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The Simplest D_4-octics

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Abstract

We show that $\{x^8 + (t^2 + 2)x^4 + 1 \mid t \in \mathbb{N}\}$ is the "simplest" infinite parametric family of octic polynomials having the dihedral group of order 8 as their Galois group. We also determine the discriminant of the octic field $\mathbb{Q}(\theta)$, where θ is a root of $x^8 + (t^2 + 2)x^4 + 1$. Under the assumption that $t^2 + 4$ is squarefree we give an integral basis for $\mathbb{Q}(\theta)$.

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1 Introduction

Let D_4 denote the dihedral group of order 8. G. W. Smith [5, p. 791] has given a parametric family of octic polynomials having D_4 as their Galois group. This family comprises the irreducible polynomials of the form

$$x^8 - 10(t^2 - 4)x^6 + (33t^4 - 208t^2 + 472)x^4$$

$$-40(t^2 - 4)(t^4 - t^2 + 9)x^2 + 16(t^4 + 17t^2 - 9)^2,$$

where $t \in \mathbb{Q}$. It is our purpose to present a simpler family of octic polynomials having D_4 as Galois group. We show that if $x^8 + ax^4 + 1$ is irreducible over

$\mathbb{Q}$ for some rational a then its Galois group is D_4 if and only if $a = t^2 + 2$ for some $t \in \mathbb{Q}$. Moreover, if $t \in \mathbb{N}$, we show that $x^8 + (t^2+2)x^4 + 1$ is irreducible over $\mathbb{Q}$ and thus $\{x^8 + (t^2+2)x^4 + 1 \mid t \in \mathbb{N}\}$ is an infinite parametric family of D_4-octic polynomials. For $t \in N$ let $\theta \in \mathbb{C}$ be a root of $x^8 + (t^2+2)x^4 + 1$ and set $K = \mathbb{Q}(\theta)$ so that $[K : \mathbb{Q}] = 8$. We give an explicit formula for the discriminant $d(K)$ of K in Theorem 3.1. Further, when $t^2 + 4$ $(t \in \mathbb{N})$ is squarefree, we give an explicit integral basis for K in Theorem 3.2. Finally we show in a simple manner that an irreducible polynomial $x^8 + a$ $(a \in \mathbb{Q})$ cannot have D_4 as Galois group, thus justifying that the octic polynomials $x^8 + (t^2+2)x^4 + 1$ are the simplest ones with Galois group D_4.

2 The family $x^8 + (t^2+2)x^4 + 1$

In this section we determine those irreducible octic polynomials $x^8 + ax^4 + 1$ $(a \in \mathbb{Q})$ having D_4 as Galois group.

Theorem 2.1. *Let $a \in \mathbb{Q}$ be such that x^8+ax^4+1 is irreducible over $\mathbb{Q}$. Then*

$$\mathrm{Gal}(x^8 + ax^4 + 1) \simeq D_4 \Longleftrightarrow a = t^2 + 2 \text{ for some } t \in \mathbb{Q}.$$

Proof. Throughout this proof we assume that $a \in \mathbb{Q}$ is such that $x^8 + ax^4 + 1$ is irreducible over $\mathbb{Q}$.

Suppose first that $a = t^2 + 2$ for some $t \in \mathbb{Q}$. If $t = 0$ then $a = 2$ so $x^8 + ax^4 + 1 = (x^4+1)^2$ is reducible, a contradiction. Hence $t \neq 0$. Let $\theta \in \mathbb{C}$ be a root of $x^8 + (t^2+2)x^4 + 1 = 0$. Clearly $\theta \neq 0$. Then

$$\left(\frac{\theta^4+1}{t\theta^2}\right)^2 = -1 \tag{2.1}$$

so that $i \in \mathbb{Q}(\theta)$. The eight roots of $x^8 + (t^2+2)x^4 + 1$ are

$$\pm\theta,\ \pm i\theta,\ \pm\frac{1}{\theta},\ \pm\frac{i}{\theta}.$$

Hence $\mathbb{Q}(\theta)$ is a normal extension of $\mathbb{Q}$. From (2.1) we obtain

$$\frac{\theta^4+1}{t\theta^2} = \varepsilon i$$

for some $\varepsilon \in \{-1, +1\}$. Replacing θ by $i\theta$, if necessary, we may suppose that $\varepsilon = 1$. Then

$$-2 + ti = -2 + \frac{\theta^4+1}{\theta^2} = \left(\theta - \frac{1}{\theta}\right)^2$$

so that

$$\sqrt{-2+ti} \in \mathbb{Q}(\theta).$$

Next we prove that

$$t^2+4 \notin \mathbb{Q}^2.$$

Suppose on the contrary that $t^2+4=u^2$ for some $u \in \mathbb{Q}$. Then there exists $v \in \mathbb{Q}^*$ such that

$$t = \frac{v}{2} - \frac{2}{v},\ u = \frac{v}{2} + \frac{2}{v}.$$

Hence

$$a = t^2+2 = \frac{v^2}{4} + \frac{4}{v^2},$$

and so

$$x^8+ax^4+1 = x^8 + \left(\frac{v^2}{4}+\frac{4}{v^2}\right)x^4+1 = \left(x^4+\frac{v^2}{4}\right)\left(x^4+\frac{4}{v^2}\right),$$

contradicting that x^8+ax^4+1 is irreducible over $\mathbb{Q}$. As $(-2+ti)(-2-ti) = t^2+4 \notin \mathbb{Q}^2$ we see that $(-2+ti) \neq (x+yi)^2$ for any $x,y \in \mathbb{Q}$. Thus $\big[\mathbb{Q}(\sqrt{-2+ti}) : \mathbb{Q}(i)\big] = 2$ and so $\big[\mathbb{Q}(\sqrt{-2+ti}) : \mathbb{Q}\big] = 4$. By [3, Theorem 3, p. 135] the quartic field $\mathbb{Q}(\sqrt{-2+ti})$ is a dihedral extension of $\mathbb{Q}$. By Galois theory the normal closure of $\mathbb{Q}(\sqrt{-2+ti})$ is of degree 8 over $\mathbb{Q}$. However $\mathbb{Q}(\theta)$ is a normal extension of degree 8 over $\mathbb{Q}$ containing $\mathbb{Q}(\sqrt{-2+ti})$. Thus $\mathbb{Q}(\theta)$ is the normal closure of $\mathbb{Q}(\sqrt{-2+ti})$ and thus is dihedral with Galois group D_4. Hence $\mathrm{Gal}(x^8+ax^4+1) \simeq D_4$.

Now suppose that $\mathrm{Gal}(x^8+ax^4+1) \simeq D_4$. Let $\theta \in \mathbb{C}$ be a root of x^8+ax^4+1. Then the eight roots of x^8+ax^4+1 are

$$\pm\theta,\ \pm i\theta,\ \pm\frac{1}{\theta},\ \pm\frac{i}{\theta}.$$

Thus the normal closure of $\mathbb{Q}(\theta)$ contains i. As x^8+ax^4+1 is irreducible over $\mathbb{Q}$ so are x^2+ax+1 and x^4+ax^2+1. As x^2+ax+1 is irreducible over $\mathbb{Q}$ we must have $a^2-4 \notin \mathbb{Q}$. Hence $\mathbb{Q}(\sqrt{a^2-4})$ is a quadratic subfield of the splitting field L of x^4+ax^2+1. The discriminant of x^4+ax^2+1 is $2^4(a^2-4)^2$, which is a square in $\mathbb{Q}$. Thus $\mathrm{Gal}(x^4+ax^2+1)$ is either $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or A_4. The latter cannot occur [3, Theorem 3, p. 135] so $\mathrm{Gal}(x^4+ax^2+1) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The four roots of x^4+ax^2+1 are

$$\pm\frac{1}{2}\sqrt{-2a \pm 2\sqrt{a^2-4}}.$$

As

$$2-a=\frac{1}{4}\left(\sqrt{-2a+2\sqrt{a^2-4}}+\varepsilon\sqrt{-2a-2\sqrt{a^2-4}}\right)^2$$

for some $\varepsilon \in \{-1,+1\}$ then

$$-2-a=\frac{1}{4}\left(\sqrt{-2a+2\sqrt{a^2-4}}-\varepsilon\sqrt{-2a-2\sqrt{a^2-4}}\right)^2,$$

so that

$$\sqrt{2-a},\ \sqrt{-2-a}\in L.$$

Since $(2-a)(-2-a)=a^2-4\notin\mathbb{Q}^2$ at least one of $2-a$ and $-2-a\notin\mathbb{Q}^2$. If $2-a\notin\mathbb{Q}^2$ then $\mathbb{Q}\left(\sqrt{2-a}\right)$ is a quadratic subfield of L. Suppose that $\mathbb{Q}\left(\sqrt{a^2-4}\right)=\mathbb{Q}\left(\sqrt{2-a}\right)$. Then there exists $b\in\mathbb{Q}^*$ such that $a^2-4=b^2(2-a)$. As $a^2-4\notin\mathbb{Q}^2$ we have $a-2\neq 0$ and so $a+2=-b^2$. Thus $a=-\left(b^2+2\right)$. Hence

$$x^4+ax^2+1=x^4-(b^2+2)x^2+1=x^4-2x^2+1-b^2x^2$$

$$=(x^2-1)^2-b^2x^2=(x^2-bx-1)(x^2+bx-1),$$

contradicting that x^4+ax^2+1 is irreducible over $\mathbb{Q}$. Thus $\mathbb{Q}\left(\sqrt{a^2-4}\right)\neq\mathbb{Q}\left(\sqrt{2-a}\right)$. Since $(a^2-4)(2-a)=(2-a)^2(-2-a)$ it is easy to check that $\mathbb{Q}\left(\sqrt{-2-a}\right)$ is the third quadratic subfield of L. The argument is similar if $-2-a\notin\mathbb{Q}^2$. As $\mathrm{Gal}(x^8+ax^4+1)\simeq D_4$, the splitting field M of x^8+ax^4+1 contains exactly three quadratic subfields. These must be $\mathbb{Q}\left(\sqrt{a^2-4}\right)$, $\mathbb{Q}\left(\sqrt{2-a}\right)$ and $\mathbb{Q}\left(\sqrt{-2-a}\right)$. However $\mathbb{Q}\left(\sqrt{-1}\right)\subseteq M$. Thus (i) $a^2-4=-b^2$ or (ii) $2-a=-b^2$ or (iii) $-2-a=-b^2$ for some $b\in\mathbb{Q}^*$. First we show that (i) cannot occur. If $a^2-4=-b^2$ then $a^2+b^2=4$ and so

$$a=\frac{4v}{v^2+1},\quad b=\frac{2(v^2-1)}{v^2+1}$$

for some $v\in\mathbb{Q}^*$. If $2(v^2+1)\in\mathbb{Q}^2$ then

$$v=\frac{2t^2-4t+1}{2t^2-1}$$

for some $t\in\mathbb{Q}$. In this case

$$x^8+ax^4+1 \quad = \quad x^8+\frac{4v}{v^2+1}x^4+1$$

$$
\begin{aligned}
&= x^8 + \frac{2(2t^2-1)(2t^2-4t+1)}{(2t^2-2t+1)^2}x^4+1 \\
&= \left(x^4 - \frac{(4t-2)}{(2t^2-2t+1)}x^2+1\right)\left(x^4 + \frac{(4t-2)}{(2t^2-2t+1)^2}x^2+1\right)
\end{aligned}
$$

is reducible over $\mathbb{Q}$. Hence $2(v^2+1) \notin \mathbb{Q}^2$. In particular we have $v \neq \pm 1$. Let $\theta_1, \ldots, \theta_8$ be the eight complex roots of $f(x) = x^8 + \frac{4v}{v^2+1}x^4 + 1$. Set

$$
g(x) := \prod_{i,j=1}^{8} (x - (\theta_i + \theta_j)) \in \mathbb{Q}[x].
$$

Using MAPLE we can calculate $g(x)$ by means of

$$
g(x) = \mathrm{Resultant}\left(f(x-X), f(X)\right).
$$

One factor of $g(x)$ is found to be

$$
h(x) := x^4 - 4x^2 + \frac{2(v+1)^2}{v^2+1}.
$$

If $h(x)$ is reducible over $\mathbb{Q}$ then

$$
h(x) = (x^2 + ax + b)(x^2 + cx + d)
$$

for some $a, b, c, d \in \mathbb{Q}$. Thus

$$
\begin{aligned}
a + c &= 0, \\
ac + b + d &= -4, \\
ad + bc &= 0, \\
bd &= \frac{2(v+1)^2}{v^2+1}.
\end{aligned}
$$

Clearly $c = -a$ so that

$$
a(d-b) = 0.
$$

If $a = 0$ then $c = 0$ and

$$
b + d = -4, \quad bd = \frac{2(v+1)^2}{v^2+1}.
$$

Thus

$$
(b-d)^2 = (b+d)^2 - 4bd = 16 - \frac{8(v+1)^2}{v^2+1} = \frac{8(v-1)^2}{v^2+1},
$$

so that $2(v^2+1) \in \mathbb{Q}^2$, a contradiction. Hence $a \neq 0$ so $b = d$. Thus

$$b^2 = \frac{2(v+1)^2}{v^2+1}$$

so $2(v^2+1) \in \mathbb{Q}^2$, a contradiction. Thus $h(x)$ is irreducible over $\mathbb{Q}$. Let α be one of the four roots of $h(x)$. Set $E = \mathbb{Q}(\alpha)$. Then $[E : \mathbb{Q}] = 4$. All four roots of $h(x)$ are

$$\pm\alpha,\ \pm\left(\frac{(v^2+1)}{(v-1)}\alpha^3 - \frac{(3v^2-2v+3)}{(v^2-1)}\alpha\right).$$

Thus E is a normal extension of $\mathbb{Q}$. The discriminant of $h(x)$ is

$$\operatorname{disc}(h) = \frac{2^{11}(v-1)^4(v+1)^2}{(v^2+1)^3}.$$

As $2(v^2+1) \notin \mathbb{Q}^2$ we see that $\operatorname{disc}(h) \notin \mathbb{Q}^2$. Thus E is not a bicyclic extension of $\mathbb{Q}$. Hence E is a cyclic quartic extension of $\mathbb{Q}$. Such an extension cannot be a subfield of an octic field with Galois group D_4 [1, p. 291]. Thus $a^2 - 4 \neq -b^2$ showing that (i) does not occur. Suppose now that (ii) occurs. If $-2-a = -b^2$ then $a = b^2 - 2$ and we set

$$h_1(x) = x^4 + (b+2)^2$$

and

$$h_2(x) = x^4 + (b-2)^2.$$

We note that $b \neq \pm 2$ as $a \neq 2$. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of h_1 and $\beta_1, \beta_2, \beta_3, \beta_4$ the roots of h_2. Then the polynomial with roots $(\alpha_1 - \beta_1), (\alpha_1 - \beta_2), \ldots, (\alpha_4 - \beta_4)$ is found using MAPLE to be

$$\left(x^8 + (64 - 8b^2)x^4 + 16b^4\right)\left(x^8 + (16b^2 - 32)x^4 + 256\right).$$

As θ is a root of $x^8 + (b^2-2)x^4 + 1$, 2θ is a root of $x^8 + (16b^2 - 32)x^4 + 256$, and so $2\theta = \alpha_i - \beta_j$ for some $i, j \in \{1, 2, 3, 4\}$. Thus $\theta = \frac{\alpha_i}{2} - \frac{\beta_j}{2}$ belongs to the compositum of the splitting fields of h_1 and h_2. Hence

$$\theta \in \mathbb{Q}\left(\sqrt{2(b+2)}, \sqrt{-2(b+2)}, \sqrt{2(b-2)}, \sqrt{-2(b-2)}\right).$$

Thus θ belongs to an abelian extension of $\mathbb{Q}$, contradicting that $\mathrm{Gal}(f)$ is D_4. Hence (ii) does not occur. This leaves only the possibility (iii) $2 - a = -b^2$, that is $a = b^2 + 2$ as required.

This completes the proof of Theorem 2.1. $\square$

Next we show that $x^8+(t^2+2)x^4+1$ is irreducible over $\mathbb{Q}$ for $t \in \mathbb{N}$.

Theorem 2.2. *If $t \in \mathbb{N}$ then $x^8+(t^2+2)x^4+1$ is irreducible over $\mathbb{Q}$.*

Proof. Let θ be a root of $x^8+(t^2+2)x^4+1$. As in the proof of Theorem 2.1 we can choose θ so that

$$\sqrt{-2+ti} \in \mathbb{Q}(\theta).$$

Suppose that

$$-2+ti=(r+si)^2$$

for some $r,s \in \mathbb{Z}$. Then

$$4+t^2=(r^2+s^2)^2.$$

However $4+t^2 \in \mathbb{N}^2$ for $t \in \mathbb{N}$ only for $t=0$, a contradiction. Thus $-2+ti \notin (\mathbb{Z}+\mathbb{Z}i)^2$ so $\mathbb{Q}\left(\sqrt{-2+ti}\right)$ is a dihedral quartic field [3, Theorem 3, p. 135]. $\mathbb{Q}(\theta)$ is a normal extension of $\mathbb{Q}$ so $\mathbb{Q}(\theta)$ contains the normal closure of $\mathbb{Q}\left(\sqrt{-2+ti}\right)$. The field $\mathbb{Q}\left(\sqrt{-2+ti}\right)$ is a dihedral quartic field so $\mathbb{Q}(\theta)$ is an octic field. Thus $x^8+(t^2+2)x^4+1$ is irreducible over $\mathbb{Q}$. $\square$

We observe that there are $t \in \mathbb{Q}^+ \setminus \mathbb{N}$ for which $x^8+(t^2+2)x^4+1$ is reducible as well as $t \in \mathbb{Q}^+ \setminus \mathbb{N}$ for which $x^8+(t^2+2)x^4+1$ is irreducible. For example for $t=3/2$ we have

$$x^8+\left(\left(\frac{3}{2}\right)^2+2\right)x^4+1=x^8+\frac{17}{4}x^4+1=\left(x^4+\frac{1}{4}\right)\left(x^4+4\right),$$

whereas for $t=8/3$ the octic polynomial

$$x^8+\left(\left(\frac{8}{3}\right)^2+2\right)x^4+1=x^8+\frac{82}{9}x^4+1$$

is irreducible over $\mathbb{Q}$.

3 The field $\mathbb{Q}(\theta)$, $\theta^8+(t^2+2)\theta^4+1=0$, $t \in \mathbb{N}$

First we determine the discriminant of the octic field $\mathbb{Q}(\theta)$, where $\theta^8+(t^2+2)\theta^4+1=0$ and $t \in \mathbb{N}$.

Theorem 3.1. *Let $t \in \mathbb{N}$. Let s denote the squarefree part of the positive integer t^2+4. Let θ be a root of $x^8+(t^2+2)x^4+1$. Then*

$$d(\mathbb{Q}(\theta)) = 2^\gamma s^4,$$

where

$$\gamma = \begin{cases} 16, & \text{if } t \equiv 1 \pmod 2, \\ 18, & \text{if } t \equiv 2 \pmod 4, \\ 16, & \text{if } t \equiv 0 \pmod 4. \end{cases}$$

Proof. $\mathbb{Q}(\theta)$ is the normal closure of the dihedral quartic field $\mathbb{Q}(\sqrt{-2+ti})$, we can apply Theorem 1 of [1]. In the notation of [1] we have

$$a=-2, \quad b=t, \quad c \equiv -1 \,(\mathrm{mod}\ 4)$$

so

$$a \equiv 2 \,(\mathrm{mod}\ 4), \quad c \equiv 3 \,(\mathrm{mod}\ 4).$$

Thus the only cases that arise are cases B_1, B_7 and B_8, that is

$$\begin{array}{lll} B_1: & t \equiv 1 \,(\mathrm{mod}\ 2), & \theta = 16, \quad 2 \nmid s, \\ B_7: & t \equiv 2 \,(\mathrm{mod}\ 4), & \theta = 18, \quad 2 \parallel s, \\ B_8: & t \equiv 0 \,(\mathrm{mod}\ 4), & \theta = 12, \quad 2 \nmid s. \end{array}$$

Then, by Theorem 1 of [1], we obtain

$$\gamma = \begin{cases} 16, & \text{if } t \equiv 1 \pmod 2, \\ 18, & \text{if } t \equiv 2 \pmod 4, \\ 16, & \text{if } t \equiv 0 \pmod 4. \end{cases}$$

This completes the proof of Theorem 3.1. □

Next we determine an integral basis for the field $\mathbb{Q}(\theta)$, where $\theta^8+(t^2+2)\theta^4+1=0$, under the assumption that t^2+4 is squarefree.

Theorem 3.2. *Let $t \in \mathbb{N}$ be such that t^2+4 is squarefree. Then an integral basis for $\mathbb{Q}(\theta)$, where $\theta^8+(t^2+2)\theta^4+1=0$, is*

$$\left\{1, \theta, \theta^2, \theta^3, \frac{\theta^4+1}{t}, \frac{\theta(\theta^4+1)}{t}, \frac{\theta^2(\theta^4+1)}{t}, \frac{\theta^3(\theta^4+1)}{t}\right\}.$$

Proof. By Theorem 3.1, as t is odd, we have

$$d(\mathbb{Q}(\theta)) = 2^{16}(t^2+4)^4.$$

As

$$\left(\frac{\theta^4+1}{t}\right)^2 = -\theta^2$$

we have

$$\frac{\theta^4+1}{t} = \pm i\theta \tag{3.1}$$

so that $\frac{\theta^4+1}{t}$ is an integer of $\mathbb{Q}(\theta)$. Finally

$$\begin{aligned} & d\left(1,\theta,\theta^2,\theta^3,\frac{\theta^4+1}{t},\frac{\theta(\theta^4+1)}{t},\frac{\theta^2(\theta^4+1)}{t},\frac{\theta^3(\theta^4+1)}{t}\right) \\ &= \frac{1}{t^8}\, d(1,\theta,\theta^2,\theta^3,\theta^4,\theta^5,\theta^6,\theta^7) \\ &= \frac{1}{t^8} 2^{16} t^8 (t^2+4)^4 = 2^{16}(t^2+4)^4 = d(\mathbb{Q}(\theta)) \end{aligned}$$

so $\left\{1,\theta,\theta^2,\theta^3,\frac{\theta^4+1}{t},\frac{\theta(\theta^4+1)}{t},\frac{\theta^2(\theta^4+1)}{t},\frac{\theta^3(\theta^4+1)}{t}\right\}$ is an integral basis. □

By Nagel's theorem [4] there exist infinitely many $t \in \mathbb{N}$ such that t^2+4 is squarefree.

In view of (3.1)

$$\left\{1,\theta,\theta^2,\theta^3,i,i\theta,i\theta^2,i\theta^3\right\}$$

is also an integral basis for $\mathbb{Q}(\theta)$.

Taking $t=1$ in Theorem 3.2 we see that the octic field $\mathbb{Q}(\theta)$, where $\theta^8 + 3\theta^4 + 1 = 0$, has a power basis.

4 The polynomials $x^8 + a$, $a \in \mathbb{Z}$

Finally we show that the polynomials $x^8 + (t^2+2)x^4 + 1$ are the simplest ones that give rise to octic fields with Galois group D_4 by giving a simple direct proof that the binomials $x^8 + a$ ($a \in \mathbb{Z}$) do not have Galois group D_4. We note that Jacobson and Vélez [2] treat the problem of determining the Galois group of the more general polynomial $x^{2^e} + a$.

Theorem 4.1. *Let $a \in \mathbb{Z}$ be such that the polynomial x^8+a is irreducible over $\mathbb{Q}$. Then*

$$\mathrm{Gal}(x^8+a) \not\simeq D_4.$$

Proof. Suppose $a \in \mathbb{Z}$ is such that x^8+a is irreducible over $\mathbb{Q}$ and $\mathrm{Gal}(x^8+a) \simeq D_4$. Let θ be a root of x^8+a so that $\theta^8+a=0$ and $\mathbb{Q}(\theta)$ is a normal octic extension of $\mathbb{Q}$. Now $\mathrm{disc}(x^8+a) = 2^{24}a^7$ and by [1, Theorem] $d(\mathbb{Q}(\theta)) \in \mathbb{Z}^2$. Thus $a \in \mathbb{Z}^2$, say $a=b^2$ for $b \in \mathbb{N}$. Thus $x^8+a = x^8+b^2$. As the eighth roots of unity belong to $\mathbb{Q}(\theta)$ we have

$$\mathbb{Q}(\theta) \supset \mathbb{Q}(\sqrt{2}, \sqrt{-1}).$$

As $\theta^8+b^2=0$ we have

$$b = \pm\left(\frac{\theta^2(1+i)^2}{\sqrt{2}}\right)^2$$

so that

$$\mathbb{Q}(\sqrt{b}) \subset \mathbb{Q}(\theta), \quad \mathbb{Q}(\sqrt{-b}) \subset \mathbb{Q}(\theta).$$

If $[\mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{b}) : \mathbb{Q}] = 8$ then $\mathbb{Q}(\theta) = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{b})$ and so

$$\mathrm{Gal}(\mathbb{Q}(\theta)) = \mathrm{Gal}\left(\mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{b})\right) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

a contradiction. Thus $\sqrt{b} \in \mathbb{Q}(\sqrt{2}, \sqrt{-1})$ so that b, $2b$, $-b$ or $-2b \in \mathbb{Q}^2$. If $\pm 2b \in \mathbb{Q}^2$ then $b = \pm 2c^2$ $(c \in \mathbb{Q})$ and

$$x^8+a = x^8+b^2 = x^8+4c^4 = \left(x^4-2cx^2+2c^2\right)\left(x^4+2cx^2+2c^2\right),$$

contradicting that x^8+a is irreducible over $\mathbb{Q}$. If $\pm b \in \mathbb{Q}^2$ then $b = \pm c^2$ $(c \in \mathbb{Q})$ so $x^8+a = x^8+c^4$. Then the polynomial with roots $\theta_i+\theta_j$ $(i,j \in \{1,2,\ldots,8\})$ is by MAPLE

$$x^8\left(x^8+256c^4\right)(x^8+16c^4)^2(x^4+4cx^2+2c^4)^2(x^4-4cx^2+2c^2)^2(x^8+12c^2x^4+4c^4)^2.$$

The factor $x^4-4cx^2+2c^2$ of this polynomial has roots

$$\pm\sqrt{2c \pm c\sqrt{2}}.$$

The field $\mathbb{Q}\left(\sqrt{2c+c\sqrt{2}}\right)$ is a cyclic quartic field, which cannot occur as a subfield of $\mathbb{Q}(\theta)$. This completes the proof. $\square$

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