^{2\alpha+2}F(N)\\ &= -\left(2^{2\alpha+4}-4\left(\frac{-4}{N}\right)\right)F(N).\end{aligned}$$

(ii), (iii), (iv) In a similar manner, we can prove parts (ii), (iii), and (iv). For part (ii), we use

$$q\frac{E_1^8E_8^4}{E_2^4E_4^2}=\left(-\varphi^2(q)+2\varphi^2(q^2)\right)^2\frac{1}{4}\left(\varphi^2(q)-\varphi^2(q^2)\right);$$

for part (iii), we use

$$q^2\frac{E_1^4E_8^8}{E_2^2E_4^4}=\left(-\varphi^2(q)+2\varphi^2(q^2)\right)\left(\frac{1}{4}\left(\varphi^2(q)-\varphi^2(q^2)\right)\right)^2;$$

and for part (iv), we use

$$q^3\frac{E_8^{12}}{E_4^6}=\left(\frac{1}{4}\left(\varphi^2(q)-\varphi^2(q^2)\right)\right)^3.$$

8 Proof of Theorem 1.5

(i) We have by (2.1) and Theorem 2.1(v)

$$\begin{aligned}\frac{E_2^5E_{10}^7}{E_1E_4E_5^3E_{20}^3}&=\frac{E_2^5}{E_1^2E_4^2}\cdot\frac{E_{10}^5}{E_5^2E_{20}^2}\cdot\frac{E_1E_4E_{10}^2}{E_5E_{20}}\\&=\varphi(q)\varphi(q^5)\frac{1}{4}\left(-\varphi^2(q)+5\varphi^2(q^5)\right)\\&=-\frac{1}{4}\varphi^3(q)\varphi(q^5)+\frac{5}{4}\varphi(q)\varphi^3(q^5)\\&=1+\frac{1}{4}\sum_{n=1}^{\infty}\left(-N(1,1,1,5;n)+5N(1,5,5,5;n)\right)q^n\end{aligned}$$

so that by (3.19) and (3.20) we have

$$\begin{aligned}\left[\frac{E_2^5E_{10}^7}{E_1E_4E_5^3E_{20}^3}\right]_n&=\frac{1}{4}\left(-N(1,1,1,5;n)+5N(1,5,5,5;n)\right)\\&=\sum_{\substack{d\in\mathbb{N}\\ d\mid N}}(-1)^{n+d}d\left(\frac{5}{d}\right).\end{aligned}$$

(ii) Similarly, we obtain

$$\begin{aligned}\left[q\frac{E_2^7E_{10}^5}{E_1^3E_4^3E_5E_{20}}\right]_n&=\frac{1}{4}\left(N(1,1,1,5;n)-N(1,5,5,5;n)\right)\\&=\sum_{\substack{d\in\mathbb{N}\\ d\mid N}}(-1)^{n+d}d\left(\frac{5}{n/d}\right).\end{aligned}$$

This completes the proof of Theorem 1.5.

We conclude this section by noting that if q is replaced by $-q$ in parts (i) and (ii) of Theorem 1.5, as $E_1(-q)=\frac{E_2^3}{E_1E_4}$, $E_5(-q)=\frac{E_{10}^3}{E_5E_{20}}$ and $E_{2m}(-q)=E_{2m}$ ($m\in\mathbb{N}$), we obtain

$$\left[\frac{E_1E_2^2E_5^3}{E_{10}^2}\right]_n=\sum_{\substack{d\in\mathbb{N}\\ d\mid N}}(-1)^d d\left(\frac{5}{d}\right)$$

and

$$\left[q\frac{E_1^3E_5E_{10}^2}{E_2^2}\right]_n=\sum_{\substack{d\in\mathbb{N}\\ d\mid N}}(-1)^{d+1} d\left(\frac{5}{n/d}\right),$$

which are Theorems 4.2 and 4.3 in [6], respectively.

9 Proof of Theorem 1.6

We have from (2.1) and Theorem 2.1(vii)

$$\begin{aligned} q\frac{E_2^4E_3E_4^4E_{24}}{E_1^3E_8^3} &= \frac{E_2^5}{E_1^2E_4^2}\cdot\frac{E_4^5}{E_2^2E_8^2}.q\frac{E_2E_3E_4E_{24}}{E_1E_8} \\ &= \varphi(q)\varphi(q^2)\frac{1}{2}\big(\varphi(q)\varphi(q^2)-\varphi(q^3)\varphi(q^6)\big) \\ &= \frac{1}{2}\varphi^2(q)\varphi^2(q^2)-\frac{1}{2}\varphi(q)\varphi(q^2)\varphi(q^3)\varphi(q^6) \\ &= \frac{1}{2}\sum_{n=0}^{\infty}N(1,1,2,2;n)q^n-\frac{1}{2}\sum_{n=0}^{\infty}N(1,2,3,6;n)q^n \\ &= \sum_{n=1}^{\infty}\frac{1}{2}\big(N(1,1,2,2;n)-N(1,2,3,6;n)\big)q^n \end{aligned}$$

so that for $n\in\mathbb{N}$

$$\left[q\frac{E_2^4E_3E_4^4E_{24}}{E_1^3E_8^3}\right]_n=\frac{1}{2}\big(N(1,1,2,2;n)-N(1,2,3,6;n)\big). \tag{9.1}$$

The asserted formula now follows from (3.8) and (3.18).

10 An application

Let $n\in\mathbb{N}$. In [31, pp. 117, 240] (see also [16]) it is shown that

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}}\left(\frac{-4}{ab}\right)=\frac{1}{2}\sigma(n)-2\sigma(n/4)-\frac{1}{2}\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right) \tag{10.1}$$

and

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+2by=n}} \left(\frac{-4}{ab}\right) = \frac{1}{4}\sigma(n) - \frac{1}{4}\sigma(n/2) + \frac{1}{2}\sigma(n/4) - 2\sigma(n/8) \qquad (10.2)$$

$$- \frac{1}{4}\sum_{\substack{d\in\mathbb{N}\\ d|n}} \left(\frac{-4}{d}\right) - \frac{1}{4}\sum_{\substack{d\in\mathbb{N}\\ d|n/2}} \left(\frac{-4}{d}\right).$$

As an application of our results, we use Theorems 1.1(iv) and 1.3(xvii) in conjunction with a technique of Jacobi to evaluate the similar sum

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+3by=n}} \left(\frac{-4}{ab}\right) \qquad (10.3)$$

for all $n \in \mathbb{N}$. To do this, we consider the infinite product

$$q\frac{E_2^{13}E_6E_{12}^2}{E_1^5E_3E_4^6}. \qquad (10.4)$$

By Theorem 1.3(xvii), we obtain after a short calculation

$$\left[q\frac{E_2^{13}E_6E_{12}^2}{E_1^5E_3E_4^6}\right] = \sigma(n) + 2\sigma(n/2) + 3\sigma(n/3) \qquad (10.5)$$

$$- 12\sigma(n/4) - 6\sigma(n/6) + 12\sigma(n/12)$$

in agreement with [32, Table 1, No. 75, p. 1000], so that

$$q\frac{E_2^{13}E_6E_{12}^2}{E_1^5E_3E_4^6} = \sum_{n=1}^{\infty}\big(\sigma(n) + 2\sigma(n/2) + 3\sigma(n/3) \qquad (10.6)$$

$$- 12\sigma(n/4) - 6\sigma(n/6) + 12\sigma(n/12)\big)q^n.$$

Next we express the product (10.4) as

$$q\frac{E_2^{13}E_6E_{12}^2}{E_1^5E_3E_4^6} = \frac{E_2^{10}}{E_1^4E_4^4} \cdot q\frac{E_2^3E_6E_{12}^2}{E_1E_3E_4^2}$$

and determine the power series expansion of each member of the product. First, we have

$$\frac{E_2^{10}}{E_1^4E_4^4} = \varphi^2(q) = 1 + \sum_{n_1=1}^{\infty} N(1,1;n_1)q^{n_1},$$

so that by (3.4)

$$\frac{E_2^{10}}{E_1^4 E_4^4} = 1 + 4 \sum_{n_1=1}^{\infty} \sum_{\substack{d_1 \in \mathbb{N} \\ d_1 | n_1}} \left(\frac{-4}{d_1} \right) q^{n_1}. \tag{10.7}$$

Secondly, by Theorem 1.1(iv), we have

$$q \frac{E_2^3 E_6 E_{12}^2}{E_1 E_3 E_4^2} = \sum_{n_2=1}^{\infty} \left(\sum_{\substack{d_2 \in \mathbb{N} \\ d_2 | n_2}} \left(\frac{-4}{d_2} \right) - \sum_{\substack{d_2 \in \mathbb{N} \\ d_2 | n_2/3}} \left(\frac{-4}{d_2} \right) \right) q^{n_2}. \tag{10.8}$$

Multiplying (10.7) and (10.8) together, we obtain

$$\begin{aligned} q \frac{E_2^{13} E_6 E_{12}^2}{E_1^5 E_3 E_4^6} &= \sum_{n_2=1}^{\infty} \left(\sum_{\substack{d_2 \in \mathbb{N} \\ d_2 | n_2}} \left(\frac{-4}{d_2} \right) - \sum_{\substack{d_2 \in \mathbb{N} \\ d_2 | n_2/3}} \left(\frac{-4}{d_2} \right) \right) q^{n_2} \\ &\quad + 4 \sum_{n=1}^{\infty} \left(\sum_{\substack{n_1, n_2 \in \mathbb{N} \\ n_1 + n_2 = n}} \sum_{\substack{d_1 | n_1 \\ d_2 | n_2}} \left(\frac{-4}{d_1 d_2} \right) - \sum_{\substack{n_1, n_2 \in \mathbb{N} \\ n_1 + 3n_2 = n}} \sum_{\substack{d_1 | n_1 \\ d_2 | n_2}} \left(\frac{-4}{d_1 d_2} \right) \right) q^n. \end{aligned} \tag{10.9}$$

Equating coefficients of q^n $(n \in \mathbb{N})$ in (10.6) and (10.9), we deduce

$$\begin{aligned} &\sum_{\substack{d \in \mathbb{N} \\ d | n}} \left(\frac{-4}{d} \right) - \sum_{\substack{d \in \mathbb{N} \\ d | n/3}} \left(\frac{-4}{d} \right) + 4 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax + by = n}} \left(\frac{-4}{ab} \right) - 4 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax + 3by = n}} \left(\frac{-4}{ab} \right) \\ &= \sigma(n) + 2\sigma(n/2) + 3\sigma(n/3) - 12\sigma(n/4) - 6\sigma(n/6) + 12\sigma(n/12). \end{aligned}$$

Appealing to (10.1), we have the following result.

Theorem 10.1 *Let* $n \in \mathbb{N}$. *Then*

$$\begin{aligned} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax + 3by = n}} \left(\frac{-4}{ab} \right) &= \frac{1}{4}\sigma(n) - \frac{1}{2}\sigma(n/2) - \frac{3}{4}\sigma(n/3) + \sigma(n/4) + \frac{3}{2}\sigma(n/6) \\ &\quad - 3\sigma(n/12) - \frac{1}{4} \sum_{\substack{d \in \mathbb{N} \\ d | n}} \left(\frac{-4}{d} \right) - \frac{1}{4} \sum_{\substack{d \in \mathbb{N} \\ d | n/3}} \left(\frac{-4}{d} \right). \end{aligned}$$

By the same method, we can obtain many other arithmetic identities similar to Theorem 10.1.

References

1. Alaca, A.: Representations by quaternary quadratic forms whose coefficients are 1, 3 and 9. Acta Arith. **136**, 151–166 (2009)
2. Alaca, A., Alaca, Ş., Lemire, M.F., Williams, K.S.: Nineteen quaternary quadratic forms. Acta Arith. **130**, 277–310 (2007)
3. Alaca, A., Alaca, Ş., Lemire, M.F., Williams, K.S.: Jacobi's identity and representations of integers by certain quaternary quadratic forms. Int. J. Mod. Math. **2**, 143–176 (2007)
4. Alaca, A., Alaca, Ş., Lemire, M.F., Williams, K.S.: The number of representations of a positive integer by certain quaternary quadratic forms. Int. J. Number Theory **5**, 13–40 (2009)
5. Alaca, A., Alaca, Ş., Williams, K.S.: On the two-dimensional theta functions of the Borweins. Acta Arith. **124**, 177–195 (2006)
6. Alaca, A., Alaca, Ş., Williams, K.S.: On the quaternary forms $x^2+y^2+z^2+5t^2$, $x^2+y^2+5z^2+5t^2$ and $x^2+5y^2+5z^2+5t^2$. JP J. Algebra Number Theory Appl. **9**, 37–53 (2007)
7. Alaca, A., Alaca, Ş., Williams, K.S.: Liouville's sextanary quadratic forms $x^2+y^2+z^2+t^2+2u^2+2v^2$, $x^2+y^2+2z^2+2t^2+2u^2+2v^2$ and $x^2+2y^2+2z^2+2t^2+2u^2+4v^2$, Far East. J. Math. Sci. **30**, 547–556 (2008)
8. Alaca, Ş., Williams, K.S.: The number of representations of a positive integer by certain octonary quadratic forms. Funct. Approx. Comment. Math. **43**, 45–54 (2010)
9. Berndt, B.C.: Ramanujan's Notebooks Part III. Springer, New York (1991)
10. Berndt, B.C.: Number Theory in the Spirit of Ramanujan. Am. Math. Soc., Providence (2006)
11. Bhargava, S.: Some applications of Ramanujan's remarkable summation formula. In: Ramanujan Rediscovered. Proceedings of a Conference on Elliptic Functions, Partitions, and q-Series in Memory of K. Venkatachaliengar, Bangalore, 1–5 June 2009. Ramanujan Mathematical Society Lecture Notes Series, vol. 14, pp. 63–72. Ramanujan Mathematical Society, Thiruchirappalli (2010)
12. Blecksmith, R., Brillhart, J., Gerst, I.: Parity results for certain partition functions and identities similar to theta function identities. Math. Comput. **48**, 29–38 (1987)
13. Chen, S.-L., Huang, S.-S.: Identities for certain products of theta functions. Ramanujan J. **8**, 5–12 (2004)
14. Cooper, S., Hirschhorn, M.: On some sum-to-product identities. Bull. Aust. Math. Soc. **63**, 353–365 (2001)
15. Hardy, G.H., Wright, E.M.: An Introduction to the Theory of Numbers, 4th edn. Oxford University Press, London (1960)
16. Huard, J.G., Ou, Z.M., Spearman, B.K., Williams, K.S.: Elementary evaluation of certain convolution sums involving divisor functions. In: Bennett, M.A., et al. (eds.) Number Theory for the Millennium, II, pp. 229–274. A K Peters, Natick (2002)
17. Liouville, J.: Sur la forme $x^2+y^2+3(z^2+t^2)$. J. Math. Pures Appl. **5**, 147–152 (1860)
18. Liouville, J.: Sur la forme $x^2+y^2+2(z^2+t^2)$. J. Math. Pures Appl. **5**, 269–272 (1860)
19. Liouville, J.: Sur la forme $x^2+y^2+4(z^2+t^2)$. J. Math. Pures Appl. **5**, 305–308 (1860)
20. Liouville, J.: Sur la forme $x^2+2y^2+4z^2+8t^2$. J. Math. Pures Appl. **6**, 409–416 (1861)
21. Liouville, J.: Sur la forme $x^2+2y^2+4z^2+4t^2$. J. Math. Pures Appl. **7**, 62–64 (1862)
22. Liouville, J.: Sur la forme $x^2+2y^2+8z^2+8t^2$. J. Math. Pures Appl. **7**, 65–68 (1862)
23. Liouville, J.: Sur la forme $x^2+y^2+2z^2+4t^2$. J. Math. Pures Appl. **7**, 99–100 (1862)
24. Liouville, J.: Sur la forme $x^2+y^2+4z^2+8t^2$. J. Math. Pures Appl. **7**, 103–104 (1862)
25. Liouville, J.: Sur la forme $x^2+y^2+8z^2+8t^2$. J. Math. Pures Appl. **7**, 109–112 (1862)
26. Liouville, J.: Sur la forme $x^2+xy+y^2+z^2+zt+t^2$. J. Math. Pures Appl. **8**, 141–144 (1863)
27. Liouville, J.: Sur la forme $x^2+xy+y^2+2z^2+2zt+2t^2$. J. Math. Pures Appl. **8**, 308–310 (1863)
28. Liouville, J.: Sur la forme $2x^2+2xy+2y^2+3z^2+3zt+3t^2$. J. Math. Pures Appl. **9**, 183–184 (1864)
29. Liouville, J.: Sur la forme $x^2+xy+y^2+3z^2+3zt+3t^2$. J. Math. Pures Appl. **9**, 223–224 (1864)
30. Williams, K.S.: On the representations of a positive integer by the forms $x^2+y^2+z^2+2t^2$ and $x^2+2y^2+2z^2+2t^2$. Int. J. Mod. Math. **3**, 225–230 (2008)
31. Williams, K.S.: Number Theory in the Spirit of Liouville. Cambridge University Press, Cambridge (2011)
32. Williams, K.S.: Fourier series of a class of eta quotients. Int. J. Number Theory **8**, 993–1004 (2012)