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THE SIMPLEST PROOF OF JACOBI'S SIX SQUARES THEOREM

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Abstract

A very simple proof is given of Jacobi's formula for the number of representations of a positive integer as the sum of six squares.

1. Introduction

For a nonnegative integer n and a positive integer k, we let $r_k(n)$ denote the number of representations of n as the sum of k squares, that is,

$$r_k(n) := \text{card}\{(x_1, \ldots, x_k) \in \mathbb{Z}^k \mid n = x_1^2 + \cdots + x_k^2\}.$$

Clearly $r_k(0) = 1$. Jacobi [11] gave formulae for $r_k(n)$ $(n \in \mathbb{N})$ for $k = 2, 4, 6$ and 8. Berndt [2, p. 63] has expressed the view that perhaps the most difficult of these formulae to prove is that for $r_6(n)$. We present

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what we believe to be the simplest proof of Jacobi's formula for $r_6(n)$. Analytic proofs of Jacobi's formula are given in [2, pp. 63-67], [4], [5], [6], [7] and [9, p. 121], and arithmetic proofs in [12] and [13, pp. 436-440]. None of these proofs is particularly simple and most are quite long.

Jacobi's formula for $r_6(n)$ $(n \in \mathbb{N})$ is (see for example [10, p. 314])

$$r_6(n) = 16 \sum_{d \mid n} \left(\frac{-4}{n/d}\right) d^2 - 4 \sum_{d \mid n} \left(\frac{-4}{d}\right) d^2, \tag{1}$$

where d runs through the positive integers dividing n, and for a positive integer k

$$\left(\frac{-4}{k}\right) = \begin{cases} +1, & \text{if } k \equiv 1 \pmod 4, \\ -1, & \text{if } k \equiv 3 \pmod 4, \\ 0, & \text{if } k \equiv 0 \pmod 2, \end{cases}$$

is the Legendre-Jacobi-Kronecker symbol for discriminant -4.

2. Notation and Elementary Properties

Our proof of (1) is based on elementary properties of the theta functions φ and ψ defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}, \quad q \in \mathbb{C}, \quad |q| < 1,$$

see [2, p. 6], as well as a classical identity due to Carlitz [3, eq. (1.3), p. 168], which is a limiting form of an identity of Bailey [1, eq. (5), p. 159], namely,

$$\frac{a(1+a)}{(1-a)^3} + \sum_{n=1}^{\infty} \frac{n^2 z^n}{1-z^n} (a^n - a^{-n})$$

$$= \frac{a(1+a)}{(1-a)^3} \prod_{n=1}^{\infty} \frac{(1-z^n a^2)(1-z^n a^{-2})(1-z^n)^6}{(1-z^n a)^4 (1-z^n a^{-1})^4},$$

$$a \in \mathbb{C},\ a \neq 0, 1,\ z \in \mathbb{C},\ |z| < \min\left(\frac{1}{|a|}, |a|\right). \tag{2}$$

An elementary proof of (2) has been given by Dobbie [8, pp. 194-195].

The basic properties of $\varphi(q)$ and $\psi(q)$ that we need are the following

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \quad [2, \text{ p. } 71], \tag{3}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \quad [2, \text{ p. } 72], \tag{4}$$

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8), \quad [2, \text{ p. } 71]. \tag{5}$$

In addition we need Jacobi's infinite product representations of $\varphi(q)$ and $\psi(q)$. It is convenient to set

$$P_k = P_k(q) := \prod_{n=1}^{\infty}(1 - q^{kn}), \quad k \in \mathbb{N}. \tag{6}$$

In the notation of (6), Jacobi proved

$$\varphi(q) = \frac{P_2^5}{P_1^2 P_4^2}, \quad \psi(q) = \frac{P_2^2}{P_1}, \quad \varphi(-q) = \frac{P_1^2}{P_2}, \tag{7}$$

see for example [10, pp. 283-284].

3. Two Lemmas

Our proof of (1), which is given in Section 4, will follow immediately from the two lemmas proved in this section.

Lemma 1.

$$\varphi^2(q)\varphi^4(-q) = 1 - 4\sum_{n=1}^{\infty}\left(\sum_{d \mid n}\left(\frac{-4}{d}\right)d^2\right)q^n.$$

Proof. Taking $z = q$ and $a = i$ in (2), we obtain after a little simplification

$$1 - 4\sum_{n=1}^{\infty}\left(\frac{-4}{n}\right)\frac{n^2 q^n}{1 - q^n} = \prod_{n=1}^{\infty}\frac{(1 - q^n)^4(1 - q^{2n})^6}{(1 - q^{4n})^4} = \frac{P_1^4 P_2^6}{P_4^4}. \tag{8}$$

Using $\frac{q^n}{1 - q^n} = \sum_{m=1}^{\infty} q^{mn}$ in the left hand side of (8), we see that the left hand side of (8) is the right hand side of the asserted formula. The right hand side of (8) is equal to the left hand side of the asserted formula by (7).

Lemma 2.

$$\varphi^6(q) - \varphi^2(q)\varphi^4(-q) = 16\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-4}{n/d}\right)d^2\right)q^n.$$

Proof. Taking $z = q^4$ and $a = q$ in (2), we obtain after a little simplification

$$\frac{q(1+q)}{(1-q)^3} + \sum_{n=1}^{\infty}\frac{n^2q^{4n}}{1-q^{4n}}(q^n - q^{-n})$$

$$= q\prod_{n=1}^{\infty}\frac{(1-q^{2n})^6(1-q^{4n})^4}{(1-q^n)^4} = q\,\frac{P_2^6P_4^4}{P_1^4}. \tag{9}$$

The left hand side of (9) is

$$\sum_{n=1}^{\infty}n^2q^n + \sum_{n=1}^{\infty}\frac{n^2q^{4n}}{1-q^{4n}}(q^n - q^{-n})$$

$$= \sum_{n=1}^{\infty}n^2\left(\frac{q^n}{1-q^{4n}} - \frac{q^{3n}}{1-q^{4n}}\right)$$

$$= \sum_{d=1}^{\infty}d^2\left(\sum_{\substack{e=1\\ e\equiv 1 \pmod 4}}^{\infty}q^{de} - \sum_{\substack{e=1\\ e\equiv 3 \pmod 4}}^{\infty}q^{de}\right)$$

$$= \sum_{d,e=1}^{\infty}d^2\left(\frac{-4}{e}\right)q^{de}$$

$$= \sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-4}{n/d}\right)d^2\right)q^n.$$

Thus (9) becomes

$$\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-4}{n/d}\right)d^2\right)q^n = q\,\frac{P_2^6P_4^4}{P_1^4}. \tag{10}$$

Next, appealing to (3), (4), (5) and (7), we obtain

$$
\begin{aligned}
16qP_1^{24} &= 16q\,\frac{P_2^{20}}{P_1^8P_4^8}\,\frac{P_4^{10}}{P_2^4P_8^4}\,\frac{P_8^5}{P_4^2P_{16}^2}\,\frac{P_1^{32}}{P_2^{16}}\,\frac{P_{16}^2}{P_8} \\
&= 16q\varphi^4(q)\varphi^2(q^2)\varphi(q^4)\varphi^{16}(-q)\psi(q^8) \\
&= \varphi^4(q)(\varphi^2(q)+\varphi^2(-q))(\varphi(q)+\varphi(-q))\varphi^{16}(-q)(\varphi(q)-\varphi(-q)) \\
&= \varphi^4(q)\varphi^{16}(-q)(\varphi^4(q)-\varphi^4(-q)) \\
&= \varphi^8(q)\varphi^{16}(-q)-\varphi^4(q)\varphi^{20}(-q) \\
&= \frac{P_1^{16}P_2^{24}}{P_4^{16}}-\frac{P_1^{32}}{P_4^8}
\end{aligned}
$$

so that

$$
16q = \frac{P_2^{24}}{P_1^8P_4^{16}}-\frac{P_1^8}{P_4^8}. \tag{11}
$$

Finally, by (10), (11) and (7), we deduce

$$
\begin{aligned}
16\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-4}{n/d}\right)d^2\right)q^n &= 16q\,\frac{P_2^6P_4^4}{P_1^4} = \left(\frac{P_2^{24}}{P_1^8P_4^{16}}-\frac{P_1^8}{P_4^8}\right)\frac{P_2^6P_4^4}{P_1^4} \\
&= \frac{P_2^{30}}{P_1^{12}P_4^{12}}-\frac{P_1^4P_2^6}{P_4^4} = \varphi^6(q)-\varphi^2(q)\varphi^4(-q).
\end{aligned}
$$

4. Proof of Jacobi's Six Squares Formula

Appealing to Lemmas 1 and 2, we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} r_6(n)q^n &= \varphi^6(q) = \varphi^2(q)\varphi^4(-q)+\varphi^6(q)-\varphi^2(q)\varphi^4(-q) \\
&= 1-4\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-4}{d}\right)d^2\right)q^n+16\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-4}{n/d}\right)d^2\right)q^n.
\end{aligned}
$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain (1).

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