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RESTRICTED EISENSTEIN SERIES AND CERTAIN CONVOLUTION SUMS

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Abstract

We parameterize the restricted Eisenstein series $E_{a,m}(q) = \sum_{\substack{n=1 \\ n\equiv a \ (\mathrm{mod}\ m)}}^{\infty} \sigma(n)q^n$ in terms of certain theta functions for $m = 8$ and then use this parameterization to evaluate the convolution sum $\sum_{\substack{m=1 \\ m\equiv a \ (\mathrm{mod}\ 8)}}^{n-1} \sigma(m)\sigma(n-m)$ for all $n \in \mathbb{N}$ and all $a \in \{0,1,2,3,4,5,6,7\}$.

Key words and phrases: convolution sums, sum of divisors function, theta functions, Eisenstein series.

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1. Introduction

Throughout this paper q denotes a complex variable satisfying $|q| < 1$. Let $a \in \mathbb{Z}$ and $m \in \mathbb{N}$. We define the restricted Eisenstein series $E_{a,m}(q)$ by

$$E_{a,m}(q) := \sum_{\substack{n=1 \\ n\equiv a \ (\mathrm{mod}\ m)}}^{\infty} \sigma(n)q^n,$$

where $\sigma(n)$ denotes the sum of the positive divisors of n. An easy calculation shows that for all $a, b \in \mathbb{Z}$ and $m \in \mathbb{N}$ we have

$$E_{a,m}(q)E_{b-a,m}(q) = \sum_{\substack{n=1 \\ n\equiv b \ (\mathrm{mod}\ m)}}^{\infty} S_{a,m}(n)q^n, \tag{1.1}$$

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where the convolution sum $S_{a,m}(n)$ is given by

$$S_{a,m}(n) := \sum_{\substack{k=1 \\ k \equiv a \pmod{m}}}^{n-1} \sigma(k)\sigma(n-k), \quad n \in \mathbb{N}. \tag{1.2}$$

Thus, we can evaluate $S_{a,m}(n)$ for all $n \in \mathbb{N}$ with $n \equiv b \pmod{m}$ if we can determine the power series expansion of $E_{a,m}(q)E_{b-a,m}(q)$ in powers of q in a different way. We show that this can be done in the case $m = 8$ by parameterizing $E_{a,8}(q)$ in terms of the theta functions. We remark that the sums $S_{a,m}(n)$ have been evaluated for $m = 1$ in [6, eq. (3.10)], for $m = 2$ in [6, eqs. (5.3), (5.4)], for $m = 3$ in [7, Theorem 1.2] and for $m = 4$ in [4, Theorem 1.1]. We prove the following theorem in Section 3. For brevity we abbreviate $S_{a,8}(n)$ to $S_a(n)$.

Theorem. *Let $n \in \mathbb{N}$. For $k \in \mathbb{N}$ define*

$$\sigma_k(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d^k, \quad \sigma_1(n) = \sigma(n), \quad \sigma_k(t) = 0, \quad \text{if } t \notin \mathbb{N},$$

and

$$E_k = E_k(q) := \prod_{n=1}^{\infty}(1 - q^{kn}).$$

Define integers $c(n), d(n), e(n)$ $(n \in \mathbb{N})$ by

$$\sum_{n=1}^{\infty} c(n)q^n := qE_2^4E_4^4, \quad \sum_{n=1}^{\infty} d(n)q^n := q^2E_8^{10}E_{16}^{-2},$$

$$\sum_{n=1}^{\infty} e(n)q^n := q^3E_2^{-4}E_4^{12}E_8^{-4}E_{16}^4.$$

(i) *If* $n \equiv 0 \pmod{8}$, *then*

$$S_0(n) = \frac{85}{1536}\sigma_3(n) + \frac{185}{512}\sigma_3(n/2) + \left(\frac{1}{12} - \frac{1}{2}n\right)\sigma(n), \tag{1.3}$$

$$S_1(n) = S_7(n) = \frac{1}{32}\sigma_3(n) - \frac{1}{32}\sigma_3(n/2) - 8c(n/8), \tag{1.4}$$

$$S_2(n) = S_6(n) = \frac{9}{128}\sigma_3(n) - \frac{9}{128}\sigma_3(n/2), \tag{1.5}$$

$$S_3(n) = S_5(n) = \frac{1}{32}\sigma_3(n) - \frac{1}{32}\sigma_3(n/2) + 8c(n/8), \tag{1.6}$$

$$S_4(n) = \frac{49}{512}\sigma_3(n) - \frac{49}{512}\sigma_3(n/2). \tag{1.7}$$

(ii) *If* $n \equiv 1 \pmod{8}$, *then*

$$S_0(n) = S_1(n) = \frac{23}{384}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n) + \frac{19}{128}c(n), \tag{1.8}$$

$$S_2(n)=S_7(n)=\frac{3}{64}\sigma_3(n)-\frac{3}{64}c(n)+\frac{3}{2}e(n), \tag{1.9}$$

$$S_3(n)=S_6(n)=\frac{3}{64}\sigma_3(n)-\frac{3}{64}c(n)-\frac{3}{2}e(n), \tag{1.10}$$

$$S_4(n)=S_5(n)=\frac{7}{128}\sigma_3(n)-\frac{7}{128}c(n). \tag{1.11}$$

(iii) *If* $n\equiv 2 \pmod 8$, *then*

$$S_0(n)=S_2(n)=\frac{23}{32}\sigma_3(n/2)+\left(\frac{1}{8}-\frac{3}{4}n\right)\sigma(n/2)+\frac{21}{32}c(n/2), \tag{1.12}$$

$$S_1(n)=\frac{1}{4}\sigma_3(n/2)+\frac{1}{4}c(n/2)+\frac{1}{4}d(n), \tag{1.13}$$

$$S_3(n)=S_7(n)=\frac{1}{4}\sigma_3(n/2)-\frac{1}{4}c(n/2), \tag{1.14}$$

$$S_4(n)=S_6(n)=\frac{21}{32}\sigma_3(n/2)-\frac{21}{32}c(n/2), \tag{1.15}$$

$$S_5(n)=\frac{1}{4}\sigma_3(n/2)+\frac{1}{4}c(n/2)-\frac{1}{4}d(n). \tag{1.16}$$

(iv) *If* $n\equiv 3 \pmod 8$, *then*

$$S_0(n)=S_3(n)=\frac{23}{384}\sigma_3(n)+\left(\frac{1}{24}-\frac{1}{4}n\right)\sigma(n)-\frac{37}{128}c(n), \tag{1.17}$$

$$S_1(n)=S_2(n)=\frac{3}{64}\sigma_3(n)-\frac{3}{64}c(n)+\frac{3}{2}e(n), \tag{1.18}$$

$$S_4(n)=S_7(n)=\frac{7}{128}\sigma_3(n)+\frac{49}{128}c(n), \tag{1.19}$$

$$S_5(n)=S_6(n)=\frac{3}{64}\sigma_3(n)-\frac{3}{64}c(n)-\frac{3}{2}e(n). \tag{1.20}$$

(v) *If* $n\equiv 4 \pmod 8$, *then*

$$S_0(n)=S_4(n)=\frac{161}{24}\sigma_3(n/4)+\left(\frac{7}{24}-\frac{7}{4}n\right)\sigma(n/4), \tag{1.21}$$

$$S_1(n)=S_3(n)=2\sigma_3(n/4)+2(-1)^{(n-4)/8}c(n/4), \tag{1.22}$$

$$S_2(n)=\frac{9}{2}\sigma_3(n/4)+\frac{9}{2}c(n/4), \tag{1.23}$$

$$S_5(n)=S_7(n)=2\sigma_3(n/4)-2(-1)^{(n-4)/8}c(n/4), \tag{1.24}$$

$$S_6(n)=\frac{9}{2}\sigma_3(n/4)-\frac{9}{2}c(n/4). \tag{1.25}$$

(vi) *If* $n \equiv 5 \pmod 8$, *then*

$$S_0(n) = S_5(n) = \frac{23}{384}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n) + \frac{19}{128}c(n), \tag{1.26}$$

$$S_1(n) = S_4(n) = \frac{7}{128}\sigma_3(n) - \frac{7}{128}c(n), \tag{1.27}$$

$$S_2(n) = S_3(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) + \frac{3}{2}e(n), \tag{1.28}$$

$$S_6(n) = S_7(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) - \frac{3}{2}e(n). \tag{1.29}$$

(vii) *If* $n \equiv 6 \pmod 8$, *then*

$$S_0(n) = S_6(n) = \frac{23}{32}\sigma_3(n/2) + \left(\frac{1}{8} - \frac{3}{4}n\right)\sigma(n/2) + \frac{21}{32}c(n/2), \tag{1.30}$$

$$S_1(n) = S_5(n) = \frac{1}{4}\sigma_3(n/2) + \frac{1}{4}c(n/2), \tag{1.31}$$

$$S_2(n) = S_4(n) = \frac{21}{32}\sigma_3(n/2) - \frac{21}{32}c(n/2), \tag{1.32}$$

$$S_3(n) = \frac{1}{4}\sigma_3(n/2) - \frac{1}{4}c(n/2) + 8e(n/2), \tag{1.33}$$

$$S_7(n) = \frac{1}{4}\sigma_3(n/2) - \frac{1}{4}c(n/2) - 8e(n/2). \tag{1.34}$$

(viii) *If* $n \equiv 7 \pmod 8$, *then*

$$S_0(n) = S_7(n) = \frac{23}{384}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n) - \frac{37}{128}c(n), \tag{1.35}$$

$$S_1(n) = S_6(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) - \frac{3}{2}e(n), \tag{1.36}$$

$$S_2(n) = S_5(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) + \frac{3}{2}e(n), \tag{1.37}$$

$$S_3(n) = S_4(n) = \frac{7}{128}\sigma_3(n) + \frac{49}{128}c(n). \tag{1.38}$$

We remark that

$$c(n) = e(n) = 0, \text{ if } n \equiv 0 \pmod 2, \ \ d(n) = 0, \text{ if } n \not\equiv 2 \pmod 8.$$

There are 8 sums $S_a(n)$ $(a = 0, 1, 2, \ldots, 7)$ to be evaluated in 8 cases depending upon $n \pmod 8$. Thus there are $8 \times 8 = 64$ formulae in total. It is interesting to note that of these 64 formulae just 6 require only the divisor functions σ_3 and σ, while of the remaining 58 formulae, in addition to the divisor functions σ_3 and σ, 38 require c, 18 require c and e, and 2 require c and d. It is simple to check that no linear relation of the type

$$d(n) = \left(A + B(-1)^{(n-2)/8}\right)\sigma_3(n/2) + \left(C + D(-1)^{(n-2)/8}\right)c(n/2)$$

$$+\big(E+F(-1)^{(n-2)/8}\big)e(n/2),\ n\equiv 2\ (\mathrm{mod}\ 8), \tag{1.39}$$

exists so that d is really required.

In proving our theorem we make use of the following result of Williams [8, Theorem 1]

$$\sum_{\substack{m\in\mathbb{N}\\ m<n/8}}\sigma(m)\sigma(n-8m)=\frac{1}{192}\sigma_3(n)+\frac{1}{64}\sigma_3(n/2)+\frac{1}{16}\sigma_3(n/4)+\frac{1}{3}\sigma_3(n/8)$$
$$+\Big(\frac{1}{24}-\frac{1}{32}n\Big)\sigma(n)+\Big(\frac{1}{24}-\frac{1}{4}n\Big)\sigma(n/8)-\frac{1}{64}c(n).$$

2. Parameterization of Theta Functions and Restricted Eisenstein Series

The theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q):=\sum_{n=-\infty}^{\infty}q^{n^2},\ \ \psi(q):=\sum_{n=0}^{\infty}q^{n(n+1)/2},\ \ q\in\mathbb{C},\ \ |q|<1. \tag{2.1}$$

The basic properties of these functions are

$$\begin{cases}\varphi(q)+\varphi(-q)=2\varphi(q^4),\ \text{[5, eq. (3.6.1)]},\\ \varphi^2(q)+\varphi^2(-q)=2\varphi^2(q^2),\ \ \text{[5, eq. (3.6.7)]},\\ \varphi(q)\varphi(-q)=\varphi^2(-q^2),\ \ \text{[5, eq. (1.3.32)]},\\ \varphi(q)-\varphi(-q)=4q\psi(q^8),\ \text{[5, eq. (3.6.2)]}.\end{cases} \tag{2.2}$$

Using the first three of these relations we can parameterize $\varphi(\pm q),\varphi(\pm q^2),\varphi(\pm q^4)$, $\varphi(\pm q^8)$ and $\varphi(\pm q^{16})$ in terms of the parameters A,B and X defined by

$$A=A(q):=\varphi(q),\ B=B(q):=\varphi(-q),\ X=X(q):=\frac{1}{2}AB(A^2+B^2), \tag{2.3}$$

namely

$$\begin{cases}\varphi(q)=A,\ \ \varphi(-q)=B,\\ \varphi(q^2)=\left(\dfrac{A^2+B^2}{2}\right)^{\frac{1}{2}},\ \ \varphi(-q^2)=(AB)^{\frac{1}{2}},\\ \varphi(q^4)=\dfrac{1}{2}(A+B),\\ \varphi(-q^4)=X^{\frac{1}{4}},\\ \varphi(q^8)=\dfrac{1}{2}\left(\left(\dfrac{A^2+B^2}{2}\right)^{\frac{1}{2}}+(AB)^{\frac{1}{2}}\right),\\ \varphi(-q^8)=\left(\dfrac{A+B}{2}\right)^{\frac{1}{2}}X^{\frac{1}{8}},\\ \varphi(q^{16})=\dfrac{1}{2}\left(\dfrac{A+B}{2}+X^{\frac{1}{4}}\right).\end{cases} \tag{2.4}$$

For future use, we note that under the transformation $q \mapsto -q$, we have

$$A(-q) = B,\ B(-q) = A,\ X(-q) = X, \tag{2.5}$$

and under the transformation $q \mapsto q^2$, we have

$$\begin{cases} A(q^2) = \left(\dfrac{A^2+B^2}{2}\right)^{\frac{1}{2}},\ B(q^2) = (AB)^{\frac{1}{2}}, \\ A(q^2)B(q^2) = X^{\frac{1}{2}}, \\ A^2(q^2)+B^2(q^2) = \dfrac{(A+B)^2}{2}, \\ A^2(q^2)-B^2(q^2) = \dfrac{(A-B)^2}{2}, \\ (A(q^2)+B(q^2))^2 = \dfrac{(A+B)^2}{2}+2X^{\frac{1}{2}}, \\ (A(q^2)-B(q^2))^2 = \dfrac{(A+B)^2}{2}-2X^{\frac{1}{2}}, \\ X(q^2) = \dfrac{(A+B)^2}{4}X^{\frac{1}{2}}. \end{cases} \tag{2.6}$$

We recall next how $E_{1,2}(q) = \displaystyle\sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} \sigma(n)q^n$ can be parameterized in terms of A and B. Jacobi proved that the number $r_4(n)$ of representations of an odd positive integer n as the sum of four integral squares is given by $r_4(n) = 8\sigma(n)$. Hence

$$\begin{aligned} A^4 - B^4 &= \varphi^4(q) - \varphi^4(-q) = \sum_{n=0}^{\infty} r_4(n)q^n - \sum_{n=0}^{\infty} r_4(n)(-q)^n \\ &= 2 \sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} r_4(n)q^n = 16 \sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} \sigma(n)q^n \end{aligned}$$

so that

$$E_{1,2}(q) = \frac{1}{16}(A^4 - B^4). \tag{2.7}$$

It follows from (2.4), (2.6), (2.7), and [1, Theorem 2.4] that

$$\begin{cases} E_{1,4}(q) = \dfrac{1}{2^5}(A-B)(A+B)^3, \\ E_{2,4}(q) = 3E_{1,2}(q^2) = \dfrac{3}{2^6}(A-B)^2(A+B)^2, \\ E_{3,4}(q) = \dfrac{1}{2^5}(A-B)^3(A+B). \end{cases} \tag{2.8}$$

Also from (2.4), (2.6), (2.7), (2.8) and [1, Theorem 2.4] it can be deduced that

$$\begin{cases} E_{1,8}(q) = \frac{1}{2^6}((A-B)(A+B)^3+8(A-B)X^{\frac{3}{4}}), \\ E_{2,8}(q) = 3E_{1,4}(q^2) = \frac{3}{2^7}((A-B)^2(A+B)^2+4(A-B)^2X^{\frac{1}{2}}), \\ E_{3,8}(q) = \frac{1}{2^6}((A-B)^3(A+B)+2(A-B)^3X^{\frac{1}{4}}), \\ E_{4,8}(q) = 7E_{1,2}(q^4) = \frac{7}{2^8}(A-B)^4, \\ E_{5,8}(q) = \frac{1}{2^6}((A-B)(A+B)^3-8(A-B)X^{\frac{3}{4}}), \\ E_{6,8}(q) = 3E_{3,4}(q^2) = \frac{3}{2^7}((A-B)^2(A+B)^2-4(A-B)^2X^{\frac{1}{2}}), \\ E_{7,8}(q) = \frac{1}{2^6}((A-B)^3(A+B)-2(A-B)^3X^{\frac{1}{4}}). \end{cases} \tag{2.9}$$

Jacobi has shown that

$$A = \frac{E_2^5}{E_1^2E_4^2}, \ \ B = \frac{E_1^2}{E_2}, \ \ X = \frac{E_4^8}{E_8^4}, \tag{2.10}$$

so that

$$\sum_{n=1}^{\infty} c(n)q^n = \frac{1}{2^4}A^2B^2(A-B)(A+B)(A^2+B^2), \tag{2.11}$$

$$\sum_{n=1}^{\infty} d(n)q^n = \frac{1}{2^7}(A-B)^2(A+B)^3X^{\frac{3}{4}}, \tag{2.12}$$

$$\sum_{n=1}^{\infty} e(n)q^n = \frac{1}{2^8}(A-B)^3(A+B)(A^2+B^2)X^{\frac{1}{2}}. \tag{2.13}$$

Applying the mapping $q \mapsto q^2$ successively three times to (2.11), and appealing to (2.6), we obtain

$$\sum_{n=1}^{\infty} c(n)q^{2n} = \frac{1}{2^7}AB(A-B)^2(A+B)^2(A^2+B^2), \tag{2.14}$$

$$\sum_{n=1}^{\infty} c(n)q^{4n} = \frac{1}{2^{10}}(A-B)^4(A+B)^2X^{\frac{1}{2}}, \tag{2.15}$$

$$\sum_{n=1}^{\infty} c(n)q^{8n} = \frac{1}{2^{14}}((A-B)^4(A+B)^3X^{\frac{1}{4}}-4(A-B)^4(A+B)X^{\frac{3}{4}}). \tag{2.16}$$

Applying the transformation $q \mapsto q^2$ to (2.13), and appealing to (2.6), we obtain

$$\sum_{n=1}^{\infty} e(n)q^{2n} = \frac{1}{2^{12}}((A-B)^2(A+B)^5X^{\frac{1}{4}}-4(A-B)^2(A+B)^3X^{\frac{3}{4}}). \tag{2.17}$$

Finally, we require two other parameterizations, namely

$$\sum_{\substack{n=1 \\ n\equiv 4 \pmod 8}}^{\infty} (-1)^{(n-4)/8}\, c(n/4)q^n$$
$$= \frac{1}{2^{12}}\left((A-B)^4(A+B)^3X^{\frac{1}{4}} + 4(A-B)^4(A+B)X^{\frac{3}{4}}\right) \quad (2.18)$$

and

$$\sum_{\substack{n=1 \\ n\equiv 6 \pmod 8}}^{\infty} e(n/2)q^n = \frac{1}{2^{13}}(A-B)^6(A+B)X^{\frac{1}{4}}. \quad (2.19)$$

We just prove (2.18) as the proof of (2.19) can be carried out in a similar manner. Since (see for example [1, eq. (2.11)])

$$\varphi(iq) = \varphi(q^4) + \frac{i}{2}(\varphi(q) - \varphi(-q))$$

and

$$\varphi(-iq) = \varphi(q^4) - \frac{i}{2}(\varphi(q) - \varphi(-q)),$$

under the transformation $q \mapsto iq$, we have

$$\begin{cases} A(iq) = \dfrac{(A+B)}{2} + i\dfrac{(A-B)}{2}, \\ B(iq) = \dfrac{(A+B)}{2} - i\dfrac{(A-B)}{2}, \\ A(iq)B(iq) = \dfrac{A^2+B^2}{2}, \\ A^2(iq) + B^2(iq) = 2AB, \\ X(iq) = X. \end{cases} \quad (2.20)$$

Mapping $q \mapsto iq$ in (2.11), and appealing to (2.20), we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} c(n)(iq)^n \\ &\quad = \frac{1}{2^4}A^2(iq)B^2(iq)(A(iq)-B(iq))(A(iq)+B(iq))(A^2(iq)+B^2(iq)) \\ &\quad = \frac{1}{2^4}\left(\frac{A^2+B^2}{2}\right)^2 i(A-B)(A+B)2AB \\ &\quad = \frac{i}{2^5}AB(A-B)(A+B)(A^2+B^2)^2. \end{aligned}$$

Then, applying the transformation $q \mapsto q^2$ and appealing to (2.6), we deduce

$$\begin{aligned} \sum_{n=1}^{\infty} c(n)(iq^2)^n &= \frac{i}{2^5}A(q^2)B(q^2)\left(A^2(q^2)-B^2(q^2)\right)\left(A^2(q^2)+B^2(q^2)\right)^2 \\ &= \frac{i}{2^5}X^{\frac{1}{2}}\frac{(A-B)^2}{2}\frac{(A+B)^4}{2^2} \end{aligned}$$

$$= \frac{i}{2^8}(A-B)^2(A+B)^4 X^{\frac{1}{2}}.$$

Applying the transformation $q \mapsto q^2$ again, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} c(n)(iq^4)^n &= \frac{i}{2^8}(A(q^2)-B(q^2))^2(A(q^2)+B(q^2))^4 X(q^2)^{\frac{1}{2}} \\
&= \frac{i}{2^8}(A^2(q^2)-B^2(q^2))^2(A(q^2)+B(q^2))^2 X(q^2)^{\frac{1}{2}} \\
&= \frac{i}{2^8}\frac{(A-B)^4}{2^2}\Big(\frac{(A+B)^2}{2}+2X^{\frac{1}{2}}\Big)\frac{(A+B)}{2}X^{\frac{1}{4}} \\
&= \frac{i}{2^{12}}\big((A-B)^4(A+B)^3 X^{\frac{1}{4}}+4(A-B)^4(A+B)X^{\frac{3}{4}}\big).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\sum_{\substack{n=1 \\ n\equiv 4 \ (\mathrm{mod}\ 8)}}^{\infty} (-1)^{(n-4)/8} c(n/4)q^n &= \sum_{\substack{n=1 \\ n\equiv 1 \ (\mathrm{mod}\ 2)}}^{\infty} (-1)^{(n-1)/2} c(n)q^{4n} \\
&= i^{-1} \sum_{\substack{n=1 \\ n\equiv 1 \ (\mathrm{mod}\ 2)}}^{\infty} c(n)(iq^4)^n = i^{-1}\sum_{n=1}^{\infty} c(n)(iq^4)^n \\
&= \frac{1}{2^{12}}((A-B)^4(A+B)^3 X^{\frac{1}{4}}+4(A-B)^4(A+B)X^{\frac{3}{4}}),
\end{aligned}$$

which is (2.18).

3. Proof of Theorem

We begin by determining $S_0(n)$ for all $n \in \mathbb{N}$. If $m \in \mathbb{N}$, it is easy to show that $\sigma(8m) = 15\sigma(m) - 14\sigma(m/2)$ so that

$$\begin{aligned}
S_0(n) &= \sum_{\substack{m=1 \\ m\equiv 0 \ (\mathrm{mod}\ 8)}}^{n-1} \sigma(m)\sigma(n-m) \\
&= \sum_{\substack{m\in\mathbb{N} \\ m<n/8}} \sigma(8m)\sigma(n-8m) \\
&= \sum_{\substack{m\in\mathbb{N} \\ m<n/8}} (15\sigma(m)-14\sigma(m/2))\sigma(n-8m),
\end{aligned}$$

that is

$$S_0(n) = 15T_8(n) - 14T_{16}(n), \tag{3.1}$$

where

$$T_k(n) := \sum_{\substack{m \in \mathbb{N} \\ m < n/k}} \sigma(m)\sigma(n-km), \quad k,n \in \mathbb{N}. \tag{3.2}$$

As we mentioned in the introduction, Williams [8, Theorem 1] has shown that

$$\begin{aligned} T_8(n) = {} & \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3(n/2) + \frac{1}{16}\sigma_3(n/4) + \frac{1}{3}\sigma_3(n/8) \\ & + \left(\frac{1}{24} - \frac{1}{32}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n/8) - \frac{1}{64}c(n). \end{aligned} \tag{3.3}$$

Further Alaca, Alaca and Williams in [2, Theorem 1.1] have shown that

$$\begin{aligned} T_{16}(n) = {} & \frac{1}{768}\sigma_3(n) + \frac{1}{256}\sigma_3(n/2) + \frac{1}{64}\sigma_3(n/4) + \frac{1}{16}\sigma_3(n/8) + \frac{1}{3}\sigma_3(n/16) \\ & + \left(\frac{1}{24} - \frac{1}{64}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n/16) - \frac{7}{256}c_{16}(n), \end{aligned} \tag{3.4}$$

where [3, Theorem 3.1]

$$c_{16}(n) = \begin{cases} \frac{12}{7}\, c(n/2), & \text{if } n \equiv 0 \pmod 2, \\ c(n), & \text{if } n \equiv 1 \pmod 4, \\ -\frac{1}{7}\, c(n), & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{3.5}$$

Putting (3.3), (3.4) and (3.5) into (3.1), we obtain

$$\begin{aligned} S_0(n) = {} & \frac{23}{384}\sigma_3(n) + \frac{23}{128}\sigma_3(n/2) + \frac{23}{32}\sigma_3(n/4) + \frac{33}{8}\sigma_3(n/8) - \frac{14}{3}\sigma_3(n/16) \\ & + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n) + \left(\frac{5}{8} - \frac{15}{4}n\right)\sigma(n/8) - \left(\frac{7}{12} - \frac{7}{2}n\right)\sigma(n/16) \\ & + \begin{cases} \frac{21}{32}c(n/2), & \text{if } n \equiv 0 \pmod 2, \\ \frac{19}{28}c(n), & \text{if } n \equiv 1 \pmod 4, \\ -\frac{37}{128}c(n), & \text{if } n \equiv 3 \pmod 4. \end{cases} \end{aligned} \tag{3.6}$$

This formula for $S_0(n)$ reduces to those given in the Theorem in the eight cases $n \equiv 0,1,\ldots,7 \pmod 8$.

We now return to the evaluation of $S_a(n)$ for $a = 1,2,\ldots,7$. It is easy to prove that

$$S_a(n) = S_{c-a}(n), \quad \text{if } n \equiv c \pmod 8, \tag{3.7}$$

and

$$S_a(n) + S_{a+4}(n) = S_{a,4}(n). \tag{3.8}$$

As the $S_{a,4}(n)$ have been determined in [4] these formulae reduce the number of cases we must consider. In some cases they give us the value of $S_a(n)$ immediately. For example if $n \equiv 2 \pmod 8$ then by (3.7) we have

$$S_3(n) = S_7(n) \tag{3.9}$$

and by (3.8)

$$S_3(n) + S_7(n) = S_{3,4}(n). \tag{3.10}$$

From [4, Theorem 1.1] we have

$$S_{3,4}(n) = \frac{1}{18}\sigma_3(n) - \frac{1}{2}c(n/2) \tag{3.11}$$

so that (3.9), (3.10) and (3.11) yield

$$\begin{aligned} S_3(n) &= S_7(n) = \frac{1}{2}S_{3,4}(n) = \frac{1}{36}\sigma_3(n) - \frac{1}{4}c(n/2) \\ &= \frac{1}{4}\sigma_3(n/2) - \frac{1}{4}c(n/2). \end{aligned} \tag{3.12}$$

However, in many cases they do not suffice and we use the parameterization of theta functions and restricted Eisenstein series given in Section 2. We illustrate the ideas involved in three cases.

First we show that for $n \equiv 0 \pmod 8$

$$\begin{cases} S_1(n) = \dfrac{1}{32}\sigma_3(n) - \dfrac{1}{32}\sigma_3(n/2) - 8c(n/8), \\ S_5(n) = \dfrac{1}{32}\sigma_3(n) - \dfrac{1}{32}\sigma_3(n/2) + 8c(n/8). \end{cases} \tag{3.13}$$

Using (1.1) and (2.9), we have

$$\begin{aligned} \sum_{\substack{n=1 \\ n \equiv 0 \pmod 8}}^{\infty} (S_1(n) - S_5(n))q^n &= E_{1,8}(q)E_{7,8}(q) - E_{3,8}(q)E_{5,8}(q) \\ &= \frac{1}{2^{12}}\left((A-B)(A+B)^3 + 8(A-B)X^{\frac{3}{4}}\right)\left((A-B)^3(A+B) - 2(A-B)^3X^{\frac{1}{4}}\right) \\ &\quad - \frac{1}{2^{12}}\left((A-B)^3(A+B) + 2(A-B)^3X^{\frac{1}{4}}\right)\left((A-B)(A+B)^3 - 8(A-B)X^{\frac{3}{4}}\right). \end{aligned}$$

Using the identity

$$(R+S)(T-U) - (R-S)(T+U) = 2ST - 2RU$$

with $R = (A-B)(A+B)^3$, $S = 8(A-B)X^{\frac{3}{4}}$, $T = (A-B)^3(A+B)$, $U = 2(A-B)^3X^{\frac{1}{4}}$, and the formula (2.16), we obtain

$$\sum_{\substack{n=1 \\ n \equiv 0 \pmod 8}}^{\infty} (S_1(n) - S_5(n))q^n$$

$$
\begin{aligned}
&= \frac{1}{2^{11}}\big(8(A-B)^4(A+B)X^{\frac{3}{4}} - 2(A-B)^4(A+B)^3X^{\frac{1}{4}}\big)\\
&= -\frac{1}{2^{10}}\big((A-B)^4(A+B)^3X^{\frac{1}{4}} - 4(A-B)^4(A+B)X^{\frac{3}{4}}\big)\\
&= -2^4\sum_{n=1}^{\infty} c(n)q^{8n} = -16 \sum_{\substack{n=1\\ n\equiv 0 \ (\mathrm{mod}\ 8)}}^{\infty} c(n/8)q^n.
\end{aligned}
$$

Thus

$$
S_1(n)-S_5(n) = -16c(n/8),\ n\equiv 0 \ (\mathrm{mod}\ 8). \tag{3.14}
$$

By (3.8) and [4, Theorem 1.1], we have

$$
S_1(n)+S_5(n) = S_{1,4}(n) = \frac{1}{16}\sigma_3(n) - \frac{1}{16}\sigma_3(n/2),\ \ n\equiv 0 \ (\mathrm{mod}\ 4). \tag{3.15}
$$

Adding and subtracting (3.14) and (3.15) we obtain (3.13).

Secondly we prove for $n\equiv 2 \ (\mathrm{mod}\ 8)$.

$$
\begin{cases}
S_1(n) = \frac{1}{4}\sigma_3(n/2) + \frac{1}{4}c(n/2) + \frac{1}{2}d(n),\\
S_5(n) = \frac{1}{4}\sigma_3(n/2) + \frac{1}{4}c(n/2) - \frac{1}{2}d(n).
\end{cases} \tag{3.16}
$$

Appealing to (1.1), (2.9) and (2.12) we obtain

$$
\begin{aligned}
\sum_{\substack{n=1\\ n\equiv 2 \ (\mathrm{mod}\ 8)}}^{\infty} (S_1(n)-S_5(n))q^n &= L_{1,8}^2(q) - L_{5,8}^2(q)\\
&= \frac{1}{2^{12}}((A-B)(A+B)^3 + 8(A-B)X^{\frac{3}{4}})^2\\
&\quad -\frac{1}{2^{12}}((A-B)(A+B)^3 - 8(A-B)X^{\frac{3}{4}})^2\\
&= \frac{1}{2^7}(A-B)^2(A+B)^3X^{\frac{3}{4}}\\
&= \sum_{n=1}^{\infty} d(n)q^n\\
&= \sum_{\substack{n=1\\ n\equiv 2 \ (\mathrm{mod}\ 8)}}^{\infty} d(n)q^n,
\end{aligned}
$$

as $d(n)=0$ for $n\not\equiv 2 \ (\mathrm{mod}\ 8)$, so

$$
S_1(n)-S_5(n) = d(n),\ n\equiv 2 \ (\mathrm{mod}\ 8). \tag{3.17}
$$

Now, by (3.8) and [4, Theorem 1.1], we have

$$
S_1(n)+S_5(n) = S_{1,4}(n) = \frac{1}{18}\sigma_3(n) + \frac{1}{2}c(n/2),\ n\equiv 2 \ (\mathrm{mod}\ 4). \tag{3.18}
$$

Adding and subtracting (3.17) and (3.18), we obtain the formulae of (3.16).

Finally we prove for $n \equiv 3 \pmod 8$

$$\begin{cases} S_1(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) + \frac{3}{2}e(n), \\ S_5(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) - \frac{3}{2}e(n). \end{cases} \tag{3.19}$$

Now appealing to (1.1), (2.7), (2.9) and (2.13) we obtain

$$\begin{aligned}
&\sum_{\substack{n=1 \\ n \equiv 1 \ (\mathrm{mod}\ 2)}}^{\infty} \big(S_2(n) - S_6(n)\big)q^n \\
&= \sum_{j=1,3,5,7} \Big(\sum_{\substack{n=1 \\ n \equiv j \ (\mathrm{mod}\ 8)}}^{\infty} S_2(n)q^n - \sum_{\substack{n=1 \\ n \equiv j \ (\mathrm{mod}\ 8)}}^{\infty} S_6(n)q^n \Big) \\
&= \sum_{j=1,3,5,7} \big(E_{2,8}(q)E_{j-2,8}(q) - E_{6,8}(q)E_{j-6,8}(q)\big) \\
&= E_{2,8}(q) \sum_{j=1,3,5,7} E_{j-2,8}(q) - E_{6,8}(q) \sum_{j=1,3,5,7} E_{j-6,8}(q) \\
&= E_{2,8}(q)E_{1,2}(q) - E_{6,8}(q)E_{1,2}(q) \\
&= (E_{2,8}(q) - E_{6,8}(q))E_{1,2}(q) \\
&= \frac{3}{2^7}\, 8(A-B)^2 X^{\frac{1}{2}} \Big(\frac{1}{2^5}(A-B)(A+B)^3 + \frac{1}{2^5}(A-B)^3(A+B) \Big) \\
&= \frac{3}{2^9}(A-B)^3(A+B)((A+B)^2 + (A-B)^2)X^{\frac{1}{2}} \\
&= \frac{3}{2^8}(A-B)^3(A+B)(A^2+B^2)X^{\frac{1}{2}} \\
&= 3\sum_{n=1}^{\infty} e(n)q^n \\
&= 3 \sum_{\substack{n=1 \\ n \equiv 1 \ (\mathrm{mod}\ 2)}}^{\infty} e(n)q^n,
\end{aligned}$$

so $S_2(n) - S_6(n) = 3e(n), \ n \equiv 1 \pmod 2$. For $n \equiv 3 \pmod 8$, by (3.7) we have

$$S_1(n) = S_2(n) \quad \text{and} \quad S_5(n) = S_6(n),$$

so

$$S_1(n) - S_5(n) = 3e(n), \ \ n \equiv 3 \pmod 8. \tag{3.20}$$

By (3.8) and [4, Theorem 1.1], we have

$$\begin{aligned} S_1(n) + S_5(n) &= S_{1,4}(n) \\ &= \frac{3}{32}\sigma_3(n) - \frac{3}{32}c(n), \ \ n \equiv 3 \pmod 4. \end{aligned} \tag{3.21}$$

Adding and subtracting (3.20) and (3.21), we obtain (3.19). □

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