Functiones et Approximatio
43.1 (2010), 45–54

THE NUMBER OF REPRESENTATIONS OF A POSITIVE INTEGER BY CERTAIN OCTONARY QUADRATIC FORMS

Şaban Alaca, Kenneth S. Williams

Abstract: The number of representations of a positive integer by each of the octonary quadratic forms $x_1^2+x_2^2+3x_3^2+3x_4^2+3x_5^2+3x_6^2+3x_7^2+3x_8^2$, $x_1^2+x_2^2+x_3^2+x_4^2+3x_5^2+3x_6^2+3x_7^2+3x_8^2$, $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+3x_7^2+3x_8^2$ is determined.

Keywords: octonary quadratic forms, representations, theta functions, Eisenstein series.

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, rational numbers, real numbers and complex numbers respectively so that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a_1, \dots, a_8 \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define

$$N(a_1, \dots, a_8; n) := \text{card}\{(x_1, \dots, x_8) \in \mathbb{Z}^8 \mid n = a_1x_1^2 + \cdots + a_8x_8^2\}. \tag{1.1}$$

Clearly

$$N(a_1, \dots, a_8; 0) = 1. \tag{1.2}$$

For $k \in \mathbb{Z}$ and $n \in \mathbb{Q}$ we set

$$\sigma_k(n) := \begin{cases} \sum\limits_{\substack{d \in \mathbb{N} \\ d \mid n}} d^k, & \text{if } n \in \mathbb{N}, \\ 0, & \text{if } n \in \mathbb{Q} \setminus \mathbb{N}. \end{cases} \tag{1.3}$$

We write $\sigma(n)$ for $\sigma_1(n)$.

For $a, n \in \mathbb{N}$ we set

$$N_{1,a}(n) := N(1,1,1,1,a,a,a,a;n). \tag{1.4}$$

The research of both authors was supported by research grants from the Natural Sciences and Engineering Research Council of Canada.

2000 Mathematics Subject Classification: primary: 11E25, 11F27

It is a classical result of Jacobi [5, §§40-42, pp. 159-170] that for all $n \in \mathbb{N}$

$$N_{1,1}(n) = 16\sigma_3(n) - 32\sigma_3(n/2) + 256\sigma_3(n/4).$$

In [7, Theorem 2, p. 388] it was shown that for all $n \in \mathbb{N}$

$$N_{1,2}(n) = 4\sigma_3(n) - 4\sigma_3(n/2) - 16\sigma_3(n/4) + 256\sigma_3(n/8) + 4c_8(n)$$

and in [2, Theorem 1.2, p. 4] (see [3, Theorem 3.1, p. 348]), [3, Theorem 1.1(iv)], [4, Theorem 1.1(vi)] that

$$\begin{aligned} N_{1,4}(n) &= \sigma_3(n) + 3\sigma_3(n/2) - 68\sigma_3(n/4) + 48\sigma_3(n/8) \\ &\quad + 256\sigma_3(n/16) + \Big(3 + 4\Big(\frac{-4}{n}\Big)\Big)c_8(n) + 12c_8(n/2), \end{aligned}$$

where

$$\Big(\frac{-4}{n}\Big) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2, \\ (-1)^{(n-1)/2}, & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

and the integers $c_8(n)$ $(n \in \mathbb{N})$ are given by

$$\sum_{n=1}^{\infty} c_8(n)q^n = q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4, \qquad q \in \mathbb{C}, \quad |q| < 1.$$

Clearly $c_8(n) = 0$ for $n \equiv 0 \pmod 2$.

In this paper we determine $N_{1,3}(n) = N(1^4 3^4; n) = N(1,1,1,1,3,3,3,3; n)$ for all $n \in \mathbb{N}$. We also find formulae for $N(1^2 3^6; n) := N(1,1,3,3,3,3,3,3; n)$ and $N(1^6 3^2; n) := N(1,1,1,1,1,1,3,3; n)$. These formulae are given in terms of $\sigma_3(n)$, $\sigma_3(n/2)$, $\sigma_3(n/3)$, $\sigma_3(n/4)$, $\sigma_3(n/6)$, $\sigma_3(n/12)$ and the integers $b(n)$ and $c(n)$ $(n \in \mathbb{N})$ defined by

$$\sum_{n=1}^{\infty} b(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})^2(1-q^{3n})^2(1-q^{6n})^2 \tag{1.5}$$

and

$$\sum_{n=1}^{\infty} c(n)q^n = q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^{-3}(1-q^{6n})^9, \tag{1.6}$$

where $q \in \mathbb{C}$ and $|q| < 1$. The first 12 values of $b(n)$ and $c(n)$ are given in Table 1.

Table 1

n	$b(n)$	$c(n)$	n	$b(n)$	$c(n)$
1	1	0	7	−16	−4
2	−2	1	8	−8	4
3	−3	−1	9	9	0
4	4	−2	10	−12	6
5	6	4	11	12	−4
6	6	−3	12	−12	6

It is known from the work of Mason [6, p. 189] that $b(n)$ is a multiplicative function of n. Clearly $c(n)$ is not multiplicative. We prove the following theorem in Section 3 after some preliminary results are given in Section 2.

Theorem 1.1. *Let $n \in \mathbb{N}$. Then*

(i)
$$N(1^2 3^6; n) = \frac{4}{5}\sigma_3(n) - \frac{84}{5}\sigma_3(n/3) - \frac{64}{5}\sigma_3(n/4) + \frac{1344}{5}\sigma_3(n/12) + \frac{16}{5}b(n) - \frac{32}{5}c(n) + \frac{48}{5}(-1)^n c(n),$$

(ii)
$$N(1^4 3^4; n) = \frac{8}{5}\sigma_3(n) - \frac{16}{5}\sigma_3(n/2) + \frac{72}{5}\sigma_3(n/3) + \frac{128}{5}\sigma_3(n/4) - \frac{144}{5}\sigma_3(n/6) + \frac{1152}{5}\sigma_3(n/12) + \frac{32}{5}(-1)^{n-1}b(n),$$

(iii)
$$N(1^6 3^2; n) = \frac{28}{5}\sigma_3(n) - \frac{108}{5}\sigma_3(n/3) - \frac{448}{5}\sigma_3(n/4) + \frac{1728}{5}\sigma_3(n/12) + \frac{32}{5}b(n) - \frac{64}{5}c(n) + \frac{176}{5}(-1)^n c(n).$$

2. Notation and Preliminary Results

The theta function $\varphi(q)$ is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad q \in \mathbb{C}, \quad |q| < 1,$$

so that for $a_1, \ldots, a_8 \in \mathbb{N}$ we have

$$\sum_{n=0}^{\infty} N(a_1, \ldots, a_8; n) q^n = \varphi(q^{a_1}) \cdots \varphi(q^{a_8}), \qquad q \in \mathbb{C}, \quad |q| < 1. \tag{2.1}$$

The Eisenstein series $E_k(q)$ is given by

$$E_k(q) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \qquad q \in \mathbb{C}, \quad |q| < 1, \ k \in \mathbb{N}, \tag{2.2}$$

where B_m ($m \in \mathbb{N}_0$) is the m-th Bernoulli number given by

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!}, \qquad x \in \mathbb{R}, \quad |x| < 2\pi,$$

so that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30, \ldots$. For our purposes we require

$$M(q) := E_2(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \qquad q \in \mathbb{C}, \quad |q| < 1. \tag{2.3}$$

As in [1, pp. 32, 33] we define

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \tag{2.4}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{2.5}$$

From (2.4) and (2.5) we deduce

$$\varphi(q) = (1+2p)^{3/4} k^{1/2} \tag{2.6}$$

and

$$\varphi(q^3) = (1+2p)^{1/4} k^{1/2}. \tag{2.7}$$

It was shown in [1, eqs. (3.14)-(3.19), p. 34] that

$$\begin{aligned} M(q) &= (1 + 124p + 964p^2 + 2788p^3 + 3910p^4 \\ &\quad + 2788p^5 + 964p^6 + 124p^7 + p^8)k^4, \end{aligned} \tag{2.8}$$

$$\begin{aligned} M(q^2) &= (1 + 4p + 64p^2 + 178p^3 + 235p^4 \\ &\quad + 178p^5 + 64p^6 + 4p^7 + p^8)k^4, \end{aligned} \tag{2.9}$$

$$\begin{aligned} M(q^3) &= (1 + 4p + 4p^2 + 28p^3 + 70p^4 \\ &\quad + 28p^5 + 4p^6 + 4p^7 + p^8)k^4, \end{aligned} \tag{2.10}$$

$$\begin{aligned} M(q^4) &= (1 + 4p + 4p^2 - 2p^3 + 10p^4 + 28p^5 \\ &\quad + \frac{31}{4}p^6 - \frac{29}{4}p^7 + \frac{1}{16}p^8)k^4, \end{aligned} \tag{2.11}$$

$$\begin{aligned} M(q^6) &= (1 + 4p + 4p^2 - 2p^3 - 5p^4 \\ &\quad - 2p^5 + 4p^6 + 4p^7 + p^8)k^4, \end{aligned} \tag{2.12}$$

$$\begin{aligned} M(q^{12}) &= (1 + 4p + 4p^2 - 2p^3 - 5p^4 - 2p^5 \\ &\quad + \frac{1}{4}p^6 + \frac{1}{4}p^7 + \frac{1}{16}p^8)k^4. \end{aligned} \tag{2.13}$$

From [1, eqs. (2.1) and (3.28)-(3.33)] it can be deduced that

$$\begin{aligned} &\prod_{n=1}^{\infty}(1-q^n) \\ &\quad = q^{-1/24} 2^{-1/6} p^{1/24} (1+p)^{1/6} (1-p)^{1/2} (1+2p)^{1/8} (2+p)^{1/8} k^{1/2}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} &\prod_{n=1}^{\infty}(1-q^{2n}) \\ &\quad = q^{-1/12} 2^{-1/3} p^{1/12} (1+p)^{1/12} (1-p)^{1/4} (1+2p)^{1/4} (2+p)^{1/4} k^{1/2}, \end{aligned} \tag{2.15}$$

$$\prod_{n=1}^{\infty}(1-q^{3n})$$
$$= q^{-1/8}2^{-1/6}p^{1/8}(1+p)^{1/2}(1-p)^{1/6}(1+2p)^{1/24}(2+p)^{1/24}k^{1/2}, \quad (2.16)$$

$$\prod_{n=1}^{\infty}(1-q^{4n})$$
$$= q^{-1/6}2^{-2/3}p^{1/6}(1+p)^{1/24}(1-p)^{1/8}(1+2p)^{1/8}(2+p)^{1/2}k^{1/2}, \quad (2.17)$$

$$\prod_{n=1}^{\infty}(1-q^{6n})$$
$$= q^{-1/4}2^{-1/3}p^{1/4}(1+p)^{1/4}(1-p)^{1/12}(1+2p)^{1/12}(2+p)^{1/12}k^{1/2}, \quad (2.18)$$

$$\prod_{n=1}^{\infty}(1-q^{12n})$$
$$= q^{-1/2}2^{-2/3}p^{1/2}(1+p)^{1/8}(1-p)^{1/24}(1+2p)^{1/24}(2+p)^{1/6}k^{1/2}. \quad (2.19)$$

From (1.5), (2.14), (2.15), (2.16) and (2.18), we deduce

$$\begin{aligned}\sum_{n=1}^{\infty} b(n)q^n &= \frac{1}{4}p(1+p)^2(1-p)^2(1+2p)(2+p)k^4 \quad (2.20)\\ &= \frac{1}{4}(2p+5p^2-2p^3-10p^4-2p^5+5p^6+2p^7)k^4.\end{aligned}$$

From (1.6), (2.14), (2.15), (2.16) and (2.18), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} c(n)q^n &= \frac{1}{8}p^2(1+p)(1-p)(1+2p)(2+p)k^4 \quad (2.21)\\ &= \frac{1}{8}(2p^2+5p^3-5p^5-2p^6)k^4.\end{aligned}$$

A simple calculation shows that

$$\prod_{n=1}^{\infty}\Big(1-(-q)^n\Big) = \prod_{n=1}^{\infty}\frac{\big(1-q^{2n}\big)^3}{(1-q^n)(1-q^{4n})}. \quad (2.22)$$

Replacing q by $-q$ in (1.5), and appealing to (2.22), we obtain

$$\sum_{n=1}^{\infty} b(n)(-q)^n = -q\prod_{n=1}^{\infty}\frac{\big(1-q^{2n}\big)^8\big(1-q^{6n}\big)^8}{\big(1-q^n\big)^2\big(1-q^{3n}\big)^2\big(1-q^{4n}\big)^2\big(1-q^{12n}\big)^2}. \quad (2.23)$$

Replacing q by $-q$ in (1.6), and appealing to (2.22), we deduce

$$\sum_{n=1}^{\infty} c(n)(-q)^n = q^2 \prod_{n=1}^{\infty} \frac{\left(1-q^{2n}\right)^4\left(1-q^{3n}\right)^3\left(1-q^{12n}\right)^3}{\left(1-q^n\right)\left(1-q^{4n}\right)}. \qquad (2.24)$$

Then, from (2.14)-(2.19) and (2.23), we deduce

$$\begin{aligned}\sum_{n=1}^{\infty} b(n)(-q)^n &= -\frac{1}{4}p(1+p)(1-p)(1+2p)^2(2+p)k^4 \qquad (2.25)\\ &= -\frac{1}{4}(2p+9p^2+10p^3-5p^4-12p^5-4p^6)k^4,\end{aligned}$$

and, from (2.14)-(2.19) and (2.24), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} c(n)(-q)^n &= \frac{1}{8}p^2(1+p)^2(1-p)(1+2p)(2+p)k^4 \qquad (2.26)\\ &= \frac{1}{8}(2p^2+7p^3+5p^4-5p^5-7p^6-27p^7)k^4.\end{aligned}$$

We are now in a position to prove Theorem 1.1.

3. Proof of Theorem 1.1

(i) We have

$$\sum_{n=0}^{\infty} N(1^2 3^6; n)q^n \overset{(2.1)}{=} \varphi^2(q)\varphi^6(q^3) \overset{(2.6)\text{ and }(2.7)}{=} (1+2p)^3k^4.$$

Then by (2.8), (2.10), (2.11), (2.13), (2.20), (2.21) and (2.26)

$$\begin{aligned}\sum_{n=0}^{\infty} N(1^2 3^6; n)q^n &= \frac{1}{300}M(q) - \frac{7}{100}M(q^3) - \frac{4}{75}M(q^4) + \frac{28}{25}M(q^{12})\\ &\quad + \frac{16}{5}\sum_{n=1}^{\infty} b(n)q^n - \frac{32}{5}\sum_{n=1}^{\infty} c(n)q^n + \frac{48}{5}\sum_{n=1}^{\infty} c(n)(-q)^n\\ &\overset{(2.3)}{=} 1 + \sum_{n=1}^{\infty}\Big(\frac{4}{5}\sigma_3(n) - \frac{84}{5}\sigma_3(n/3) - \frac{64}{5}\sigma_3(n/4) + \frac{1344}{5}\sigma_3(n/12)\\ &\quad + \frac{16}{5}b(n) - \frac{32}{5}c(n) + \frac{48}{5}(-1)^n c(n)\Big)q^n\end{aligned}$$

Equating coefficients of q^n $(n \in \mathbb{N})$ in the above equation, we obtain formula (i) of Theorem 1.1.

Table 2 illustrates part (i) of Theorem 1.1. We set

$$\begin{aligned}F_1(n) := \frac{4}{5}\sigma_3(n) - \frac{84}{5}\sigma_3(n/3) - \frac{64}{5}\sigma_3(n/4) + \frac{1344}{5}\sigma_3(n/12)\\ + \frac{16}{5}b(n) - \frac{32}{5}c(n) + \frac{48}{5}(-1)^n c(n).\end{aligned}$$

Table 2

n	$N(1^2 3^6; n)$	$\sigma_3(n)$	$\sigma_3(\frac{n}{3})$	$\sigma_3(\frac{n}{4})$	$\sigma_3(\frac{n}{12})$	$b(n)$	$c(n)$	$(-1)^n c(n)$	$F_1(n)$
1	4	1	0	0	0	1	0	0	4
2	4	9	0	0	0	−2	1	1	4
3	12	28	1	0	0	−3	−1	1	12
4	52	73	0	1	0	4	−2	−2	52
5	56	126	0	0	0	6	4	−4	56
6	60	252	9	0	0	6	−3	−3	60
7	288	344	0	0	0	−16	−4	4	288
8	340	585	0	9	0	−8	4	4	340
9	164	757	28	0	0	9	0	0	164
10	888	1134	0	0	0	−12	6	6	888
11	1168	1332	0	0	0	12	−4	4	1168
12	300	2044	73	28	1	−12	6	6	300

(ii) We have

$$\sum_{n=0}^{\infty} N(1^4 3^4; n) q^n \overset{(2.1)}{=} \varphi^4(q)\varphi^4(q^3) \overset{(2.6)\text{ and }(2.7)}{=} (1+2p)^4 k^4$$
$$\overset{(2.8)-(2.13)\text{ and }(2.25)}{=} \frac{1}{150}M(q) - \frac{1}{75}M(q^2) + \frac{3}{50}M(q^3) + \frac{8}{75}M(q^4) - \frac{3}{25}M(q^6) + \frac{24}{25}M(q^{12}) - \frac{32}{5}\sum_{n=1}^{\infty} b(n)(-q)^n$$
$$\overset{(2.3)}{=} 1 + \sum_{n=1}^{\infty}\Big(\frac{8}{5}\sigma_3(n) - \frac{16}{5}\sigma_3(n/2) + \frac{72}{5}\sigma_3(n/3) + \frac{128}{5}\sigma_3(n/4) - \frac{144}{5}\sigma_3(n/6) + \frac{1152}{5}\sigma_3(n/12) + \frac{32}{5}(-1)^{n-1}b(n)\Big)q^n.$$

Equating coefficients of q^n $(n \in \mathbb{N})$ in the above equation, we obtain formula (ii) of Theorem 1.1.

Table 3 illustrates part (ii) of Theorem 1.1. We set

$$F_2(n) := \frac{8}{5}\sigma_3(n) - \frac{16}{5}\sigma_3(n/2) + \frac{72}{5}\sigma_3(n/3) + \frac{128}{5}\sigma_3(n/4) - \frac{144}{5}\sigma_3(n/6) + \frac{1152}{5}\sigma_3(n/12) + \frac{32}{5}(-1)^{n-1}b(n).$$

Table 3

n	$N(1^4 3^4;n)$	$\sigma_3(n)$	$\sigma_3(\frac{n}{2})$	$\sigma_3(\frac{n}{3})$	$\sigma_3(\frac{n}{4})$	$\sigma_3(\frac{n}{6})$	$\sigma_3(\frac{n}{12})$	$(-1)^{n-1}b(n)$	$F_2(n)$
1	8	1	0	0	0	0	0	1	8
2	24	9	1	0	0	0	0	2	24
3	40	28	0	1	0	0	0	−3	40
4	88	73	9	0	1	0	0	−4	88
5	240	126	0	0	0	0	0	6	240
6	376	252	28	9	0	1	0	−6	376
7	448	344	0	0	0	0	0	−16	448
8	984	585	73	0	9	0	0	8	984
9	1672	757	0	28	0	0	0	9	1672
10	1488	1134	126	0	0	0	0	12	1488
11	2208	1332	0	0	0	0	0	12	2208
12	420	2044	252	73	28	9	1	12	4280

(iii) We have

$$\sum_{n=0}^{\infty} N(1^6 3^2;n)q^n \overset{(2.1)}{=} \varphi^6(q)\varphi^2(q^3) \overset{(2.6) \text{ and } (2.7)}{=} (1+2p)^5 k^4.$$

Then by (2.8), (2.10), (2.11), (2.13), (2.20), (2.21) and (2.26)

$$\begin{aligned}\sum_{n=0}^{\infty} N(1^6 3^2;n)q^n &= \frac{7}{300}M(q) - \frac{9}{100}M(q^3) - \frac{28}{75}M(q^4) + \frac{36}{25}M(q^{12}) \\ &\quad + \frac{32}{5}\sum_{n=1}^{\infty} b(n)q^n - \frac{64}{5}\sum_{n=1}^{\infty} c(n)q^n + \frac{176}{5}\sum_{n=1}^{\infty} c(n)(-q)^n \\ &\overset{(2.3)}{=} 1 + \sum_{n=1}^{\infty}\Big(\frac{28}{5}\sigma_3(n) - \frac{108}{5}\sigma_3(n/3) - \frac{448}{5}\sigma_3(n/4) \\ &\quad + \frac{1728}{5}\sigma_3(n/12) + \frac{32}{5}b(n) - \frac{64}{5}c(n) + \frac{176}{5}(-1)^n c(n)\Big)q^n.\end{aligned}$$

Equating coefficients of q^n ($n \in \mathbb{N}$) in the above equation, we obtain formula (iii) of Theorem 1.1.

Table 4 illustrates part (iii) of Theorem 1.1. We set

$$\begin{aligned}F_3(n) := \frac{28}{5}\sigma_3(n) - \frac{108}{5}\sigma_3(n/3) - \frac{448}{5}\sigma_3(n/4) + \frac{1728}{5}\sigma_3(n/12) \\ + \frac{32}{5}b(n) - \frac{64}{5}c(n) + \frac{176}{5}(-1)^n c(n).\end{aligned}$$

Table 4

n	$N(1^6 3^2; n)$	$\sigma_3(n)$	$\sigma_3(\frac{n}{3})$	$\sigma_3(\frac{n}{4})$	$\sigma_3(\frac{n}{12})$	$b(n)$	$c(n)$	$(-1)^n c(n)$	$F_3(n)$
1	12	1	0	0	0	1	0	0	12
2	60	9	0	0	0	-2	1	1	60
3	164	28	1	0	0	-3	-1	1	164
4	300	73	0	1	0	4	-2	-2	300
5	552	126	0	0	0	6	4	-4	552
6	1188	252	9	0	0	6	-3	-3	1188
7	2016	344	0	0	0	-16	-4	4	2016
8	2508	585	0	9	0	-8	4	4	2508
9	3692	757	28	0	0	9	0	0	3692
10	6408	1134	0	0	0	-12	6	6	6408
11	7728	1332	0	0	0	12	-4	4	7728
12	7764	2044	73	28	1	-12	6	6	7764

This completes the proof of Theorem 1.1. ■

4. Concluding remarks

From (2.20), (2.21), (2.25) and (2.26), we deduce

$$2\sum_{\substack{n=1\\ n \text{ even}}}^{\infty} (b(n)+2c(n))q^n = \sum_{n=1}^{\infty} b(n)q^n + \sum_{n=1}^{\infty} b(n)(-q)^n + 2\sum_{n=1}^{\infty} c(n)q^n + 2\sum_{n=1}^{\infty} c(n)(-q)^n = 0$$

so that

$$b(n) = -2c(n), \qquad n \in \mathbb{N}, \qquad n \equiv 0 \pmod 2.$$

It is not possible to use the method of this paper to determine formulae for $N(1^1 3^7; n)$, $N(1^3 3^5; n)$, $N(1^5 3^3; n)$ and $N(1^7 3^1; n)$ as $\varphi(q)\varphi^7(q^3)$, $\varphi^3(q)\varphi^5(q^3)$, $\varphi^5(q)\varphi^3(q^3)$ and $\varphi^7(q)\varphi(q^3)$ are respectively $(1+2p)^{5/2}k^4$, $(1+2p)^{7/2}k^4$, $(1+2p)^{9/2}k^4$ and $(1+2p)^{11/2}k^4$ and so are not of the form $f(p)k^4$, where $f(p)$ is a polynomial in p.

References

[1] A. Alaca, Ş. Alaca and K. S. Williams, *Evaluation of the convolution sums* $\sum_{l+12m=n}\sigma(l)\sigma(m)$ *and* $\sum_{3l+4m=n}\sigma(l)\sigma(m)$, Adv. Theoretical Appl. Math. **1** (2006), 27–48.

[2] A. Alaca, Ş. Alaca and K. S. Williams, *The convolution sum* $\sum_{m<n/16}\sigma(m)\sigma(n-16m)$, Canad. Math. Bull. **51** (2008), 3–14.

[3] A. Alaca, Ş. Alaca and K. S. Williams, *Seven octonary quadratic forms,* Acta Arith. **135** (2008), 339–350.

[4] A. Alaca, Ş. Alaca and K. S. Williams, *Fourteen octonary quadratic forms,* Int. J. Number Theory **6** (2010), 37–50.

[5] C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum,* Bornträger, Regiomonti, 1829. (Gesammelte Werke, Erster Band, Chelsea Publishing Co., New York, 1969, pp. 49–239.)

[6] G. Mason, *On a system of elliptic modular forms attached to the large Mathieu group,* Nagoya Math. J. **118** (1990), 177–193.

[7] K. S. Williams, *The convolution sum* $\sum_{m<n/8}\sigma(m)\sigma(n-8m)$, Pacific J. Math. **228** (2006), 387–396.

Address: Centre for Research in Algebra and Number Theory, School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6.

E-mail: salaca@connect.carleton.ca, kwilliam@connect.carleton.ca

Received: 2 April 2010