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ON INTEGERS WITH PRIME FACTORS RESTRICTED TO CERTAIN CONGRUENCE CLASSES

BLAIR K. SPEARMAN and KENNETH S. WILLIAMS

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Abstract

An asymptotic formula is given for the sum $\sum_{n\leq x}^{*} n^{a}$, where $a \geq -1$, and the asterisk indicates that the summation is restricted to those positive integers n whose prime factors belong to certain congruence classes.

1. Introduction

Let $k, r \in \mathbb{N}$ with $1 \leq r \leq \phi(k)$, where $\phi(k)$ denotes Euler's phi function. Let $l_1, \ldots, l_r$ be integers such that $1 \leq l_1 < \ldots < l_r \leq k$ and $(l_1, k) = (l_2, k) = \cdots = (l_r, k) = 1$. Let

$$T(l_1, l_2, \ldots, l_r, k) = \{p(\text{prime}) \mid p \equiv l_1, l_2, \ldots, l_r \pmod{k}\} \tag{1.1}$$

and

$$S(l_1, l_2, \ldots, l_r, k) = \{n \in \mathbb{N} \mid p(\text{prime}) \mid n \Rightarrow p \equiv l_1, l_2, \ldots, l_r \pmod{k}\}. \tag{1.2}$$

Let χ be a Dirichlet character modulo k. The character $\overline{\chi}$ is defined

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by $\bar{\chi}(n) = \overline{\chi(n)}$ the complex conjugate of $\chi(n)$. The principal character (mod k) is denoted by χ_0. It is well known that

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \neq 0 \quad \text{for } \chi \neq \chi_0. \tag{1.3}$$

For each character χ (mod k) and each prime p, we set

$$k_\chi(p) = p\left\{1 - \left(1 - \frac{\chi(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-\chi(p)}\right\}, \tag{1.4}$$

and define a completely multiplicative function $k_\chi : \mathbb{N} \to \mathbb{C}$ by

$$k_\chi(n) = \prod_{p^\alpha \| n} k_\chi(p)^\alpha, \quad n \in \mathbb{N}. \tag{1.5}$$

It is shown in [5, p. 355] that for all χ

$$K(1, \chi) = \sum_{n=1}^{\infty} \frac{k_\chi(n)}{n} = \prod_p \left(1 - \frac{k_\chi(p)}{p}\right)^{-1} \neq 0. \tag{1.6}$$

We prove the following asymptotic formula. As usual $\Gamma(x)$ denotes the gamma function. We write S for $S(l_1, l_2, \ldots, l_r, k)$.

Theorem 1.1. *Let* $\varepsilon \in \mathbb{R}$ *be such that* $0 < \varepsilon < 1$. *Set*

$$C := \frac{1}{\Gamma(1 + (r/\phi(k)))}\left(\frac{\phi(k)}{k}\right)^{r/\phi(k)} \prod_{\chi \neq \chi_0} \left(\frac{L(1, \chi)}{K(1, \chi)}\right)^{\sum_{j=1}^{r} \bar{\chi}(l_j)/\phi(k)}. \tag{1.7}$$

Then, as $x \to \infty$, *we have*

$$\sum_{\substack{n \leq x \\ n \in S}} \frac{1}{n} = C(\log x)^{r/\phi(k)} + O((\log x)^{r/\phi(k) - 1 + \varepsilon}) \tag{1.8}$$

and for each fixed $\lambda > 0$

$$\sum_{\substack{n \leq x \\ n \in S}} n^{\lambda - 1} = \frac{1}{\lambda} \frac{r}{\phi(k)} C x^\lambda (\log x)^{r/\phi(k) - 1} + O(x^\lambda (\log x)^{r/\phi(k) - 2 + \varepsilon}), \tag{1.9}$$

where the constants implied by the O-symbols depend at most on ε, k, $l_1, \ldots, l_r$ *and* λ.

2. Proof of Theorem 1.1

Define the constant C as in (1.7). Let $\varepsilon \in \mathbb{R}$ be such that $0 < \varepsilon < 1$. We begin by recalling a theorem due to Odoni [2, Theorem II, p. 205; Note added in proof, p. 216].

Proposition 2.1. *Let* $f : \mathbb{N} \to \mathbb{R}$ *be multiplicative with* $f(n) \geq 0$, *for all* $n \in \mathbb{N}$. *Suppose that there exist constants* $a_1 > 1$ *and* $a_2 > 1$ *such that*

$$0 \leq f(p^k) \leq a_1 k^{a_2}, \tag{2.1}$$

for all primes p *and all* $k \in \mathbb{N}$, *and also that there exist constants* τ *and* β *with* $\tau > 0$ *and* $0 < \beta < 1$ *such that*

$$\sum_{p \leq x} f(p) = \tau \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{1+\beta}}\right), \tag{2.2}$$

as $x \to \infty$, *then there is a constant* $B > 0$ *such that*

$$\sum_{n \leq x} f(n) n^{-1} = B(\log x)^{\tau} + O((\log x)^{\tau - \beta}), \tag{2.3}$$

as $x \to \infty$. *Further, for each fixed* $\lambda > 0$, *we have*

$$\sum_{n \leq x} f(n) n^{\lambda - 1} = \lambda^{-1} \tau B x^{\lambda} (\log x)^{\tau - 1} + O(x^{\lambda} (\log x)^{\tau - 1 - \beta}), \tag{2.4}$$

as $x \to \infty$.

We define

$$f(n) = \begin{cases} 1, & \text{if } n \in S, \\ 0, & \text{if } n \notin S, \end{cases}$$

where $S = S(l_1, l_2, \ldots, l_r, k)$ is defined in (1.2). Clearly $f : \mathbb{N} \to \{0, 1\}$ is a multiplicative function, which satisfies (2.1) with $a_1 = a_2 = 2$. Further, by the prime number theorem for arithmetic progressions [3, p. 139], (2.2) is satisfied with $\tau = r/\phi(k)$ and $\beta = 1 - \varepsilon$. Appealing to Proposition 2.1, we obtain from (2.3)

$$\sum_{\substack{n \le x \\ n \in S}} \frac{1}{n} = B(\log x)^{r/\phi(k)} + O((\log x)^{r/\phi(k)-1+\varepsilon}), \tag{2.5}$$

as $x \to \infty$, for some constant $B > 0$, and then from (2.4) we obtain for $\lambda > 0$

$$\sum_{\substack{n \le x \\ n \in S}} n^{\lambda-1} = \frac{1}{\lambda}\frac{r}{\phi(k)} Bx^{\lambda}(\log x)^{r/\phi(k)-1} + O(x^{\lambda}(\log x)^{r/\phi(k)-2+\varepsilon}), \tag{2.6}$$

as $x \to \infty$. Rieger [4, Satz 1, p. 247] has proved the following theorem.

Proposition 2.2. *Let T be an infinite set of prime numbers p such that*

$$\sum_{\substack{p \le x \\ p \in T}} \frac{\log p}{p} = \tau\left(1 + O\left(\frac{1}{(\log x)^{\delta}}\right)\right)\log x, \tag{2.7}$$

as $x \to \infty$, for some $\tau = \tau(T) > 0$ and some $\delta = \delta(T) > 0$. Then

$$\sum_{\substack{n \le x \\ p \mid n \Rightarrow p \in T}} \frac{1}{n} = \frac{e^{-\gamma\tau}}{\Gamma(1+\tau)}\left(1 + O\left(\frac{1}{\log\log x}\right)\right)\prod_{\substack{p \le x \\ p \in T}}\left(1 - \frac{1}{p}\right)^{-1}, \tag{2.8}$$

as $x \to \infty$, where γ is Euler's constant.

Taking $T = T(l_1, l_2, \ldots, l_r, k)$ (defined in (1.1)), we see that (2.7) is satisfied with $\tau = r/\phi(k)$ and $\delta = 1$, see for example [1, p. 449]. Then (2.8) gives

$$\sum_{\substack{n \le x \\ n \in S}} \frac{1}{n} = \frac{e^{-\gamma r/\phi(k)}}{\Gamma(1 + (r/\phi(k)))}\left(1 + O\left(\frac{1}{\log\log x}\right)\right)\prod_{\substack{p \le x \\ p \in T}}\left(1 - \frac{1}{p}\right)^{-1}, \tag{2.9}$$

as $x \to \infty$. Williams [5, Theorem 1, p. 356] has shown the following result.

Proposition 2.3.

$$\prod_{\substack{p\le x\\ p\equiv l \pmod k}}\left(1-\frac{1}{p}\right)=\left(e^{-\gamma}\frac{k}{\phi(k)}\prod_{\chi\ne\chi_0}\left(\frac{K(1,\chi)}{L(1,\chi)}\right)^{\overline{\chi}(l)}\right)^{1/\phi(k)}(\log x)^{-1/\phi(k)}$$

$$+\,O((\log x)^{-1/\phi(k)-1}),$$

as $x\to\infty$, *where the product on the right-hand side is taken over all characters* $\chi \pmod k$ *different from the principal character* $\chi_0 \pmod k$, *and the constant implied by the O-symbol depends only on k.*

From Proposition 2.3 we obtain

$$\prod_{\substack{p\le x\\ p\in T}}\left(1-\frac{1}{p}\right)=M(\log x)^{-r/\phi(k)}+O((\log x)^{-r/\phi(k)-1}) \tag{2.10}$$

as $x\to\infty$, where

$$M=e^{-\gamma r/\phi(k)}\left(\frac{k}{\phi(k)}\right)^{r/\phi(k)}\prod_{\chi\ne\chi_0}\left(\frac{K(1,\chi)}{L(1,\chi)}\right)^{\sum_{j=1}^{r}\overline{\chi}(l_j)/\phi(k)} \tag{2.11}$$

Appealing to (1.7) and (2.11), we see that

$$M^{-1}=\Gamma(1+(r/\phi(k))e^{\gamma r/\phi(k)}C. \tag{2.12}$$

Then, using (2.10) and (2.12) in (2.9), we obtain

$$\sum_{\substack{n\le x\\ n\in S}}\frac{1}{n}=C(\log x)^{r/\phi(k)}\left(1+O\left(\frac{1}{\log\log x}\right)\right). \tag{2.13}$$

as $x\to\infty$. Comparing (2.5) and (2.13) we deduce that $B=C$. Then (2.5) gives (1.8) and (2.6) gives (1.9).

We note that (1.8) is stronger than the estimate (2.13) derived from Rieger's work (Proposition 2.2).

3. Applications

We give explicit versions of Theorem 1.1 when $k=3$ and $k=4$. If χ

is a real character (mod k) and p denotes a prime, then

$$k_\chi(p) = \begin{cases} 0, & \text{if } \chi(p) = 1 \text{ or } 0, \\ \frac{1}{p}, & \text{if } \chi(p) = -1, \end{cases}$$

so that

$$K(1, \chi) = \prod_{\chi(p)=-1} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

It is convenient to define

$$l(a, b) = \prod_{p \equiv a \pmod b} \left(1 - \frac{1}{p^2}\right), \quad 1 \le a \le b, \quad (a, b) = 1.$$

First we treat $k = 3$. There is exactly one nonprincipal character (mod 3), namely

$$\chi(n) = \left(\frac{-3}{n}\right).$$

Thus

$$K(1, \chi) = l(2, 3)^{-1} = \frac{4\pi^2}{27} l(1, 3).$$

Also

$$L(1, \chi) = \sum_{n=1}^{\infty} \left(\frac{-3}{n}\right)\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \cdots = \frac{\pi}{3\sqrt{3}}.$$

With $r = 1$ and $l_1 = 1$, we have

$$\prod_{\chi \ne \chi_0} \left(\frac{L(1, \chi)}{K(1, \chi)}\right)^{\sum_{j=1}^{r} \bar{\chi}(l_j)/\phi(k)} = \frac{\pi^{1/2}}{3^{3/4}} l(2, 3)^{1/2}.$$

Thus, by Theorem 1.1, we have

$$\sum_{\substack{n \le x \\ p \mid n \Rightarrow p \equiv 1 \pmod 3}} \frac{1}{n} = \frac{2^{3/2}}{3^{5/4}} l(2, 3)^{1/2} (\log x)^{1/2} + O((\log x)^{-1/2+\varepsilon})$$

and for $\lambda > 0$

$$\sum_{\substack{n\le x\\ p\mid n\Rightarrow p\equiv 1\,(\mathrm{mod}\,3)}} n^{\lambda-1} = \frac{2^{1/2}}{\lambda 3^{5/4}}\, l(2,\,3)^{1/2}\,\frac{x^{\lambda}}{(\log x)^{1/2}} + O\left(\frac{x^{\lambda}}{(\log x)^{3/2-\varepsilon}}\right).$$

In particular with $\lambda = 1$, we have

$$\sum_{\substack{n\le x\\ p\mid n\Rightarrow p\equiv 1\,(\mathrm{mod}\,3)}} 1 = \frac{2^{1/2}}{3^{5/4}}\, l(2,\,3)^{1/2}\,\frac{x}{(\log x)^{1/2}} + O\left(\frac{x}{(\log x)^{3/2-\varepsilon}}\right).$$

With $r = 1$ and $l_1 = 2$, we have

$$\prod_{\chi\ne\chi_0}\left(\frac{L(1,\,\chi)}{K(1,\,\chi)}\right)^{\sum_{j=1}^{r}\overline{\chi}(l_j)/\phi(k)} = \frac{2\pi^{1/2}}{3^{3/4}}\cdot l(1,\,3)^{1/2}.$$

Then, by Theorem 1.1, we obtain

$$\sum_{\substack{n\le x\\ p\mid n\Rightarrow p\equiv 2\,(\mathrm{mod}\,3)}} \frac{1}{n} = \frac{2^{5/2}}{3^{5/4}}\, l(1,\,3)^{1/2}(\log x)^{1/2} + O((\log x)^{-1/2+\varepsilon})$$

and for $\lambda > 0$

$$\sum_{\substack{n\le x\\ p\mid n\Rightarrow p\equiv 2\,(\mathrm{mod}\,3)}} n^{\lambda-1} = \frac{2^{3/2}}{\lambda 3^{5/4}}\, l(1,\,3)^{1/2}\,\frac{x^{\lambda}}{(\log x)^{1/2}} + O\left(\frac{x^{\lambda}}{(\log x)^{3/2-\varepsilon}}\right).$$

In particular with $\lambda = 1$, we have

$$\sum_{\substack{n\le x\\ p\mid n\Rightarrow p\equiv 2\,(\mathrm{mod}\,3)}} 1 = \frac{2^{3/2}}{3^{5/4}}\, l(1,\,3)^{1/2}\,\frac{x}{(\log x)^{1/2}} + O\left(\frac{x}{(\log x)^{3/2-\varepsilon}}\right).$$

Finally we treat the case $k = 4$. There is exactly one nonprincipal character (mod 4), namely

$$\chi(n) = \left(\frac{-4}{n}\right).$$

Thus

$$K(1,\ \chi) = \prod_{p \equiv 3 \pmod 4} \left(1 - \frac{1}{p^2}\right)^{-1} = l(3,\ 4)^{-1} = \frac{\pi^2}{8} l(1,\ 4).$$

Also

$$L(1,\ \chi) = \sum_{n=1}^{\infty} \left(\frac{-4}{n}\right)\frac{1}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

With $r = 1$ and $l_1 = 1$, we have

$$\prod_{\chi \neq \chi_0} \left(\frac{L(1,\ \chi)}{K(1,\ \chi)}\right)^{\sum_{j=1}^{r} \bar{\chi}(l_j)/\phi(k)} = \frac{\pi^{1/2}}{2} l(3,\ 4)^{1/2}.$$

Thus, by Theorem 1.1, we have

$$\sum_{\substack{n \leq x \\ p \mid n \Rightarrow p \equiv 1 \pmod 4}} \frac{1}{n} = \frac{1}{2^{1/2}} l(3,\ 4)^{1/2} (\log x)^{1/2} + O((\log x)^{-1/2+\varepsilon})$$

and for $\lambda > 0$

$$\sum_{\substack{n \leq x \\ p \mid n \Rightarrow p \equiv 1 \pmod 4}} n^{\lambda - 1} = \frac{1}{\lambda 2^{3/2}} l(3,\ 4)^{1/2} \frac{x^{\lambda}}{(\log x)^{1/2}} + O\left(\frac{x^{\lambda}}{(\log x)^{3/2-\varepsilon}}\right).$$

In particular with $\lambda = 1$, we have

$$\sum_{\substack{n \leq x \\ p \mid n \Rightarrow p \equiv 1 \pmod 4}} 1 = \frac{1}{2^{3/2}} l(3,\ 4)^{1/2} \frac{x}{(\log x)^{1/2}} + O\left(\frac{x}{(\log x)^{3/2-\varepsilon}}\right).$$

With $r = 1$ and $l_1 = 3$, we have

$$\prod_{\chi \neq \chi_0} \left(\frac{L(1,\ \chi)}{K(1,\ \chi)}\right)^{\sum_{j=1}^{r} \bar{\chi}(l_j)/\phi(k)} = \frac{\pi^{1/2}}{2^{1/2}} l(1,\ 4)^{1/2}.$$

Thus, by Theorem 1.1, we have

$$\sum_{\substack{n \leq x \\ p \mid n \Rightarrow p \equiv 3 \pmod 4}} \frac{1}{n} = l(1,\ 4)^{1/2} (\log x)^{1/2} + O((\log x)^{-1/2+\varepsilon})$$

and for $\lambda > 0$

$$\sum_{\substack{n \le x \\ p \mid n \Rightarrow p \equiv 3 \pmod 4}} n^{\lambda-1} = \frac{1}{2\lambda} l(1, 4)^{1/2} \frac{x^{\lambda}}{(\log x)^{1/2}} + O\left(\frac{x^{\lambda}}{(\log x)^{3/2-\varepsilon}}\right).$$

In particular with $\lambda = 1$, we have

$$\sum_{\substack{n \le x \\ p \mid n \Rightarrow p \equiv 3 \pmod 4}} 1 = \frac{1}{2} l(1, 4)^{1/2} \frac{x}{(\log x)^{1/2}} + O\left(\frac{x}{(\log x)^{3/2-\varepsilon}}\right).$$

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Department of Mathematics and Statistics
University of British Columbia Okanagan
Kelowna, B. C. Canada V1V 1V7
e-mail: blair.spearman@ubc.ca

School of Mathematics and Statistics
Carleton University
Ottawa, Ontario Canada K1S 5B6
e-mail: kwilliam@connect.carleton.ca