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Evaluation of the convolution sums $\sum_{l+6m=n}\sigma(l)\sigma(m)$ and $\sum_{2l+3m=n}\sigma(l)\sigma(m)$

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Abstract

The sums $\sum_{(l,m)\in\mathbb{N}^2, l+6m=n}\sigma(l)\sigma(m)$ and $\sum_{(l,m)\in\mathbb{N}^2, 2l+3m=n}\sigma(l)\sigma(m)$ are evaluated for all $n\in\mathbb{N}$, and their evaluations used to determine the number of representations of a positive integer n by the form

$$x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2+2\left(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2\right).$$

MSC: 11A25; 11E20; 11E25

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1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of natural numbers, integers, real numbers, complex numbers respectively. For $k,n\in\mathbb{N}$ we set

$$\sigma_k(n)=\sum_{d\mid n}d^k,$$

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where d runs through the positive divisors of n. If $n \notin \mathbb{N}$ we set $\sigma_k(n) = 0$. We write $\sigma(n)$ for $\sigma_1(n)$. We define the convolution sum $W_k(n)$ by

$$W_k(n) := \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ l+km=n}} \sigma(l)\sigma(m) = \sum_{\substack{m\in\mathbb{N} \\ m<n/k}} \sigma(m)\sigma(n-km). \tag{1.1}$$

The sum $W_k(n)$ has been evaluated for $k = 1, 2, 3, 4, 8, 9$ for all $n \in \mathbb{N}$ and for $k = 5$ for $n \equiv 8 \pmod{16}$, $n \not\equiv 0 \pmod 5$, see [3,5,8] ($k = 1$); [10,12,13] ($k = 2$); [10,12,13,18] ($k = 3$); [6,10, 12,13] ($k = 4$); [12,13] ($k = 5$); [20] ($k = 8$); [19] ($k = 9$). In this paper we evaluate $W_k(n)$ for $k = 6$ and all $n \in \mathbb{N}$. In addition we evaluate the complementary sum

$$\sum_{\substack{(l,m)\in\mathbb{N}^2 \\ 2l+3m=n}} \sigma(l)\sigma(m) \tag{1.2}$$

for all $n \in \mathbb{N}$, see Theorem 1. We use our evaluations to determine the number of $(x_1, \ldots, x_8) \in \mathbb{Z}^8$ satisfying

$$n = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 2\left(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2\right),$$

see Theorem 2.

2. Statement of Theorems 1 and 2

Let $q \in \mathbb{C}$ be such that $|q| < 1$. Ramanujan's discriminant function $\Delta(q)$ is defined by

$$\Delta(q) := q\prod_{n=1}^{\infty}\left(1-q^n\right)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n, \tag{2.1}$$

where $\tau(n)$ is Ramanujan's tau function [11], [15, Eq. (92)], [17, p. 151]. From (2.1) we deduce that

$$\begin{aligned}&\left(\Delta(q)\Delta\left(q^2\right)\Delta\left(q^3\right)\Delta\left(q^6\right)\right)^{1/12} \\ &\quad = q\prod_{n=1}^{\infty}\left(1-q^n\right)^2\left(1-q^{2n}\right)^2\left(1-q^{3n}\right)^2\left(1-q^{6n}\right)^2.\end{aligned} \tag{2.2}$$

In view of (2.2) we can define integers $c_6(n)$ ($n \in \mathbb{N}$) by

$$\left(\Delta(q)\Delta\left(q^2\right)\Delta\left(q^3\right)\Delta\left(q^6\right)\right)^{1/12} = \sum_{n=1}^{\infty}c_6(n)q^n. \tag{2.3}$$

The first fifty values of $c_6(n)$ are given in Table 1.

Table 1

n	$c_6(n)$	n	$c_6(n)$	n	$c_6(n)$	n	$c_6(n)$	n	$c_6(n)$
1	1	11	12	21	48	31	−88	41	42
2	−2	12	−12	22	−24	32	−32	42	−96
3	−3	13	38	23	168	33	−36	43	−52
4	4	14	32	24	24	34	252	44	48
5	6	15	−18	25	−89	35	−96	45	54
6	6	16	16	26	−76	36	36	46	−336
7	−16	17	−126	27	−27	37	254	47	−96
8	−8	18	−18	28	−64	38	−40	48	−48
9	9	19	20	29	30	39	−114	49	−87
10	−12	20	24	30	36	40	−48	50	178

We prove

Theorem 1. *Let $n \in \mathbb{N}$. Then*

$$\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+6m=n}} \sigma(l)\sigma(m) = \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3(n/2) + \frac{3}{40}\sigma_3(n/3) + \frac{3}{10}\sigma_3(n/6) + \left(\frac{1}{24} - \frac{n}{24}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n/6) - \frac{1}{120}c_6(n)$$

and

$$\sum_{\substack{(l,m)\in\mathbb{N}^2\\ 2l+3m=n}} \sigma(l)\sigma(m) = \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3(n/2) + \frac{3}{40}\sigma_3(n/3) + \frac{3}{10}\sigma_3(n/6) + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma(n/2) + \left(\frac{1}{24} - \frac{n}{8}\right)\sigma(n/3) - \frac{1}{120}c_6(n).$$

As an application of Theorem 1, we prove

Theorem 2. *Let $n \in \mathbb{N}$. The number of $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8$ such that*

$$n = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 2\left(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2\right)$$

is

$$\frac{1}{5}\left(24\sigma_3(n) + 96\sigma_3(n/2) + 216\sigma_3(n/3) + 864\sigma_3(n/6) + 36c_6(n)\right).$$

3. Proof of Theorem 1

For $z \in \mathbb{C}$ with $|z| < 1$ we set

$$w_2(z) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{2})_n}{(1)_n} \frac{z^n}{n!} \tag{3.1}$$

and

$$w_3(z) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right) = \sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n (\frac{2}{3})_n}{(1)_n} \frac{z^n}{n!}, \tag{3.2}$$

where ${}_2F_1$ is the Gaussian hypergeometric function and $(a)_n$ is the Pochhammer symbol, see for example [7, p. 247], [14, p. 45]. Clearly

$$w_2(0) = w_3(0) = 1. \tag{3.3}$$

The infinite series (3.1) and (3.2) diverge at $z = 1$ [7, p. 249] so that

$$w_2(1) = w_3(1) = +\infty. \tag{3.4}$$

For $x \in \mathbb{R}$ with $0 \leqslant x < 1$ we have

$$w_2(x) = 1 + \sum_{n=1}^{\infty} \frac{2n!^2}{n!^4 2^{4n}} x^n \geqslant 1$$

and

$$w_3(x) = 1 + \sum_{n=1}^{\infty} \frac{3n!}{n!^3 3^{3n}} x^n \geqslant 1$$

so that

$$w_2(x) > 0, \quad w_3(x) > 0, \quad 0 \leqslant x < 1. \tag{3.5}$$

The derivatives with respect to x of the functions

$$y_2(x) := \pi \frac{w_2(1-x)}{w_2(x)}, \quad y_3(x) := \frac{2\pi}{\sqrt{3}} \frac{w_3(1-x)}{w_3(x)}, \quad 0 < x < 1, \tag{3.6}$$

are [1, p. 87]

$$y_2'(x) = \frac{-1}{x(1-x)w_2(x)^2}, \quad y_3'(x) = \frac{-1}{x(1-x)w_3(x)^2}, \quad 0 < x < 1. \tag{3.7}$$

Thus, by (3.5)–(3.7), we have

$$y_2'(x) < 0, \quad y_3'(x) < 0, \quad 0 < x < 1. \tag{3.8}$$

Hence, as x increases from 0 to 1, $y_2(x)$ and $y_3(x)$ strictly decrease from $y_2(0) = y_3(0) = +\infty$ to $y_2(1) = y_3(1) = 0$. Let $q \in \mathbb{R}$ be such that $0 < q < 1$. Then $0 < -\log q < +\infty$. Hence there are unique real numbers $x_2(q)$ and $x_3(q)$ between 0 and 1 such that

$$y_2\big(x_2(q)\big) = y_3\big(x_3(q)\big) = -\log q. \tag{3.9}$$

Hence

$$\pi \frac{w_2(1 - x_2(q))}{w_2(x_2(q))} = \frac{2\pi}{\sqrt{3}} \frac{w_3(1 - x_3(q))}{w_3(x_3(q))} = -\log q. \tag{3.10}$$

Ramanujan [16, p. 258] stated and Berndt, Bhargava and Garvan [4, p. 4184] proved that there exists a real number p depending on q such that $0 < p < 1$ and

$$x_2\big(q^3\big) = \frac{p^3(2+p)}{1+2p} \tag{3.11}$$

and

$$x_3\big(q^2\big) = \frac{27p^2(1+p)^2}{4(1+p+p^2)^3}. \tag{3.12}$$

Berndt, Bhargava and Garvan give p in terms of theta functions. As $0 < p < 1$ we have $1 + p + p^2 > 0$, and, as $0 < x_3(q^2) < 1$, we have $w_3(x_3(q^2)) > 0$. Thus we can define a positive real number k depending on q by

$$k = \frac{w_3(x_3(q^2))}{1+p+p^2}.$$

Then, from the identity [16, p. 258], [4, p. 4184], [3, p. 112]

$$\big(1+p+p^2\big)_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x_2\big(q^3\big)\right) = (1+2p)^{1/2}{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x_3\big(q^2\big)\right),$$

we deduce that

$$k = \frac{w_2(x_2(q^3))}{(1+2p)^{1/2}}.$$

Hence there exists a positive real number k depending upon q such that

$$w_2\big(x_2\big(q^3\big)\big) = (1+2p)^{1/2}k \tag{3.13}$$

and

$$w_3\big(x_3\big(q^2\big)\big) = \big(1+p+p^2\big)k. \tag{3.14}$$

The principle of duplication [2, p. 125] asserts that

$$x_2\big(q^2\big) = \left(\frac{1-(1-x_2(q))^{1/2}}{1+(1+x_2(q))^{1/2}}\right)^2 \tag{3.15}$$

and

$$w_2(x_2(q^2)) = \left(\frac{1+(1-x_2(q))^{1/2}}{2}\right) w_2(x_2(q)), \tag{3.16}$$

and the principle of triplication [4, p. 4174], [3, p. 102] asserts that

$$x_3(q^3) = \left(\frac{1-(1-x_3(q))^{1/3}}{1+2(1-x_3(q))^{1/3}}\right)^3 \tag{3.17}$$

and

$$w_3(x_3(q^3)) = \left(\frac{1+2(1-x_3(q))^{1/3}}{3}\right) w_3(x_3(q)). \tag{3.18}$$

We define the Eisenstein series $L(q)$, $M(q)$, $N(q)$ [1, p. 318], [15, Eq. (25)], [17, p. 144] by

$$L(q) := 1 - 24\sum_{n=1}^{\infty} \sigma(n) q^n, \tag{3.19}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{3.20}$$

$$N(q) := 1 - 504\sum_{n=1}^{\infty} \sigma_5(n) q^n. \tag{3.21}$$

Ramanujan gave in his notebooks [16] the following formulae for $L(q)$, $M(q)$ and $N(q)$:

$$L(q) = (1-5x_2(q)) w_2(x_2(q))^2 + 12 x_2(q)(1-x_2(q)) w_2(x_2(q)) w_2'(x_2(q)), \tag{3.22}$$

$$M(q) = (1+14x_2(q)+x_2(q)^2) w_2(x_2(q))^4, \tag{3.23}$$

$$N(q) = (1-33x_2(q)-33x_2(q)^2+x_2(q)^3) w_2(x_2(q))^6, \tag{3.24}$$

where ($'$) denotes differentiation with respect to x_2, and

$$L(q) = (1-4x_3(q)) w_3(x_3(q))^2 + 12 x_3(q)(1-x_3(q)) w_3(x_3(q)) w_3'(x_3(q)), \tag{3.25}$$

$$M(q) = (1+8x_3(q)) w_3(x_3(q))^4, \tag{3.26}$$

$$N(q) = (1-20x_3(q)-8x_3(q)^2) w_3(x_3(q))^6, \tag{3.27}$$

where ($'$) denotes differentiation with respect to x_3. Formulae (3.22)–(3.24) are proved in [2, pp. 126–129] and (3.25)–(3.27) in [4, pp. 4178–4179], [3, pp. 105–106]. Mapping $q \to q^3$ in (3.23) and (3.24) we obtain

$$M(q^3) = (1 + 14x_2(q^3) + x_2(q^3)^2)w_2(x_2(q^3))^4, \tag{3.28}$$

$$N(q^3) = (1 + x_2(q^3))(1 - 34x_2(q^3) + x_2(q^3)^2)w_2(x_2(q^3))^6. \tag{3.29}$$

Mapping $q \to q^2$ in (3.26) and (3.27) we obtain

$$M(q^2) = (1 + 8x_3(q^2))w_3(x_3(q^2))^4, \tag{3.30}$$

$$N(q^2) = (1 - 20x_3(q^2) - 8x_3(q^2)^2)w_3(x_3(q^2))^6. \tag{3.31}$$

Applying the duplication principle to (3.23) and (3.24) we obtain

$$M(q^2) = (1 - x_2(q) + x_2(q)^2)w_2(x_2(q))^4, \tag{3.32}$$

$$N(q^2) = (1 + x_2(q))\left(1 - \frac{1}{2}x_2(q)\right)(1 - 2x_2(q))w_2(x_2(q))^6, \tag{3.33}$$

see for example [2, p. 126], [6, p. 564]. Mapping $q \to q^3$ in (3.32) and (3.33) we have

$$M(q^6) = (1 - x_2(q^3) + x_2(q^3)^2)w_2(x_2(q^3))^4, \tag{3.34}$$

$$N(q^6) = (1 + x_2(q^3))\left(1 - \frac{1}{2}x_2(q^3)\right)(1 - 2x_2(q^3))w_2(x_2(q^3))^6. \tag{3.35}$$

Applying the triplication principle to (3.26) and (3.27) we obtain

$$M(q^3) = \left(1 - \frac{8}{9}x_3(q)\right)w_3(x_3(q))^4, \tag{3.36}$$

$$N(q^3) = \left(1 - \frac{4}{3}x_3(q) + \frac{8}{27}x_3(q)^2\right)w_3(x_3(q))^6, \tag{3.37}$$

see for example [3, p. 107], [18, p. 533]. Mapping $q \to q^2$ in (3.36) and (3.37) we have

$$M(q^6) = \left(1 - \frac{8}{9}x_3(q^2)\right)w_3(x_3(q^2))^4, \tag{3.38}$$

$$N(q^6) = \left(1 - \frac{4}{3}x_3(q^2) + \frac{8}{27}x_3(q^2)^2\right)w_3(x_3(q^2))^6. \tag{3.39}$$

It is now convenient to set

$$A := x_2(q), \tag{3.40}$$

$$B := x_3(q), \tag{3.41}$$

$$C := x_2(q^3), \tag{3.42}$$

$$D := x_3(q^2), \tag{3.43}$$

$$W := w_2(x_2(q)), \tag{3.44}$$

$$X := w_3\big(x_3(q)\big), \tag{3.45}$$

$$Y := w_2\big(x_2\big(q^3\big)\big), \tag{3.46}$$

$$Z := w_3\big(x_3\big(q^2\big)\big). \tag{3.47}$$

We note that

$$0 < A < 1, \qquad 0 < B < 1, \qquad 0 < C < 1, \qquad 0 < D < 1,$$
$$W > 0, \qquad X > 0, \qquad Y > 0, \qquad Z > 0.$$

With this notation (3.11)–(3.14), (3.23), (3.24), (3.26)–(3.39) become

$$C = \frac{p^3(2+p)}{1+2p}, \tag{3.48}$$

$$D = \frac{27p^2(1+p)^2}{4(1+p+p^2)^3}, \tag{3.49}$$

$$Y = (1+2p)^{1/2}k, \tag{3.50}$$

$$Z = \big(1+p+p^2\big)k, \tag{3.51}$$

$$M(q) = \big(1+14A+A^2\big)W^4 = (1+8B)X^4, \tag{3.52}$$

$$M\big(q^2\big) = \big(1-A+A^2\big)W^4 = (1+8D)Z^4, \tag{3.53}$$

$$M\big(q^3\big) = \left(1-\frac{8}{9}B\right)X^4 = \big(1+14C+C^2\big)Y^4, \tag{3.54}$$

$$M\big(q^6\big) = \left(1-\frac{8}{9}D\right)Z^4 = \big(1-C+C^2\big)Y^4, \tag{3.55}$$

$$N(q) = (1+A)\big(1-34A+A^2\big)W^6 = \big(1-20B-8B^2\big)X^6, \tag{3.56}$$

$$N\big(q^2\big) = (1+A)\left(1-\frac{1}{2}A\right)(1-2A)W^6 = \big(1-20D-8D^2\big)Z^6, \tag{3.57}$$

$$N\big(q^3\big) = \left(1-\frac{4}{3}B+\frac{8}{27}B^2\right)X^6 = (1+C)\big(1-34C+C^2\big)Y^6, \tag{3.58}$$

$$N\big(q^6\big) = \left(1-\frac{4}{3}D+\frac{8}{27}D^2\right)Z^6 = (1+C)\left(1-\frac{1}{2}C\right)(1-2C)Y^6. \tag{3.59}$$

Subtracting (3.53) from (3.52), we deduce

$$A = \frac{(1+8B)X^4-(1+8D)Z^4}{15W^4}. \tag{3.60}$$

Appealing to (3.49) and (3.51), we obtain

$$A = \frac{(1+8B)X^4-((1+p+p^2)^4+54p^2(1+p)^2(1+p+p^2))k^4}{15W^4}. \tag{3.61}$$

From (3.54) we have

$$B = \frac{9}{8} - \frac{9(1+14C+C^2)Y^4}{8X^4}. \tag{3.62}$$

Appealing to (3.48) and (3.50), (3.62) becomes

$$B = \frac{9}{8} - \frac{9((1+2p)^2 + 14p^3(2+p)(1+2p) + p^6(2+p)^2)k^4}{8X^4}. \tag{3.63}$$

Using (3.48), (3.50) and (3.63) in (3.58), we obtain an equation for X. Using MAPLE to solve this equation, we find (remembering that X is real and positive)

$$X = \left(1+4p+p^2\right)k. \tag{3.64}$$

Using (3.64) in (3.63) and (3.61), we find

$$B = \frac{27(1+p)^4 p}{2(1+4p+p^2)^3} \tag{3.65}$$

and

$$A = \frac{p(2+p)^3(1+2p)^3k^4}{W^4}. \tag{3.66}$$

Next, using (3.49), (3.51) and (3.66) in (3.53), we obtain an equation for W. Using MAPLE to solve this equation we find (remembering that W is real and positive)

$$W = (1+2p)^{3/2}k \quad \text{or} \quad (2+p)\big(p(p+2)\big)^{1/2}k. \tag{3.67}$$

Then, appealing to (3.66), we obtain

$$A = \frac{p(2+p)^3}{(1+2p)^3} \quad \text{or} \quad \frac{(1+2p)^3}{p(2+p)^3} \tag{3.68}$$

respectively. As $0 < p < 1$ we have $1 \pm p > 0$ so that

$$(1+2p)^3 - p(2+p)^3 = (1+p)(1-p)^3 > 0.$$

Hence

$$0 < \frac{p(2+p)^3}{(1+2p)^3} < 1, \qquad \frac{(1+2p)^3}{p(2+p)^3} > 1.$$

Since $0 < A < 1$ we must have

$$A = \frac{p(2+p)^3}{(1+2p)^3}, \qquad W = (1+2p)^{3/2}k.$$

Thus

$$(1+A)W^2 = \left(1+14p+24p^2+14p^3+p^4\right)k^2.$$

Appealing to (3.52), (3.64) and (3.65), we obtain

$$M(q) = \left(1+124p+964p^2+2788p^3+3910p^4 + 2788p^5+964p^6+124p^7+p^8\right)k^4. \tag{3.69}$$

From (3.49), (3.51) and (3.53), we have

$$M\left(q^2\right) = \left(1+4p+64p^2+178p^3+235p^4+178p^5+64p^6+4p^7+p^8\right)k^4. \tag{3.70}$$

From (3.48), (3.50) and (3.54), we get

$$M\left(q^3\right) = \left(1+4p+4p^2+28p^3+70p^4+28p^5+4p^6+4p^7+p^8\right)k^4. \tag{3.71}$$

From (3.49), (3.51) and (3.55), we deduce

$$M\left(q^6\right) = \left(1+4p+4p^2-2p^3-5p^4-2p^5+4p^6+4p^7+p^8\right)k^4. \tag{3.72}$$

From (3.56), (3.64) and (3.65), we get

$$N(q) = \left(1-246p-5532p^2-38\,614p^3-135\,369p^4-276\,084p^5-348\,024p^6 - 276\,084p^7-135\,369p^8-38\,614p^9-5532p^{10}-246p^{11}+p^{12}\right)k^6. \tag{3.73}$$

From (3.49), (3.51) and (3.57), we have

$$N\left(q^2\right) = \left(1+6p-114p^2-625p^3-\frac{4059}{2}p^4-4302p^5-5556p^6 - 4302p^7-\frac{4059}{2}p^8-625p^9-114p^{10}+6p^{11}+p^{12}\right)k^6. \tag{3.74}$$

From (3.58), (3.64) and (3.65), we have

$$N\left(q^3\right) = \left(1+6p+12p^2-58p^3-297p^4-396p^5-264p^6 - 396p^7-297p^8-58p^9+12p^{10}+6p^{11}+p^{12}\right)k^6. \tag{3.75}$$

From (3.49), (3.51) and (3.59), we deduce

$$N\left(q^6\right) = \left(1+6p+12p^2+5p^3-\frac{27}{2}p^4-18p^5-12p^6 - 18p^7-\frac{27}{2}p^8+5p^9+12p^{10}+6p^{11}+p^{12}\right)k^6. \tag{3.76}$$

Now Ramanujan [15, Eq. (44)], [17, p. 144] has shown that

$$\Delta(q) = \frac{1}{1728}\big(M(q)^3 - N(q)^2\big). \tag{3.77}$$

Hence, by (3.69), (3.73) and (3.77), we obtain

$$\Delta(q) = \frac{1}{16} p(1+p)^4(1-p)^{12}(1+2p)^3(2+p)^3k^{12}. \tag{3.78}$$

From (3.70), (3.74) and (3.77), we have

$$\Delta\big(q^2\big) = \frac{1}{256} p^2(1+p)^2(1-p)^6(1+2p)^6(2+p)^6k^{12}. \tag{3.79}$$

From (3.71), (3.75) and (3.77), we have

$$\Delta\big(q^3\big) = \frac{1}{16} p^3(1+p)^{12}(1-p)^4(1+2p)(2+p)k^{12}. \tag{3.80}$$

From (3.72), (3.76) and (3.77), we have

$$\Delta\big(q^6\big) = \frac{1}{256} p^6(1+p)^6(1-p)^2(1+2p)^2(2+p)^2k^{12}. \tag{3.81}$$

Thus, by (2.1), (2.3) and (3.78)–(3.81), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} c_6(n)q^n &= \frac{1}{4} p(1+p)^2(1-p)^2(1+2p)(2+p)k^4 \\ &= \frac{1}{4}\big(2p + 5p^2 - 2p^3 - 10p^4 - 2p^5 + 5p^6 + 2p^7\big)k^4. \end{aligned} \tag{3.82}$$

Next, applying the duplication principle to (3.22), we obtain

$$L\big(q^2\big) = \big(1 - 2x_2(q)\big)w_2\big(x_2(q)\big)^2 + 6x_2(q)\big(1 - x_2(q)\big)w_2\big(x_2(q)\big)w_2'\big(x_2(q)\big), \tag{3.83}$$

where $(')$ denotes differentiation with respect to x_2, see [2, p. 126], [6, p. 564]. From (3.22) and (3.83) we deduce

$$\begin{aligned}L(q) - 2L\big(q^2\big) &= -\big(1 + x_2(q)\big)w_2\big(x_2(q)\big)^2 \\ &= -(1+A)W^2 \\ &= -\big(1 + 14p + 24p^2 + 14p^3 + p^4\big)k^2. \end{aligned} \tag{3.84}$$

Mapping $q \to q^3$ in (3.84) we obtain

$$\begin{aligned} L(q^3) - 2L(q^6) &= -(1 + x_2(q^3))w_2(x_2(q^3))^2 \\ &= -(1 + C)Y^2 \\ &= -(1 + 2p + 2p^3 + p^4)k^2. \end{aligned} \tag{3.85}$$

Applying the triplication principle to (3.25), we have

$$L(q^3) = \left(1 - \frac{4}{3}x_3(q)\right)w_3(x_3(q))^2 + 4x_3(q)(1 - x_3(q))w_3(x_3(q))w_3'(x_3(q)), \tag{3.86}$$

see for example [3, p. 178], [18, p. 533]. From (3.25) and (3.86) we obtain

$$\begin{aligned} L(q) - 3L(q^3) &= -2w_3(x_3(q))^2 \\ &= -2X^2 \\ &= -2(1 + 4p + p^2)^2k^2 \\ &= (-2 - 16p - 36p^2 - 16p^3 - 2p^4)k^2, \end{aligned} \tag{3.87}$$

by (3.45) and (3.64). Mapping $q \to q^2$ in (3.87) we have

$$\begin{aligned} L(q^2) - 3L(q^6) &= -2w_3(x_3(q^2))^2 \\ &= -2Z^2 \\ &= -2(1 + p + p^2)^2k^2 \\ &= (-2 - 4p - 6p^2 - 4p^3 - 2p^4)k^2, \end{aligned} \tag{3.88}$$

by (3.47) and (3.51). Then, from (3.84) and (3.88), or (3.85) and (3.87), we obtain

$$L(q) - 6L(q^6) = -(5 + 22p + 36p^2 + 22p^3 + 5p^4)k^2 \tag{3.89}$$

and

$$2L(q^2) - 3L(q^3) = -(1 + 2p + 12p^2 + 2p^3 + p^4)k^2. \tag{3.90}$$

Now, by (3.89), (3.69)–(3.72), (3.82) and (3.20), we obtain

$$\begin{aligned} &25(L(q) - 6L(q^6))^2 \\ &\quad = 25(5 + 22p + 36p^2 + 22p^3 + 5p^4)^2k^4 \\ &\quad = (625 + 5500p + 21\,100p^2 + 45\,100p^3 + 57\,850p^4 \\ &\qquad + 45\,100p^5 + 21\,100p^6 + 5500p^7 + 625p^8)k^4 \end{aligned}$$

$$
\begin{aligned}
&= \bigl(19 + 2356p + 18\,316p^2 + 52\,972p^3 + 74\,290p^4 \\
&\quad + 52\,972p^5 + 18\,316p^6 + 2356p^7 + 19p^8\bigr)k^4 \\
&\quad - \bigl(24 + 96p + 1536p^2 + 4272p^3 + 5640p^4 + 4272p^5 + 1536p^6 + 96p^7 + 24p^8\bigr)k^4 \\
&\quad - \bigl(54 + 216p + 216p^2 + 1512p^3 + 3780p^4 + 1512p^5 + 216p^6 + 216p^7 + 54p^8\bigr)k^4 \\
&\quad + \bigl(684 + 2736p + 2736p^2 - 1368p^3 - 3420p^4 \\
&\quad - 1368p^5 + 2736p^6 + 2736p^7 + 684p^8\bigr)k^4 \\
&\quad + \bigl(720p + 1800p^2 - 720p^3 - 3600p^4 - 720p^5 + 1800p^6 + 720p^7\bigr)k^4 \\
&= 19M(q) - 24M\bigl(q^2\bigr) - 54M\bigl(q^3\bigr) + 684M\bigl(q^6\bigr) + 1440\sum_{n=1}^{\infty} c_6(n)q^n \\
&= 19\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right) - 24\left(1 + 240\sum_{n=1}^{\infty}\sigma_3\left(\frac{n}{2}\right)q^n\right) \\
&\quad - 54\left(1 + 240\sum_{n=1}^{\infty}\sigma_3\left(\frac{n}{3}\right)q^n\right) + 684\left(1 + 240\sum_{n=1}^{\infty}\sigma_3\left(\frac{n}{6}\right)q^n\right) \\
&\quad + 1440\sum_{n=1}^{\infty} c_6(n)q^n,
\end{aligned}
$$

that is

$$
\begin{aligned}
25\bigl(L(q) - 6L\bigl(q^6\bigr)\bigr)^2 = 625 + 240\sum_{n=1}^{\infty}\biggl(&19\sigma_3(n) - 24\sigma_3\left(\frac{n}{2}\right) - 54\sigma_3\left(\frac{n}{3}\right) \\
&+ 684\sigma_3\left(\frac{n}{6}\right) + 6c_6(n)\biggr)q^n.
\end{aligned}
$$

Recalling the classical result [8,9]

$$
L(q)^2 = 1 + \sum_{n=1}^{\infty}\bigl(240\sigma_3(n) - 288n\sigma(n)\bigr)q^n,
$$

so that

$$
L\bigl(q^6\bigr)^2 = 1 + \sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{6}\right) - 48n\sigma\left(\frac{n}{6}\right)\right)q^n,
$$

we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\biggl(\sum_{\substack{l,m\geqslant 1\\ l+6m=n}}\sigma(l)\sigma(m)\biggr)q^n\\
&\quad=\sum_{l,m=1}^{\infty}\sigma(l)\sigma(m)q^{l+6m}\\
&\quad=\biggl(\sum_{l=1}^{\infty}\sigma(l)q^l\biggr)\biggl(\sum_{m=1}^{\infty}\sigma(m)q^{6m}\biggr)\\
&\quad=\biggl(\frac{1-L(q)}{24}\biggr)\biggl(\frac{1-L(q^6)}{24}\biggr)\\
&\quad=\frac{1}{576}-\frac{1}{576}L(q)-\frac{1}{576}L\bigl(q^6\bigr)+\frac{1}{576}L(q)L\bigl(q^6\bigr)\\
&\quad=\frac{1}{576}-\frac{1}{576}L(q)-\frac{1}{576}L\bigl(q^6\bigr)+\frac{1}{6912}L(q)^2+\frac{1}{192}L\bigl(q^6\bigr)^2-\frac{1}{6912}\bigl(L(q)-6L\bigl(q^6\bigr)\bigr)^2\\
&\quad=\frac{1}{576}-\frac{1}{576}\biggl(1-24\sum_{n=1}^{\infty}\sigma(n)q^n\biggr)-\frac{1}{576}\biggl(1-24\sum_{n=1}^{\infty}\sigma\biggl(\frac{n}{6}\biggr)q^n\biggr)\\
&\qquad+\frac{1}{6912}\biggl(1+\sum_{n=1}^{\infty}\bigl(240\sigma_3(n)-288n\sigma(n)\bigr)q^n\biggr)\\
&\qquad+\frac{1}{192}\biggl(1+\sum_{n=1}^{\infty}\biggl(240\sigma_3\biggl(\frac{n}{6}\biggr)-48n\sigma\biggl(\frac{n}{6}\biggr)\biggr)q^n\biggr)\\
&\qquad-\frac{1}{6912}\biggl(25+\frac{48}{5}\sum_{n=1}^{\infty}\biggl(19\sigma_3(n)-24\sigma\biggl(\frac{n}{2}\biggr)-54\sigma_3\biggl(\frac{n}{3}\biggr)+684\sigma_3\biggl(\frac{n}{6}\biggr)+6c_6(n)\biggr)q^n\biggr)\\
&\quad=\sum_{n=1}^{\infty}\biggl(\frac{1}{120}\sigma(n)+\frac{1}{30}\sigma_3\biggl(\frac{n}{2}\biggr)+\frac{3}{40}\sigma_3\biggl(\frac{n}{3}\biggr)+\frac{3}{10}\sigma_3\biggl(\frac{n}{6}\biggr)\\
&\qquad+\biggl(\frac{1}{24}-\frac{n}{24}\biggr)\sigma(n)+\biggl(\frac{1}{24}-\frac{n}{4}\biggr)\sigma\biggl(\frac{n}{6}\biggr)-\frac{1}{120}c_6(n)\biggr)q^n.
\end{aligned}$$

Equating coefficients of q^n, we obtain

$$\begin{aligned}
\sum_{\substack{l,m\geqslant 1\\ l+6m=n}}\sigma(l)\sigma(m)&=\frac{1}{120}\sigma_3(n)+\frac{1}{30}\sigma_3\biggl(\frac{n}{2}\biggr)+\frac{3}{40}\sigma_3\biggl(\frac{n}{3}\biggr)+\frac{3}{10}\sigma_3\biggl(\frac{n}{6}\biggr)\\
&\qquad+\biggl(\frac{1}{24}-\frac{n}{24}\biggr)\sigma(n)+\biggl(\frac{1}{24}-\frac{n}{4}\biggr)\sigma\biggl(\frac{n}{6}\biggr)-\frac{1}{120}c_6(n),
\end{aligned}$$

which is the first part of Theorem 1.

Similarly we can show that

$$\begin{aligned}&25\big(2L(q^2)-3L(q^3)\big)^2\\&\quad=-6M(q)+76M(q^2)+171M(q^3)-216M(q^6)+1440\sum_{n=1}^{\infty}c_6(n)q^n\end{aligned}$$

from which we can deduce as above

$$\begin{aligned}\sum_{\substack{l,m\geqslant 1\\2l+3m=n}}\sigma(l)\sigma(m)&=\frac{1}{120}\sigma_3(n)+\frac{1}{30}\sigma_3\left(\frac{n}{2}\right)+\frac{3}{40}\sigma_3\left(\frac{n}{3}\right)+\frac{3}{10}\sigma_3\left(\frac{n}{6}\right)\\&\quad+\left(\frac{1}{24}-\frac{n}{12}\right)\sigma\left(\frac{n}{2}\right)+\left(\frac{1}{24}-\frac{n}{8}\right)\sigma\left(\frac{n}{3}\right)-\frac{1}{120}c_6(n).\end{aligned}$$

4. Proof of Theorem 2

Let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For $l\in\mathbb{N}_0$ we set

$$r(l)=\operatorname{card}\big\{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4\;\big|\;x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2=l\big\}$$

so that $r(0)=1$. It is known that [10, Theorem 13]

$$r(l)=12\sum_{\substack{d\mid l\\3\nmid d}}1=12\sigma(l)-36\sigma\left(\frac{l}{3}\right),\quad l\in\mathbb{N}.$$

Then the number $N(n)$ of representations $(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\in\mathbb{Z}^8$ of n by the form

$$x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2+2\big(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2\big)$$

is

$$\begin{aligned}N(n)&=\sum_{\substack{l,m\in\mathbb{N}_0\\l+2m=n}}\Bigg(\sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4\\x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2=l}}1\Bigg)\Bigg(\sum_{\substack{(x_5,x_6,x_7,x_8)\in\mathbb{Z}^4\\x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2=m}}1\Bigg)\\&=\sum_{\substack{l,m\in\mathbb{N}_0\\l+2m=n}}r(l)r(m)\\&=r(0)r\left(\frac{n}{2}\right)+r(n)r(0)+\sum_{\substack{l,m\in\mathbb{N}\\l+2m=n}}r(l)r(m)\\&=12\sigma\left(\frac{n}{2}\right)-36\sigma\left(\frac{n}{6}\right)+12\sigma(n)-36\sigma\left(\frac{n}{3}\right)\\&\quad+\sum_{\substack{l,m\in\mathbb{N}\\l+2m=n}}\left(12\sigma(l)-36\sigma\left(\frac{l}{3}\right)\right)\left(12\sigma(m)-36\sigma\left(\frac{m}{3}\right)\right).\end{aligned}$$

Thus

$$\begin{aligned}
&N(n) - 12\sigma(n) - 12\sigma\left(\frac{n}{2}\right) + 36\sigma\left(\frac{n}{3}\right) + 36\sigma\left(\frac{n}{6}\right) \\
&\quad = 144 \sum_{\substack{l,m\in\mathbb{N} \\ l+2m=n}} \sigma(l)\sigma(m) - 432 \sum_{l+2m=n} \sigma\left(\frac{l}{3}\right)\sigma(m) \\
&\qquad - 432 \sum_{l+2m=n} \sigma(l)\sigma\left(\frac{m}{3}\right) + 1296 \sum_{l+2m=n} \sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right).
\end{aligned}$$

The first sum is

$$\begin{aligned}
\sum_{\substack{l,m\in\mathbb{N} \\ l+2m=n}} \sigma(l)\sigma(m) &= \sum_{\substack{m\in\mathbb{N} \\ m<n/2}} \sigma(m)\sigma(n-2m) \\
&= \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{n}{8}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{2}\right),
\end{aligned}$$

see for example [10, Theorem 2].

The second sum is

$$\begin{aligned}
\sum_{\substack{(l,m)\in\mathbb{N}^2 \\ l+2m=n}} \sigma\left(\frac{l}{3}\right)\sigma(m) &= \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ 3l+2m=n}} \sigma(l)\sigma(m) \\
&= \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{40}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{6}\right) \\
&\quad + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{n}{8}\right)\sigma\left(\frac{n}{3}\right) - \frac{1}{120}c_6(n),
\end{aligned}$$

by Theorem 1.

The third sum is

$$\begin{aligned}
\sum_{\substack{(l,m)\in\mathbb{N}^2 \\ l+2m=n}} \sigma(l)\sigma\left(\frac{m}{3}\right) &= \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ l+6m=n}} \sigma(l)\sigma(m) \\
&= \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{40}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{6}\right) \\
&\quad + \left(\frac{1}{24} - \frac{n}{24}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{6}\right) - \frac{1}{120}c_6(n),
\end{aligned}$$

by Theorem 1.

The fourth sum is

$$\begin{aligned}\sum_{l+2m=n}\sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right) &= \sum_{3l+6m=n}\sigma(l)\sigma(m)\\ &= \sum_{l+2m=n/3}\sigma(l)\sigma(m) = \sum_{m<n/6}\sigma(m)\sigma\left(\frac{n}{3}-2m\right)\\ &= \frac{1}{12}\sigma_3\left(\frac{n}{3}\right) + \left(\frac{1}{24}-\frac{n}{24}\right)\sigma\left(\frac{n}{3}\right)\\ &\quad + \frac{1}{3}\sigma\left(\frac{n}{6}\right) + \left(\frac{1}{24}-\frac{n}{12}\right)\sigma\left(\frac{n}{6}\right),\end{aligned}$$

from above.

Finally

$$\begin{aligned}N(n) &= 12\sigma(n) + 12\sigma\left(\frac{n}{2}\right) - 36\sigma\left(\frac{n}{3}\right) - 36\sigma\left(\frac{n}{6}\right) + 12\sigma_3(n)\\ &\quad + 48\sigma_3\left(\frac{n}{2}\right) + (6-18n)\sigma(n) + (6-36n)\sigma\left(\frac{n}{2}\right) - \frac{18}{5}\sigma_3(n)\\ &\quad - \frac{72}{5}\sigma_3\left(\frac{n}{2}\right) - \frac{162}{5}\sigma_3\left(\frac{n}{3}\right) - \frac{648}{5}\sigma_3\left(\frac{n}{6}\right) - (18-36n)\sigma\left(\frac{n}{2}\right)\\ &\quad - (18-54n)\sigma\left(\frac{n}{3}\right) + \frac{18}{5}c_6(n) - \frac{18}{5}\sigma_3(n) - \frac{72}{5}\sigma_3\left(\frac{n}{2}\right)\\ &\quad - \frac{162}{5}\sigma_3\left(\frac{n}{3}\right) - \frac{648}{5}\sigma_3\left(\frac{n}{6}\right) - (18-18n)\sigma(n)\\ &\quad - (18-108n)\sigma\left(\frac{n}{6}\right) + \frac{18}{5}c_6(n) + 108\sigma_3\left(\frac{n}{3}\right) + (54-54n)\sigma\left(\frac{n}{3}\right)\\ &\quad + 432\sigma_3\left(\frac{n}{6}\right) + (54-108n)\sigma\left(\frac{n}{6}\right)\\ &= \frac{24}{5}\sigma_3(n) + \frac{96}{5}\sigma_3\left(\frac{n}{2}\right) + \frac{216}{5}\sigma_3\left(\frac{n}{3}\right) + \frac{864}{5}\sigma_3\left(\frac{n}{6}\right) + \frac{36}{5}c_6(n),\end{aligned}$$

as claimed.

We close this section by giving a short table (Table 2) of values of $N(n)$ (second column) as well as the values of

$$\frac{24}{5}\sigma_3(n) + \frac{96}{5}\sigma_3(n/2) + \frac{216}{5}\sigma_3(n/3) + \frac{864}{5}\sigma_3(n/6) + \frac{36}{5}c_6(n)$$

denoted by $E(n)$ (fifth column).

Table 2

n	$N(n)$	$\sigma_3(n)$	$c_6(n)$	$E(n)$
1	12	1	1	12
2	48	9	−2	48
3	156	28	−3	156
4	552	73	4	552
5	648	126	6	648
6	2352	252	6	2352
7	1536	344	−16	1536
8	4152	585	−8	4152
9	4908	757	9	4908
10	7776	1134	−12	7776

5. Two properties of $c_6(n)$

The table of values of $c_6(n)$ given in Section 2 suggests that $c_6(n)$ satisfies the relations

$$c_6(2n) = -2c_6(n), \quad n \in \mathbb{N}, \tag{5.1}$$

and

$$c_6(3n) = -3c_6(n), \quad n \in \mathbb{N}. \tag{5.2}$$

We now prove these two properties of $c_6(n)$.

First we prove (5.1). Using the elementary identity

$$\sigma(2k) = 3\sigma(k) - 2\sigma(k/2) \quad (k \in \mathbb{N}) \tag{5.3}$$

we obtain

$$\begin{aligned} W_6(2n) &= \sum_{m<2n/6} \sigma(m)\sigma(2n-6m) \\ &= 3\sum_{m<n/3} \sigma(m)\sigma(n-3m) - 2\sum_{m<n/3} \sigma(m)\sigma\big((n-3m)/2\big). \end{aligned}$$

Hence

$$3W_3(n) - W_6(2n) = 2\sum_{\substack{(k,m)\in\mathbb{N}^2 \\ 2k+3m=n}} \sigma(k)\sigma(m).$$

Appealing to the result

$$W_3(n) = \frac{1}{24}\big(\sigma_3(n) + 9\sigma_3(n/3) + (1-2n)\sigma(n) + (1-6n)\sigma(n/3)\big),$$

see for example [10, Theorem 3, p. 248], [18, Theorem 1.9, p. 528], and to Theorem 1, we obtain using (5.3) and the elementary identity

$$\sigma_3(2k) = 9\sigma_3(k) - 8\sigma_3(k/2) \quad (k \in \mathbb{N})$$

the relation (5.1).

Finally we prove (5.2). Using the elementary identity

$$\sigma(3k) = 4\sigma(k) - 3\sigma(k/3) \quad (k \in \mathbb{N}) \tag{5.4}$$

we obtain

$$\begin{aligned} W_6(3n) &= \sum_{m<3n/6} \sigma(m)\sigma(3n-6m) \\ &= 4\sum_{m<n/2} \sigma(m)\sigma(n-2m) - 3\sum_{m<n/2} \sigma(m)\sigma\big((n-2m)/3\big). \end{aligned}$$

Hence

$$4W_2(n) - W_6(3n) = 3\sum_{\substack{(k,m)\in\mathbb{N}^2 \\ 2m+3k=n}} \sigma(m)\sigma(k).$$

Appealing to the result

$$W_2(n) = \frac{1}{24}\big(2\sigma_3(n) + 8\sigma_3(n/2) + (1-3n)\sigma(n) + (1-6n)\sigma(n/2)\big),$$

see for example [10, Theorem 2, p. 247], [6, Theorem 5.2, p. 571], and to Theorem 1, we obtain using (5.4) and the elementary identity

$$\sigma_3(3k) = 28\sigma_3(k) - 27\sigma_3(k/3) \quad (k \in \mathbb{N})$$

the relation (5.2).

We leave it to the reader to deduce (5.1) and (5.2) from (2.2) and (2.3). It is clear from (5.1) and (5.2) that

$$c\big(2^r 3^s\big) = (-1)^{r+s} 2^r 3^s \quad (r, s \in \mathbb{N}_0).$$

Hence the sums $W_6(n)$ and (1.2) have elementary evaluations when $n = 2^r 3^s$.

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