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# ON THE QUATERNARY FORMS $x^2 + y^2 + 2z^2 + 3t^2$, $x^2 + 2y^2 + 2z^2 + 6t^2$, $x^2 + 3y^2 + 3z^2 + 6t^2$ AND $2x^2 + 3y^2 + 6z^2 + 6t^2$

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Formulas are proved for the number of representations of a positive integer by each of the four quaternary quadratic forms $x^2+y^2+2z^2+3t^2$, $x^2+2y^2+2z^2+6t^2$, $x^2+3y^2+3z^2+6t^2$ and $2x^2 + 3y^2 + 6z^2 + 6t^2$. As a consequence of these formulas, each of the four series

$$\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-3}{n/d}\right)\left(\frac{-8}{n/d}\right)d\right)q^n, \quad \sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-3}{n/d}\right)\left(\frac{-8}{d}\right)d\right)q^n,$$

$$\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-3}{d}\right)\left(\frac{-8}{n/d}\right)d\right)q^n, \quad \sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-3}{d}\right)\left(\frac{-8}{d}\right)d\right)q^n,$$

is determined in terms of Ramanujan's theta function.

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## 1. Introduction

In 2007, Alaca *et al.* [1] used their $(p, k)$-parametrization of Ramanujan's theta function

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q \in \mathbb{C}, \ |q| < 1, \tag{1.1}$$

to determine the number $N(a, b, c, d; n)$ of representations of a positive integer $n$ by each of the nineteen quaternary quadratic forms $ax^2 + by^2 + cz^2 + dt^2$, where

$$\begin{aligned}(a,b,c,d) = {} & (1,1,1,1),(1,1,1,4),(1,1,2,2),(1,1,3,3),\\ & (1,1,3,12),(1,1,4,4),(1,1,6,6),(1,1,12,12),\\ & (1,2,2,4),(1,2,3,6),(1,3,3,4),(1,3,4,12),\\ & (1,4,4,4),(1,4,6,6),(1,4,12,12),(2,2,3,3),\\ & (2,2,3,12),(3,3,4,4),(3,4,4,12). \end{aligned} \tag{1.2}$$

Some of these evaluations had been conjectured by Liouville in the nineteenth century and proved later by other authors, while some were new, see [1] for details. We observe that in the list (1.2), "2" and "6" only occur as "22", "26" or "66". The reason for this is that the parametrizations of $\varphi^2(q^2)$, $\varphi(q^2)\varphi(q^6)$ and $\varphi^2(q^6)$ in terms of $p$ and $k$ are fairly simple while those of $\varphi(q^2)$ and $\varphi(q^6)$ individually are more complicated. However, somewhat surprisingly, there are quaternary quadratic forms $ax^2 + by^2 + cz^2 + dt^2$ $(a,b,c,d \in \{1,2,3,4,6,12\})$ with a singleton "2" or a singleton "6", which have a simple formula for $N(a,b,c,d;n)$, namely $x^2 + y^2 + 2z^2 + 3t^2$, $x^2 + 2y^2 + 2z^2 + 6t^2$, $x^2 + 3y^2 + 3z^2 + 6t^2$ and $2x^2+3y^2+6z^2+6t^2$. These forms have not been considered before. The evaluations of $N(1,1,2,3;n)$, $N(1,2,2,6;n)$, $N(1,3,3,6;n)$ and $N(2,3,6,6;n)$ are given in the first part of Theorem 4.1. The second part of Theorem 4.1 gives alternative expressions for $N(1,1,2,3;n)$, $N(1,2,2,6;n)$, $N(1,3,3,6;n)$ and $N(2,3,6,6;n)$. Theorem 4.1 is proved in Sec. 4 using the "local densities" approach, which is described in Sec. 2. Certain Gauss sums needed in the proof of Theorem 4.1 are evaluated in Sec. 3. As an application of Theorem 4.1, we determine the series

$$\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-3}{n/d}\right)\left(\frac{-8}{n/d}\right)d\right)q^n, \quad \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-3}{n/d}\right)\left(\frac{-8}{d}\right)d\right)q^n,$$

$$\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-3}{d}\right)\left(\frac{-8}{n/d}\right)d\right)q^n, \quad \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-3}{d}\right)\left(\frac{-8}{d}\right)d\right)q^n,$$

in terms of $\varphi(q)$, $\varphi(q^2)$, $\varphi(q^3)$ and $\varphi(q^6)$, see Theorem 5.1. We remark that in the list (1.2) all the forms have $abcd =$ a square. In [2, 3, 7] the diagonal forms considered all involve $abcd = 3\times$ a square, whereas the results in the present paper are for $abcd = 6\times$ a square.

## 2. Local Densities Approach

Let $a_1, a_2, a_3, a_4 \in \mathbb{N} = \{1,2,3,\ldots\}$. The discriminant of the positive-definite, diagonal, integral, quaternary quadratic form $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$ is

$$\begin{vmatrix} 2a_1 & 0 & 0 & 0 \\ 0 & 2a_2 & 0 & 0 \\ 0 & 0 & 2a_3 & 0 \\ 0 & 0 & 0 & 2a_4 \end{vmatrix} = 16a_1a_2a_3a_4. \tag{2.1}$$

For a prime $p$ and $k, n \in \mathbb{N}$ we define

$$N_{p^k}(a_1, a_2, a_3, a_4; n) := \text{number of solutions of the congruence} \quad a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 \equiv n \pmod{p^k}. \tag{2.2}$$

If the class of the quaternary form $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$ belongs to a genus of discriminant $16a_1a_2a_3a_4$ containing one and only one form class, then Siegel's mass formula [10] asserts that

$$N(a_1, a_2, a_3, a_4; n) = \frac{\pi^2 n}{\sqrt{a_1a_2a_3a_4}} \prod_p d_p(a_1, a_2, a_3, a_4; n), \tag{2.3}$$

where $d_p(a_1, a_2, a_3, a_4; n)$ denotes the "local density" given by

$$d_p(a_1, a_2, a_3, a_4; n) = \lim_{k\to\infty} \frac{N_{p^k}(a_1, a_2, a_3, a_4; n)}{p^{3k}}. \tag{2.4}$$

If $a_1 \le a_2 \le a_3 \le a_4$ then the least positive integer represented by the form $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$ is $a_1$ so that

$$N(a_1, a_2, a_3, a_4; a_1) > 0. \tag{2.5}$$

Then, by (2.3), we have

$$N(a_1, a_2, a_3, a_4; a_1) = \frac{\pi^2 a_1}{\sqrt{a_1a_2a_3a_4}} \prod_p d_p(a_1, a_2, a_3, a_4; a_1). \tag{2.6}$$

Hence, dividing (2.3) by (2.6), we obtain in view of (2.5) the formula

$$N(a_1, a_2, a_3, a_4; n) = \frac{n}{a_1} N(a_1, a_2, a_3, a_4; a_1) \prod_p \frac{d_p(a_1, a_2, a_3, a_4; n)}{d_p(a_1, a_2, a_3, a_4; a_1)}. \tag{2.7}$$

Thus, if we can determine the local densities $d_p(a_1, a_2, a_3, a_4; n)$ for all primes $p$ and all $n \in \mathbb{N}$, then we can use the formula (2.7) to determine $N(a_1, a_2, a_3, a_4; n)$ for all $n \in \mathbb{N}$ since

$$N(a_1, a_2, a_3, a_4; a_1) = \begin{cases} 2 & \text{if } a_1 < a_2, \\ 4 & \text{if } a_1 = a_2 < a_3, \\ 6 & \text{if } a_1 = a_2 = a_3 < a_4, \\ 8 & \text{if } a_1 = a_2 = a_3 = a_4. \end{cases} \tag{2.8}$$

This is the local densities approach that we shall take to prove Theorem 4.1. In order to determine the local density $d_p(a_1, a_2, a_3, a_4; n)$ we evaluate $N_{p^k}(a_1, a_2, a_3, a_4; n)$ for all primes $p$ and all $k, n \in \mathbb{N}$ using Gauss sums. The relevant Gauss sums are evaluated in the next section.

## 3. Gauss Sums

In this section we give the evaluation of the Gauss sums that we need in Sec. 4 in order to determine the local density $d_p(1, 3, 3, 6; n)$ for all primes $p$ and all $n \in \mathbb{N}$.

**Lemma 3.1.** *If $k, l \in \mathbb{N}$ and $m \in \mathbb{Z}$ then*

$$\sum_{x=0}^{lk-1} e^{\frac{2\pi i m x^2}{k}} = l \sum_{x=0}^{k-1} e^{\frac{2\pi i m x^2}{k}}.$$

**Proof.** Each integer $x$ satisfying $0 \leq x \leq lk - 1$ can be expressed uniquely in the form

$$x = y + kz,$$

where $y$ and $z$ are integers satisfying $0 \leq y \leq k - 1$ and $0 \leq z \leq l - 1$. Thus

$$\sum_{x=0}^{lk-1} e^{\frac{2\pi i m x^2}{k}} = \sum_{z=0}^{l-1} \sum_{y=0}^{k-1} e^{\frac{2\pi i m (y+kz)^2}{k}} = \sum_{z=0}^{l-1} \sum_{y=0}^{k-1} e^{\frac{2\pi i m y^2}{k}} = l \sum_{y=0}^{k-1} e^{\frac{2\pi i m y^2}{k}},$$

which is the asserted result. □

**Lemma 3.2.** *If $k, l \in \mathbb{N}$ and $m \in \mathbb{Z}$ satisfy*

$$k \text{ odd}, \quad \gcd(k, m) = 1,$$

*then*

$$\sum_{x=0}^{lk-1} e^{\frac{2\pi i m x^2}{k}} = l \left(\frac{m}{k}\right) i^{(\frac{k-1}{2})^2} \sqrt{k}.$$

**Proof.** This follows immediately from Lemma 3.1 and Gauss' famous evaluation

$$\sum_{x=0}^{k-1} e^{\frac{2\pi i m x^2}{k}} = \left(\frac{m}{k}\right) i^{(\frac{k-1}{2})^2} \sqrt{k},$$

which is valid as $\gcd(k, m) = 1$ and $k$ is odd, see for example [6, Theorem 1.5.2, p. 26]. □

**Lemma 3.3.** *If $l \in \mathbb{N}$, $m \in \mathbb{Z}$ with $m$ odd, and $s \in \mathbb{N}$ then*

$$\sum_{x=0}^{l2^s-1} e^{\frac{2\pi i m x^2}{2^s}} = \begin{cases} 0 & \text{if } s = 1, \\ l\left(\dfrac{2}{m}\right)^s (1 + i^m) 2^{s/2} & \text{if } s \geq 2. \end{cases}$$

**Proof.** This follows immediately from Lemma 3.1 and the evaluation

$$\sum_{x=0}^{2^s-1} e^{\frac{2\pi i m x^2}{2^s}} = \begin{cases} 0 & \text{if } s = 1, \\ \left(\dfrac{2}{m}\right)^s (1 + i^m) 2^{s/2} & \text{if } s \geq 2, \end{cases}$$

given in [6, Theorem 1.5.1, Proposition 1.5.3, p. 26]. □

Our final sum that we need, although not a Gauss sum, is nevertheless an exponential sum.

**Lemma 3.4.** *Let $k, l, m$ be integers such that*

$$k \geq 1, \quad l \geq 0, \quad m \neq 0.$$

*Let $p$ be a prime with $p \nmid m$. Then*

$$\sum_{\substack{x=0 \\ p \nmid x}}^{p^{k+l}-1} e^{\frac{2\pi i m x}{p^k}} = \begin{cases} -p^l & \text{if } k = 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

**Proof.** We have

$$\sum_{\substack{x=0 \\ p \nmid x}}^{p^{k+l}-1} e^{\frac{2\pi i m x}{p^k}} = \sum_{z=0}^{p^l-1} \sum_{\substack{y=0 \\ p \nmid y}}^{p^k-1} e^{\frac{2\pi i m (y+p^k z)}{p^k}} = \sum_{z=0}^{p^l-1} \sum_{\substack{y=0 \\ p \nmid y}}^{p^k-1} e^{\frac{2\pi i m y}{p^k}}$$

$$= p^l \left( \sum_{y=0}^{p^k-1} e^{\frac{2\pi i m y}{p^k}} - \sum_{\substack{y=0 \\ p \mid y}}^{p^k-1} e^{\frac{2\pi i m y}{p^k}} \right) = p^l \left( 0 - \sum_{t=0}^{p^{k-1}-1} e^{\frac{2\pi i m t}{p^{k-1}}} \right)$$

$$= \begin{cases} -p^l & \text{if } k = 1, \\ 0 & \text{if } k \geq 2, \end{cases}$$

as claimed. □

## 4. Determination of $N(1,1,2,3;n)$, $N(1,2,2,6;n)$, $N(1,3,3,6;n)$ and $N(2,3,6,6;n)$

We see from the tables of Nipp [8] that each of the three forms $x^2 + y^2 + 2z^2 + 3t^2$ (discriminant $= 96$), $x^2 + 2y^2 + 2z^2 + 6t^2$ (discriminant $= 384$) and $x^2 + 3y^2 + 3z^2 + 6t^2$ (discriminant $= 864$) belongs to a genus containing exactly one form class so that each of $N(1,1,2,3;n)$, $N(1,2,2,6;n)$ and $N(1,3,3,6;n)$ can be determined from Siegel's formula (2.7). The fourth form $2x^2 + 3y^2 + 6z^2 + 6t^2$ (discriminant $= 3456$) is beyond the range of Nipp's tables. From the webpage http://www.kobepharma-u.ac.jp/∼math/notes/note03.html we know that it too belongs to a genus containing exactly one form class. The following result allows us to determine each of $N(1,1,2,3;n)$, $N(1,2,2,6;n)$ and $N(2,3,6,6;n)$ from $N(1,3,3,6;n)$, $N(1,3,3,6;2n)$, $N(1,3,3,6;3n)$ and $N(1,3,3,6;6n)$. Thus it suffices to determine $N(1,3,3,6;n)$ from (2.7).

**Lemma 4.1.** *For $n \in \mathbb{N}$ we have*

$$N(1,1,2,3;n) = N(1,3,3,6;3n),$$

$$N(1,2,2,6;n) = \frac{3}{2}N(1,3,3,6;n) - N(1,3,3,6;3n) + \frac{1}{2}N(1,3,3,6;6n),$$

$$N(2,3,6,6;n) = N(1,3,3,6;n) + \frac{1}{2}N(1,3,3,6;2n) - \frac{1}{2}N(1,3,3,6;3n).$$

**Proof.** We have

$$\begin{aligned}N(1,3,3,6;3n) &= \text{card}\{(x,y,z,t)\in\mathbb{Z}^4 \mid x^2+3y^2+3z^2+6t^2=3n\}\\ &= \text{card}\{(x,y,z,t)\in\mathbb{Z}^4 \mid x^2+3y^2+3z^2+6t^2=3n, x\equiv 0 \pmod 3\}\\ &= \text{card}\{(x_1,y,z,t)\in\mathbb{Z}^4 \mid 9{x_1}^2+3y^2+3z^2+6t^2=3n\}\\ &= \text{card}\{(y,z,t,x_1)\in\mathbb{Z}^4 \mid y^2+z^2+2t^2+3{x_1}^2=n\}\\ &= N(1,1,2,3;n),\end{aligned}$$

which is the first assertion of Lemma 4.1.

As in [4, p. 178; 5, p. 91], we set

$$p = p(q) := \frac{\varphi^2(q)-\varphi^2(q^3)}{2\varphi^2(q^3)}, \quad k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}.$$

From [2, pp. 222–223; 3, pp. 541–542] we have

$$\begin{cases}\varphi(q) = (1+2p)^{3/4}k^{1/2},\\ \varphi(q^2) = \dfrac{1}{\sqrt{2}}((1+2p)^{3/2}+(1-p)^{3/2}(1+p)^{1/2})^{1/2}k^{1/2},\\ \varphi(q^3) = (1+2p)^{1/4}k^{1/2},\\ \varphi(q^4) = \dfrac{1}{2}((1+2p)^{3/4}+(1-p)^{3/4}(1+p)^{1/4})k^{1/2},\\ \varphi(q^6) = \dfrac{1}{\sqrt{2}}((1+2p)^{1/2}+(1-p)^{1/2}(1+p)^{3/2})^{1/2}k^{1/2},\\ \varphi(q^{12}) = \dfrac{1}{2}((1+2p)^{1/4}+(1-p)^{1/4}(1+p)^{3/4})k^{1/2},\end{cases} \tag{4.1}$$

and

$$\begin{cases}\varphi(-q) = (1-p)^{3/4}(1+p)^{1/4}k^{1/2},\\ \varphi(-q^2) = (1+2p)^{3/8}(1-p)^{3/8}(1+p)^{1/8}k^{1/2},\\ \varphi(-q^3) = (1-p)^{1/4}(1+p)^{3/4}k^{1/2},\\ \varphi(-q^4) = 2^{-1/4}(1+2p)^{3/16}(1-p)^{3/16}(1+p)^{1/16}\\ \qquad\times((1+2p)^{3/2}+(1-p)^{3/2}(1+p)^{1/2})^{1/4}k^{1/2},\\ \varphi(-q^6) = (1+2p)^{1/8}(1-p)^{1/8}(1+p)^{3/8}k^{1/2},\\ \varphi(-q^{12}) = 2^{-1/4}(1+2p)^{1/16}(1-p)^{1/16}(1+p)^{3/16}\\ \qquad\times((1+2p)^{1/2}+(1-p)^{1/2}(1+p)^{3/2})^{1/4}k^{1/2}.\end{cases} \tag{4.2}$$

Using this parametrization we find with the help of MAPLE that the following two identities hold:

$$\begin{aligned}&\varphi^2(q)\varphi(q^3)+\varphi^2(-q)\varphi(-q^3)\\&\quad=4\varphi(q^2)\varphi(q^4)\varphi(q^6)+4\varphi^2(q^4)\varphi(q^{12})-6\varphi^2(q^6)\varphi(q^{12}),\end{aligned}\tag{4.3}$$

$$\begin{aligned}&\varphi(q)\varphi^2(q^3)+\varphi(-q)\varphi^2(-q^3)\\&\quad=2\varphi^2(q^2)\varphi(q^4)+4\varphi(q^4)\varphi^2(q^{12})-4\varphi(q^2)\varphi(q^6)\varphi(q^{12}).\end{aligned}\tag{4.4}$$

Multiplying (4.3) by $\varphi(q^2)$ and (4.4) by $\varphi(q^6)$, and then equating coefficients of $q^{2n}$ ($n \in \mathbb{N}$), we obtain

$$N(1,1,2,3;2n)=2N(1,1,2,3;n)+2N(1,2,2,6;n)-3N(1,3,3,6;n)$$

and

$$N(1,3,3,6;2n)=N(1,1,2,3;n)+2N(2,3,6,6;n)-2N(1,3,3,6;n).$$

By rearranging these two equations, we obtain the last two assertions of Lemma 4.1. □

We are now ready to determine $N(1,3,3,6;n)$. From (2.7) we have

$$N(1,3,3,6;n)=2n\prod_p \frac{d_p(1,3,3,6;n)}{d_p(1,3,3,6;1)}\tag{4.5}$$

as $N(1,3,3,6;1)=2$. We make use of the simple result: For $h \in \mathbb{N}$ and $r \in \mathbb{Z}$ we have

$$\frac{1}{h}\sum_{m=0}^{h-1} e^{2\pi irm/h}=\begin{cases}1 & \text{if } r\equiv 0 \pmod h,\\ 0 & \text{if } r\not\equiv 0 \pmod h.\end{cases}\tag{4.6}$$

**Theorem 4.1.** *Let* $n \in \mathbb{N}$. *Define* $\alpha,\beta \in \mathbb{N}_0=\mathbb{N}\cup\{0\}$ *and* $N\in\mathbb{N}$ *with* $\gcd(N,6)=1$ *uniquely by* $n=2^\alpha 3^\beta N$. *Then*

$$\begin{aligned}&N(1,1,2,3;n)\\&\quad=\frac{1}{3}\left(2^{\alpha+2}-(-1)^\alpha\left(\frac{-2}{N}\right)\right)\left(3^{\beta+1}+(-1)^\alpha\left(\frac{-3}{N}\right)\right)\prod_{p^\gamma\|N}\frac{p^{\gamma+1}-\left(\frac{6}{p}\right)^{\gamma+1}}{p-\left(\frac{6}{p}\right)},\end{aligned}$$

$$\begin{aligned}&N(1,2,2,6;n)\\&\quad=\frac{1}{3}\left(2^{\alpha+1}+(-1)^\alpha\left(\frac{-2}{N}\right)\right)\left(3^{\beta+1}-(-1)^\alpha\left(\frac{-3}{N}\right)\right)\prod_{p^\gamma\|N}\frac{p^{\gamma+1}-\left(\frac{6}{p}\right)^{\gamma+1}}{p-\left(\frac{6}{p}\right)},\end{aligned}$$

$$N(1,3,3,6;n)$$
$$= \frac{1}{3}\left(2^{\alpha+2} - (-1)^{\alpha}\left(\frac{-2}{N}\right)\right)\left(3^{\beta} + (-1)^{\alpha}\left(\frac{-3}{N}\right)\right)\prod_{p^{\gamma}\| N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)},$$

$$N(2,3,6,6;n)$$
$$= \frac{1}{3}\left(2^{\alpha+1} + (-1)^{\alpha}\left(\frac{-2}{N}\right)\right)\left(3^{\beta} - (-1)^{\alpha}\left(\frac{-3}{N}\right)\right)\prod_{p^{\gamma}\| N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)}.$$

*Further, for $n \in \mathbb{N}$ define*

$$A(n) := \sum_{d|n} \left(\frac{-3}{n/d}\right)\left(\frac{-8}{n/d}\right) d, \tag{4.7}$$

$$B(n) := \sum_{d|n} \left(\frac{-3}{n/d}\right)\left(\frac{-8}{d}\right) d, \tag{4.8}$$

$$C(n) := \sum_{d|n} \left(\frac{-3}{d}\right)\left(\frac{-8}{n/d}\right) d, \tag{4.9}$$

$$D(n) := \sum_{d|n} \left(\frac{-3}{d}\right)\left(\frac{-8}{d}\right) d. \tag{4.10}$$

*Then*

$$N(1,1,2,3;n) = 4A(n) - B(n) + \frac{4}{3}C(n) - \frac{1}{3}D(n),$$
$$N(1,2,2,6;n) = 2A(n) + B(n) - \frac{2}{3}C(n) - \frac{1}{3}D(n),$$
$$N(1,3,3,6;n) = \frac{4}{3}A(n) - \frac{1}{3}B(n) + \frac{4}{3}C(n) - \frac{1}{3}D(n),$$
$$N(2,3,6,6;n) = \frac{2}{3}A(n) + \frac{1}{3}B(n) - \frac{2}{3}C(n) - \frac{1}{3}D(n).$$

**Proof.** Let $n \in \mathbb{N}$. For each prime $p$ we define $\alpha_p \in \mathbb{N}_0$ and $n_p \in \mathbb{N}$ uniquely by

$$n = p^{\alpha_p} n_p, \quad p \nmid n_p. \tag{4.11}$$

Let $k \in \mathbb{N}$. Appealing to (2.2), (4.6) and (4.11), we have

$$N_{p^k}(1,3,3,6;n) = \frac{1}{p^k} \sum_{x_1,x_2,x_3,x_4=0}^{p^k-1} \sum_{m=0}^{p^k-1} e^{\frac{2\pi i m}{p^k}(x_1^2+3x_2^2+3x_3^2+6x_4^2-p^{\alpha_p}n_p)}.$$

Interchanging the order of summation, and isolating the terms with $m=0$, we obtain

$$N_{p^k}(1,3,3,6;n) = p^{3k} + \frac{1}{p^k}\sum_{m=1}^{p^k-1} e^{\frac{-2\pi i m n_p}{p^{k-\alpha_p}}}\left(\sum_{x_1=0}^{p^k-1} e^{\frac{2\pi i m x_1^2}{p^k}}\right)$$
$$\times\left(\sum_{x_2=0}^{p^k-1} e^{\frac{6\pi i m x_2^2}{p^k}}\right)\left(\sum_{x_3=0}^{p^k-1} e^{\frac{6\pi i m x_3^2}{p^k}}\right)\left(\sum_{x_4=0}^{p^k-1} e^{\frac{12\pi i m x_4^2}{p^k}}\right).$$

Collecting together those terms having the same power of $p$ in $m$, we obtain

$$N_{p^k}(1,3,3,6;n) = p^{3k} + \frac{1}{p^k}\sum_{u=0}^{k-1}\sum_{\substack{m=1\\ p^u\|m}}^{p^k-1} e^{\frac{-2\pi i m n_p}{p^{k-\alpha_p}}}\left(\sum_{x=0}^{p^k-1} e^{\frac{2\pi i m x^2}{p^k}}\right)$$
$$\times\left(\sum_{y=0}^{p^k-1} e^{\frac{6\pi i m y^2}{p^k}}\right)^2\left(\sum_{z=0}^{p^k-1} e^{\frac{12\pi i m z^2}{p^k}}\right),$$

that is, for $k\in\mathbb{N}$ and any prime $p$, we have

$$N_{p^k}(1,3,3,6;n) = p^{3k} + \frac{1}{p^k}\sum_{u=0}^{k-1}\sum_{\substack{v=1\\ p\nmid v}}^{p^{k-u}-1} e^{\frac{-2\pi i n_p v}{p^{k-\alpha_p-u}}}\left(\sum_{x=0}^{p^k-1} e^{\frac{2\pi i v x^2}{p^{k-u}}}\right)$$
$$\times\left(\sum_{y=0}^{p^k-1} e^{\frac{6\pi i v y^2}{p^{k-u}}}\right)^2\left(\sum_{z=0}^{p^k-1} e^{\frac{12\pi i v z^2}{p^{k-u}}}\right). \tag{4.12}$$

First we treat the case $p>3$. By Lemma 3.2 we have

$$\sum_{x=0}^{p^k-1} e^{\frac{2\pi i v x^2}{p^{k-u}}} = p^u\left(\frac{v}{p}\right)^{k-u} i^{\left(\frac{p^{k-u}-1}{2}\right)^2} p^{\frac{k-u}{2}},$$

$$\sum_{y=0}^{p^k-1} e^{\frac{6\pi i v y^2}{p^{k-u}}} = p^u\left(\frac{3v}{p}\right)^{k-u} i^{\left(\frac{p^{k-u}-1}{2}\right)^2} p^{\frac{k-u}{2}},$$

$$\sum_{z=0}^{p^k-1} e^{\frac{12\pi i v z^2}{p^{k-u}}} = p^u\left(\frac{6v}{p}\right)^{k-u} i^{\left(\frac{p^{k-u}-1}{2}\right)^2} p^{\frac{k-u}{2}},$$

so that

$$\left(\sum_{x=0}^{p^k-1} e^{\frac{2\pi i v x^2}{p^{k-u}}}\right)\left(\sum_{y=0}^{p^k-1} e^{\frac{6\pi i v y^2}{p^{k-u}}}\right)^2\left(\sum_{z=0}^{p^k-1} e^{\frac{12\pi i v z^2}{p^{k-u}}}\right) = p^{2k+2u}\left(\frac{6}{p}\right)^{k+u}. \tag{4.13}$$

Hence using (4.13) in (4.12), we obtain

$$N_{p^k}(1,3,3,6;n) = p^{3k} + p^k\left(\frac{6}{p}\right)^k \sum_{u=0}^{k-1} p^{2u}\left(\frac{6}{p}\right)^u \sum_{\substack{v=1 \\ p\nmid v}}^{p^{k-u}-1} e^{\frac{-2\pi i n_p v}{p^{k-\alpha_p-u}}}. \tag{4.14}$$

For $u \in \mathbb{N}_0$ with $0 \leq u \leq k-1$ define

$$S := \sum_{\substack{v=1 \\ p\nmid v}}^{p^{k-u}-1} e^{\frac{-2\pi i n_p v}{p^{k-\alpha_p-u}}}.$$

If $u \geq k-\alpha_p$ then $k-\alpha_p-u \leq 0$ so

$$S = \sum_{\substack{v=1 \\ p\nmid v}}^{p^{k-u}-1} 1 = \phi(p^{k-u}) = p^{k-u} - p^{k-u-1}. \tag{4.15}$$

If $u < k-\alpha_p$ then $k-\alpha_p-u \geq 1$ so by Lemma 3.4 we have

$$S = \sum_{\substack{v=0 \\ p\nmid v}}^{p^{k-u}-1} e^{\frac{-2\pi i n_p v}{p^{k-\alpha_p-u}}} = \begin{cases} 0 & \text{if } u \leq k-\alpha_p-2, \\ -p^{\alpha_p} & \text{if } u = k-\alpha_p-1. \end{cases} \tag{4.16}$$

Hence by using (4.15) and (4.16) in (4.14), we obtain

$$\begin{aligned} N_{p^k}(1,3,3,6;n) = p^{3k} &+ p^k\left(\frac{6}{p}\right)^k \sum_{\substack{u=0 \\ u\geq k-\alpha_p}}^{k-1} p^{2u}\left(\frac{6}{p}\right)^u (p^{k-u} - p^{k-u-1}) \\ &+ p^k\left(\frac{6}{p}\right)^k \sum_{\substack{u=0 \\ u=k-\alpha_p-1}}^{k-1} p^{2u}\left(\frac{6}{p}\right)^u (-p^{\alpha_p}). \end{aligned} \tag{4.17}$$

Three cases arise according as $\alpha_p = 0$, $1 \leq \alpha_p \leq k-1$ or $k \leq \alpha_p$. If $\alpha_p = 0$ then from (4.17) we have

$$N_{p^k}(1,3,3,6;n) = p^{3k} + 0 - p^k\left(\frac{6}{p}\right)^k p^{2(k-1)}\left(\frac{6}{p}\right)^{k-1}.$$

Hence

$$N_{p^k}(1,3,3,6;n) = p^{3k} - \left(\frac{6}{p}\right)p^{3k-2} \quad \text{if } \alpha_p = 0. \tag{4.18}$$

If $1 \leq \alpha_p \leq k-1$ then from (4.17) we have

$$\begin{aligned} N_{p^k}(1,3,3,6;n) = p^{3k} &+ p^k\left(\frac{6}{p}\right)^k \sum_{u=k-\alpha_p}^{k-1} p^{2u}\left(\frac{6}{p}\right)^u (p^{k-u} - p^{k-u-1}) \\ &+ p^k\left(\frac{6}{p}\right)^k p^{2(k-\alpha_p-1)}\left(\frac{6}{p}\right)^{k-\alpha_p-1} (-p^{\alpha_p}), \end{aligned}$$

that is

$$N_{p^k}(1,3,3,6;n) = p^{3k} + p^{2k}\left(\frac{6}{p}\right)^k \sum_{u=k-\alpha_p}^{k-1} \left(\frac{6}{p}\right)^u (p^u - p^{u-1}) - p^{3k-\alpha_p-2}\left(\frac{6}{p}\right)^{\alpha_p+1}. \tag{4.19}$$

Now

$$\sum_{u=k-a}^{k-1} r^u(s^u - s^{u-1}) = \frac{1-\dfrac{1}{s}}{1-rs}(rs)^{k-a}(1-(rs)^a) \tag{4.20}$$

for integers $a$ and $k$ with $1 \le a \le k$ and real numbers $r$ and $s$ with $s \neq 1$ and $rs \neq 1$. Using (4.20) to evaluate the sum in (4.19), we obtain

$$N_{p^k}(1,3,3,6;n) = p^{3k} + p^{2k}\left(\frac{6}{p}\right)^k \frac{\left(1-\dfrac{1}{p}\right)}{\left(1-\left(\dfrac{6}{p}\right)p\right)} \left(\left(\frac{6}{p}\right)p\right)^{k-\alpha_p} \left(1-\left(\frac{6}{p}\right)^{\alpha_p} p^{\alpha_p}\right) - p^{3k-\alpha_p-2}\left(\frac{6}{p}\right)^{\alpha_p+1}.$$

Thus

$$N_{p^k}(1,3,3,6;n) = p^{3k} + p^{3k-\alpha_p}\left(\frac{6}{p}\right)^{\alpha_p} \frac{\left(1-\dfrac{1}{p}\right)\left(1-\left(\dfrac{6}{p}\right)^{\alpha_p} p^{\alpha_p}\right)}{\left(1-\left(\dfrac{6}{p}\right)p\right)} - p^{3k-\alpha_p-2}\left(\frac{6}{p}\right)^{\alpha_p+1} \quad \text{if } 1 \le \alpha_p \le k-1. \tag{4.21}$$

If $k \le \alpha_p$ then from (4.17) we have

$$N_{p^k}(1,3,3,6;n) = p^{3k} + p^k\left(\frac{6}{p}\right)^k \sum_{u=0}^{k-1} p^{2u}\left(\frac{6}{p}\right)^u (p^{k-u} - p^{k-u-1}) + 0, \tag{4.22}$$

that is

$$N_{p^k}(1,3,3,6;n) = p^{3k} + p^{2k}\left(\frac{6}{p}\right)^k \sum_{u=0}^{k-1} \left(\frac{6}{p}\right)^u (p^u - p^{u-1}). \tag{4.23}$$

Using (4.20) to evaluate the sum in (4.23), we deduce

$$N_{p^k}(1,3,3,6;n) = p^{3k} + p^{2k}\left(\frac{6}{p}\right)^k \frac{\left(1-\dfrac{1}{p}\right)}{\left(1-\left(\dfrac{6}{p}\right)p\right)} \left(1-\left(\frac{6}{p}\right)^k p^k\right) \quad \text{if } \alpha_p \ge k. \tag{4.24}$$

In summary, putting together (4.18), (4.21) and (4.24), we have for $p>3$

$$N_{p^k}(1,3,3,6;n)$$
$$= \begin{cases} p^{3k} - \left(\frac{6}{p}\right) p^{3k-2} & \text{if } \alpha_p = 0, \\ p^{3k} + p^{3k-\alpha_p}\left(\frac{6}{p}\right)^{\alpha_p} \frac{\left(1-\frac{1}{p}\right)\left(1-\left(\frac{6}{p}\right)^{\alpha_p} p^{\alpha_p}\right)}{\left(1-\left(\frac{6}{p}\right)p\right)} & \\ \quad - p^{3k-\alpha_p-2}\left(\frac{6}{p}\right)^{\alpha_p+1} & \text{if } 1 \le \alpha_p \le k-1, \\ p^{3k} + p^{2k}\left(\frac{6}{p}\right)^{k} \frac{\left(1-\frac{1}{p}\right)\left(1-\left(\frac{6}{p}\right)^{k} p^{k}\right)}{\left(1-\left(\frac{6}{p}\right)p\right)} & \text{if } \alpha_p \ge k. \end{cases} \tag{4.25}$$

Thus, if $p>3$ and $p \nmid n$ (so that $\alpha_p = 0$), we have by (4.25)

$$\begin{aligned} d_p(1,3,3,6;n) &= \lim_{k\to\infty} \frac{N_{p^k}(1,3,3,6;n)}{p^{3k}} = \lim_{k\to\infty} \frac{p^{3k} - \left(\frac{6}{p}\right)p^{3k-2}}{p^{3k}} \\ &= 1 - \left(\frac{6}{p}\right)\frac{1}{p^2}, \quad p>3, \ p\nmid n. \end{aligned} \tag{4.26}$$

In particular we have

$$d_p(1,3,3,6;1) = 1 - \left(\frac{6}{p}\right)\frac{1}{p^2}, \quad p>3. \tag{4.27}$$

Hence

$$\prod_{\substack{p>3\\ p\nmid n}} \frac{d_p(1,3,3,6;n)}{d_p(1,3,3,6;1)} = \prod_{\substack{p>3\\ p\nmid n}} \frac{1-\left(\frac{6}{p}\right)\frac{1}{p^2}}{1-\left(\frac{6}{p}\right)\frac{1}{p^2}} = 1. \tag{4.28}$$

If $p>3$ and $p \mid n$ (so that $\alpha_p \ge 1$), we have by (4.25)

$$\begin{aligned} d_p(1,3,3,6;n) &= \lim_{k\to\infty} \frac{N_{p^k}(1,3,3,6;n)}{p^{3k}} \\ &= 1 + p^{-\alpha_p}\left(\frac{6}{p}\right)^{\alpha_p} \frac{\left(1-\frac{1}{p}\right)\left(1-\left(\frac{6}{p}\right)^{\alpha_p} p^{\alpha_p}\right)}{\left(1-\left(\frac{6}{p}\right)p\right)} - p^{-\alpha_p-2}\left(\frac{6}{p}\right)^{\alpha_p+1}, \end{aligned}$$

that is

$$d_p(1,3,3,6;n) = \frac{\left(1-\left(\frac{6}{p}\right)p^2\right)\left(p^{\alpha_p+1}-\left(\frac{6}{p}\right)^{\alpha_p+1}\right)}{\left(1-\left(\frac{6}{p}\right)p\right)p^{\alpha_p+2}} \quad \text{if } p>3,\ \ p\,|\,n. \tag{4.29}$$

Hence, by (4.27) and (4.29), we deduce

$$\begin{aligned}\prod_{\substack{p>3\\ p|n}} \frac{d_p(1,3,3,6;n)}{d_p(1,3,3,6;1)} &= \prod_{\substack{p>3\\ p|n}} \frac{\left(1-\left(\frac{6}{p}\right)p^2\right)\left(p^{\alpha_p+1}-\left(\frac{6}{p}\right)^{\alpha_p+1}\right)}{\left(1-\left(\frac{6}{p}\right)p\right)\left(1-\left(\frac{6}{p}\right)\frac{1}{p^2}\right)p^{\alpha_p+2}} \\ &= \prod_{\substack{p>3\\ p|n}} \frac{p^{\alpha_p+1}-\left(\frac{6}{p}\right)^{\alpha_p+1}}{\left(p-\left(\frac{6}{p}\right)\right)p^{\alpha_p}}.\end{aligned}$$

Thus

$$\prod_{\substack{p>3\\ p|n}} \frac{d_p(1,3,3,6;n)}{d_p(1,3,3,6;1)} = \frac{1}{\prod\limits_{\substack{p>3\\ p|n}} p^{\alpha_p}} \prod_{\substack{p>3\\ p|n}} \frac{p^{\alpha_p+1}-\left(\frac{6}{p}\right)^{\alpha_p+1}}{p-\left(\frac{6}{p}\right)}. \tag{4.30}$$

Secondly we treat the case $p=3$. By Lemma 3.2 we have

$$\sum_{x=0}^{3^k-1} e^{\frac{2\pi i v x^2}{3^{k-u}}} = 3^u \left(\frac{v}{3}\right)^{k-u} i^{\left(\frac{3^{k-u}-1}{2}\right)^2} 3^{\frac{k-u}{2}},$$

$$\sum_{y=0}^{3^k-1} e^{\frac{2\pi i v y^2}{3^{k-u-1}}} = 3^{u+1} \left(\frac{v}{3}\right)^{k-u-1} i^{\left(\frac{3^{k-u-1}-1}{2}\right)^2} 3^{\frac{k-u-1}{2}},$$

$$\sum_{z=0}^{3^k-1} e^{\frac{4\pi i v z^2}{3^{k-u-1}}} = 3^{u+1} \left(\frac{2v}{3}\right)^{k-u-1} i^{\left(\frac{3^{k-u-1}-1}{2}\right)^2} 3^{\frac{k-u-1}{2}},$$

so that

$$\begin{aligned}&\left(\sum_{x=0}^{3^k-1} e^{\frac{2\pi i v x^2}{3^{k-u}}}\right)\left(\sum_{y=0}^{3^k-1} e^{\frac{2\pi i v y^2}{3^{k-u-1}}}\right)^2\left(\sum_{z=0}^{3^k-1} e^{\frac{4\pi i v z^2}{3^{k-u-1}}}\right) \\ &\quad = 3^{4u+3}(-1)^{k-u-1}\left(\frac{v}{3}\right)(-1)^{k-u-1} i\, 3^{2k-2u-2}\sqrt{3} \\ &\quad = \left(\frac{v}{3}\right)3^{2k+2u+1}\sqrt{-3}.\end{aligned}$$

Hence (4.12) in this case becomes

$$N_{3^k}(1,3,3,6;n) = 3^{3k} + \frac{1}{3^k}\sum_{u=0}^{k-1}\sum_{\substack{v=1\\3\nmid v}}^{3^{k-u}-1} e^{\frac{-2\pi i n_3 v}{3^{k-\alpha_3-u}}}\left(\frac{v}{3}\right)3^{2k+2u+1}\sqrt{-3},$$

that is

$$N_{3^k}(1,3,3,6;n) = 3^{3k} + 3^{k+1}\sqrt{-3}\sum_{u=0}^{k-1}3^{2u}\sum_{\substack{v=1\\3\nmid v}}^{3^{k-u}-1}\left(\frac{v}{3}\right)e^{\frac{-2\pi i n_3 v}{3^{k-\alpha_3-u}}}. \qquad (4.31)$$

For $u \in \mathbb{N}_0$ with $0 \le u \le k-1$ define

$$T := \sum_{\substack{v=1\\3\nmid v}}^{3^{k-u}-1}\left(\frac{v}{3}\right)e^{\frac{-2\pi i n_3 v}{3^{k-\alpha_3-u}}}.$$

If $u \ge k-\alpha_3$ then $k-\alpha_3-u \le 0$ so

$$T = \sum_{\substack{v=1\\3\nmid v}}^{3^{k-u}-1}\left(\frac{v}{3}\right) = \sum_{\substack{v=1\\v\equiv 1 \pmod 3}}^{3^{k-u}-1} 1 - \sum_{\substack{v=1\\v\equiv 2 \pmod 3}}^{3^{k-u}-1} 1 = 3^{k-u-1} - 3^{k-u-1} = 0.$$

Next suppose that $u = k-\alpha_3-1$. Then $3^{k-\alpha_3-u} = 3$, $3^{k-u} = 3^{\alpha_3+1}$ and

$$\begin{aligned}
T &= \sum_{\substack{v=1\\3\nmid v}}^{3^{\alpha_3+1}-1}\left(\frac{v}{3}\right)e^{\frac{-2\pi i n_3 v}{3}}\\
&= \sum_{\substack{v=1\\v\equiv 1 \pmod 3}}^{3^{\alpha_3+1}-1} e^{\frac{-2\pi i n_3 v}{3}} - \sum_{\substack{v=1\\v\equiv 2 \pmod 3}}^{3^{\alpha_3+1}-1} e^{\frac{-2\pi i n_3 v}{3}}\\
&= \sum_{x=0}^{3^{\alpha_3}-1} e^{\frac{-2\pi i n_3(3x+1)}{3}} - \sum_{x=0}^{3^{\alpha_3}-1} e^{\frac{-2\pi i n_3(3x+2)}{3}}\\
&= e^{\frac{-2\pi i n_3}{3}}\sum_{x=0}^{3^{\alpha_3}-1} e^{-2\pi i n_3 x} - e^{\frac{-4\pi i n_3}{3}}\sum_{x=0}^{3^{\alpha_3}-1} e^{-2\pi i n_3 x}\\
&= e^{\frac{-2\pi i n_3}{3}}3^{\alpha_3} - e^{\frac{2\pi i n_3}{3}}3^{\alpha_3}\\
&= -3^{\alpha_3}2i\sin\left(\frac{2\pi n_3}{3}\right)\\
&= -\left(\frac{n_3}{3}\right)3^{\alpha_3}\sqrt{-3}.
\end{aligned}$$

Finally suppose that $u \le k-\alpha_3-2$. Then $k-\alpha_3-u-1 \ge 1$ so $e^{\frac{-2\pi i n_3}{3^{k-\alpha_3-u-1}}} \neq 1$. Hence

$$\begin{aligned}
T &= \sum_{\substack{v=1\\ 3\nmid v}}^{3^{k-u}-1} \left(\frac{v}{3}\right) e^{\frac{-2\pi i n_3 v}{3^{k-\alpha_3-u}}} \\
&= \sum_{\substack{v=1\\ v\equiv 1 \pmod 3}}^{3^{k-u}-1} e^{\frac{-2\pi i n_3 v}{3^{k-\alpha_3-u}}} - \sum_{\substack{v=1\\ v\equiv 2 \pmod 3}}^{3^{k-u}-1} e^{\frac{-2\pi i n_3 v}{3^{k-\alpha_3-u}}} \\
&= \sum_{x=0}^{3^{k-u-1}-1} e^{\frac{-2\pi i n_3 (3x+1)}{3^{k-\alpha_3-u}}} - \sum_{x=0}^{3^{k-u-1}-1} e^{\frac{-2\pi i n_3 (3x+2)}{3^{k-\alpha_3-u}}} \\
&= e^{\frac{-2\pi i n_3}{3^{k-\alpha_3-u}}} \sum_{x=0}^{3^{k-u-1}-1} e^{\frac{-2\pi i n_3 x}{3^{k-\alpha_3-u-1}}} - e^{\frac{-4\pi i n_3}{3^{k-\alpha_3-u}}} \sum_{x=0}^{3^{k-u-1}-1} e^{\frac{-2\pi i n_3 x}{3^{k-\alpha_3-u-1}}} \\
&= \left(e^{\frac{-2\pi i n_3}{3^{k-\alpha_3-u}}} - e^{\frac{-4\pi i n_3}{3^{k-\alpha_3-u}}}\right) \sum_{x=0}^{3^{k-u-1}-1} e^{\frac{-2\pi i n_3 x}{3^{k-\alpha_3-u-1}}} \\
&= 0,
\end{aligned}$$

as

$$\sum_{x=0}^{3^{k-u-1}-1} e^{\frac{-2\pi i n_3 x}{3^{k-\alpha_3-u-1}}} = \frac{1-e^{-2\pi i n_3 3^{\alpha_3}}}{1-e^{\frac{-2\pi i n_3}{3^{k-\alpha_3-u-1}}}} = 0.$$

Summarizing we have

$$T = \begin{cases} 0 & \text{if } u \ge k-\alpha_3, \\ -\left(\frac{n_3}{3}\right) 3^{\alpha_3}\sqrt{-3} & \text{if } u = k-\alpha_3-1, \\ 0 & \text{if } u \le k-\alpha_3-2. \end{cases}$$

Using this evaluation in (4.31), we have

$$N_{3^k}(1,3,3,6;n) = 3^{3k} + 3^{k+1}\sqrt{-3} \sum_{\substack{u=0\\ u=k-\alpha_3-1}}^{k-1} 3^{2u}\left(-\left(\frac{n_3}{3}\right)3^{\alpha_3}\sqrt{-3}\right).$$

Hence if $\alpha_3 \le k-1$ then $0 \le k-\alpha_3-1 \le k-1$ so

$$\begin{aligned}
N_{3^k}(1,3,3,6;n) &= 3^{3k} + 3^{k+1}\sqrt{-3}\, 3^{2(k-\alpha_3-1)}\left(-\left(\frac{n_3}{3}\right) 3^{\alpha_3}\sqrt{-3}\right) \\
&= 3^{3k} + \left(\frac{n_3}{3}\right) 3^{3k-\alpha_3},
\end{aligned}$$

whereas if $\alpha_3 \geq k$ then $k - \alpha_3 - 1 < 0$ so

$$N_{3^k}(1,3,3,6;n) = 3^{3k}.$$

Summarizing we have

$$N_{3^k}(1,3,3,6;n) = \begin{cases} 3^{3k} + \left(\dfrac{-3}{n_3}\right) 3^{3k-\alpha_3} & \text{if } \alpha_3 \leq k-1, \\ 3^{3k} & \text{if } \alpha_3 \geq k. \end{cases} \tag{4.32}$$

Hence for all $n \in \mathbb{N}$, we have from (4.32)

$$\begin{aligned} d_3(1,3,3,6;n) &= \lim_{k\to\infty} \frac{N_{3^k}(1,3,3,6;n)}{3^{3k}} = \lim_{k\to\infty} \frac{3^{3k} + \left(\dfrac{-3}{n_3}\right) 3^{3k-\alpha_3}}{3^{3k}} \\ &= 1 + \left(\frac{-3}{n_3}\right)\frac{1}{3^{\alpha_3}}. \end{aligned}$$

In particular we have

$$d_3(1,3,3,6;1) = 2.$$

Thus we have

$$\frac{d_3(1,3,3,6;n)}{d_3(1,3,3,6;1)} = \frac{1}{2\cdot 3^{\alpha_3}}\left(3^{\alpha_3} + \left(\frac{-3}{n_3}\right)\right), \quad n \in \mathbb{N}. \tag{4.33}$$

Finally we treat the case $p = 2$. By Lemma 3.3 we have for $0 \leq u \leq k-1$ and $v$ odd

$$\sum_{x=0}^{2^k-1} e^{\frac{2\pi i v x^2}{2^{k-u}}} = \begin{cases} 0 & \text{if } u = k-1, \\ 2^{\frac{k+u}{2}} \left(\dfrac{2}{v}\right)^{k-u} (1+i^v) & \text{if } 0 \leq u \leq k-2, \end{cases}$$

$$\sum_{y=0}^{2^k-1} e^{\frac{6\pi i v y^2}{2^{k-u}}} = \sum_{y=0}^{2^k-1} e^{\frac{2\pi i (3v) y^2}{2^{k-u}}} = \begin{cases} 0 & \text{if } u = k-1, \\ 2^{\frac{k+u}{2}} \left(\dfrac{2}{3v}\right)^{k-u} (1+i^{3v}) & \text{if } 0 \leq u \leq k-2, \end{cases}$$

$$\sum_{z=0}^{2^k-1} e^{\frac{12\pi i v z^2}{2^{k-u}}} = \sum_{z=0}^{2^k-1} e^{\frac{2\pi i (3v) z^2}{2^{k-u-1}}} = \begin{cases} 2^k & \text{if } u = k-1, \\ 0 & \text{if } u = k-2, \\ 2^{\frac{k+u+1}{2}} \left(\dfrac{2}{3v}\right)^{k-u-1} (1+i^{3v}) & \text{if } 0 \leq u \leq k-3. \end{cases}$$

Hence

$$\left(\sum_{x=0}^{2^k-1} e^{\frac{2\pi i v x^2}{2^{k-u}}}\right)\left(\sum_{y=0}^{2^k-1} e^{\frac{6\pi i v y^2}{2^{k-u}}}\right)^2\left(\sum_{z=0}^{2^k-1} e^{\frac{12\pi i v z^2}{2^{k-u}}}\right)$$
$$= \begin{cases} 0 & \text{if } u = k-1, k-2, \\ 2^{2k+2u+\frac{1}{2}}\left(\frac{2}{v}\right)(-1)^{k-u-1}(1+i^v)(1+i^{3v})^3 & \text{if } 0 \le u \le k-3. \end{cases}$$

Thus, by (4.12), we have

$$N_{2^k}(1,3,3,6;n)$$
$$= 2^{3k} + \frac{1}{2^k}\sum_{u=0}^{k-3}\sum_{\substack{v=1 \\ v \text{ odd}}}^{2^{k-u}-1} e^{\frac{-2\pi i n_2 v}{2^{k-\alpha_2-u}}} 2^{2k+2u+\frac{1}{2}}\left(\frac{2}{v}\right)(-1)^{k-u-1}(1+i^v)(1+i^{3v})^3.$$

As $(1+i^v)(1+i^{3v})^3 = -(\frac{-1}{v})4i$ for $v$ odd, we deduce

$$N_{2^k}(1,3,3,6;n) = 2^{3k} + 2^{k+2}(-1)^k\sqrt{-2}\sum_{u=0}^{k-3}(-1)^u 2^{2u}\sum_{\substack{v=1 \\ v \text{ odd}}}^{2^{k-u}-1}\left(\frac{-2}{v}\right)e^{\frac{-2\pi i n_2 v}{2^{k-\alpha_2-u}}}. \tag{4.34}$$

If $k = 1$ we have

$$N_2(1,3,3,6;n) = 8.$$

If $k = 2$ we have

$$N_4(1,3,3,6;n) = 64.$$

Thus we may suppose $k \ge 3$. For $0 \le u \le k-3$ set

$$T := \sum_{\substack{v=1 \\ v \text{ odd}}}^{2^{k-u}-1}\left(\frac{-2}{v}\right)e^{\frac{-2\pi i n_2 v}{2^{k-\alpha_2-u}}}.$$

If $u \ge k - \alpha_2$ then $k - \alpha_2 - u \le 0$ so

$$T = \sum_{\substack{v=1 \\ v \text{ odd}}}^{2^{k-u}-1}\left(\frac{-2}{v}\right) \overset{v=8x+y}{=} \sum_{y=1,3,5,7}\left(\frac{-2}{y}\right)\sum_{x=0}^{2^{k-u-3}-1} 1$$
$$= 2^{k-u-3}\left(\left(\frac{-2}{1}\right)+\left(\frac{-2}{3}\right)+\left(\frac{-2}{5}\right)+\left(\frac{-2}{7}\right)\right)$$
$$= 2^{k-u-3}(1+1-1-1) = 0$$

so that

$$T = 0 \quad \text{if } u \geq k - \alpha_2.$$

If $u \leq k - \alpha_2 - 1$ then $k - \alpha_2 - u \geq 1$ so

$$\begin{aligned} T &= \sum_{\substack{v=1 \\ v \text{ odd}}}^{2^{k-u}-1} \left(\frac{-2}{v}\right) e^{\frac{-2\pi i n_2 v}{2^{k-\alpha_2-u}}} \\ &\overset{v=8x+y}{=} \sum_{y=1,3,5,7} \left(\frac{-2}{y}\right) e^{\frac{-2\pi i n_2 y}{2^{k-\alpha_2-u}}} \sum_{x=0}^{2^{k-u-3}-1} e^{\frac{-2\pi i n_2 x}{2^{k-\alpha_2-u-3}}}. \end{aligned}$$

We now consider cases according as $u \geq k-\alpha_2-3$ or $u \leq k-\alpha_2-4$. If $u \geq k-\alpha_2-3$ then $k - \alpha_2 - u - 3 \leq 0$ so

$$\begin{aligned} T &= \sum_{y=1,3,5,7} \left(\frac{-2}{y}\right) e^{\frac{-2\pi i n_2 y}{2^{k-\alpha_2-u}}} \sum_{x=0}^{2^{k-u-3}-1} 1 \\ &= 2^{k-u-3}\left(e^{\frac{-2\pi i n_2}{2^{k-\alpha_2-u}}} + e^{\frac{-6\pi i n_2}{2^{k-\alpha_2-u}}} - e^{\frac{-10\pi i n_2}{2^{k-\alpha_2-u}}} - e^{\frac{-14\pi i n_2}{2^{k-\alpha_2-u}}}\right). \end{aligned}$$

As $k - \alpha_2 - 3 \leq u \leq k - \alpha_2 - 1$ we have

$$u = k - \alpha_2 - 3, \quad k - \alpha_2 - 2 \quad \text{or} \quad k - \alpha_2 - 1.$$

If $u = k - \alpha_2 - 3$ we have

$$T = 2^{\alpha_2}\left(e^{\frac{-2\pi i n_2}{8}} + e^{\frac{-6\pi i n_2}{8}} - e^{\frac{-10\pi i n_2}{8}} - e^{\frac{-14\pi i n_2}{8}}\right).$$

As

$$\cos\frac{k\pi}{4} = \left(\frac{2}{k}\right)\frac{1}{\sqrt{2}}, \quad \sin\frac{k\pi}{4} = \left(\frac{-2}{k}\right)\frac{1}{\sqrt{2}},$$

for $k$ odd, we have after a short calculation

$$e^{\frac{-2\pi i n_2}{8}} + e^{\frac{-6\pi i n_2}{8}} - e^{\frac{-10\pi i n_2}{8}} - e^{\frac{-14\pi i n_2}{8}} = -2\left(\frac{-2}{n_2}\right)\sqrt{-2}$$

so that

$$T = -2^{\alpha_2+1}\left(\frac{-2}{n_2}\right)\sqrt{-2} \quad \text{if } u = k - \alpha_2 - 3.$$

If $u = k - \alpha_2 - 2$ we have

$$\begin{aligned} T &= 2^{\alpha_2-1}\left(e^{\frac{-2\pi i n_2}{4}} + e^{\frac{-6\pi i n_2}{4}} - e^{\frac{-10\pi i n_2}{4}} - e^{\frac{-14\pi i n_2}{4}}\right) \\ &= 2^{\alpha_2-1}\left((-i)^{n_2} + (-i)^{3n_2} - (-i)^{5n_2} - (-i)^{7n_2}\right) \\ &= -2^{\alpha_2-1}\left(i^{n_2} + i^{3n_2} - i^{5n_2} - i^{7n_2}\right) \end{aligned}$$

so that

$$T = 0 \quad \text{if } u = k - \alpha_2 - 2.$$

If $u = k - \alpha_2 - 1$ we have

$$\begin{aligned} T &= 2^{\alpha_2-2}(e^{\frac{-2\pi i n_2}{2}} + e^{\frac{-6\pi i n_2}{2}} - e^{\frac{-10\pi i n_2}{2}} - e^{\frac{-14\pi i n_2}{2}}) \\ &= 2^{\alpha_2-2}(-1-1+1+1) \end{aligned}$$

so that

$$T = 0 \quad \text{if } u = k - \alpha_2 - 1.$$

Finally we treat $u \le k - \alpha_2 - 4$ in which case $k - \alpha_2 - u - 3 \ge 1$ and

$$T = \sum_{y=1,3,5,7} \left(\frac{-2}{y}\right) e^{\frac{-2\pi i n_2 y}{2^{k-\alpha_2-u}}} \sum_{x=0}^{2^{k-u-3}-1} \left(e^{\frac{-2\pi i n_2}{2^{k-\alpha_2-u-3}}}\right)^x .$$

As $e^{\frac{-2\pi i n_2}{2^{k-\alpha_2-u-3}}} \neq 1$ we have

$$\sum_{x=0}^{2^{k-u-3}-1} \left(e^{\frac{-2\pi i n_2}{2^{k-\alpha_2-u-3}}}\right)^x = \frac{1 - e^{-2^{\alpha_2+1}\pi i n_2}}{1 - e^{\frac{-2\pi i n_2}{2^{k-\alpha_2-u-3}}}} = 0$$

so

$$T = 0 \quad \text{if } u \le k - \alpha_2 - 4.$$

Summarizing we have: For $k \ge 3$ and $0 \le u \le k-3$ we have

$$T = 0 \quad \text{in all cases except when } \alpha_2 \le k-3 \quad \text{and} \quad u = k - \alpha_2 - 3$$

in which case

$$T = -2^{\alpha_2+1}\left(\frac{-2}{n_2}\right)\sqrt{-2}.$$

Thus appealing to (4.34), we obtain for $k \ge 3$

$$N_{2^k}(1,3,3,6;n) = \begin{cases} 2^{3k} & \text{if } k \le \alpha_2 + 2, \\ 2^{3k} - (-1)^{\alpha_2}\left(\frac{-2}{n_2}\right)2^{3k-\alpha_2-2} & \text{if } k \ge \alpha_2 + 3. \end{cases} \tag{4.35}$$

It is clear that (4.35) is also true for $k = 1$ and $k = 2$. From (4.35) we deduce

$$d_2(1,3,3,6;n) = \lim_{k\to\infty} \frac{N_{2^k}(1,3,3,6;n)}{2^{3k}} = 1 - (-1)^{\alpha_2}\left(\frac{-2}{n_2}\right)\frac{1}{2^{\alpha_2+2}}, \quad n \in \mathbb{N}.$$

In particular for $n = 1$ (so that $\alpha_2 = 0$, $n_2 = 1$) we have

$$d_2(1,3,3,6;n) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Hence

$$\frac{d_2(1,3,3,6;n)}{d_2(1,3,3,6;1)} = \frac{1}{3\cdot 2^{\alpha_2}}\left(2^{\alpha_2+2} - (-1)^{\alpha_2}\left(\frac{-2}{n_2}\right)\right), \quad n \in \mathbb{N}. \tag{4.36}$$

Appealing to (4.5), (4.28), (4.30), (4.33) and (4.36), we obtain for $n \in \mathbb{N}$

$$N(1,3,3,6;n) = \frac{1}{3}\left(2^{\alpha_2+2} - (-1)^{\alpha_2}\left(\frac{-2}{n_2}\right)\right)\left(3^{\alpha_3} + \left(\frac{-3}{n_3}\right)\right) \times \prod_{\substack{p>3\\ p\mid n}} \frac{p^{\alpha_p+1} - \left(\frac{6}{p}\right)^{\alpha_p+1}}{p - \left(\frac{6}{p}\right)}. \tag{4.37}$$

Now we express $n$ in the form

$$n = 2^{\alpha}3^{\beta}N, \quad \text{where } \alpha,\beta \in \mathbb{N}_0,\ \ N \in \mathbb{N},\ \ \gcd(N,6) = 1.$$

Then we have

$$\alpha_2 = \alpha, \quad \alpha_3 = \beta, \quad n_2 = 3^{\beta}N, \quad n_3 = 2^{\alpha}N.$$

Hence

$$\left(\frac{-2}{n_2}\right) = \left(\frac{-2}{N}\right), \quad \left(\frac{-3}{n_3}\right) = (-1)^{\alpha}\left(\frac{-3}{N}\right).$$

Thus (4.37) becomes

$$N(1,3,3,6;n) = \frac{1}{3}\left(2^{\alpha+2} - (-1)^{\alpha}\left(\frac{-2}{N}\right)\right)\left(3^{\beta} + (-1)^{\alpha}\left(\frac{-3}{N}\right)\right) \times \prod_{p^{\gamma}\parallel N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)}, \tag{4.38}$$

as asserted.

Then, by Lemma 4.1 and (4.38), we have

$$\begin{aligned} N(1,1,2,3;n) &= N(1,3,3,6;3n) \\ &= \frac{1}{3}\left(2^{\alpha+2} - (-1)^{\alpha}\left(\frac{-2}{N}\right)\right)\left(3^{\beta+1} + (-1)^{\alpha}\left(\frac{-3}{N}\right)\right) \\ &\quad \times \prod_{p^{\gamma}\parallel N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)}, \end{aligned}$$

as claimed.

Also, by Lemma 4.1 and (4.38), we have

$$\begin{aligned}
&N(1,2,2,6;n)\\
&\quad= \frac{3}{2}N(1,3,3,6;n) - N(1,3,3,6;3n) + \frac{1}{2}N(1,3,3,6;6n)\\
&\quad= \frac{1}{3}\Bigg[\frac{3}{2}\left(2^{\alpha+2} - (-1)^{\alpha}\left(\frac{-2}{N}\right)\right)\left(3^{\beta} + (-1)^{\alpha}\left(\frac{-3}{N}\right)\right)\\
&\qquad - \left(2^{\alpha+2} - (-1)^{\alpha}\left(\frac{-2}{N}\right)\right)\left(3^{\beta+1} + (-1)^{\alpha}\left(\frac{-3}{N}\right)\right)\\
&\qquad + \frac{1}{2}\left(2^{\alpha+3} + (-1)^{\alpha}\left(\frac{-2}{N}\right)\right)\left(3^{\beta+1} - (-1)^{\alpha}\left(\frac{-3}{N}\right)\right)\Bigg]\\
&\qquad \times \prod_{p^{\gamma}\| N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)}\\
&\quad= \frac{1}{3}\left(2^{\alpha+1}3^{\beta+1} + (-1)^{\alpha}\left(\frac{-2}{N}\right)3^{\beta+1} - 2^{\alpha+1}(-1)^{\alpha}\left(\frac{-3}{N}\right) - \left(\frac{6}{N}\right)\right)\\
&\qquad \times \prod_{p^{\gamma}\| N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)}\\
&\quad= \frac{1}{3}\left(2^{\alpha+1} + (-1)^{\alpha}\left(\frac{-2}{N}\right)\right)\left(3^{\beta+1} - (-1)^{\alpha}\left(\frac{-3}{N}\right)\right)\prod_{p^{\gamma}\| N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)},
\end{aligned}$$

as asserted.

Further, by Lemma 4.1 and (4.38), we have

$$\begin{aligned}
&N(2,3,6,6;n)\\
&\quad= N(1,3,3,6;n) + \frac{1}{2}N(1,3,3,6;2n) - \frac{1}{2}N(1,3,3,6;3n)\\
&\quad= \frac{1}{3}\Bigg[\left(2^{\alpha+2} - (-1)^{\alpha}\left(\frac{-2}{N}\right)\right)\left(3^{\beta} + (-1)^{\alpha}\left(\frac{-3}{N}\right)\right)\\
&\qquad + \frac{1}{2}\left(2^{\alpha+3} + (-1)^{\alpha}\left(\frac{-2}{N}\right)\right)\left(3^{\beta} - (-1)^{\alpha}\left(\frac{-3}{N}\right)\right)\\
&\qquad - \frac{1}{2}\left(2^{\alpha+2} - (-1)^{\alpha}\left(\frac{-2}{N}\right)\right)\left(3^{\beta+1} + (-1)^{\alpha}\left(\frac{-3}{N}\right)\right)\Bigg]\prod_{p^{\gamma}\| N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)}
\end{aligned}$$

$$= \frac{1}{3}\left(2^{\alpha+1}3^{\beta} + (-1)^{\alpha}\left(\frac{-2}{N}\right)3^{\beta} - 2^{\alpha+1}(-1)^{\alpha}\left(\frac{-3}{N}\right) - \left(\frac{6}{N}\right)\right)$$

$$\times \prod_{p^{\gamma}\|N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)}$$

$$= \frac{1}{3}\left(2^{\alpha+1} + (-1)^{\alpha}\left(\frac{-2}{N}\right)\right)\left(3^{\beta} - (-1)^{\alpha}\left(\frac{-3}{N}\right)\right)\prod_{p^{\gamma}\|N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)},$$

as claimed. This completes the proof of the first part of the theorem.

We now turn to the proof of the final four formulas of the theorem. From (4.7)–(4.10) it is easy to verify that for $n \in \mathbb{N}$ with $\gcd(N,6) = 1$ and $\alpha, \beta \in \mathbb{N}_0$ we have

$$\begin{aligned}
A(2^{\alpha}3^{\beta}N) &= 2^{\alpha}3^{\beta}A(N),\\
B(2^{\alpha}3^{\beta}N) &= (-1)^{\alpha}3^{\beta}B(N),\\
C(2^{\alpha}3^{\beta}N) &= (-1)^{\alpha}2^{\alpha}C(N),\\
D(2^{\alpha}3^{\beta}N) &= D(N),
\end{aligned}$$

and

$$\begin{aligned}
A(N) &= \left(\frac{-3}{N}\right)\left(\frac{-8}{N}\right)D(N) = \left(\frac{-3}{N}\right)\left(\frac{-2}{N}\right)D(N) = \left(\frac{6}{N}\right)D(N),\\
B(N) &= \left(\frac{-3}{N}\right)D(N),\\
C(N) &= \left(\frac{-8}{N}\right)D(N) = \left(\frac{-2}{N}\right)D(N),
\end{aligned}$$

and

$$A(N) = \prod_{p^{\gamma}\|N}\left(\sum_{d|p^{\gamma}}\left(\frac{6}{p^{\gamma}/d}\right)d\right) = \prod_{p^{\gamma}\|N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)}.$$

Hence

$$A(2^{\alpha}3^{\beta}N) = 2^{\alpha}3^{\beta}\prod_{p^{\gamma}\|N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)},$$

$$B(2^\alpha 3^\beta N) = (-1)^\alpha 3^\beta \left(\frac{-2}{N}\right) \prod_{p^\gamma \| N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)},$$

$$C(2^\alpha 3^\beta N) = (-1)^\alpha 2^\alpha \left(\frac{-3}{N}\right) \prod_{p^\gamma \| N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)},$$

$$D(2^\alpha 3^\beta N) = \left(\frac{6}{N}\right) \prod_{p^\gamma \| N} \frac{p^{\gamma+1} - \left(\frac{6}{p}\right)^{\gamma+1}}{p - \left(\frac{6}{p}\right)}.$$

Thus, from these and the first four formulas of the theorem, we obtain

$$\begin{aligned}
N(1,1,2,3;n) &= 4A(n) - B(n) + \frac{4}{3}C(n) - \frac{1}{3}D(n),\\
N(1,2,2,6;n) &= 2A(n) + B(n) - \frac{2}{3}C(n) - \frac{1}{3}D(n),\\
N(1,3,3,6;n) &= \frac{4}{3}A(n) - \frac{1}{3}B(n) + \frac{4}{3}C(n) - \frac{1}{3}D(n),\\
N(2,3,6,6;n) &= \frac{2}{3}A(n) + \frac{1}{3}B(n) - \frac{2}{3}C(n) - \frac{1}{3}D(n),
\end{aligned}$$

as claimed. □

## 5. Evaluation of the Generating Functions of $A(n)$, $B(n)$, $C(n)$ and $D(n)$

Recall that Ramanujan's theta function $\varphi(q)$ was defined in (1.1). In [2, pp. 225–227], it was shown that the identities

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left( \sum_{d \mid n} \left(\frac{-3}{n/d}\right) \left(\frac{-4}{n/d}\right) d \right) q^n \\
&\quad = 1 - \frac{1}{4}\varphi^2(q)\varphi(-q)\varphi(-q^3) - \frac{3}{4}\varphi^2(q^3)\varphi(-q)\varphi(-q^3),\\
&\sum_{n=1}^{\infty} \left( \sum_{d \mid n} \left(\frac{-3}{n/d}\right) \left(\frac{-4}{d}\right) d \right) q^n \\
&\quad = \frac{1}{4}\varphi^2(q)\varphi(-q)\varphi(-q^3) - \frac{1}{4}\varphi^2(q^3)\varphi(-q)\varphi(-q^3),
\end{aligned}$$

$$\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-3}{d}\right)\left(\frac{-4}{n/d}\right)d\right)q^n = \frac{3}{8}\varphi(q)\varphi^3(q^3) - \frac{1}{8}\varphi^3(q)\varphi(q^3) + \frac{1}{8}\varphi^2(q)\varphi(-q)\varphi(-q^3) - \frac{3}{8}\varphi^2(q^3)\varphi(-q)\varphi(-q^3),$$

$$\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-3}{d}\right)\left(\frac{-4}{d}\right)d\right)q^n = \frac{1}{8}\varphi(q)\varphi^3(q^3) + \frac{1}{8}\varphi^3(q)\varphi(q^3) - \frac{1}{8}\varphi^2(q)\varphi(-q)\varphi(-q^3) - \frac{3}{8}\varphi^2(q^3)\varphi(-q)\varphi(-q^3),$$

follow easily from some identities of Petr [9]. A different proof of these identities has been given by Cooper [7].

As an application of our theorem (Theorem 4.1), we evaluate the analogous infinite series with the discriminant $-4$ replaced by $-8$. We prove the following result.

**Theorem 5.1.** *For $q \in \mathbb{C}$ with $|q| < 1$ we have*

$$\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-3}{n/d}\right)\left(\frac{-8}{n/d}\right)d\right)q^n = \frac{1}{4}\varphi^2(q)\varphi(q^2)\varphi(q^3) + \frac{1}{4}\varphi(q)\varphi^2(q^2)\varphi(q^6) - \frac{1}{4}\varphi(q)\varphi^2(q^3)\varphi(q^6) - \frac{1}{4}\varphi(q^2)\varphi(q^3)\varphi^2(q^6), \tag{5.1}$$

$$\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-3}{n/d}\right)\left(\frac{-8}{d}\right)d\right)q^n = -\frac{1}{2}\varphi^2(q)\varphi(q^2)\varphi(q^3) + \varphi(q)\varphi^2(q^2)\varphi(q^6) + \frac{1}{2}\varphi(q)\varphi^2(q^3)\varphi(q^6) - \varphi(q^2)\varphi(q^3)\varphi^2(q^6), \tag{5.2}$$

$$\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-3}{d}\right)\left(\frac{-8}{n/d}\right)d\right)q^n = -\frac{1}{4}\varphi^2(q)\varphi(q^2)\varphi(q^3) + \frac{1}{4}\varphi(q)\varphi^2(q^2)\varphi(q^6) + \frac{3}{4}\varphi(q)\varphi^2(q^3)\varphi(q^6) - \frac{3}{4}\varphi(q^2)\varphi(q^3)\varphi^2(q^6), \tag{5.3}$$

$$3 - \sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-3}{d}\right)\left(\frac{-8}{d}\right)d\right)q^n$$
$$= -\frac{1}{2}\varphi^2(q)\varphi(q^2)\varphi(q^3) - \varphi(q)\varphi^2(q^2)\varphi(q^6)$$
$$+ \frac{3}{2}\varphi(q)\varphi^2(q^3)\varphi(q^6) + 3\varphi(q^2)\varphi(q^3)\varphi^2(q^6). \qquad (5.4)$$

**Proof.** From the second part of Theorem 4.1 we obtain

$$A(n) = \frac{1}{4}N(1,1,2,3;n) + \frac{1}{4}N(1,2,2,6;n)$$
$$- \frac{1}{4}N(1,3,3,6;n) - \frac{1}{4}N(2,3,6,6;n),$$
$$B(n) = -\frac{1}{2}N(1,1,2,3;n) + N(1,2,2,6;n)$$
$$+ \frac{1}{2}N(1,3,3,6;n) - N(2,3,6,6;n),$$
$$C(n) = -\frac{1}{4}N(1,1,2,3;n) + \frac{1}{4}N(1,2,2,6;n)$$
$$+ \frac{3}{4}N(1,3,3,6;n) - \frac{3}{4}N(2,3,6,6;n),$$
$$D(n) = \frac{1}{2}N(1,1,2,3;n) + N(1,2,2,6;n)$$
$$- \frac{3}{2}N(1,3,3,6;n) - 3N(2,3,6,6;n).$$

Hence

$$\sum_{n=1}^{\infty} D(n)q^n = \frac{1}{2}\sum_{n=1}^{\infty} N(1,1,2,3;n)q^n + \sum_{n=1}^{\infty} N(1,2,2,6;n)q^n$$
$$- \frac{3}{2}\sum_{n=1}^{\infty} N(1,3,3,6;n)q^n - 3\sum_{n=1}^{\infty} N(2,3,6,6;n)q^n$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} N(1,1,2,3;n)q^n + \sum_{n=0}^{\infty} N(1,2,2,6;n)q^n$$
$$- \frac{3}{2}\sum_{n=0}^{\infty} N(1,3,3,6;n)q^n - 3\sum_{n=0}^{\infty} N(2,3,6,6;n)q^n$$
$$- \left(\frac{1}{2} + 1 - \frac{3}{2} - 3\right)$$

$$= \frac{1}{2}\varphi^2(q)\varphi(q^2)\varphi(q^3) + \varphi(q)\varphi^2(q^2)\varphi(q^6) - \frac{3}{2}\varphi(q)\varphi^2(q^3)\varphi(q^6) - 3\varphi(q^2)\varphi(q^3)\varphi^2(q^6) + 3,$$

from which (5.4) follows. Identities (5.1)–(5.3) follow in a similar manner. □

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