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Some Infinite Products of Ramanujan Type

Ayşe Alaca, Şaban Alaca, and Kenneth S. Williams

Abstract. In his "lost" notebook, Ramanujan stated two results, which are equivalent to the identities

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^5}{(1-q^{5n})} = 1 - 5 \sum_{n=1}^{\infty} \Big(\sum_{d \mid n} \Big(\frac{5}{d}\Big) d \Big) q^n$$

and

$$q \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)} = \sum_{n=1}^{\infty} \Big(\sum_{d \mid n} \Big(\frac{5}{n/d}\Big) d \Big) q^n.$$

We give several more identities of this type.

1 Introduction

A nonsquare integer Δ is called a *discriminant* if $\Delta \equiv 0$ or $1 \pmod 4$. A discriminant Δ is said to be *fundamental* if the largest integer m such that Δ/m^2 is also a discriminant is $m = 1$. The Legendre–Jacobi–Kronecker symbol corresponding to the discriminant Δ is denoted by $\big(\frac{\Delta}{*}\big)$. Throughout this paper q denotes a complex variable satisfying $|q| < 1$. The expansions of the infinite products $\prod_{n=1}^{\infty} \frac{(1-q^n)^5}{(1-q^{5n})}$ and $q \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)}$ as power series in q, namely,

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^5}{(1-q^{5n})} = 1 - 5 \sum_{n=1}^{\infty} \Big(\sum_{d \mid n} \Big(\frac{5}{d}\Big) d \Big) q^n \tag{1.1}$$

and

$$q \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)} = \sum_{n=1}^{\infty} \Big(\sum_{d \mid n} \Big(\frac{5}{n/d}\Big) d \Big) q^n \tag{1.2}$$

are due to Ramanujan [18, (1.51) and (1.52), p. 354]. Proofs have been given by Bailey [4, 5], Darling [9], Farkas and Kra [11], and Mordell [16]. In this note we give several more infinite products similar to the left-hand sides of (1.1) and (1.2), whose power series expansions are of the form

$$1 + a \sum_{n=1}^{\infty} \Big(\sum_{d \mid n} \Big(\frac{\Delta}{d}\Big) d^b \Big) q^n \quad \text{or} \quad \sum_{n=1}^{\infty} \Big(\sum_{d \mid n} \Big(\frac{\Delta}{n/d}\Big) d^b \Big) q^n, \tag{1.3}$$

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where a is an integer, $b \in \{0, 1, 2\}$, and Δ is a fundamental discriminant, see Theorems 2.2–2.5, 3.2–3.7, and 4.2–4.6. We note that when $b = 0$, we have

$$\sum_{d|n} \Big(\frac{\Delta}{d}\Big) d^b = \sum_{d|n} \Big(\frac{\Delta}{n/d}\Big) d^b.$$

To do this we make use of identities due to Carlitz [8], Bailey [5], and Andrews, Lewis, and Liu [3, Theorem 1] in conjunction with the classical Gauss sum

$$\sum_{\substack{t=1 \\ \gcd(t,|\Delta|)=1}}^{|\Delta|} \Big(\frac{\Delta}{t}\Big) \omega_{|\Delta|}^{dt} = \Big(\frac{\Delta}{d}\Big) \sqrt{\Delta}, \tag{1.4}$$

where $\omega_{|\Delta|} = e^{2\pi i/|\Delta|}$, which is valid for any positive integer d and any fundamental discriminant Δ [15, Theorem 215, p. 221]. The number of terms in the sum on the left hand side of (1.4) is $\phi(|\Delta|)$, where ϕ is Euler's phi function. We recall that $\phi(n) = 2$ if and only if $n = 3, 4, 6$, $\phi(n) = 4$ if and only if $n = 5, 8, 10, 12$, and $\phi(n) = 8$ if and only if $n = 15, 16, 20, 24, 30$.

2 Carlitz's Formula

The following formula is due to Carlitz [8, (1.3), p. 168], who derived it from a well-known formula in the theory of elliptic functions for the derivative of the Weierstrass $\wp$-function. An elementary proof has been given by Dobbie [10].

Theorem 2.1 *Let a be a complex number such that $a \neq 0$, $a \neq -1$, and $a \neq q^n$ for any integer n. Then*

$$\prod_{n=1}^{\infty} \frac{(1 - a^2q^n)(1 - a^{-2}q^n)(1 - q^n)^6}{(1 - aq^n)^4(1 - a^{-1}q^n)^4} = 1 + \frac{(1 - a)^3}{a(1 + a)} \sum_{n=1}^{\infty} \Big(\sum_{d|n} (a^d - a^{-d}) d^2 \Big) q^n.$$

We wish to choose a to be a $|\Delta|$-th root of unity in such a way that $a^d - a^{-d}$ is a Gaussian sum (1.4) for a suitable fundamental discriminant Δ. Clearly we must have $\phi(|\Delta|) = 2$ so that $\Delta = -3$ or -4, as $3, 4, 6$, and -6 are not discriminants.

With $\Delta = -3$, we can choose $a = \omega_3$ so that by (1.4)

$$a^d - a^{-d} = \Big(\frac{-3}{1}\Big) \omega_3^d + \Big(\frac{-3}{2}\Big) \omega_3^{2d} = \Big(\frac{-3}{d}\Big) \sqrt{-3}.$$

As $\frac{(1-a)^3}{a(1+a)} = 3\sqrt{-3}$, Theorem 2.1 gives the following identity, see [8, (3.1), p. 170].

Theorem 2.2

$$\prod_{n=1}^{\infty} \frac{(1 - q^n)^9}{(1 - q^{3n})^3} = 1 - 9 \sum_{n=1}^{\infty} \Big(\sum_{d|n} \Big(\frac{-3}{d}\Big) d^2 \Big) q^n.$$

Guided by this choice, if we replace q by q^3 in Theorem 2.1 and take $a = q$, we obtain the following result after a little simplification, see [8, (2.1), p. 169 ($x^n + x^{-n}$ should be replaced by $x^n - x^{-n}$)].

Theorem 2.3

$$q\prod_{n=1}^{\infty}\frac{(1-q^{3n})^9}{(1-q^n)^3} = \sum_{n=1}^{\infty}\Big(\sum_{d|n}\Big(\frac{-3}{n/d}\Big)d^2\Big)q^n.$$

With $\Delta = -4$ we can choose $a = \omega_4$ so that by (1.4)

$$a^d - a^{-d} = \Big(\frac{-4}{1}\Big)\omega_4^d + \Big(\frac{-4}{3}\Big)\omega_4^{3d} = \Big(\frac{-4}{d}\Big)\sqrt{-4}.$$

As $\frac{(1-a)^3}{a(1+a)} = \sqrt{-4}$, Theorem 2.1 gives the following result, see [8, (4.3), p. 170].

Theorem 2.4

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^4(1-q^{2n})^6}{(1-q^{4n})^4} = 1 - 4\sum_{n=1}^{\infty}\Big(\sum_{d|n}\Big(\frac{-4}{d}\Big)d^2\Big)q^n.$$

Again, guided by this choice, we replace q by q^4 in Theorem 2.1 and take $a = q$. After a little simplification, we obtain the following result, see [8, (4.1), p. 170].

Theorem 2.5

$$q\prod_{n=1}^{\infty}\frac{(1-q^{2n})^6(1-q^{4n})^4}{(1-q^n)^4} = \sum_{n=1}^{\infty}\Big(\sum_{d|n}\Big(\frac{-4}{n/d}\Big)d^2\Big)q^n.$$

3 Bailey's Formula

The following formula is implicit in the work of Bailey [5, (4) and (5)], who obtained it from a formula for the difference of two values of the Weierstrass $\wp$-function. An elementary proof has been given by Dobbie [10].

Theorem 3.1 *Let a and b be complex numbers such that $a \neq 0$, $b \neq 0$, $a \neq b$, $ab \neq 1$, $a \neq q^n$ for any integer n and $b \neq q^n$ for any integer n. Then*

$$\prod_{n=1}^{\infty}\frac{(1-abq^n)(1-a^{-1}b^{-1}q^n)(1-ab^{-1}q^n)(1-a^{-1}bq^n)(1-q^n)^4}{(1-aq^n)^2(1-a^{-1}q^n)^2(1-bq^n)^2(1-b^{-1}q^n)^2} =$$

$$1 + \frac{(1-a)^2(1-b)^2}{(a-b)(1-ab)}\sum_{n=1}^{\infty}\Big(\sum_{d|n}(a^d + a^{-d} - b^d - b^{-d})d\Big)q^n.$$

Carlitz [8] noted that if we divide $a^d + a^{-d} - b^d - b^{-d}$ by $a - b$ in Theorem 3.1 and let $b \to a$, we obtain Theorem 2.1.

We wish to choose a and b to be $|\Delta|$-th roots of unity so that $a^d + a^{-d} - b^d - b^{-d}$ is a Gauss sum for an appropriate fundamental discriminant Δ. We must have $\phi(|\Delta|) = 4$ so that $\Delta = 5, 8, 12$, or -8 (as $-5, 10$, and -10 are not discriminants and -12 is not a fundamental discriminant).

With $\Delta = 5$ we can choose $a = \omega_5$ and $b = \omega_5^2$ so that by (1.4)

$$a^d + a^{-d} - b^d - b^{-d} = \left(\frac{5}{1}\right)\omega_5^d + \left(\frac{5}{2}\right)\omega_5^{2d} + \left(\frac{5}{3}\right)\omega_5^{3d} + \left(\frac{5}{4}\right)\omega_5^{4d} = \left(\frac{5}{d}\right)\sqrt{5}.$$

As $\frac{(1-a)^2(1-b)^2}{(a-b)(1-ab)} = -\sqrt{5}$, appealing to Theorem 3.1 we obtain Ramanujan's identity (1.1).

Theorem 3.2

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^5}{(1-q^{5n})} = 1 - 5\sum_{n=1}^{\infty}\Big(\sum_{d|n}\left(\frac{5}{d}\right)d\Big)q^n.$$

Guided by this choice, we replace q by q^5 and choose $a = q$ and $b = q^2$ in Theorem 3.1. After a little simplification we obtain Ramanujan's identity (1.2).

Theorem 3.3

$$q\prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)} = \sum_{n=1}^{\infty}\Big(\sum_{d|n}\left(\frac{5}{n/d}\right)d\Big)q^n.$$

With $\Delta = 8$, we can choose $a = \omega_8$ and $b = \omega_8^3$ so that by (1.4)

$$a^d + a^{-d} - b^d - b^{-d} = \left(\frac{8}{1}\right)\omega_8^d + \left(\frac{8}{3}\right)\omega_8^{3d} + \left(\frac{8}{5}\right)\omega_8^{5d} + \left(\frac{8}{7}\right)\omega_8^{7d} = \left(\frac{8}{d}\right)\sqrt{8}.$$

Then, as

$$\frac{(1-a)^2(1-b)^2}{(a-b)(1-ab)} = -\frac{1}{\sqrt{2}},$$

Theorem 3.1 gives the following result, see ([8, (6.2), p. 172] and [17, (19′), p. 8 (with an obvious misprint corrected)].

Theorem 3.4

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-q^{2n})(1-q^{4n})^3}{(1-q^{8n})^2} = 1 - 2\sum_{n=1}^{\infty}\Big(\sum_{d|n}\left(\frac{8}{d}\right)d\Big)q^n.$$

Guided by this choice, we replace q by q^8 and take $a = q$ and $b = q^3$ in Theorem 3.1. After a little simplification we obtain the following identity.

Theorem 3.5

$$q\prod_{n=1}^{\infty} \frac{(1-q^{2n})^3(1-q^{4n})(1-q^{8n})^2}{(1-q^n)^2} = \sum_{n=1}^{\infty}\Big(\sum_{d|n}\left(\frac{8}{n/d}\right)d\Big)q^n.$$

With $\Delta = 12$, we can choose $a = \omega_{12}$ and $b = \omega_{12}^5$ so that

$$\begin{aligned} a^d + a^{-d} - b^d - b^{-d} &= \left(\frac{12}{1}\right)\omega_{12}^d + \left(\frac{12}{5}\right)\omega_{12}^{5d} + \left(\frac{12}{7}\right)\omega_{12}^{7d} + \left(\frac{12}{11}\right)\omega_{12}^{11d} \\ &= \left(\frac{12}{d}\right)\sqrt{12}. \end{aligned}$$

Then, as

$$\frac{(1-a)^2(1-b)^2}{(a-b)(1-ab)} = -\frac{1}{\sqrt{12}},$$

Theorem 3.1 gives the following result, which was not given in [8].

Theorem 3.6

$$\prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{3n})(1-q^{4n})^2(1-q^{6n})^2}{(1-q^{12n})^2} = 1 - \sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\left(\frac{12}{d}\right)d\Big)q^n.$$

Again, guided by the above choice, we replace q by q^{12} and take $a = q$ and $b = q^5$ in Theorem 3.1. After some simplification we obtain the following identity.

Theorem 3.7

$$q\prod_{n=1}^{\infty} \frac{(1-q^{2n})^2(1-q^{3n})^2(1-q^{4n})(1-q^{12n})}{(1-q^n)^2} = \sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\left(\frac{12}{n/d}\right)d\Big)q^n.$$

We show that Theorem 3.6 also follows from a classical identity due to Petr [17] and a recent identity of the authors [2]. The authors proved the following result in [2], where $\mathbb{N}_0$ denotes the set of nonnegative integers.

Theorem 3.8 *Suppose that $a(k_1,k_2,k_3,k_4,k_5)$ $((k_1,k_2,k_3,k_4,k_5) \in \mathbb{N}_0^5)$ are complex numbers (not all zero and nonzero for only finitely many $(k_1,k_2,k_3,k_4,k_5) \in \mathbb{N}_0^5$) such that*

$$\sum_{(k_1,k_2,k_3,k_4,k_5)\in\mathbb{N}_0^5} a(k_1,k_2,k_3,k_4,k_5)x^{k_1}(1+x)^{k_2}(1-x)^{k_3}(1+2x)^{k_4}(2+x)^{k_5} = 0$$

holds identically in x. Then

$$\begin{aligned} \sum_{(k_1,k_2,k_3,k_4,k_5)\in\mathbb{N}_0^5} & a(k_1,k_2,k_3,k_4,k_5)2^{k_1+k_5}q^{k_1}\prod_{n=1}^{\infty}(1-q^n)^{-k_1-2k_2+2k_3-4k_4-k_5} \\ &\times (1-q^{2n})^{3k_1+3k_2+k_3+10k_4+k_5}(1-q^{3n})^{3k_1+6k_2+2k_3+4k_4+3k_5} \\ &\times (1-q^{4n})^{-2k_1-k_2-k_3-4k_4+2k_5}(1-q^{6n})^{-9k_1-9k_2-7k_3-10k_4-7k_5} \\ &\times (1-q^{12n})^{6k_1+3k_2+3k_3+4k_4+2k_5} = 0. \end{aligned}$$

Choosing in Theorem 3.8

$$a(0,0,0,0,0) = 3, \quad a(0,0,0,1,0) = 1, \quad a(0,0,0,0,1) = -2,$$

and $a(k_1, k_2, k_3, k_4, k_5) = 0$, otherwise, we obtain, after a short calculation using Jacobi's identity

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty}(1-q^n)^{-2}(1-q^{2n})^5(1-q^{4n})^{-2}, \tag{3.1}$$

that

$$\frac{1}{4}\varphi^2(q)\varphi(-q)\varphi(-q^3) + \frac{3}{4}\varphi^3(q^3)\varphi(-q)\varphi(-q^3) = \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{3n})(1-q^{4n})^2(1-q^{6n})^2}{(1-q^{12n})^2}.$$

But Petr [17, (30), p. 15] has shown that the left-hand side of this identity is

$$1 - \sum_{n=1}^{\infty}\Big(\sum_{d \mid n}\Big(\frac{12}{d}\Big) d\Big) q^n$$

and Theorem 3.6 follows. Theorem 3.7 follows in a similar way from [17, (30), p. 15]. Petr's work also gives

$$1 - \sum_{n=1}^{\infty}\Big(\sum_{d \mid n}\Big(\frac{-3}{d}\Big)\Big(\frac{-4}{n/d}\Big) d\Big) q^n \quad \text{and} \quad \sum_{n=1}^{\infty}\Big(\sum_{d \mid n}\Big(\frac{-4}{d}\Big)\Big(\frac{-3}{n/d}\Big) d\Big) q^n,$$

in terms of $\varphi(q)$, see [1, Section 3].

Finally, with $\Delta = -8$, it is easy to check that no choice of a and b as $|\Delta|$-th roots of unity makes $a^d + a^{-d} - b^d - b^{-d}$ into a Gauss sum.

4 The Identity of Andrews, Lewis, and Liu

The following identity was proved recently by Andrews, Lewis, and Liu [3, Theorem 1].

Theorem 4.1 *Let a, b, and c be complex numbers such that $a \neq 0$, $b \neq 0$, $c \neq 0$, $ab \neq 1$, $bc \neq 1$, $ca \neq 1$, $a \neq q^n$ for any integer n, $b \neq q^n$ for any integer n, $c \neq q^n$ for any integer n, and $abc \neq q^n$ for any integer n. Then*

$$\prod_{n=1}^{\infty} \frac{\begin{array}{r}(1-abq^n)(1-a^{-1}b^{-1}q^n)(1-bcq^n)(1-b^{-1}c^{-1}q^n)\\ \times (1-caq^n)(1-c^{-1}a^{-1}q^n)(1-q^n)^2\end{array}}{\begin{array}{r}(1-aq^n)(1-a^{-1}q^n)(1-bq^n)(1-b^{-1}q^n)(1-cq^n)\\ \times (1-c^{-1}q^n)(1-abcq^n)(1-a^{-1}b^{-1}c^{-1}q^n)\end{array}}$$
$$= 1 + \frac{(1-a)(1-b)(1-c)(1-abc)}{(1-ab)(1-bc)(1-ca)}$$
$$\times \sum_{n=1}^{\infty} \left(\sum_{d \mid n} (a^d - a^{-d} + b^d - b^{-d} + c^d - c^{-d} - (abc)^d + (abc)^{-d} \right) q^n.$$

Andrews, Lewis, and Liu [3, Theorems 2 and 3] used their theorem to reprove classical theorems of Jacobi, Dirichlet, Lorenz, and Ramanujan in a uniform manner. They did not deduce any new identities from their result. We deduce three new identities from Theorem 4.1, see Theorems 4.2, 4.3, and 4.4.

Andrews, Lewis, and Liu [3, Lemma 4] noted that the limiting case $c \to 1/a$ of their theorem is Bailey's formula (Theorem 3.1). As both Bailey's formula and Carlitz's formula can be obtained from identities involving the Weierstrass $\wp$-function, it would be interesting to know if there is a property of the $\wp$-function from which the identity of Andrews, Lewis, and Liu can be deduced.

We wish to choose a, b, and c to be $|\Delta|$-th roots of unity so that

$$a^d - a^{-d} + b^d - b^{-d} + c^d - c^{-d} - (abc)^d + (abc)^{-d} \tag{4.1}$$

is a Gauss sum for an appropriate fundamental discriminant Δ. We must have $\phi(|\Delta|) = 8$ so that $\Delta = -15, -20, -24$, or 24 as $-30, 15, 16$, and 30 are not discriminants and -16 and 20 are not fundamental discriminants.

With $\Delta = -15$, we can choose $a = \omega_{15}$, $b = \omega_{15}^2$ and $c = \omega_{15}^4$ so that $abc = \omega_{15}^7$ and by (1.4) we have

$$\begin{aligned} a^d &- a^{-d} + b^d - b^{-d} + c^d - c^{-d} - (abc)^d + (abc)^{-d} \\ &= \left(\frac{-15}{1}\right)\omega_{15}^{d} + \left(\frac{-15}{2}\right)\omega_{15}^{2d} + \left(\frac{-15}{4}\right)\omega_{15}^{4d} + \left(\frac{-15}{7}\right)\omega_{15}^{7d} \\ &\quad + \left(\frac{-15}{8}\right)\omega_{15}^{8d} + \left(\frac{-15}{11}\right)\omega_{15}^{11d} + \left(\frac{-15}{13}\right)\omega_{15}^{13d} + \left(\frac{-15}{14}\right)\omega_{15}^{14d} \\ &= \left(\frac{-15}{d}\right)\sqrt{-15}. \end{aligned}$$

Then, as

$$\frac{(1-a)(1-b)(1-c)(1-abc)}{(1-ab)(1-bc)(1-ca)} = \frac{1}{\sqrt{-15}},$$

we obtain the following result from Theorem 4.1.

Theorem 4.2

$$\prod_{n=1}^{\infty}\frac{(1-q^{3n})^2(1-q^{5n})^2}{(1-q^n)(1-q^{15n})}=1+\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\Big(\frac{-15}{d}\Big)\Big)q^n.$$

With $\Delta=-20$, we can choose $a=\omega_{20}$, $b=\omega_{20}^3$, and $c=\omega_{20}^7$ so that $abc=\omega_{20}^{11}$, and by (1.3) we have

$$\begin{aligned}
&a^d-a^{-d}+b^d-b^{-d}+c^d-c^{-d}-(abc)^d+(abc)^{-d}\\
&\quad=\Big(\frac{-20}{1}\Big)\omega_{20}^{d}+\Big(\frac{-20}{3}\Big)\omega_{20}^{3d}+\Big(\frac{-20}{7}\Big)\omega_{20}^{7d}+\Big(\frac{-20}{9}\Big)\omega_{20}^{9d}\\
&\qquad+\Big(\frac{-20}{11}\Big)\omega_{20}^{11d}+\Big(\frac{-20}{13}\Big)\omega_{20}^{13d}+\Big(\frac{-20}{17}\Big)\omega_{20}^{17d}+\Big(\frac{-20}{19}\Big)\omega_{20}^{19d}\\
&\quad=\Big(\frac{-20}{d}\Big)\sqrt{-20}.
\end{aligned}$$

As

$$\frac{(1-a)(1-b)(1-c)(1-abc)}{(1-ab)(1-bc)(1-ca)}=\frac{1}{\sqrt{-20}},$$

we obtain the following result from Theorem 4.1.

Theorem 4.3

$$\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{4n})(1-q^{5n})(1-q^{10n})}{(1-q^n)(1-q^{20n})}=1+\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\Big(\frac{-20}{d}\Big)\Big)q^n.$$

With $\Delta=-24$, we can choose $a=\omega_{24}$, $b=\omega_{24}^5$, and $c=\omega_{24}^7$ so that $abc=\omega_{24}^{13}$ and by (1.3) we have

$$\begin{aligned}
&a^d-a^{-d}+b^d-b^{-d}+c^d-c^{-d}-(abc)^d+(abc)^{-d}\\
&\quad=\Big(\frac{-24}{1}\Big)\omega_{24}^{d}+\Big(\frac{-24}{5}\Big)\omega_{24}^{5d}+\Big(\frac{-24}{7}\Big)\omega_{24}^{7d}+\Big(\frac{-24}{11}\Big)\omega_{24}^{11d}\\
&\qquad+\Big(\frac{-24}{13}\Big)\omega_{24}^{13d}+\Big(\frac{-24}{17}\Big)\omega_{24}^{17d}+\Big(\frac{-24}{19}\Big)\omega_{24}^{19d}+\Big(\frac{-24}{23}\Big)\omega_{24}^{23d}\\
&\quad=\Big(\frac{-24}{d}\Big)\sqrt{-24}.
\end{aligned}$$

As

$$\frac{(1-a)(1-b)(1-c)(1-abc)}{(1-ab)(1-bc)(1-ca)}=\frac{1}{\sqrt{-24}},$$

we obtain the following identity from Theorem 4.1.

Theorem 4.4

$$\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{3n})(1-q^{8n})(1-q^{12n})}{(1-q^n)(1-q^{24n})}=1+\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\Big(\frac{-24}{d}\Big)\Big)q^n.$$

We note that for $\Delta = 24$ there are no values of a, b, and c as 24-th roots of unity which make $a^d - a^{-d} + b^d - b^{-d} + c^d - c^{-d} - (abc)^d + (abc)^{-d}$ a Gauss sum.

By allowing equalities between a, b, and c, it is possible to make (4.1) a multiple of a Gauss sum for certain fundamental discriminants Δ. This occurs for $\Delta = -4$ and $\Delta = -8$.

With $\Delta = -4$, we can choose $a = b = c = \omega_4$ so that $abc = \omega_4^3 = \omega_4^{-1} = a^{-1}$. Then, by (1.4), we have

$$\begin{aligned} a^d - a^{-d} + b^d - b^{-d} + c^d - c^{-d} - (abc)^d + (abc)^{-d} &= 4(a^d - a^{-d}) \\ = 4(\omega_4^d - \omega_4^{3d}) = 4\left(\left(\frac{-4}{1}\right)\omega_4^d + \left(\frac{-4}{3}\right)\omega_4^{3d}\right) &= 4\left(\frac{-4}{d}\right)\sqrt{-4}. \end{aligned}$$

Then, as

$$\frac{(1-a)(1-b)(1-c)(1-abc)}{(1-ab)(1-bc)(1-ca)} = \frac{-i}{2},$$

we obtain the following result from Theorem 4.1.

Theorem 4.5

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^{10}}{(1-q^n)^4(1-q^{4n})^4} = 1 + 4\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-4}{d}\right)\right)q^n.$$

By (3.1), the left‘hand side of Theorem 4.5 is $\varphi^2(q)$. Thus Theorem 4.5 gives the well-known Lambert series expansion

$$\varphi^2(q) = 1 + 4\sum_{n=1}^{\infty}\left(\frac{-4}{n}\right)\frac{q^n}{1-q^n},$$

see for example [6, (3.2.8), p. 58].

With $\Delta = -8$, we can choose $a = b = \omega_8$ and $c = \omega_8^3$ so that $abc = \omega_8^5 = \omega_8^{-3} = c^{-1}$. Then, by (1.4), we have

$$\begin{aligned} & a^d - a^{-d} + b^d - b^{-d} + c^d - c^{-d} - (abc)^d + (abc)^{-d} \\ &= 2(a^d + c^d - c^{-d} - a^{-d}) \\ &= 2(\omega_8^d + \omega_8^{3d} - \omega_8^{5d} - \omega_8^{7d}) \\ &= 2\left(\left(\frac{-8}{1}\right)\omega_8^d + \left(\frac{-8}{3}\right)\omega_8^{3d} + \left(\frac{-8}{5}\right)\omega_8^{5d} + \left(\frac{-8}{7}\right)\omega_8^{7d}\right) \\ &= 2\left(\frac{-8}{d}\right)\sqrt{-8}. \end{aligned}$$

Then, as

$$\frac{(1-a)(1-b)(1-c)(1-abc)}{(1-ab)(1-bc)(1-ca)} = -\frac{1}{4}i\sqrt{2},$$

we obtain the following result from Theorem 4.1.

Theorem 4.6

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^3(1-q^{4n})^3}{(1-q^n)^2(1-q^{8n})^2} = 1 + 2\sum_{n=1}^{\infty}\Big(\sum_{d|n}\Big(\frac{-8}{d}\Big)\Big)q^n.$$

By (3.1), the left-hand side of Theorem 4.6 is $\varphi(q)\varphi(q^2)$. Thus Theorem 4.6 gives the well-known Lambert series expansion

$$\varphi(q)\varphi(q^2) = 1 + 2\sum_{n=1}^{\infty}\Big(\frac{-8}{n}\Big)\frac{q^n}{1-q^n},$$

see for example [6, Theorem 3.7.2, p. 73].

We close this section by relating Theorems 4.2, 4.3, and 4.4 to binary quadratic forms. Suppose $D < 0$ is a fundamental discriminant. Let

$$A = \{a_1x^2 + b_1xy + c_1y^2, \ldots, a_hx^2 + b_hxy + c_hy^2\}$$

be a representative set of inequivalent, primitive, integral, positive-definite, binary quadratic forms of discriminant D. The number of representations of $n \in \mathbb{N}$ by the forms in the set A is given by

$$w(D)\sum_{d|n}\Big(\frac{D}{d}\Big),$$

where $w(D) = 6$, 4, or 2, according as $D = -3$, $D = -4$, or $D < -4$, respectively, see for example [13, p. 294] or [14]. Representative sets of forms for $D = -15, -20$, and -24 are $\{x^2 + xy + 4y^2,\ 2x^2 + xy + 2y^2\}$, $\{x^2 + 5y^2,\ 2x^2 + 2xy + 3y^2\}$, and $\{x^2+6y^2,\ 2x^2+3y^2\}$, respectively. Theorems 4.2, 4.3, and 4.4 then give the identities of our final theorem.

Theorem 4.7

$$\sum_{x,y=-\infty}^{\infty}(q^{x^2+xy+4y^2} + q^{2x^2+xy+2y^2}) = 2\prod_{n=1}^{\infty}\frac{(1-q^{3n})^2(1-q^{5n})^2}{(1-q^n)(1-q^{15n})},$$

$$\sum_{x,y=-\infty}^{\infty}(q^{x^2+5y^2} + q^{2x^2+2xy+3y^2}) = 2\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{4n})(1-q^{5n})(1-q^{10n})}{(1-q^n)(1-q^{20n})},$$

$$\sum_{x,y=-\infty}^{\infty}(q^{x^2+6y^2} + q^{2x^2+3y^2}) = 2\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{3n})(1-q^{8n})(1-q^{12n})}{(1-q^n)(1-q^{24n})}.$$

The left-hand side of the third identity in Theorem 4.7 is $\varphi(q)\varphi(q^6)+\varphi(q^2)\varphi(q^3)$.

Appealing to (3.1), we obtain the identity

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^5(1-q^{12n})^5}{(1-q^n)^2(1-q^{4n})^2(1-q^{6n})^2(1-q^{24n})^2} + \prod_{n=1}^{\infty} \frac{(1-q^{4n})^5(1-q^{6n})^5}{(1-q^{2n})^2(1-q^{3n})^2(1-q^{8n})^2(1-q^{12n})^2} = 2\prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{3n})(1-q^{8n})(1-q^{12n})}{(1-q^n)(1-q^{24n})}.$$

5 Conclusion

The negative fundamental discriminants are $-3, -4, -7, -8, \ldots$. In view of Theorems 2.2 and 2.4 it is natural to ask if there is an identity of the form

$$\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{7n})^b = 1 + c\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\Big(\frac{-7}{d}\Big)d^2\Big)q^n \tag{5.1}$$

for some integers a, b and $c \neq 0$. Equating the coefficients of q, q^2 and q^3, we obtain

$$-a = c, \quad \frac{a(a-1)}{2} - a = 5c,$$
$$-\frac{a(a-1)(a-2)}{6} + a^2 - a = -8c.$$

As $c \neq 0$ the first two equations give $(a, c) = (-7, 7)$, which do not satisfy the third equation. Hence no such identity of the form (5.1) exists. Similarly there are no integers a and b such that

$$q\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{7n})^b = \sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\Big(\frac{-7}{n/d}\Big)d^2\Big)q^n.$$

The positive fundamental discriminants are $5, 8, 12, 13, \ldots$. Theorems 3.2, 3.4 and 3.6 give identities involving the discriminants 5, 8 and 12. Thus one can ask if there is a similar identity for discriminant 13, that is, are there integers a, b and $c \neq 0$ such that

$$\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{13n})^b = 1 + c\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\Big(\frac{13}{d}\Big)d\Big)q^n? \tag{5.2}$$

Again it is easy to check that no such identity of the form (5.2) exists. Similarly there are no integers a and b such that

$$q\prod_{n=1}^{\infty}(1-q^n)^a(1-q^{13n})^b = \sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\Big(\frac{13}{n/d}\Big)d\Big)q^n.$$

In view of Theorems 4.5 and 4.6 it is natural to ask about the sum

$$\sum_{n=1}^{\infty}\Big(\sum_{d|n}\Big(\frac{-3}{d}\Big)\Big)q^n.$$

In this case we have from the work of Borwein, Borwein, and Garvan [7, Proposition 2.2, (2.21) and (2.1)] that

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^3}{1-q^{3n}}+9q\prod_{n=1}^{\infty}\frac{(1-q^{9n})^3}{1-q^{3n}}=1+6\sum_{n=1}^{\infty}\Big(\sum_{d|n}\Big(\frac{-3}{d}\Big)\Big)q^n. \tag{5.3}$$

Formula (5.3) is implicit in the work of Ramanujan [18, (1.41), p. 353 and (1.42), p. 354]. Although we have obtained a number of formulae of Ramanujan type in a uniform manner, clearly much still remains to be discovered.

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Centre for Research in Algebra and Number Theory, School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6

e-mail: aalaca@math.carleton.ca
salaca@math.carleton.ca
kwilliam@connect.carleton.ca