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AN INFINITE CLASS OF IDENTITIES

AYŞE ALACA, ŞABAN ALACA AND KENNETH S. WILLIAMS

An infinite class of identities relating infinite products is proved. It is shown that this class contains a famous identity of Jacobi.

1. INTRODUCTION

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Let $\mathbb{C}$ denote the field of complex numbers. Throughout this paper $q \in \mathbb{C}$ is such that $|q| < 1$.

Let $a(k_1, k_2, k_3, k_4, k_5)$ $\big((k_1, k_2, k_3, k_4, k_5) \in \mathbb{N}_0^5\big)$ be complex numbers (not all zero and nonzero for only finitely many $(k_1, k_2, k_3, k_4, k_5) \in \mathbb{N}_0^5$) such that

$$(1.1) \qquad \sum_{(k_1,k_2,k_3,k_4,k_5)\in\mathbb{N}_0^5} a(k_1, k_2, k_3, k_4, k_5)x^{k_1}(1+x)^{k_2}(1-x)^{k_3}(1+2x)^{k_4}(2+x)^{k_5} = 0$$

identically in x. Examples are

$$(1.2) \qquad \begin{cases} a(0, 1, 3, 0, 0) = 1, \\ a(1, 0, 0, 0, 3) = 1, a(0, 0, 0, 3, 0) = -1, \\ a(k_1, k_2, k_3, k_4, k_5) = 0, \text{ otherwise,} \end{cases}$$

as

$$(1+x)(1-x)^3 + x(2+x)^3 - (1+2x)^3 = 0;$$

$$(1.3) \qquad \begin{cases} a(0, 1, 1, 0, 0) = 1, \\ a(0, 0, 0, 1, 0) = -1, \\ a(1, 0, 0, 0, 1) = 1, \\ a(k_1, k_2, k_3, k_4, k_5) = 0, \text{ otherwise,} \end{cases}$$

as

$$(1+x)(1-x) - (1+2x) + x(2+x) = 0;$$

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$$(1.4)\qquad \begin{cases} a(0,0,2,0,0) = 1, \\ a(1,0,0,0,0) = 4, \\ a(0,2,0,0,0) = -1, \\ a(k_1,k_2,k_3,k_4,k_5) = 0, \text{ otherwise,} \end{cases}$$

as

$$(1-x)^2 + 4x - (1+x)^2 = 0;$$

and

$$(1.5)\qquad \begin{cases} a(0,0,0,1,0) = 1, \\ a(0,0,0,0,1) = 1, \\ a(0,1,0,0,0) = -3, \\ a(k_1,k_2,k_3,k_4,k_5) = 0, \text{ otherwise,} \end{cases}$$

as

$$(1+2x) + (2+x) - 3(1+x) = 0.$$

The following example shows that there are infinitely many choices for the $a(k_1,k_2,k_3,k_4,k_5)$. For each $m \in \mathbb{N}$ we can choose

$$(1.6)\qquad a(k_1,k_2,k_3,k_4,k_5) = \begin{cases} \binom{m}{k_1}, & \text{if } k_1 + k_2 = m,\ k_3 = k_4 = k_5 = 0, \\ -1, & \text{if } k_4 = m,\ k_1 = k_2 = k_3 = k_5 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

as

$$\sum_{\substack{(k_1,k_2) \in \mathbb{N}_0^2 \\ k_1 + k_2 = m}} \binom{m}{k_1} x^{k_1}(1+x)^{k_2} - (1+2x)^m = 0$$

by the binomial theorem.

In Section 2 we prove the following identity relating infinite products.

THEOREM 1.1. *Suppose that* $a(k_1,k_2,k_3,k_4,k_5)$ $\left((k_1,k_2,k_3,k_4,k_5) \in \mathbb{N}_0^5\right)$ *are complex numbers (not all zero and nonzero for only finitely many* $(k_1,k_2,k_3,k_4,k_5) \in \mathbb{N}_0^5$*) satisfying* (1.1). *Then*

$$\sum_{(k_1,k_2,k_3,k_4,k_5)\in\mathbb{N}_0^5} a(k_1,k_2,k_3,k_4,k_5) 2^{k_1+k_5} q^{k_1} \prod_{n=1}^{\infty} (1-q^n)^{-k_1-2k_2+2k_3-4k_4-k_5}$$
$$\times (1-q^{2n})^{3k_1+3k_2+k_3+10k_4+k_5} (1-q^{3n})^{3k_1+6k_2+2k_3+4k_4+3k_5}$$
$$\times (1-q^{4n})^{-2k_1-k_2-k_3-4k_4+2k_5} (1-q^{6n})^{-9k_1-9k_2-7k_3-10k_4-7k_5}$$
$$\times (1-q^{12n})^{6k_1+3k_2+3k_3+4k_4+2k_5} = 0.$$

In Section 3 we show that Jacobi's famous "aequatio identica satis abstrusa" [**2**, p. 147]

$$\prod_{n=1}^{\infty}(1+q^{2n-1})^8 = \prod_{n=1}^{\infty}(1-q^{2n-1})^8 + 16q\prod_{n=1}^{\infty}(1+q^{2n})^8$$

is a special case of Theorem 1.1, see Corollary 3.1. Other identities which follow from Theorem 1.1 are given in Corollaries 3.2, 3.3, 3.4 and 3.5.

2. Proof of Theorem 1.1

Jacobi's theta function $\varphi(q)$ and Ramanujan's discriminant function $\Delta(q)$ are defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}. \tag{2.1}$$

Set

$$p := p(q) = \frac{\varphi^2(q)-\varphi^2(q^3)}{2\varphi^2(q^3)}, \quad k := k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{2.2}$$

Then, as we showed in [**1**, equations (3.28)–(3.33)],

$$\begin{aligned}
\Delta(q) &= 2^{-4}p(1+p)^4(1-p)^{12}(1+2p)^3(2+p)^3k^{12},\\
\Delta(q^2) &= 2^{-8}p^2(1+p)^2(1-p)^6(1+2p)^6(2+p)^6k^{12},\\
\Delta(q^3) &= 2^{-4}p^3(1+p)^{12}(1-p)^4(1+2p)(2+p)k^{12},\\
\Delta(q^4) &= 2^{-16}p^4(1+p)(1-p)^3(1+2p)^3(2+p)^{12}k^{12},\\
\Delta(q^6) &= 2^{-8}p^6(1+p)^6(1-p)^2(1+2p)^2(2+p)^2k^{12},\\
\Delta(q^{12}) &= 2^{-16}p^{12}(1+p)^3(1-p)(1+2p)(2+p)^4k^{12}.
\end{aligned} \tag{2.3}$$

If we write

$$\begin{aligned}
&p^{k_1}(1+p)^{k_2}(1-p)^{k_3}(1+2p)^{k_4}(2+p)^{k_5}\\
&\qquad = C\Delta(q)^{l_1}\Delta(q^2)^{l_2}\Delta(q^3)^{l_3}\Delta(q^4)^{l_4}\Delta(q^6)^{l_6}\Delta(q^{12})^{l_{12}}
\end{aligned}$$

then $C = 2^{k_1+k_5}$ and

$$\begin{aligned}
l_1 &= \frac{1}{24}(-k_1-2k_2+2k_3-4k_4-k_5),\\
l_2 &= \frac{1}{24}(3k_1+3k_2+k_3+10k_4+k_5),\\
l_3 &= \frac{1}{24}(3k_1+6k_2+2k_3+4k_4+3k_5),\\
l_4 &= \frac{1}{24}(-2k_1-k_2-k_3-4k_4+2k_5),\\
l_6 &= \frac{1}{24}(-9k_1-9k_2-7k_3-10k_4-7k_5),\\
l_{12} &= \frac{1}{24}(6k_1+3k_2+3k_3+4k_4+2k_5),
\end{aligned}$$

and so

$$\begin{aligned}(2.4)\qquad & p^{k_1}(1+p)^{k_2}(1-p)^{k_3}(1+2p)^{k_4}(2+p)^{k_5} \\ &= 2^{k_1+k_5}q^{k_1}\prod_{n=1}^{\infty}(1-q^n)^{-k_1-2k_2+2k_3-4k_4-k_5} \\ &\quad\times(1-q^{2n})^{3k_1+3k_2+k_3+10k_4+k_5}(1-q^{3n})^{3k_1+6k_2+2k_3+4k_4+3k_5} \\ &\quad\times(1-q^{4n})^{-2k_1-k_2-k_3-4k_4+2k_5}(1-q^{6n})^{-9k_1-9k_2-7k_3-10k_4-7k_5} \\ &\quad\times(1-q^{12n})^{6k_1+3k_2+3k_3+4k_4+2k_5}.\end{aligned}$$

Taking $x = p$ in (1.1), and appealing to (2.4), we obtain the asserted identity. □

3. Examples

Our first corollary is the famous identity of Jacobi mentioned in the Introduction [**2**, p. 147].

Corollary 3.1.

$$\prod_{n=1}^{\infty}(1+q^{2n-1})^8 = \prod_{n=1}^{\infty}(1-q^{2n-1})^8 + 16q\prod_{n=1}^{\infty}(1+q^{2n})^8.$$

Proof: With the choice (1.2), Theorem 1.1 gives

$$\begin{aligned}&\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^6(1-q^{3n})^{12}(1-q^{4n})^{-4}(1-q^{6n})^{-30}(1-q^{12n})^{12} \\ &\quad+16q\prod_{n=1}^{\infty}(1-q^n)^{-4}(1-q^{2n})^6(1-q^{3n})^{12}(1-q^{4n})^4(1-q^{6n})^{-30}(1-q^{12n})^{12} \\ &\quad-\prod_{n=1}^{\infty}(1-q^n)^{-12}(1-q^{2n})^{30}(1-q^{3n})^{12}(1-q^{4n})^{-12}(1-q^{6n})^{-30}(1-q^{12n})^{12} = 0.\end{aligned}$$

Multiplying by

$$\prod_{n=1}^{\infty}(1-q^n)^{12}(1-q^{2n})^{-6}(1-q^{3n})^{-12}(1-q^{4n})^{12}(1-q^{6n})^{30}(1-q^{12n})^{-12},$$

we obtain

$$(3.1)\qquad \prod_{n=1}^{\infty}(1-q^n)^{16}(1-q^{4n})^8 + 16q\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{4n})^{16} = \prod_{n=1}^{\infty}(1-q^{2n})^{24}.$$

Set

$$A := \prod_{n=1}^{\infty}(1+q^{2n-1}),\ B := \prod_{n=1}^{\infty}(1-q^{2n-1}),\ C := \prod_{n=1}^{\infty}(1+q^{2n}),$$

and

$$X := \prod_{n=1}^{\infty}(1-q^{2n}).$$

We have

$$\begin{aligned} ABC &= \prod_{n=1}^{\infty}(1-q^{4n-2})(1+q^{2n}) \\ &= \prod_{n=1}^{\infty}(1-q^{4n-2})(1-q^{4n})\frac{(1+q^{2n})}{(1-q^{4n})} \\ &= \prod_{n=1}^{\infty}(1-q^{2n})\frac{1}{(1-q^{2n})} \\ &= 1. \end{aligned}$$

Also

$$\prod_{n=1}^{\infty}(1-q^n)^{16}(1-q^{4n})^8 = \prod_{n=1}^{\infty}(1-q^{2n})^{16}(1-q^{2n-1})^{16}(1-q^{2n})^8(1+q^{2n})^8 = X^{24}B^{16}C^8,$$

$$\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{4n})^{16} = \prod_{n=1}^{\infty}(1-q^{2n})^8(1-q^{2n-1})^8(1-q^{2n})^{16}(1+q^{2n})^{16} = X^{24}B^8C^{16},$$

and

$$\prod_{n=1}^{\infty}(1-q^{2n})^{24} = X^{24} = X^{24}A^8B^8C^8.$$

From (3.1) we deduce

$$X^{24}B^{16}C^8 + 16qX^{24}B^8C^{16} = X^{24}A^8B^8C^8,$$

so that

$$B^8 + 16qC^8 = A^8$$

as asserted. □

COROLLARY 3.2.

$$\prod_{n=1}^{\infty}(1-q^{2n})^6(1-q^{6n})^6 = \prod_{n=1}^{\infty}(1-q^n)^4(1-q^{3n})^4(1-q^{4n})^2(1-q^{12n})^2 + 4q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{3n})^2(1-q^{4n})^4(1-q^{12n})^4.$$

PROOF: Using the choice (1.3) in Theorem 1.1, we obtain

$$\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{3n})^8(1-q^{4n})^{-2}(1-q^{6n})^{-16}(1-q^{12n})^6$$
$$-\prod_{n=1}^{\infty}(1-q^n)^{-4}(1-q^{2n})^{10}(1-q^{3n})^4(1-q^{4n})^{-4}(1-q^{6n})^{-10}(1-q^{12n})^4$$
$$+4q\prod_{n=1}^{\infty}(1-q^n)^{-2}(1-q^{2n})^4(1-q^{3n})^6(1-q^{6n})^{-16}(1-q^{12n})^8=0.$$

Multiplying by

$$\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^{-4}(1-q^{3n})^{-4}(1-q^{4n})^4(1-q^{6n})^{16}(1-q^{12n})^{-4},$$

we obtain the asserted result. □

COROLLARY 3.3.

$$\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{3n})^9=\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{3n})(1-q^{6n})^4$$
$$+8q\prod_{n=1}^{\infty}(1-q^n)^3(1-q^{2n})(1-q^{6n})^9.$$

PROOF: Using the choice (1.4) in Theorem 1.1, we obtain

$$\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^2(1-q^{3n})^4(1-q^{4n})^{-2}(1-q^{6n})^{-14}(1-q^{12n})^6$$
$$+8q\prod_{n=1}^{\infty}(1-q^n)^{-1}(1-q^{2n})^3(1-q^{3n})^3(1-q^{4n})^{-2}(1-q^{6n})^{-9}(1-q^{12n})^6$$
$$-\prod_{n=1}^{\infty}(1-q^n)^{-4}(1-q^{2n})^6(1-q^{3n})^{12}(1-q^{4n})^{-2}(1-q^{6n})^{-18}(1-q^{12n})^6=0.$$

Multiplying by

$$\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^{-2}(1-q^{3n})^{-3}(1-q^{4n})^2(1-q^{6n})^{18}(1-q^{12n})^{-6},$$

we obtain the asserted result. □

COROLLARY 3.4.

$$\prod_{n=1}^{\infty}(1-q^{2n})^9(1-q^{3n})(1-q^{12n})^2+2\prod_{n=1}^{\infty}(1-q^n)^3(1-q^{4n})^6(1-q^{6n})^3$$
$$=3\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})^2(1-q^{3n})^3(1-q^{4n})^3(1-q^{6n})(1-q^{12n}).$$

PROOF: Using the choice (1.5) in Theorem 1.1, we obtain

$$\prod_{n=1}^{\infty}(1-q^n)^{-4}(1-q^{2n})^{10}(1-q^{3n})^4(1-q^{4n})^{-4}(1-q^{6n})^{-10}(1-q^{12n})^4$$
$$+2\prod_{n=1}^{\infty}(1-q^n)^{-1}(1-q^{2n})(1-q^{3n})^3(1-q^{4n})^2(1-q^{6n})^{-7}(1-q^{12n})^2$$
$$-3\prod_{n=1}^{\infty}(1-q^n)^{-2}(1-q^{2n})^3(1-q^{3n})^6(1-q^{4n})^{-1}(1-q^{6n})^{-9}(1-q^{12n})^3=0.$$

Multiplying by

$$\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^{-1}(1-q^{3n})^{-3}(1-q^{4n})^4(1-q^{6n})^{10}(1-q^{12n})^{-2},$$

we obtain the asserted result. □

COROLLARY 3.5. *For $m \in \mathbb{N}$*

$$\sum_{k=0}^{m}\binom{m}{k}2^kq^k\prod_{n=1}^{\infty}(1-q^n)^{2m+k}(1-q^{3n})^{2m-3k}(1-q^{4n})^{3m-k}(1-q^{12n})^{3k}$$
$$=\left(\prod_{n=1}^{\infty}(1-q^{2n})^7(1+q^{6n})\right)^m.$$

PROOF: Using the choice (1.6) in Theorem 1.1, we obtain

$$\sum_{k=0}^{m}\binom{m}{k}2^kq^k\prod_{n=1}^{\infty}(1-q^n)^{-2m+k}(1-q^{2n})^{3m}(1-q^{3n})^{6m-3k}$$
$$\times(1-q^{4n})^{-m-k}(1-q^{6n})^{-9m}(1-q^{12n})^{3m+3k}$$
$$=\prod_{n=1}^{\infty}(1-q^n)^{-4m}(1-q^{2n})^{10m}(1-q^{3n})^{4m}(1-q^{4n})^{-4m}$$
$$\times(1-q^{6n})^{-10m}(1-q^{12n})^{4m}.$$

Multiplying by

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^{4m}(1-q^{4n})^{4m}(1-q^{6n})^{10m}}{(1-q^{2n})^{3m}(1-q^{3n})^{4m}(1-q^{12n})^{3m}}$$

we obtain

$$\sum_{k=0}^{m}\binom{m}{k}2^kq^k\prod_{n=1}^{\infty}(1-q^n)^{2m+k}(1-q^{3n})^{2m-3k}(1-q^{4n})^{3m-k}(1-q^{6n})^m(1-q^{12n})^{3k}$$
$$=\prod_{n=1}^{\infty}(1-q^{2n})^{7m}(1-q^{12n})^m.$$

Then, multiplying both sides of the equation by

$$\prod_{n=1}^{\infty}(1-q^{6n})^{-m}$$

we obtained the asserted result. □

References

[1] A. Alaca, Ş. Alaca and K.S. Williams, 'Evaluation of the convolution sums $\sum_{l+12m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+4m=n} \sigma(l)\sigma(m)$', *Adv. Theoretical Appl. Math.* **1** (2006), 27–48.

[2] C.G.J. Jacobi, 'Fundamenta nova theoriae functionum ellipticarum (1829)', in *C.G.J. Jacobi's Gesammelte Werke, Volume I* (Chelsea Publ, Co., New York, 1969), pp. 49-239.

School of Mathematics and Statistics
Carleton University
Ottawa
Ontario
Canada K1S 5B5
e-mail: aalaca@math.carleton.ca
salaca@math.carleton.ca
kwilliam@connect.carleton.ca