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Nineteen quaternary quadratic forms

by

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1. Introduction. Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers and complex numbers respectively. Throughout this paper q denotes a complex variable such that $|q| < 1$. For $a, b, c, d \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we define

$$(1.1)\qquad N(a,b,c,d;n) = \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = ax^2 + by^2 + cz^2 + dt^2\}.$$

Clearly

$$(1.2)\qquad N(a,b,c,d;0) = 1.$$

In this paper we determine in a uniform manner a formula for $N(a,b,c,d;n)$, valid for all $n \in \mathbb{N}$, for each of the nineteen quaternary quadratic forms given by

$$(1.3)\qquad \begin{aligned}(a,b,c,d) = {} & (1,1,1,1),\ (1,1,1,4),\ (1,1,2,2),\ (1,1,3,3),\ (1,1,3,12),\\ & (1,1,4,4),\ (1,1,6,6),\ (1,1,12,12),\ (1,2,2,4),\ (1,2,3,6),\\ & (1,3,3,4),\ (1,3,4,12),\ (1,4,4,4),\ (1,4,6,6),\ (1,4,12,12),\\ & (2,2,3,3),\ (2,2,3,12),\ (3,3,4,4),\ (3,4,4,12).\end{aligned}$$

We show that for each of the forms listed in (1.3), formulae for $N(a,b,c,d;n)$ ($n \in \mathbb{N}$) can be given in terms of the sum of divisors function

$$(1.4)\qquad \sigma(n) = \begin{cases} \sum_{d \mid n} d, & n \in \mathbb{N},\\ 0, & n \notin \mathbb{N},\end{cases}$$

where d runs through the positive integers dividing n, and the integers $a(n)$, which are defined for $n \in \mathbb{N}_0$ by

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$$(1.5)\qquad \sum_{n=0}^{\infty} a(n)q^n = q\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{4n})(1-q^{6n})(1-q^{12n}),$$

so that $a(n) = 0$ for $n \equiv 0 \pmod 2$. The formulae for $N(a,b,c,d;n)$ for the seven quaternary forms given by

$$(a,b,c,d) = (1,1,1,1),\ (1,1,1,4),\ (1,1,2,2),\ (1,1,3,3),\ (1,1,4,4),\ (1,2,2,4),\ (1,4,4,4)$$

depend only on $\sigma(n)$, whereas those for the remaining twelve forms in (1.3) depend upon $\sigma(n)$ and $a(n)$. Numerical evidence suggests that the nineteen forms listed in (1.3) are the only ones satisfying $1 \le a \le b \le c \le d \le 12$, $\gcd(a,b,c,d) = 1$ and $a,b,c,d \mid 12$ for which $N(a,b,c,d;n)$ can be expressed in terms of $\sigma(n)$ and $a(n)$ for all $n \in \mathbb{N}$.

We remark that the q-series with coefficients $a(n)$ is a cusp form of weight 2. In terms of Dedekind's eta function $\eta(z)$ we have

$$\sum_{n=1}^{\infty} a(n)q^n = \eta(2z)\eta(4z)\eta(6z)\eta(12z),$$

where $q = e^{2\pi iz}$ and $\mathrm{Re}(z) > 0$. It is known that $a(n)$ is a multiplicative function of n [25, p. 4853].

It is also convenient for our purposes to define the integers $b(n)$ $(n \in \mathbb{N}_0)$ by

$$(1.6)\qquad \sum_{n=0}^{\infty} b(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{10}(1-q^{6n})^4}{(1-q^n)^4(1-q^{4n})^4(1-q^{12n})^2},$$

and the integers $c(n)$ $(n \in \mathbb{N}_0)$ by

$$(1.7)\qquad \sum_{n=0}^{\infty} c(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^4(1-q^{6n})^{10}}{(1-q^{3n})^4(1-q^{4n})^2(1-q^{12n})^4}.$$

Jacobi's theta function $\varphi(q)$ is defined by

$$(1.8)\qquad \varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Using Jacobi's infinite product expansions

$$\varphi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2}$$

and

$$\varphi(-q) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^{2n})},$$

we see that

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^{10}(1-q^{6n})^4}{(1-q^n)^4(1-q^{4n})^4(1-q^{12n})^2} = \varphi^2(q)\varphi^2(-q^6),$$

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^{4}(1-q^{6n})^{10}}{(1-q^{3n})^4(1-q^{4n})^2(1-q^{12n})^4} = \varphi^2(-q^2)\varphi^2(q^3),$$

so that for $n \in \mathbb{N}_0$,

$$b(n) = \sum_{\substack{(x,y,z,t)\in\mathbb{Z}^4 \\ x^2+y^2+6z^2+6t^2=n}} (-1)^{z+t}, \qquad c(n) = \sum_{\substack{(x,y,z,t)\in\mathbb{Z}^4 \\ 2x^2+2y^2+3z^2+3t^2=n}} (-1)^{x+y}.$$

Hence $b(n)$ and $c(n)$ can be viewed as "twists" of $N(1,1,6,6;n)$ and $N(2,2,3,3;n)$ respectively.

The values of $a(n)$, $b(n)$ and $c(n)$ for $n \in \{0,1,2,\ldots,20\}$ are given in Table 1.

Table 1

n	$a(n)$	$b(n)$	$c(n)$
0	0	1	1
1	1	4	0
2	0	4	−4
3	−1	0	4
4	0	4	4
5	−2	8	−16
6	0	−4	4
7	0	−16	16
8	0	−12	−12
9	1	4	0
10	0	−8	8
11	4	−32	16
12	0	4	4
13	−2	24	−32
14	0	0	0
15	2	−16	8
16	0	−12	−12
17	2	40	−32
18	0	4	−4
19	−4	−32	48
20	0	24	24

We now describe how the formulae for $N(a,b,c,d;n)$ for the nineteen quaternary quadratic forms $ax^2+by^2+cz^2+dt^2$ listed in (1.3) are proved. From (1.8) we see that

$$\sum_{n=0}^{\infty} N(a,b,c,d;n)q^n = \varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d). \tag{1.9}$$

Following [4, p. 178] we set

$$p = p(q) = \frac{\varphi^2(q)-\varphi^2(q^3)}{2\varphi^2(q^3)}, \tag{1.10}$$

$$k = k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}, \tag{1.11}$$

which, as functions in the variable z, are modular forms of weights 0 and 2. We begin by determining the series expansions of k^2, pk^2, p^2k^2, p^3k^2 and p^4k^2 in powers of q. We prove the following theorem in Section 2.

THEOREM 1.1.

(a) $$k^2 = 1+\sum_{n=1}^{\infty}(-4\sigma(n)+24\sigma(n/2)-4\sigma(n/3)-16\sigma(n/4) - 8\sigma(n/6)-16\sigma(n/12))q^n.$$

(b) $$pk^2 = \sum_{n=1}^{\infty}(2\sigma(n)-12\sigma(n/2)+6\sigma(n/3)+8\sigma(n/4) + 4\sigma(n/6)-8\sigma(n/12))q^n.$$

(c) $$p^2k^2 = \sum_{n=1}^{\infty}(4\sigma(n/2)-8\sigma(n/3)+4\sigma(n/6))q^n.$$

(d) $$p^3k^2 = \sum_{n=1}^{\infty}(8\sigma(n/3)-8\sigma(n/4)-8\sigma(n/6)+8\sigma(n/12))q^n.$$

(e) $$p^4k^2 = \sum_{n=1}^{\infty}(16\sigma(n/4)-32\sigma(n/6)+16\sigma(n/12))q^n.$$

The following corollary follows immediately from Theorem 1.1.

COROLLARY 1.1.

(a) $$(1+2p)^2k^2 = 1+\sum_{n=1}^{\infty}(4\sigma(n)-8\sigma(n/2)-12\sigma(n/3)+16\sigma(n/4) + 24\sigma(n/6)-48\sigma(n/12))q^n.$$

(b) $$(1+2p)^3k^2 = 1+\sum_{n=1}^{\infty}(8\sigma(n)-32\sigma(n/4))q^n.$$

(c) $$(1+2p)(1+p+p^2)k^2 = 1+\sum_{n=1}^{\infty}(2\sigma(n)+6\sigma(n/3)-8\sigma(n/4)-24\sigma(n/12))q^n.$$

(d) $$(1-p)(1+p)(1+2p)k^2 = 1+\sum_{n=1}^{\infty}(-4\sigma(n/2)+16\sigma(n/4)+12\sigma(n/6)-48\sigma(n/12))q^n.$$

Then in Section 3 we determine the series expansions in powers of q of other functions of p and k for which the coefficients of the powers of q can also be expressed in terms of σ. Recall that the Legendre–Jacobi–Kronecker symbol for discriminant -4 is defined for $n \in \mathbb{N}$ by

(1.12) $$\left(\frac{-4}{n}\right) = \begin{cases} (-1)^{(n-1)/2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

THEOREM 1.2.

(a) $$(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 = 1+\sum_{n=1}^{\infty}(-8\sigma(n/2)+48\sigma(n/4)-64\sigma(n/8))q^n.$$

(b) $$(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{1/2}k^2 = 1+\sum_{n=1}^{\infty}(-8\sigma(n/6)+48\sigma(n/12)-64\sigma(n/24))q^n.$$

(c) $$(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2 = 1+\sum_{n=1}^{\infty}\left(4\left(\frac{-4}{n}\right)\sigma(n)-8\sigma(n/4)+48\sigma(n/8)-64\sigma(n/16)\right)q^n.$$

(d) $$(1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2 = 1+\sum_{n=1}^{\infty}\left(-4\left(\frac{-4}{n}\right)\sigma(n)-8\sigma(n/4)+48\sigma(n/8)-64\sigma(n/16)\right)q^n.$$

In Section 4 we determine each of the series $\sum_{n=0}^{\infty} a(n)q^n$, $\sum_{n=0}^{\infty} b(n)q^n$ and $\sum_{n=0}^{\infty} c(n)q^n$ in terms of p and k.

THEOREM 1.3.

(a) $$\frac{1}{4}p(2+p)(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}k^2 = \sum_{n=1}^{\infty} a(n)q^n.$$

(b) $$(1-p)^{1/4}(1+p)^{3/4}(1+2p)^{7/4}k^2 = \sum_{n=0}^{\infty} b(n)q^n.$$

(c) $$(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{5/4}k^2 = \sum_{n=0}^{\infty} c(n)q^n.$$

(d) $$(1-p)^{7/4}(1+p)^{5/4}(1+2p)^{1/4}k^2 = \sum_{n=0}^{\infty} b(n)(-1)^n q^n.$$

(e) $$(1-p)^{5/4}(1+p)^{7/4}(1+2p)^{3/4}k^2 = \sum_{n=0}^{\infty} c(n)(-1)^n q^n.$$

The next theorem shows that each of the quantities

$$p^j(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}k^2, \qquad j \in \{0,1,2\},$$

has a power series expansion in q with coefficients depending only on σ and a. This result is proved in Section 5.

THEOREM 1.4.

(a) $$\begin{aligned}(1-p)^{1/2}&(1+p)^{1/2}(1+2p)^{1/2}k^2\\ &= 1+\sum_{n=1}^{\infty}(2\sigma(n/2)-12\sigma(n/4)-10\sigma(n/6)+16\sigma(n/8)\\ &\qquad +60\sigma(n/12)-80\sigma(n/24)-2a(n))q^n.\end{aligned}$$

(b) $$\begin{aligned}p(1-p)^{1/2}&(1+p)^{1/2}(1+2p)^{1/2}k^2\\ &= \sum_{n=1}^{\infty}(-2\sigma(n/2)+12\sigma(n/4)+2\sigma(n/6)-16\sigma(n/8)\\ &\qquad -12\sigma(n/12)+16\sigma(n/24)+2a(n))q^n.\end{aligned}$$

(c) $$\begin{aligned}p^2(1-p)^{1/2}&(1+p)^{1/2}(1+2p)^{1/2}k^2\\ &= \sum_{n=1}^{\infty}(4\sigma(n/2)-24\sigma(n/4)-4\sigma(n/6)+32\sigma(n/8)\\ &\qquad +24\sigma(n/12)-32\sigma(n/24))q^n.\end{aligned}$$

As

$$\begin{aligned}(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{3/2} &= (1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}\\ &\quad + 2p(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2},\\ (1-p)^{1/2}(1+p)^{3/2}(1+2p)^{3/2} &= (1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}\\ &\quad + 3p(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}\\ &\quad + 2p^2(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2},\\ (1-p)^{3/2}(1+p)^{1/2}(1+2p)^{1/2} &= (1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}\\ &\quad - p(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2},\end{aligned}$$

we obtain the following result from Theorem 1.4.

COROLLARY 1.2.

(a)
$$(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{3/2}k^2 = 1+\sum_{n=1}^{\infty}(4\sigma(n/2)-34\sigma(n/4)-12\sigma(n/6)+32\sigma(n/8) + 72\sigma(n/12)-96\sigma(n/24)+4a(n))q^n.$$

(b)
$$(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 = 1+\sum_{n=1}^{\infty}(-2\sigma(n/2)+12\sigma(n/4)-6\sigma(n/6)-16\sigma(n/8) + 36\sigma(n/12)-48\sigma(n/24)+2a(n))q^n.$$

(c)
$$(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{1/2}k^2 = 1+\sum_{n=1}^{\infty}(4\sigma(n/2)-24\sigma(n/4)-12\sigma(n/6)+32\sigma(n/8) + 72\sigma(n/12)-96\sigma(n/24)-4a(n))q^n.$$

Our next theorem, which is proved in [2, Theorem 2.4], expresses each of $\varphi(q^k)$ $(k\in\{1,2,3,4,6,12\})$ in terms of p and k.

THEOREM 1.5.

(a)
$$\varphi(q) = (1+2p)^{3/4}k^{1/2},$$

(b)
$$\varphi(q^2) = \frac{1}{\sqrt{2}}\,((1+2p)^{3/2}+(1-p)^{3/2}(1+p)^{1/2})^{1/2}k^{1/2},$$

(c)
$$\varphi(q^3) = (1+2p)^{1/4}k^{1/2},$$

(d)
$$\varphi(q^4) = \frac{1}{2}\,((1+2p)^{3/4}+(1-p)^{3/4}(1+p)^{1/4})k^{1/2},$$

(e)
$$\varphi(q^6) = \frac{1}{\sqrt{2}}\,((1+2p)^{1/2}+(1-p)^{1/2}(1+p)^{3/2})^{1/2}k^{1/2},$$

(f)
$$\varphi(q^{12}) = \frac{1}{2}\,((1+2p)^{1/4}+(1-p)^{1/4}(1+p)^{3/4})k^{1/2}.$$

From Theorem 1.5(b)(e) we deduce

$$\varphi(q^2)\varphi(q^6) = \frac{1}{2}\,(1+p+p^2+(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2})k. \tag{1.13}$$

We are now in a position to determine a formula for $N(a,b,c,d;n)$ $(n\in\mathbb{N})$ for each of the nineteen forms given in (1.3). For each (a,b,c,d) in (1.3) we use (1.9) and Theorem 1.5 to express $\sum_{n=0}^{\infty}N(a,b,c,d;n)q^n$ as a function of p and k. Then, using Theorems 1.1–1.4 and Corollaries 1.1 and 1.2, we express this function in terms of $\sum_{n=0}^{\infty}\sigma(n)q^{kn}$ $(k\in\{1,2,3,4,6,12\})$, $\sum_{n=0}^{\infty}a(n)q^n$, $\sum_{n=0}^{\infty}b(n)q^n$ and $\sum_{n=0}^{\infty}c(n)q^n$. The formula for $N(a,b,c,d;n)$ is then found by equating coefficients of q^n $(n\in\mathbb{N})$. Each such formula depends at most on σ, a, b and c. The details are carried out in Section 6. The formulae are given in Theorems 1.6–1.24. In the course of the proofs of these theorems, we obtain some relationships between $a(n)$, $b(n)$ and $c(n)$. These are stated in Theorem 1.25. Finally, we use Theorem 1.25 to give all the formulae of Theorems 1.6–1.24 in terms only of σ and a. The reader is advised that the final form of those theorems marked with an asterisk is given after Theorem 1.25.

THEOREM 1.6. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,2) = 1$. *Then*

$$N(1,1,1,1;n) = \begin{cases} 8\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

This is Jacobi's famous formula for the number of representations of a positive integer as the sum of four squares.

THEOREM 1.7. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,2) = 1$. *Then*

$$N(1,1,1,4;n) = \begin{cases} (4+2(-1)^{(N-1)/2})\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 12\sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod 8, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 8. \end{cases}$$

This formula was stated without proof by Liouville [20] and proved by Pepin [27].

THEOREM 1.8. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,2) = 1$. *Then*

$$N(1,1,2,2;n) = \begin{cases} 4\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 8\sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

This formula was stated without proof by Liouville [17] and proved by Pepin [26], [27], Bachmann [8], and Deutsch [10].

THEOREM 1.9. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,1,3,3;n) = \begin{cases} 4\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 4(2^{\alpha+1}-3)\sigma(N) & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

This formula was stated without proof by Liouville [16] (see also [22]) and proved by Pepin [27], Bachmann [8] and Kloosterman [14].

THEOREM 1.10.* *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,1,3,12;n)=\begin{cases}2\sigma(N)+\frac{1}{2}b(n) & \textit{if } n\equiv 1 \pmod 2,\\ 2(2^{\alpha+1}-3)\sigma(N)+\frac{1}{2}b(n) & \textit{if } n\equiv 0 \pmod 2.\end{cases}$$

Liouville did not consider this form.

THEOREM 1.11. *Let* $n\in\mathbb{N}$. *Set* $n=2^{\alpha}N$, *where* $\alpha\in\mathbb{N}_0$, $N\in\mathbb{N}$ *and* $\gcd(N,2)=1$. *Then*

$$N(1,1,4,4;n)=\begin{cases}4\sigma(N) & \textit{if } n\equiv 1 \pmod 4,\\ 0 & \textit{if } n\equiv 3 \pmod 4,\\ 4\sigma(N) & \textit{if } n\equiv 2 \pmod 4,\\ 8\sigma(N) & \textit{if } n\equiv 4 \pmod 8,\\ 24\sigma(N) & \textit{if } n\equiv 0 \pmod 8.\end{cases}$$

This formula was stated without proof by Liouville [18] and proved by Pepin [26], [27] and Bachmann [8, p. 417].

THEOREM 1.12. *Let* $n\in\mathbb{N}$. *Set* $n=2^{\alpha}3^{\beta}N$, *where* $\alpha,\beta\in\mathbb{N}_0$, $N\in\mathbb{N}$ *and* $\gcd(N,6)=1$. *Then*

$$N(1,1,6,6;n)=\begin{cases}2\sigma(N)+2a(n) & \textit{if } n\equiv 1 \pmod 2,\\ 4\sigma(N) & \textit{if } n\equiv 2 \pmod 4,\\ 4(2^{\alpha}-3)\sigma(N) & \textit{if } n\equiv 0 \pmod 4.\end{cases}$$

This form has been considered by Liouville [24], Kloosterman [14], Gongadze [11] and Köhler [15].

THEOREM 1.13.* *Let* $n\in\mathbb{N}$. *Set* $n=2^{\alpha}3^{\beta}N$, *where* $\alpha,\beta\in\mathbb{N}_0$, $N\in\mathbb{N}$ *and* $\gcd(N,6)=1$. *Then*

$$N(1,1,12,12;n)=\begin{cases}\sigma(N)+a(n)+\frac{1}{2}b(n) & \textit{if } n\equiv 1 \pmod 4,\\ 0 & \textit{if } n\equiv 3 \pmod 4,\\ 2\sigma(N)+\frac{1}{2}b(n) & \textit{if } n\equiv 2 \pmod 4,\\ 2(2^{\alpha}-3)\sigma(N)+\frac{1}{2}b(n) & \textit{if } n\equiv 0 \pmod 4.\end{cases}$$

This form was not considered by Liouville.

THEOREM 1.14. *Let* $n\in\mathbb{N}$. *Set* $n=2^{\alpha}N$, *where* $\alpha\in\mathbb{N}_0$, $N\in\mathbb{N}$ *and* $\gcd(N,2)=1$. *Then*

$$N(1,2,2,4;n)=\begin{cases}2\sigma(N) & \textit{if } n\equiv 1 \pmod 2,\\ 4\sigma(N) & \textit{if } n\equiv 2 \pmod 4,\\ 8\sigma(N) & \textit{if } n\equiv 4 \pmod 8,\\ 24\sigma(N) & \textit{if } n\equiv 0 \pmod 8.\end{cases}$$

This form has been treated by Liouville [21], Pepin [26, p. 177], [27, pp. 31, 42] and Bachmann [8, p. 418].

THEOREM 1.15. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,2,3,6;n) = \begin{cases} (3^{\beta+1}-2)\sigma(N) + a(n) & \text{if } n \equiv 1 \pmod 2, \\ 2(3^{\beta+1}-2)\sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 6(3^{\beta+1}-2)\sigma(N) & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

This form has been considered by Liouville [23], Bachmann [8, p. 423], Griffiths [12], [13], Gongadze [11] and Köhler [15].

THEOREM 1.16.* *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,3,3,4;n) = \begin{cases} 2\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 1 \pmod 2, \\ 2(2^{\alpha+1}-3)\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

This form was not treated by Liouville.

THEOREM 1.17.* *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,3,4,12;n) = \begin{cases} \sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 1 \pmod 2, \\ 0 & \text{if } n \equiv 2 \pmod 4, \\ 6(2^{\alpha-1}-1)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Elementary remarks on this form were given by Liouville [19] and Pepin [27, p. 24].

THEOREM 1.18. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha N$, *where* $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,2) = 1$. *Then*

$$N(1,4,4,4;n) = \begin{cases} 2\sigma(N) & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 2,3 \pmod 4, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod 8, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 8. \end{cases}$$

This result was stated by Liouville without proof in [20] and proved by Pepin [27, p. 43].

THEOREM 1.19.* *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,4,6,6;n) = \begin{cases} \sigma(N) + a(n) & \text{if } n \equiv 1 \pmod 2, \\ 2\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 2 \pmod 4, \\ 2(2^{\alpha}-3)\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Liouville did not consider this form.

THEOREM 1.20.* *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,4,12,12;n) = \begin{cases} \frac{1}{2}\sigma(N) + \frac{1}{2}a(n) + \frac{1}{4}b(n) & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 2,3 \pmod 4, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Liouville did not consider this form.

THEOREM 1.21. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(2,2,3,3;n) = \begin{cases} 2\sigma(N) - 2a(n) & \text{if } n \equiv 1 \pmod 2, \\ 4\sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 4(2^\alpha - 3)\sigma(N) & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Partial results were stated by Liouville [24] without proof.

THEOREM 1.22.* *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(2,2,3,12;n) = \begin{cases} \sigma(N) - a(n) & \text{if } n \equiv 1 \pmod 2, \\ 2\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 2 \pmod 4, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Liouville did not consider this form.

THEOREM 1.23.* *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(3,3,4,4;n) = \begin{cases} \sigma(N) - a(n) + \frac{1}{2}c(n) & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{if } n \equiv 1 \pmod 4, \\ 2\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 2 \pmod 4, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Liouville did not treat this form.

THEOREM 1.24.* *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(3,4,4,12;n) = \begin{cases} \frac{1}{2}\sigma(N) - \frac{1}{2}a(n) + \frac{1}{4}c(n) & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{if } n \equiv 1,2 \pmod 4, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

This form was not considered by Liouville.

In the course of proving Theorems 1.6–1.24 we establish the following result.

THEOREM 1.25. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

(a) $b(n) = 2\sigma(N) + 2a(n) \quad \textit{if } n \equiv 1 \pmod 4,$
(b) $b(n) = -2\sigma(N) - 2a(n) \quad \textit{if } n \equiv 3 \pmod 4,$
(c) $c(n) = -2\sigma(N) + 2a(n) \quad \textit{if } n \equiv 1 \pmod 4,$
(d) $c(n) = 2\sigma(N) - 2a(n), \quad \textit{if } n \equiv 3 \pmod 4,$
(e) $b(n) = -c(n) = 4a(n/2) \quad \textit{if } n \equiv 2 \pmod 4,$
(f) $b(n) = c(n) = 4\sigma(N) \quad \textit{if } n \equiv 4 \pmod 8,$
(g) $b(n) = c(n) = -12\sigma(N) \quad \textit{if } n \equiv 0 \pmod 8.$

Applying Theorem 1.25 to Theorems 1.10*, 1.13*, 1.16*, 1.17*, 1.19*, 1.20*, 1.22*, 1.23* and 1.24*, we obtain formulae which do not depend upon b and c.

THEOREM 1.10. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,1,3,12;n) = \begin{cases} 3\sigma(N) + a(n) & \textit{if } n \equiv 1 \pmod 4, \\ \sigma(N) - a(n) & \textit{if } n \equiv 3 \pmod 4, \\ 2\sigma(N) + 2a(n/2) & \textit{if } n \equiv 2 \pmod 4, \\ 12\sigma(N) & \textit{if } n \equiv 4 \pmod 8, \\ 4(2^\alpha - 3)\sigma(N) & \textit{if } n \equiv 0 \pmod 8. \end{cases}$$

THEOREM 1.13. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,1,12,12;n) = \begin{cases} 2\sigma(N) + 2a(n) & \textit{if } n \equiv 1 \pmod 4, \\ 0 & \textit{if } n \equiv 3 \pmod 4, \\ 2\sigma(N) + 2a(n/2) & \textit{if } n \equiv 2 \pmod 4, \\ 4\sigma(N) & \textit{if } n \equiv 4 \pmod 8, \\ 4(2^{\alpha-1} - 3)\sigma(N) & \textit{if } n \equiv 0 \pmod 8. \end{cases}$$

THEOREM 1.16. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,3,3,4;n) = \begin{cases} \sigma(N) + a(n) & \textit{if } n \equiv 1 \pmod 4, \\ 3\sigma(N) - a(n) & \textit{if } n \equiv 3 \pmod 4, \\ 2\sigma(N) - 2a(n/2) & \textit{if } n \equiv 2 \pmod 4, \\ 12\sigma(N) & \textit{if } n \equiv 4 \pmod 8, \\ 4(2^\alpha - 3)\sigma(N) & \textit{if } n \equiv 0 \pmod 8. \end{cases}$$

THEOREM 1.17. *Let* $n \in \mathbb{N}$. *Set* $n = 2^\alpha 3^\beta N$, *where* $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ *and* $\gcd(N,6) = 1$. *Then*

$$N(1,3,4,12;n)=\begin{cases}\sigma(N)+a(n) & \text{if } n\equiv 1 \pmod 4,\\ \sigma(N)-a(n) & \text{if } n\equiv 3 \pmod 4,\\ 0 & \text{if } n\equiv 2 \pmod 4,\\ 8\sigma(N) & \text{if } n\equiv 4 \pmod 8,\\ 12(2^{\alpha-2}-1)\sigma(N) & \text{if } n\equiv 0 \pmod 8.\end{cases}$$

THEOREM 1.19. *Let* $n\in\mathbb{N}$. *Set* $n=2^\alpha 3^\beta N$, *where* $\alpha,\beta\in\mathbb{N}_0$, $N\in\mathbb{N}$ *and* $\gcd(N,6)=1$. *Then*

$$N(1,4,6,6;n)=\begin{cases}\sigma(N)+a(n) & \text{if } n\equiv 1 \pmod 2,\\ 2\sigma(N)-2a(n/2) & \text{if } n\equiv 2 \pmod 4,\\ 4\sigma(N) & \text{if } n\equiv 4 \pmod 8,\\ 4(2^{\alpha-1}-3)\sigma(N) & \text{if } n\equiv 0 \pmod 8.\end{cases}$$

THEOREM 1.20. *Let* $n\in\mathbb{N}$. *Set* $n=2^\alpha 3^\beta N$, *where* $\alpha,\beta\in\mathbb{N}_0$, $N\in\mathbb{N}$ *and* $\gcd(N,6)=1$. *Then*

$$N(1,4,12,12;n)=\begin{cases}\sigma(N)+a(n) & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 2,3 \pmod 4,\\ 4\sigma(N) & \text{if } n\equiv 4 \pmod 8,\\ 4(2^{\alpha-1}-3)\sigma(N) & \text{if } n\equiv 0 \pmod 8.\end{cases}$$

THEOREM 1.22. *Let* $n\in\mathbb{N}$. *Set* $n=2^\alpha 3^\beta N$, *where* $\alpha,\beta\in\mathbb{N}_0$, $N\in\mathbb{N}$ *and* $\gcd(N,6)=1$. *Then*

$$N(2,2,3,12;n)=\begin{cases}\sigma(N)-a(n) & \text{if } n\equiv 1 \pmod 2,\\ 2\sigma(N)+2a(n/2) & \text{if } n\equiv 2 \pmod 4,\\ 4\sigma(N) & \text{if } n\equiv 4 \pmod 8,\\ 4(2^{\alpha-1}-3)\sigma(N) & \text{if } n\equiv 0 \pmod 8.\end{cases}$$

THEOREM 1.23. *Let* $n\in\mathbb{N}$. *Set* $n=2^\alpha 3^\beta N$, *where* $\alpha,\beta\in\mathbb{N}_0$, $N\in\mathbb{N}$ *and* $\gcd(N,6)=1$. *Then*

$$N(3,3,4,4;n)=\begin{cases}0 & \text{if } n\equiv 1 \pmod 4,\\ 2\sigma(N)-2a(n) & \text{if } n\equiv 3 \pmod 4,\\ 2\sigma(N)-2a(n/2) & \text{if } n\equiv 2 \pmod 4,\\ 4\sigma(N) & \text{if } n\equiv 4 \pmod 8,\\ 4(2^{\alpha-1}-3)\sigma(N) & \text{if } n\equiv 0 \pmod 8.\end{cases}$$

THEOREM 1.24. *Let* $n\in\mathbb{N}$. *Set* $n=2^\alpha 3^\beta N$, *where* $\alpha,\beta\in\mathbb{N}_0$, $N\in\mathbb{N}$ *and* $\gcd(N,6)=1$. *Then*

$$N(3,4,4,12;n) = \begin{cases} 0 & \text{if } n \equiv 1,2 \pmod 4, \\ \sigma(N) - a(n) & \text{if } n \equiv 3 \pmod 4, \\ 4\sigma(N) & \text{if } n \equiv 4 \pmod 8, \\ 4(2^{\alpha-1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod 8. \end{cases}$$

The authors have treated other quaternary quadratic forms in [1]–[4], [6] and [28]. These forms require functions other than $\sigma(n)$ and $a(n)$ for their evaluation.

2. Proof of Theorem 1.1. We begin by recalling the duplication, triplication and change of sign principles for p and k, which were proved in [4, Theorems 9, 10, 11].

THEOREM 2.1 (Duplication principle).

$$p(q^2) = \frac{1 + p - p^2 - ((1-p)(1+p)(1+2p))^{1/2}}{p^2},$$

$$k(q^2) = \frac{(1 + p - p^2 + ((1-p)(1+p)(1+2p))^{1/2})k}{2}.$$

THEOREM 2.2 (Triplication principle).

$$\begin{aligned} p(q^3) = 3^{-1}((-4 - 3p + 6p^2 + 4p^3) \\ + 2^{2/3}(1 - 2p - 2p^2)((1-p)(1+2p)(2+p))^{1/3} \\ + 2^{1/3}(1+2p)((1-p)(1+2p)(2+p))^{2/3}), \\ k(q^3) = 3^{-2}(3 + 2^{2/3}(1+2p)((1-p)(1+2p)(2+p))^{1/3} \\ + 2^{4/3}((1-p)(1+2p)(2+p))^{2/3})k. \end{aligned}$$

THEOREM 2.3 (Change of sign principle).

$$p(-q) = \frac{-p}{1+p}, \qquad k(-q) = (1+p)^2 k.$$

The Eisenstein series $L(q)$ is defined by

$$L(q) = 1 - 24\sum_{n=1}^{\infty} \sigma(n) q^n. \tag{2.1}$$

We define

$$A(q) := 2L(q^2) - L(q) = 1 + 24\sum_{n=1}^{\infty} (\sigma(n) - 2\sigma(n/2)) q^n, \tag{2.2}$$

$$B(q) := 3L(q^3) - L(q) = 2 + 24\sum_{n=1}^{\infty} (\sigma(n) - 3\sigma(n/3)) q^n. \tag{2.3}$$

Proof of Theorem 1.1. It was shown in [7, eq. (3.84)] that

$$A(q) = (1 + 14p + 24p^2 + 14p^3 + p^4)k^2. \tag{2.4}$$

Applying the duplication principle (Theorem 2.1) to (2.4), we obtain

$$A(q^2) = \left(1 + 2p + 6p^2 + 5p^3 - \frac{1}{2}\,p^4\right)k^2. \tag{2.5}$$

From [7, eq. (3.87)] we have

$$B(q) = (2 + 16p + 36p^2 + 16p^3 + 2p^4)k^2. \tag{2.6}$$

Applying the duplication principle (Theorem 1.1) twice to (2.6), we deduce

$$B(q^2) = (2 + 4p + 6p^2 + 4p^3 + 2p^4)k^2, \tag{2.7}$$

$$B(q^4) = \left(2 + 4p - 2p^3 + \frac{1}{2}\,p^4\right)k^2. \tag{2.8}$$

From (2.4)–(2.8) we obtain

$$k^2 = -\frac{2}{9}\,A(q) + \frac{4}{9}\,A(q^2) + \frac{1}{18}\,B(q) + \frac{1}{9}\,B(q^2) + \frac{2}{9}\,B(q^4), \tag{2.9}$$

$$pk^2 = \frac{1}{6}\,A(q) - \frac{1}{9}\,A(q^2) - \frac{1}{12}\,B(q) - \frac{1}{18}\,B(q^2) + \frac{1}{9}\,B(q^4), \tag{2.10}$$

$$p^2k^2 = -\frac{1}{9}\,A(q) + \frac{1}{9}\,B(q) - \frac{1}{18}\,B(q^2), \tag{2.11}$$

$$p^3k^2 = \frac{1}{9}\,A(q) + \frac{1}{9}\,A(q^2) - \frac{1}{9}\,B(q) + \frac{1}{9}\,B(q^2) - \frac{1}{9}\,B(q^4), \tag{2.12}$$

$$p^4k^2 = -\frac{4}{9}\,A(q^2) + \frac{4}{9}\,B(q^2) - \frac{2}{9}\,B(q^4). \tag{2.13}$$

Appealing to (2.2) and (2.3) we obtain the assertions of Theorem 1.1.

3. Proof of Theorem 1.2. Applying the duplication principle to (2.5) we have

$$\begin{aligned} A(q^4) = {} & \left(\frac{1}{4} + \frac{1}{2}\,p + \frac{3}{2}\,p^2 + \frac{5}{4}\,p^3 - \frac{1}{8}\,p^4\right)k^2 \\ & + \frac{3}{4}\,(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2. \end{aligned} \tag{3.1}$$

From (3.1), (2.5) and (2.2), we obtain

$$\begin{aligned} & (1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 \\ & \qquad = \frac{4}{3}\,A(q^4) - \frac{1}{3}\,A(q^2) = \frac{8}{3}\,L(q^8) - 2L(q^4) + \frac{1}{3}\,L(q^2) \\ & \qquad = 1 + \sum_{n=1}^{\infty}(-8\sigma(n/2) + 48\sigma(n/4) - 64\sigma(n/8))q^n, \end{aligned}$$

which is part (a).

Next, applying the duplication principle to (3.1), we obtain

$$A(q^8) = \left(\frac{1}{16} + \frac{1}{8}\,p + \frac{3}{8}\,p^2 + \frac{5}{16}\,p^3 - \frac{1}{32}\,p^4\right)k^2 + \frac{3}{16}\,(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 + \frac{3}{8}\,(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2 + \frac{3}{8}\,(1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2, \tag{3.2}$$

and to (2.8) we obtain

$$B(q^8) = \left(\frac{5}{4} + \frac{5}{2}\,p - \frac{3}{4}\,p^2 - 2p^3 + \frac{1}{8}\,p^4\right)k^2 - \frac{3}{8}\,(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 + \frac{9}{8}\,(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{1/2}k^2. \tag{3.3}$$

From (3.3), Theorem 1.1, part (a) and (2.3), we deduce

$$\begin{aligned}(1-p)^{1/2}&(1+p)^{3/2}(1+2p)^{1/2}k^2\\ &= \frac{8}{9}\,B(q^8) - \frac{10}{9}\,k^2 - \frac{20}{9}\,pk^2 + \frac{2}{3}\,p^2k^2 + \frac{16}{9}\,p^3k^2 - \frac{1}{9}\,p^4k^2\\ &\quad + \frac{1}{3}\,(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2\\ &= 1 + \sum_{n=1}^{\infty}(-8\sigma(n/6) + 48\sigma(n/12) - 64\sigma(n/24))q^n,\end{aligned}$$

which is part (b).

Next, from (3.2), we obtain

$$\begin{aligned}(1-p)^{3/4}&(1+p)^{1/4}(1+2p)^{9/4}k^2 + (1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2\\ &= \frac{8}{3}\,A(q^8) - \frac{1}{6}\,k^2 - \frac{1}{3}\,pk^2 - p^2k^2 - \frac{5}{6}\,p^3k^2 + \frac{1}{12}\,p^4k^2\\ &\quad - \frac{1}{2}\,(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2.\end{aligned}$$

Then, appealing to (2.2), Theorem 1.1 and part (a), we deduce

$$\begin{aligned}(1-p)^{3/4}&(1+p)^{1/4}(1+2p)^{9/4}k^2 + (1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2\\ &= 2 + \sum_{n=1}^{\infty}(-16\sigma(n/4) + 96\sigma(n/8) - 128\sigma(n/16))q^n.\end{aligned} \tag{3.4}$$

Our next objective is to determine

$$(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2 - (1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2.$$

To do this we make use of the following two sums, which are closely related to $L(q)$:

$$(3.5)\qquad L_{1,4}(q) = \sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} \sigma(n)q^n, \qquad L_{3,4}(q) = \sum_{\substack{n=1 \\ n\equiv 3 \pmod 4}}^{\infty} \sigma(n)q^n.$$

These sums have been evaluated by Cheng [9]. From [9, Theorem 3.5.1] we have

$$(3.6)\qquad L_{1,4}(q) = \frac{1}{32}(1-g)(1+g)^3w^2,$$

$$(3.7)\qquad L_{3,4}(q) = \frac{1}{32}(1-g)^3(1+g)w^2,$$

where

$$(3.8)\qquad g = (1-x)^{1/4}, \quad 1-x = \frac{(1-p)^3(1+p)}{(1+2p)^3}, \quad w = (1+2p)^{3/2}k$$

(see [7]). From (3.6) and (3.7) we deduce

$$(3.9)\qquad L_{1,4}(q) - L_{3,4}(q) = \frac{1}{8}gw^2 - \frac{1}{8}g^3w^2.$$

By (3.8) we see that

$$(3.10)\qquad (1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2 = gw^2,$$

$$(3.11)\qquad (1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2 = g^3w^2.$$

From (3.5), (3.9), (3.10) and (3.11), we deduce

$$\begin{aligned}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2 &- (1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2 \\ &= gw^2 - g^3w^2 = 8L_{1,4}(q) - 8L_{3,4}(q) \\ &= 8\sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} \sigma(n)q^n - 8\sum_{\substack{n=1 \\ n\equiv 3 \pmod 4}}^{\infty} \sigma(n)q^n,\end{aligned}$$

that is,

$$(3.12)\qquad \begin{aligned}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2 &- (1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2 \\ &= 8\sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} (-1)^{(n-1)/2}\sigma(n)q^n.\end{aligned}$$

Adding and subtracting (3.4) and (3.12), we obtain

$$(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2 = 1+\sum_{n=1}^{\infty}\Big(4\Big(\frac{-4}{n}\Big)\sigma(n)-8\sigma(n/4)+48\sigma(n/8)-64\sigma(n/16)\Big)q^n$$

and

$$(1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2 = 1+\sum_{n=1}^{\infty}\Big(-4\Big(\frac{-4}{n}\Big)\sigma(n)-8\sigma(n/4)+48\sigma(n/8)-64\sigma(n/16)\Big)q^n,$$

which are parts (c) and (d).

4. Proof of Theorem 1.3. Ramanujan's discriminant function $\Delta(q)$ is defined by

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}. \tag{4.1}$$

Alaca, Alaca and Williams [5, eqns. (3.28)–(3.33)] have shown that

$$\Delta(q) = 2^{-4}p(1+p)^4(1-p)^{12}(1+2p)^3(2+p)^3k^{12}, \tag{4.2}$$

$$\Delta(q^2) = 2^{-8}p^2(1+p)^2(1-p)^6(1+2p)^6(2+p)^6k^{12}, \tag{4.3}$$

$$\Delta(q^3) = 2^{-4}p^3(1+p)^{12}(1-p)^4(1+2p)(2+p)k^{12}, \tag{4.4}$$

$$\Delta(q^4) = 2^{-16}p^4(1+p)(1-p)^3(1+2p)^3(2+p)^{12}k^{12}, \tag{4.5}$$

$$\Delta(q^6) = 2^{-8}p^6(1+p)^6(1-p)^2(1+2p)^2(2+p)^2k^{12}, \tag{4.6}$$

$$\Delta(q^{12}) = 2^{-16}p^{12}(1+p)^3(1-p)(1+2p)(2+p)^4k^{12}. \tag{4.7}$$

Hence, by (1.5), (4.1), (4.3), (4.5), (4.6) and (4.7), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}a(n)q^n &= q\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{4n})(1-q^{6n})(1-q^{12n})\\ &= \Delta(q^2)^{1/24}\Delta(q^4)^{1/24}\Delta(q^6)^{1/24}\Delta(q^{12})^{1/24}\\ &= \frac{1}{4}p(2+p)(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}k^2,\end{aligned}$$

which is part (a).

Also, by (1.6), (4.1), (4.2), (4.3), (4.5), (4.6) and (4.7), we have

$$\begin{aligned}\sum_{n=0}^{\infty}b(n)q^n &= \prod_{n=1}^{\infty}(1-q^n)^{-4}(1-q^{2n})^{10}(1-q^{4n})^{-4}(1-q^{6n})^{4}(1-q^{12n})^{-2}\\ &= \Delta(q)^{-1/6}\Delta(q^2)^{5/12}\Delta(q^4)^{-1/6}\Delta(q^6)^{1/6}\Delta(q^{12})^{-1/12}\\ &= (1-p)^{1/4}(1+p)^{3/4}(1+2p)^{7/4}k^2\end{aligned}$$

and, by (1.7), (4.1), (4.3), (4.4), (4.5), (4.6) and (4.7),

$$\begin{aligned}\sum_{n=0}^{\infty} c(n)q^n &= \prod_{n=1}^{\infty} (1-q^{2n})^4(1-q^{3n})^{-4}(1-q^{4n})^{-2}(1-q^{6n})^{10}(1-q^{12n})^{-4}\\ &= \Delta(q^2)^{1/6}\Delta(q^3)^{-1/6}\Delta(q^4)^{-1/12}\Delta(q^6)^{5/12}\Delta(q^{12})^{-1/6}\\ &= (1-p)^{3/4}(1+p)^{1/4}(1+2p)^{5/4}k^2,\end{aligned}$$

which are parts (b) and (c).

Applying the change of sign principle (Theorem 2.3) to parts (b) and (c), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} b(n)(-q)^n &= (1-p(-q))^{1/4}(1+p(-q))^{3/4}(1+2p(-q))^{7/4}k(-q)^2\\ &= \left(1+\frac{p}{1+p}\right)^{1/4}\left(1-\frac{p}{1+p}\right)^{3/4}\left(1-\frac{2p}{1+p}\right)^{7/4}(1+p)^4k^2\\ &= (1-p)^{7/4}(1+p)^{5/4}(1+2p)^{1/4}k^2\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} c(n)(-q)^n &= (1-p(-q))^{3/4}(1+p(-q))^{1/4}(1+2p(-q))^{5/4}k(-q)^2\\ &= \left(1+\frac{p}{1+p}\right)^{3/4}\left(1-\frac{p}{1+p}\right)^{1/4}\left(1-\frac{2p}{1+p}\right)^{5/4}(1+p)^4k^2\\ &= (1-p)^{5/4}(1+p)^{7/4}(1+2p)^{3/4}k^2,\end{aligned}$$

proving parts (d) and (e).

5. Proof of Theorem 1.4. By Theorem 1.3(a) and Table 1 we have

$$(2p+p^2)(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}k^2 = \sum_{n=1}^{\infty} 4a(n)q^n.$$

By Theorem 1.2(a) we have

$$\begin{aligned}&(1+p-2p^2)(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}k^2\\ &\qquad = 1+\sum_{n=1}^{\infty}(-8\sigma(n/2)+48\sigma(n/4)-64\sigma(n/8))q^n.\end{aligned}$$

By Theorem 1.2(b) we have

$$\begin{aligned}&(1+p)(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}k^2\\ &\qquad = 1+\sum_{n=1}^{\infty}(-8\sigma(n/6)+48\sigma(n/12)-64\sigma(n/24))q^n.\end{aligned}$$

The three assertions of Theorem 1.4 now follow using

$$\begin{aligned} 1 &= -\frac{1}{2}(2p+p^2) - \frac{1}{4}(1+p-2p^2) + \frac{5}{4}(1+p), \\ p &= \frac{1}{2}(2p+p^2) + \frac{1}{4}(1+p-2p^2) - \frac{1}{4}(1+p), \\ p^2 &= -\frac{1}{2}(1+p-2p^2) + \frac{1}{2}(1+p). \end{aligned}$$

6. Proofs of Theorems 1.6–1.25. Let $n \in \mathbb{N}$. We note that in the proofs of Theorems 1.6–1.8, 1.11, 1.14 and 1.18 we set $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $\gcd(N,2) = 1$, whereas in Theorems 1.9, 1.10, 1.12, 1.13, 1.15–1.17 and 1.19–1.24 we set $n = 2^\alpha 3^\beta N$, where $\alpha \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $\gcd(N,6) = 1$. As $\sigma(n)$ is a multiplicative function of n, we have

$$\begin{aligned} \sigma(2^\alpha N) &= \sigma(2^\alpha)\sigma(N) = (2^{\alpha+1}-1)\sigma(N), \\ \sigma(2^\alpha 3^\beta N) &= \sigma(2^\alpha)\sigma(3^\beta)\sigma(N) = \frac{1}{2}(2^{\alpha+1}-1)(3^{\beta+1}-1)\sigma(N). \end{aligned}$$

Proof of Theorem 1.6. We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(1,1,1,1;n)q^n &= \varphi(q)^4 && \text{(by (1.9))} \\ &= (1+2p)^3k^2 && \text{(by Theorem 1.5(a))} \\ &= 1 + \sum_{n=1}^{\infty}(8\sigma(n) - 32\sigma(n/4)) && \text{(by Corollary 1.1(b))}, \end{aligned}$$

so for $n \in \mathbb{N}$,

$$N(1,1,1,1;n) = 8\sigma(n) - 32\sigma(n/4) = \begin{cases} 8\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Proof of Theorem 1.7. We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(1,1,1,4;n)q^n &= \varphi(q)^3\varphi(q^4) \quad \text{(by (1.9))} \\ &= \frac{1}{2}(1+2p)^3k^2 + \frac{1}{2}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2 \\ &\hphantom{=} \text{(by Theorem 1.5(a)(d))} \\ &= 1 + \sum_{n=1}^{\infty}\bigg(2\Big(\frac{-4}{n}\Big)\sigma(n) + 4\sigma(n) - 20\sigma(n/4) \\ &\qquad + 24\sigma(n/8) - 32\sigma(n/16)\bigg)q^n \\ &\hphantom{=} \text{(by Corollary 1.1(b) and Theorem 1.2(c))}, \end{aligned}$$

so for $n \in \mathbb{N}$,

$$N(1,1,1,4;n) = \left(4 + 2\left(\frac{-4}{n}\right)\right)\sigma(n) - 20\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16)$$
$$= \begin{cases} (4 + 2(-1)^{(N-1)/2})\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 12\sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod 8, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 8. \end{cases}$$

Proof of Theorem 1.8. We have

$$\sum_{n=0}^{\infty} N(1,1,2,2;n)q^n = \varphi(q)^2\varphi(q^2)^2 \quad \text{(by (1.9))}$$
$$= \frac{1}{2}(1+2p)^3k^2 + \frac{1}{2}(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2$$
$$\text{(by Theorem 1.5(a)(b))}$$
$$= 1 + \sum_{n=1}^{\infty}(4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8))q^n$$
$$\text{(by Corollary 1.1(b) and Theorem 1.2(a))},$$

so for $n \in \mathbb{N}$,

$$N(1,1,2,2;n) = 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8)$$
$$= \begin{cases} 4\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 8\sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Proof of Theorem 1.9. We have

$$\sum_{n=0}^{\infty} N(1,1,3,3;n)q^n = \varphi(q)^2\varphi(q^3)^2 \quad \text{(by (1.9))}$$
$$= (1+2p)^2k^2 \quad \text{(by Theorem 1.5(a)(c))}$$
$$= 1 + \sum_{n=1}^{\infty}(4\sigma(n) - 8\sigma(n/2) - 12\sigma(n/3) + 16\sigma(n/4)$$
$$+ 24\sigma(n/6) - 48\sigma(n/12))q^n$$
$$\text{(by Corollary 1.1(a))},$$

so for $n \in \mathbb{N}$,

$$N(1,1,3,3;n) = 4\sigma(n) - 8\sigma(n/2) - 12\sigma(n/3) + 16\sigma(n/4)$$
$$+ 24\sigma(n/6) - 48\sigma(n/12)$$
$$= \begin{cases} 4\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 4(2^{\alpha+1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Proof of Theorem 1.10. We have

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,1,3,12;n)q^n &= \varphi(q)^2\varphi(q^3)\varphi(q^{12}) \quad \text{(by (1.9))}\\ &= \frac{1}{2}\,(1+2p)^2k^2 + \frac{1}{2}\,(1-p)^{1/4}(1+p)^{3/4}(1+2p)^{7/4}k^2\\ &\qquad \text{(by Theorem 1.5(a)(c)(f))}\\ &= 1 + \sum_{n=1}^{\infty}\biggl(2\sigma(n) - 4\sigma(n/2) - 6\sigma(n/3) + 8\sigma(n/4)\\ &\qquad + 12\sigma(n/6) - 24\sigma(n/12) + \frac{1}{2}\,b(n)\biggr)q^n\\ &\qquad \text{(by Corollary 1.1(a) and Theorem 1.3(b))},\end{aligned}$$

so for $n \in \mathbb{N}$,

$$\begin{aligned}N(1,1,3,12;n) &= 2\sigma(n) - 4\sigma(n/2) - 6\sigma(n/3) + 8\sigma(n/4)\\ &\qquad + 12\sigma(n/6) - 24\sigma(n/12) + \frac{1}{2}\,b(n)\\ &= \begin{cases} 2\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 1 \pmod 2,\\ 2(2^{\alpha+1}-3)\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 0 \pmod 2.\end{cases}\end{aligned}$$

Proof of Theorem 1.11. We have

$$\begin{aligned}&\sum_{n=0}^{\infty} N(1,1,4,4;n)q^n = \varphi(q)^2\varphi(q^4)^2 \quad \text{(by (1.9))}\\ &\quad = \frac{1}{4}\,(1+2p)^3k^2 + \frac{1}{2}\,(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2\\ &\qquad + \frac{1}{4}\,(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 \quad \text{(by Theorem 1.5(a)(d))}\\ &\quad = 1 + \sum_{n=1}^{\infty}\left(\left(2 + 2\left(\frac{-4}{n}\right)\right)\sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16)\right)q^n\\ &\qquad \text{(by Corollary 1.1(b) and Theorem 1.2(a)(c))},\end{aligned}$$

so for $n \in \mathbb{N}$,

$$\begin{aligned}N(1,1,4,4;n) &= \left(2 + 2\left(\frac{-4}{n}\right)\right)\sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16)\\ &= \begin{cases} 4\sigma(N) & \text{if } n \equiv 1,2 \pmod 4,\\ 0 & \text{if } n \equiv 3 \pmod 4,\\ 8\sigma(N) & \text{if } n \equiv 4 \pmod 8,\\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 8.\end{cases}\end{aligned}$$

Proof of Theorem 1.12. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} N(1,1,6,6;n)q^n &= \varphi(q)^2\varphi(q^6)^2 \quad \text{(by (1.9))}\\
&= \frac{1}{2}(1+2p)^2k^2 + \frac{1}{2}(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{3/2}k^2 \\
&\qquad \text{(by Theorem 1.5(a)(e))}\\
&= 1 + \sum_{n=1}^{\infty}(2\sigma(n) - 2\sigma(n/2) - 6\sigma(n/3) - 4\sigma(n/4) \\
&\quad + 6\sigma(n/6) + 16\sigma(n/8) + 12\sigma(n/12) - 48\sigma(n/24) + 2a(n))q^n \\
&\qquad \text{(by Corollary 1.1(a) and Corollary 1.2(a))},
\end{aligned}$$

so for $n \in \mathbb{N}$,

$$\begin{aligned}
N(1,1,6,6;n) &= 2\sigma(n) - 2\sigma(n/2) - 6\sigma(n/3) - 4\sigma(n/4) \\
&\quad + 6\sigma(n/6) + 16\sigma(n/8) + 12\sigma(n/12) - 48\sigma(n/24) + 2a(n) \\
&= \begin{cases} 2\sigma(N) + 2a(n) & \text{if } n \equiv 1 \pmod 2, \\ 4\sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 4(2^{\alpha}-3)\sigma(N) & \text{if } n \equiv 0 \pmod 4, \end{cases}
\end{aligned}$$

as $a(n) = 0$ for $n \equiv 0 \pmod 2$.

Proof of Theorem 1.13 and Theorem 1.25(b). We have

$$\begin{aligned}
\sum_{n=0}^{\infty} N(1,1,12,12;n)q^n &= \varphi(q)^2\varphi(q^{12})^2 \quad \text{(by (1.9))}\\
&= \frac{1}{4}(1+2p)^2k^2 + \frac{1}{2}(1-p)^{1/4}(1+p)^{3/4}(1+2p)^{7/4}k^2 \\
&\quad + \frac{1}{4}(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{3/2}k^2 \quad \text{(by Theorem 1.5(a)(f))}\\
&= 1 + \sum_{n=1}^{\infty}\Big(\sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) + 3\sigma(n/6) + 8\sigma(n/8) \\
&\quad + 6\sigma(n/12) - 24\sigma(n/24) + a(n) + \frac{1}{2}b(n)\Big)q^n \\
&\qquad \text{(by Corollary 1.1(a), Theorem 1.3(b) and Corollary 1.2(a))},
\end{aligned}$$

so for $n \in \mathbb{N}$,

$$\begin{aligned}
N(1,1,12,12;n) &= \sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) \\
&\quad + 3\sigma(n/6) + 8\sigma(n/8) + 6\sigma(n/12) \\
&\quad - 24\sigma(n/24) + a(n) + \frac{1}{2}b(n)
\end{aligned}$$

$$= \begin{cases} \sigma(N) + a(n) + \frac{1}{2}b(n) & \text{if } n \equiv 1 \pmod 2, \\ 2\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 2 \pmod 4, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

As $x^2 + y^2 + 12z^2 + 12t^2 \equiv x^2 + y^2 \equiv 0, 1, 2 \pmod 4$, we have

$$N(1, 1, 12, 12; n) = 0 \quad \text{if } n \equiv 3 \pmod 4.$$

Thus

$$\sigma(n) + a(n) + \frac{1}{2}\, b(n) = 0 \quad \text{if } n \equiv 3 \pmod 4,$$

which is the assertion of Theorem 1.25(b).

Proof of Theorem 1.14. We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 2, 2, 4; n)q^n &= \varphi(q)\varphi(q^2)^2\varphi(q^4) \quad \text{(by (1.9))} \\ &= \frac{1}{4}\,(1 + 2p)^3k^2 + \frac{1}{4}\,(1 - p)^{3/2}(1 + p)^{1/2}(1 + 2p)^{3/2}k^2 \\ &\quad + \frac{1}{4}\,(1 - p)^{3/4}(1 + p)^{1/4}(1 + 2p)^{9/4}k^2 \\ &\quad + \frac{1}{4}\,(1 - p)^{9/4}(1 + p)^{3/4}(1 + 2p)^{3/4}k^2 \quad \text{(by Theorem 1.5(a)(b)(d))} \\ &= 1 + \sum_{n=1}^{\infty}(2\sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16))q^n \\ &\qquad \text{(by Corollary 1.1(b) and Theorem 1.2(a)(c)(d)),} \end{aligned}$$

so that for $n \in \mathbb{N}$,

$$\begin{aligned} N(1, 2, 2, 4; n) &= 2\sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16) \\ &= \begin{cases} 2\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 4\sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod 8, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 8. \end{cases} \end{aligned}$$

Proof of Theorem 1.15. We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 2, 3, 6; n)q^n &= \varphi(q)\varphi(q^2)\varphi(q^3)\varphi(q^6) \quad \text{(by (1.9))} \\ &= \frac{1}{2}\,(1 + 2p)(1 + p + p^2)k^2 + \frac{1}{2}\,(1 - p)^{1/2}(1 + p)^{1/2}(1 + 2p)^{3/2}k^2 \\ &\qquad \text{(by Theorem 1.5(a)(c) and (1.13))} \end{aligned}$$

$$= 1 + \sum_{n=1}^{\infty} (\sigma(n) - \sigma(n/2) + 3\sigma(n/3) + 2\sigma(n/4) - 3\sigma(n/6) - 8\sigma(n/8) + 6\sigma(n/12) - 24\sigma(n/24) + a(n))q^n$$

(by Corollary 1.1(c) and Corollary 1.2(b)),

so for $n \in \mathbb{N}$,

$$\begin{aligned} N(1,2,3,6;n) &= \sigma(n) - \sigma(n/2) + 3\sigma(n/3) + 2\sigma(n/4) - 3\sigma(n/6) \\ &\quad - 8\sigma(n/8) + 6\sigma(n/12) - 24\sigma(n/24) + a(n) \\ &= \begin{cases} (3^{\beta+1} - 2)\sigma(N) + a(n) & \text{if } n \equiv 1 \pmod 2, \\ 2(3^{\beta+1} - 2)\sigma(N) & \text{if } n \equiv 2 \pmod 4, \\ 6(3^{\beta+1} - 2)\sigma(N) & \text{if } n \equiv 0 \pmod 4, \end{cases} \end{aligned}$$

as $a(n) = 0$ when n is even.

Proof of Theorem 1.16. We have

$$\sum_{n=0}^{\infty} N(1,3,3,4;n)q^n = \varphi(q)\varphi(q^3)^2\varphi(q^4) \quad \text{(by (1.9))}$$

$$= \frac{1}{2}(1+2p)^2k^2 + \frac{1}{2}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{5/4}k^2$$

(by Theorem 1.5(a)(c)(d))

$$= 1 + \sum_{n=1}^{\infty} \left(2\sigma(n) - 4\sigma(n/2) - 6\sigma(n/3) + 8\sigma(n/4) + 12\sigma(n/6) - 24\sigma(n/12) + \frac{1}{2}c(n)\right)q^n$$

(by Corollary 1.1(a) and Theorem 1.3(c)),

so for $n \in \mathbb{N}$,

$$\begin{aligned} N(1,3,3,4;n) &= 2\sigma(n) - 4\sigma(n/2) - 6\sigma(n/3) + 8\sigma(n/4) + 12\sigma(n/6) \\ &\quad - 24\sigma(n/12) + \frac{1}{2}c(n) \\ &= \begin{cases} 2\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 1 \pmod 2, \\ 2(2^{\alpha+1} - 3)\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 0 \pmod 2. \end{cases} \end{aligned}$$

Proof of Theorem 1.17 and Theorem 1.25(e). We have

$$\sum_{n=0}^{\infty} N(1,3,4,12;n)q^n = \varphi(q)\varphi(q^3)\varphi(q^4)\varphi(q^{12}) \quad \text{(by (1.9))}$$

$$= \frac{1}{4}(1+2p)^2k^2 + \frac{1}{4}(1-p)^{1/4}(1+p)^{3/4}(1+2p)^{7/4}k^2 + \frac{1}{4}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{5/4}k^2 + \frac{1}{4}(1-p)(1+p)(1+2p)k^2$$

(by Theorem 1.5(a)(c)(d)(f))

$$= 1 + \sum_{n=1}^{\infty} \Big(\sigma(n) - 3\sigma(n/2) - 3\sigma(n/3) + 8\sigma(n/4) + 9\sigma(n/6) - 24\sigma(n/12) + \frac{1}{4}\, b(n) + \frac{1}{4}\, c(n) \Big) q^n$$

(by Corollary 1.1(a)(d) and Theorem 1.3(b)(c)),

so for $n \in \mathbb{N}$,

$$\begin{aligned} N(1,3,4,12;n) &= \sigma(n) - 3\sigma(n/2) - 3\sigma(n/3) + 8\sigma(n/4) + 9\sigma(n/6) \\ &\quad - 24\sigma(n/12) + \frac{1}{4}\, b(n) + \frac{1}{4}\, c(n) \\ &= \begin{cases} \sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 2 \pmod 4, \\ 6(2^{\alpha-1} - 1)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 0 \pmod 4. \end{cases} \end{aligned}$$

Now $x^2 + 3y^2 + 4z^2 + 12t^2 \equiv x^2 + 3y^2 \not\equiv 2 \pmod 4$ so

$$N(1,3,4,12;n) = 0 \quad \text{if } n \equiv 2 \pmod 4.$$

Thus

$$b(n) = -c(n) \quad \text{if } n \equiv 2 \pmod 4.$$

As

$$x^2 + y^2 + 12z^2 + 12t^2 \equiv 2 \pmod 4 \;\Rightarrow\; x \equiv y \pmod 2,$$

$$x^2 + y^2 + 12z^2 + 12t^2 = 2\left(\left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 + 6z^2 + 6t^2 \right),$$

$$2(x^2 + y^2 + 6z^2 + 6t^2) = (x+y)^2 + (x-y)^2 + 12z^2 + 12t^2,$$

we have

$$N(1,1,12,12;n) = N(1,1,6,6;n/2) \quad \text{if } n \equiv 2 \pmod 4.$$

Appealing to Theorems 1.13 and 1.12, we obtain

$$2\sigma(N) + \frac{1}{2}\, b(n) = 2\sigma(N) + 2a(n/2)$$

so that

$$b(n) = 4a(n/2) \quad \text{if } n \equiv 2 \pmod 4.$$

This completes the proof of Theorem 1.25(e).

Proof of Theorem 1.18. We have

$$\sum_{n=0}^{\infty} N(1,4,4,4;n) q^n = \varphi(q)\varphi(q^4)^3 \quad \text{(by (1.9))}$$

$$= \frac{1}{8}(1+2p)^3k^2 + \frac{3}{8}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2$$
$$+ \frac{3}{8}(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 + \frac{1}{8}(1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2$$
(by Theorem 1.5(a)(d))
$$= 1 + \sum_{n=1}^{\infty}\left(\left(1+\left(\frac{-4}{n}\right)\right)\sigma(n) - 3\sigma(n/2) + 10\sigma(n/4) - 32\sigma(n/16)\right)q^n$$
(by Corollary 1.1(b) and Theorem 1.2(a)(c)(d)),

so for $n \in \mathbb{N}$,

$$N(1,4,4,4;n) = \left(1+\left(\frac{-4}{n}\right)\right)\sigma(n) - 3\sigma(n/2) + 10\sigma(n/4) - 32\sigma(n/16)$$
$$= \begin{cases} 2\sigma(N) & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 2,3 \pmod 4, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod 8, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod 8. \end{cases}$$

Proof of Theorem 1.19. We have

$$\sum_{n=0}^{\infty} N(1,4,6,6;n)q^n = \varphi(q)\varphi(q^4)\varphi(q^6)^2 \quad \text{(by (1.9))}$$
$$= \frac{1}{4}(1+2p)^2k^2 + \frac{1}{4}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{5/4}k^2$$
$$+ \frac{1}{4}(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{3/2}k^2 + \frac{1}{4}(1-p)^{5/4}(1+p)^{7/4}(1+2p)^{3/4}k^2$$
(by Theorem 1.5(a)(d)(e))
$$= 1 + \sum_{n=1}^{\infty}\Big(\sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) + 3\sigma(n/6) + 8\sigma(n/8)$$
$$+ 6\sigma(n/12) - 24\sigma(n/24) + a(n) + \frac{1}{4}(1+(-1)^n)c(n)\Big)q^n$$
(by Corollary 1.1(a), Theorem 1.3(c)(e) and Corollary 1.2(a)),

so for $n \in \mathbb{N}$,

$$N(1,4,6,6;n) = \sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) + 3\sigma(n/6) + 8\sigma(n/8)$$
$$+ 6\sigma(n/12) - 24\sigma(n/24) + a(n) + \frac{1}{4}(1+(-1)^n)c(n)$$
$$= \begin{cases} \sigma(N) + a(n) & \text{if } n \equiv 1 \pmod 2, \\ 2\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 2 \pmod 4, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

as $a(n) = 0$ for n even.

Proof of Theorem 1.20 and Theorem 1.25(a)(f)(g). We have

$$\sum_{n=0}^{\infty} N(1,4,12,12;n)q^n = \varphi(q)\varphi(q^4)\varphi(q^{12})^2 \quad \text{(by (1.9))}$$
$$= \frac{1}{8}(1+2p)^2k^2 + \frac{1}{8}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{5/4}k^2$$
$$+ \frac{1}{8}(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{3/2}k^2$$
$$+ \frac{1}{8}(1-p)^{5/4}(1+p)^{7/4}(1+2p)^{3/4}k^2$$
$$+ \frac{1}{4}(1-p)^{1/4}(1+p)^{3/4}(1+2p)^{7/4}k^2 + \frac{1}{4}(1-p)(1+p)(1+2p)k^2$$
$$\text{(by Theorem 1.5(a)(d)(f))}$$
$$= 1 + \sum_{n=1}^{\infty}\Bigg(\frac{1}{2}\sigma(n) - \frac{3}{2}\sigma(n/2) - \frac{3}{2}\sigma(n/3) + 3\sigma(n/4) + \frac{9}{2}\sigma(n/6)$$
$$+ 4\sigma(n/8) - 9\sigma(n/12) - 12\sigma(n/24)$$
$$+ \frac{1}{2}a(n) + \frac{1}{4}b(n) + \frac{1}{8}(1+(-1)^n)c(n)\Bigg)q^n$$

(by Corollary 1.1(a)(d), Corollary 1.2(a) and Theorem 1.3(b)(c)(e)), so for $n \in \mathbb{N}$,

$$N(1,4,12,12;n) = \frac{1}{2}\sigma(n) - \frac{3}{2}\sigma(n/2) - \frac{3}{2}\sigma(n/3) + 3\sigma(n/4) + \frac{9}{2}\sigma(n/6)$$
$$+ 4\sigma(n/8) - 9\sigma(n/12) - 12\sigma(n/24)$$
$$+ \frac{1}{2}a(n) + \frac{1}{4}b(n) + \frac{1}{8}(1+(-1)^n)c(n)$$
$$= \begin{cases} \frac{1}{2}\sigma(N) + \frac{1}{2}a(n) + \frac{1}{4}b(n) & \text{if } n \equiv 1 \pmod 2, \\ \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 2 \pmod 4, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

as $a(n) = 0$ when n is even.

As $x^2 + 4y^2 + 6z^2 + 6t^2 \equiv 1 \pmod 4$ implies $z \equiv t \pmod 2$, and

$$x^2 + 4y^2 + 6z^2 + 6t^2 = x^2 + 4y^2 + 12\left(\frac{z+t}{2}\right)^2 + 12\left(\frac{z-t}{2}\right)^2,$$
$$x^2 + 4y^2 + 12z^2 + 12t^2 = x^2 + 4y^2 + 6(z+t)^2 + 6(z-t)^2,$$

we have

$$N(1,4,6,6;n) = N(1,4,12,12;n) \quad \text{if } n \equiv 1 \pmod 4.$$

Appealing to Theorems 1.19 and 1.20, we deduce

$$\sigma(N) + a(n) = \frac{1}{2}\sigma(N) + \frac{1}{2}a(n) + \frac{1}{4}b(n) \quad \text{if } n \equiv 1 \pmod 4,$$

so that

$$b(n) = 2\sigma(N) + 2a(n) \quad \text{if } n \equiv 1 \pmod 4,$$

which is Theorem 1.25(a).

As $x^2 + 4y^2 + 12z^2 + 12t^2 \equiv x^2 \not\equiv 2, 3 \pmod 4$ we have

$$N(1, 4, 12, 12; n) = 0 \quad \text{if } n \equiv 2, 3 \pmod 4.$$

As $x^2 + 4y^2 + 12z^2 + 12t^2 \equiv 0 \pmod 4$ implies $x \equiv 0 \pmod 2$ we have

$$N(1, 4, 12, 12; n) = N(1, 1, 3, 3; n/4) \quad \text{if } n \equiv 0 \pmod 4.$$

Appealing to Theorems 1.20 and 1.9, we obtain

$$2(2^\alpha - 3)\sigma(N) + \frac{1}{4}\, b(n) + \frac{1}{4}\, c(n) = \begin{cases} 4\sigma(N) & \text{if } n \equiv 4 \pmod 8, \\ 4(2^{\alpha-1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod 8, \end{cases}$$

so that

$$b(n) + c(n) = \begin{cases} 8\sigma(N) & \text{if } n \equiv 4 \pmod 8, \\ -24\sigma(N) & \text{if } n \equiv 0 \pmod 8. \end{cases}$$

As $x^2 + 4y^2 + 6z^2 + 6t^2 \equiv 0 \pmod 4$ implies $z \equiv t \pmod 2$, and

$$x^2 + 4y^2 + 6z^2 + 6t^2 = x^2 + 4y^2 + 12\left(\frac{z+t}{2}\right)^2 + 12\left(\frac{z-t}{2}\right)^2,$$
$$x^2 + 4y^2 + 12z^2 + 12t^2 = x^2 + 4y^2 + 6(z+t)^2 + 6(z-t)^2,$$

we have

$$N(1, 4, 6, 6; n) = N(1, 4, 12, 12; n) \quad \text{if } n \equiv 0 \pmod 4.$$

Hence, by Theorems 1.19 and 1.20, we have

$$2(2^\alpha - 3)\sigma(N) + \frac{1}{2}\, c(n) = 2(2^\alpha - 3)\sigma(N) + \frac{1}{4}\, b(n) + \frac{1}{4}\, c(n) \quad \text{if } n \equiv 0 \pmod 4,$$

so that

$$b(n) = c(n) \quad \text{if } n \equiv 0 \pmod 4.$$

Thus

$$b(n) = c(n) = 4\sigma(N) \quad \text{if } n \equiv 4 \pmod 8,$$
$$b(n) = c(n) = -12\sigma(N) \quad \text{if } n \equiv 0 \pmod 8.$$

This completes the proof of Theorem 1.25(f)(g).

Proof of Theorem 1.21. We have

$$\begin{aligned}\sum_{n=0}^{\infty} N(2,2,3,3;n)q^n &= \varphi(q^2)^2\varphi(q^3)^2 \quad \text{(by (1.9))}\\ &= \frac{1}{2}(1+2p)^2k^2 + \frac{1}{2}(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{1/2}k^2\\ &\qquad \text{(by Theorem 1.5(b)(c))}\\ &= 1 + \sum_{n=1}^{\infty}(2\sigma(n) - 2\sigma(n/2) - 6\sigma(n/3)\\ &\quad - 4\sigma(n/4) + 6\sigma(n/6) + 16\sigma(n/8) + 12\sigma(n/12)\\ &\quad - 48\sigma(n/24) - 2a(n))q^n\\ &\qquad \text{(by Corollary 1.1(a) and Corollary 1.2(c))},\end{aligned}$$

so for $n \in \mathbb{N}$,

$$\begin{aligned}N(2,2,3,3;n) &= 2\sigma(n) - 2\sigma(n/2) - 6\sigma(n/3) - 4\sigma(n/4) + 6\sigma(n/6)\\ &\quad + 16\sigma(n/8) + 12\sigma(n/12) - 48\sigma(n/24) - 2a(n)\\ &= \begin{cases} 2\sigma(N) - 2a(n) & \text{if } n \equiv 1 \pmod 2,\\ 4\sigma(N) & \text{if } n \equiv 2 \pmod 4,\\ 4(2^\alpha - 3)\sigma(N) & \text{if } n \equiv 0 \pmod 4.\end{cases}\end{aligned}$$

Proof of Theorem 1.22. We have

$$\begin{aligned}&\sum_{n=0}^{\infty} N(2,2,3,12;n)q^n = \varphi(q^2)^2\varphi(q^3)\varphi(q^{12}) \quad \text{(by (1.9))}\\ &= \frac{1}{4}(1+2p)^2k^2 + \frac{1}{4}(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{1/2}k^2\\ &\quad + \frac{1}{4}(1-p)^{1/4}(1+p)^{3/4}(1+2p)^{7/4}k^2 + \frac{1}{4}(1-p)^{7/4}(1+p)^{5/4}(1+2p)^{1/4}k^2\\ &\qquad \text{(by Theorem 1.5(b)(c)(f))}\\ &= 1 + \sum_{n=1}^{\infty}\bigg(\sigma(n) - \sigma(n/2) - 3\sigma(n/3)\\ &\quad - 2\sigma(n/4) + 3\sigma(n/6) + 8\sigma(n/8) + 6\sigma(n/12)\\ &\quad - 24\sigma(n/24) - a(n) + \frac{1}{4}(1+(-1)^n)b(n)\bigg)q^n\\ &\qquad \text{(by Corollary 1.1(a), Corollary 1.2(c) and Theorem 1.3(b)(d))},\end{aligned}$$

so for $n \in \mathbb{N}$,

$$\begin{aligned}N(2,2,3,12;n) &= \sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) + 3\sigma(n/6) + 8\sigma(n/8)\\ &\quad + 6\sigma(n/12) - 24\sigma(n/24) - a(n) + \frac{1}{4}(1+(-1)^n)b(n)\end{aligned}$$

$$= \begin{cases} \sigma(N) - a(n) & \text{if } n \equiv 1 \pmod 2, \\ 2\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 2 \pmod 4, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

as $a(n) = 0$ when n is even.

Proof of Theorem 1.23 and Theorem 1.25(c). We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(3,3,4,4;n)q^n &= \varphi(q^3)^2\varphi(q^4)^2 \quad \text{(by (1.9))} \\ &= \frac{1}{4}(1+2p)^2k^2 + \frac{1}{4}(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{1/2}k^2 \\ &\quad + \frac{1}{2}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{5/4}k^2 \quad \text{(by Theorem 1.5(c)(d))} \\ &= 1 + \sum_{n=1}^{\infty}\Big(\sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) + 3\sigma(n/6) \\ &\quad + 8\sigma(n/8) + 6\sigma(n/12) - 24\sigma(n/24) - a(n) + \frac{1}{2}c(n)\Big)q^n \\ &\qquad \text{(by Corollary 1.1(a), Corollary 1.2(c) and Theorem 1.3(c))}, \end{aligned}$$

so for $n \in \mathbb{N}$,

$$\begin{aligned} N(3,3,4,4;n) &= \sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) + 3\sigma(n/6) \\ &\quad + 8\sigma(n/8) + 6\sigma(n/12) - 24\sigma(n/24) - a(n) + \frac{1}{2}c(n) \\ &= \begin{cases} \sigma(N) - a(n) + \frac{1}{2}c(n) & \text{if } n \equiv 1 \pmod 2, \\ 2\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 2 \pmod 4, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 0 \pmod 4. \end{cases} \end{aligned}$$

As $3x^2 + 3y^2 + 4z^2 + 4t^2 \not\equiv 1 \pmod 4$ we have

$$N(3,3,4,4;n) = 0 \quad \text{if } n \equiv 1 \pmod 4.$$

Hence

$$\sigma(N) - a(n) + \frac{1}{2}c(n) = 0 \quad \text{if } n \equiv 1 \pmod 4,$$

that is,

$$c(n) = -2\sigma(N) + 2a(n) \quad \text{if } n \equiv 1 \pmod 4,$$

which is Theorem 1.25(c).

Proof of Theorem 1.24 and Theorem 1.25(d). We have

$$\sum_{n=0}^{\infty} N(3,4,4,12;n)q^n = \varphi(q^3)\varphi(q^4)^2\varphi(q^{12}) \quad \text{(by (1.9))}$$
$$= \frac{1}{8}(1+2p)^2k^2 + \frac{1}{4}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{5/4}k^2$$
$$+\frac{1}{8}(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{1/2}k^2 + \frac{1}{8}(1-p)^{1/4}(1+p)^{3/4}(1+2p)^{7/4}k^2$$
$$+\frac{1}{4}(1-p)(1+p)(1+2p)k^2 + \frac{1}{8}(1-p)^{7/4}(1+p)^{5/4}(1+2p)^{1/4}k^2$$
$$\text{(by Corollary 1.1(a)(d), Corollary 1.2(c) and Theorem 1.3(b)(c)(d))}$$
$$= 1+\sum_{n=1}^{\infty}\Big(\frac{1}{2}\sigma(n) - \frac{3}{2}\sigma(n/2) - \frac{3}{2}\sigma(n/3) + 3\sigma(n/4) + \frac{9}{2}\sigma(n/6) + 4\sigma(n/8)$$
$$- 9\sigma(n/12) - 12\sigma(n/24) - \frac{1}{2}a(n) + \frac{1}{8}(1+(-1)^n)b(n) + \frac{1}{4}c(n)\Big)q^n,$$

so for $n \in \mathbb{N}$,

$$\begin{aligned} N(3,4,4,12;n) &= \frac{1}{2}\sigma(n) - \frac{3}{2}\sigma(n/2) - \frac{3}{2}\sigma(n/3) + 3\sigma(n/4) + \frac{9}{2}\sigma(n/6) \\ &\quad + 4\sigma(n/8) - 9\sigma(n/12) - 12\sigma(n/24) \\ &\quad - \frac{1}{2}a(n) + \frac{1}{8}(1+(-1)^n)b(n) + \frac{1}{4}c(n) \\ &= \begin{cases} \frac{1}{2}\sigma(N) - \frac{1}{2}a(n) + \frac{1}{4}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 2 \pmod 4, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 0 \pmod 4. \end{cases} \end{aligned}$$

As $3x^2 + 4y^2 + 4z^2 + 12t^2 \equiv 3x^2 \equiv 0, 3 \pmod 4$, we have

$$N(3,4,4,12;n) = 0 \quad \text{if } n \equiv 1, 2 \pmod 4.$$

Finally, as $2x^2 + 2y^2 + 3z^2 + 12t^2 \equiv 3 \pmod 4$ implies $x \equiv y \pmod 2$, and

$$2x^2 + 2y^2 + 3z^2 + 12t^2 = 3z^2 + 4\left(\frac{x+y}{2}\right)^2 + 4\left(\frac{x-y}{2}\right)^2 + 12t^2,$$
$$3x^2 + 4y^2 + 4z^2 + 12t^2 = 2(y+z)^2 + 2(y-z)^2 + 3x^2 + 12t^2,$$

we have

$$N(2,2,3,12;n) = N(3,4,4,12;n) \quad \text{if } n \equiv 3 \pmod 4.$$

Appealing to Theorems 1.22 and 1.24, we obtain

$$\sigma(N) - a(n) = \frac{1}{2}\sigma(N) - \frac{1}{2}a(n) + \frac{1}{4}c(n) \quad \text{if } n \equiv 3 \pmod 4,$$

so that

$$c(n) = 2\sigma(N) - 2a(n) \quad \text{if } n \equiv 3 \pmod 4,$$

which is Theorem 1.25(d).

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