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EVALUATION OF COMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND AT SINGULAR MODULI

Habib Muzaffar and Kenneth S. Williams

Abstract. The complete elliptic integral of the first kind $K(k)$ is defined for $0 < k < 1$ by

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

The real number k is called the modulus of the elliptic integral. The complementary modulus is $k' = (1-k^2)^{\frac{1}{2}}$ $(0 < k' < 1)$. Let λ be a positive integer. The equation

$$K(k') = \sqrt{\lambda}K(k)$$

defines a unique real number $k(\lambda)$ $(0 < k(\lambda) < 1)$ called the singular modulus of $K(k)$. Let $H(D)$ denote the form class group of discriminant D. Let d be the discriminant -4λ. Using some recent results of the authors on values of the Dedekind eta function at quadratic irrationalities, a formula is given for the singular modulus $k(\lambda)$ in terms of quantities depending upon $H(4d)$ if $\lambda \equiv 0 \pmod 2$; $H(d)$ and $H(4d)$ if $\lambda \equiv 1 \pmod 4$; $H(d/4)$ and $H(4d)$ if $\lambda \equiv 3 \pmod 4$. Similarly a formula is given for the complete elliptic integral $K[\sqrt{\lambda}] := K(k(\lambda))$ in terms of quantities depending upon $H(d)$ and $H(4d)$ if $\lambda \equiv 0 \pmod 2$; $H(d)$ if $\lambda \equiv 1 \pmod 4$; $H(d/4)$ and $H(d)$ if $\lambda \equiv 3 \pmod 4$. As an example the complete elliptic integral $K[\sqrt{17}]$ is determined explicitly in terms of gamma values.

1. Introduction

Let $k \in \mathbb{R}$ be such that

$$(1.1) \qquad 0 < k < 1.$$

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The complete elliptic integral $K(k)$ of the first kind is defined by

$$(1.2) \qquad K(k) := \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Clearly

$$\lim_{k\to 0^+} K(k) = \frac{\pi}{2}, \quad \lim_{k\to 1^-} K(k) = +\infty.$$

The quantity k is called the modulus of the elliptic integral $K(k)$. The complementary modulus k' is defined by

$$(1.3) \qquad k' := \sqrt{1-k^2}.$$

From (1.1) and (1.3) we see that

$$(1.4) \qquad 0 < k' < 1.$$

The complete elliptic integral $K(k')$ of modulus k' is denoted by $K'(k)$ so that

$$(1.5) \qquad K'(k) = K(k') = K(\sqrt{1-k^2})$$

and

$$(1.6) \qquad \lim_{k\to 0^+} K'(k) = +\infty, \quad \lim_{k\to 1^-} K'(k) = \frac{\pi}{2}.$$

Let $\lambda \in \mathbb{N}$. As k increases from 0 to 1, the function $K'(k)/K(k)$ decreases from $+\infty$ to 0. Hence there is a unique modulus $k = k(\lambda)$ with $0 < k < 1$ such that

$$(1.7) \qquad \frac{K'(k)}{K(k)} = \sqrt{\lambda}.$$

The real number $k(\lambda)$ is called the singular modulus corresponding to λ. The value of the complete elliptic integral $K(k)$ of the first kind at the singular modulus $k = k(\lambda)$ is denoted by

$$(1.8) \qquad K[\sqrt{\lambda}] := K(k(\lambda)).$$

The first five singular moduli are

$$\begin{aligned} k(1) &= \frac{1}{\sqrt{2}}, \\ k(2) &= \sqrt{2}-1, \\ k(3) &= \frac{\sqrt{3}-1}{\sqrt{8}}, \end{aligned}$$

$$k(4) = 3 - 2\sqrt{2},$$
$$k(5) = \frac{\sqrt{\sqrt{5}-1} - \sqrt{3-\sqrt{5}}}{2},$$

see for example [1, p. 139]. The values of $K[\sqrt{\lambda}]$ for $\lambda = 1, 2, \ldots, 16$ are given in [1, Table 9.1, p. 298]. Other values can be found scattered in the literature. For example in [2, p. 277] the values

$$k(22) = -99 - 70\sqrt{2} + 30\sqrt{11} + 21\sqrt{22} \tag{1.9}$$

and

$$K[\sqrt{22}] = 2^{-5/2} 11^{-1/2} (7+5\sqrt{2}+3\sqrt{22})^{1/2} \pi^{1/2} \left\{ \prod_{m=1}^{88} \Gamma\left(\frac{m}{88}\right)^{\left(\frac{-88}{m}\right)} \right\}^{1/4} \tag{1.10}$$

are given, where $\Gamma(x)$ is the gamma function and $\left(\frac{d}{n}\right)$ is the Kronecker symbol. The values of $k(25)$ and $K[\sqrt{25}]$ are given in [5, p. 259].

Let $H(D)$ denote the form class group of discriminant D. Let d be the discriminant -4λ. Using some recent results of the authors on values of the Dedekind eta function at quadratic irrationalities, a formula is given for the singular modulus $k(\lambda)$ in terms of quantities depending upon $H(4d)$ if $\lambda \equiv 0 \pmod 2$; $H(d)$ and $H(4d)$ if $\lambda \equiv 1 \pmod 4$; $H(d/4)$ and $H(4d)$ if $\lambda \equiv 3 \pmod 4$, see Theorem 1 in Section 4. Similarly a formula is given for the complete elliptic integral $K[\sqrt{\lambda}] := K(k(\lambda))$ in terms of quantities depending upon $H(d)$ and $H(4d)$ if $\lambda \equiv 0 \pmod 2$; $H(d)$ if $\lambda \equiv 1 \pmod 4$; $H(d/4)$ and $H(d)$ if $\lambda \equiv 3 \pmod 4$, see Theorem 1 in Section 4. Zucker [5, p. 258] has determined but not published the values of $K[\sqrt{\lambda}]$ for $\lambda = 17$, 18, 19 and 20, so as an example we determine explicitly the complete elliptic integral $K[\sqrt{17}]$ in terms of gamma values, see Theorem 2 in Section 5. Our method is different from that of Zucker.

2. Preliminary Results

Let $\lambda \in \mathbb{N}$ and set

$$q = e^{-\pi\sqrt{\lambda}} \tag{2.1}$$

so that $0 < q < 1$. We define

$$Q_0 := \prod_{n=1}^{\infty} (1 - q^{2n}), \tag{2.2}$$

(2.3) $$Q_1 := \prod_{n=1}^{\infty}(1+q^{2n}),$$

(2.4) $$Q_2 := \prod_{n=1}^{\infty}(1+q^{2n-1}),$$

(2.5) $$Q_3 := \prod_{n=1}^{\infty}(1-q^{2n-1}).$$

Since

$$Q_1Q_2 = \prod_{n=1}^{\infty}(1+q^n), \quad Q_0Q_3 = \prod_{n=1}^{\infty}(1-q^n),$$

we have

$$Q_0Q_1Q_2Q_3 = \prod_{n=1}^{\infty}(1-q^{2n}) = Q_0,$$

so that

(2.6) $$Q_1Q_2Q_3 = 1.$$

Jacobi [3] [4, p. 147] has shown that

(2.7) $$16qQ_1^8 + Q_3^8 = Q_2^8.$$

He has also shown that the singular modulus $k = k(\lambda)$, the complementary singular modulus $k'(\lambda)$, and the complete elliptic integral $K[\sqrt{\lambda}] = K(k(\lambda))$ are given by

(2.8) $$k(\lambda) = 4\sqrt{q}\left(\frac{Q_1}{Q_2}\right)^4,$$

(2.9) $$k'(\lambda) = \left(\frac{Q_3}{Q_2}\right)^4,$$

and

(2.10) $$K[\sqrt{\lambda}] = \frac{\pi}{2}\left(\frac{Q_0Q_2}{Q_1Q_3}\right)^2,$$

see [3] [4, p. 146]. Next we recall that the Dedekind eta function $\eta(z)$ is defined by

(2.11) $$\eta(z) := e^{\pi iz/12}\prod_{m=1}^{\infty}(1-e^{2\pi imz}), \quad z \in \mathbb{C}, \ \mathrm{Im}(z) > 0,$$

and that Weber's functions $f(z)$, $f_1(z)$ and $f_2(z)$ are defined in terms of the Dedekind eta function by

(2.12) $$f(z) := e^{-\pi i/24}\frac{\eta\left(\frac{1+z}{2}\right)}{\eta(z)},$$

(2.13) $$f_1(z) := \frac{\eta\left(\frac{z}{2}\right)}{\eta(z)},$$

(2.14) $$f_2(z) := \sqrt{2}\,\frac{\eta(2z)}{\eta(z)},$$

see [9, p. 114]. It is convenient to set

$$f_0(z) := f(z)$$

so that $f_j(z)$ is defined for $j = 0, 1, 2$. From (2.1)-(2.5) and (2.11), we deduce that

(2.15) $$\eta(\sqrt{-\lambda}) = q^{1/12}Q_0,$$

(2.16) $$\eta(2\sqrt{-\lambda}) = q^{1/6}Q_0Q_1,$$

(2.17) $$\eta(\sqrt{-\lambda}/2) = q^{1/24}Q_0Q_3,$$

(2.18) $$\eta((1+\sqrt{-\lambda})/2) = e^{\pi i/24}q^{1/24}Q_0Q_2.$$

From (2.12)-(2.18) we obtain

(2.19) $$Q_0 = q^{-1/12}\eta(\sqrt{-\lambda}),$$

(2.20) $$Q_1 = 2^{-1/2}q^{-1/12}f_2(\sqrt{-\lambda}),$$

(2.21) $$Q_2 = q^{1/24}f_0(\sqrt{-\lambda}),$$

(2.22) $$Q_3 = q^{1/24}f_1(\sqrt{-\lambda}).$$

Then, from (2.6), (2.7), (2.20), (2.21) and (2.22), we obtain

(2.23) $$f_0(\sqrt{-\lambda})f_1(\sqrt{-\lambda})f_2(\sqrt{-\lambda}) = \sqrt{2}$$

and

$$f_0(\sqrt{-\lambda})^8 = f_1(\sqrt{-\lambda})^8 + f_2(\sqrt{-\lambda})^8, \tag{2.24}$$

see [9, p, 114]. Then, from (2.8), (2.10) and (2.19) – (2.23), we obtain $k(\lambda)$ and $K[\sqrt{\lambda}]$ in terms of λ, namely,

$$k(\lambda) = \left(\frac{f_2(\sqrt{-\lambda})}{f_0(\sqrt{-\lambda})}\right)^4 \tag{2.25}$$

and

$$K[\sqrt{\lambda}] = \frac{\pi}{2}\eta(\sqrt{-\lambda})^2 f_0(\sqrt{-\lambda})^4. \tag{2.26}$$

Recent results of Muzaffar and Williams [6] give the values of $\eta(\sqrt{-\lambda})$, $f_0(\sqrt{-\lambda})$, $f_1(\sqrt{-\lambda})$ and $f_2(\sqrt{-\lambda})$ for all $\lambda \in \mathbb{N}$, see Section 3. Using these values in (2.25) and (2.26), we obtain the singular modulus $k(\lambda)$ and the complete elliptic integral of the first kind $K[\sqrt{\lambda}]$ in Section 4.

3. Evaluation of $\eta(\sqrt{-\lambda})$, $f_0(\sqrt{-\lambda})$, $f_1(\sqrt{-\lambda})$ and $f_2(\sqrt{-\lambda})$

Let d be an integer satisfying

$$d < 0, \quad d \equiv 0 \text{ or } 1 \pmod 4. \tag{3.1}$$

Let f be the largest positive integer such that

$$f^2 \mid d, \;\; d/f^2 \equiv 0 \text{ or } 1 \pmod 4. \tag{3.2}$$

We set $\Delta = d/f^2 \in \mathbb{Z}$ so that

$$d = \Delta f^2, \quad \Delta \equiv 0, 1 \pmod 4 . \tag{3.3}$$

For a prime p, the nonnegative integer $v_p(f)$ is defined by $p^{v_p(f)} \mid f$, $p^{v_p(f)+1} \nmid f$. We set

$$\alpha_p(\Delta, f) = \frac{\left(p^{v_p(f)} - 1\right)\left(1 - \left(\frac{\Delta}{p}\right)\right)}{p^{v_p(f)-1}(p-1)\left(p - \left(\frac{\Delta}{p}\right)\right)}, \tag{3.4}$$

where $\left(\frac{\Delta}{p}\right)$ is the Legendre symbol modulo p. The quantity $\alpha_p(\Delta, f)$ is used in Proposition 1 below.

The positive-definite, primitive, integral, binary quadratic form $ax^2+bxy+cy^2$ is denoted by (a,b,c). Its discriminant is the quantity $d=b^2-4ac$, which satisfies (3.1). The class of the form (a,b,c) is

$$(3.5)\quad [a,b,c]=\{(a(p,r),\ b(p,q,r,s),\ c(q,s))\mid p,q,r,s\in\mathbb{Z},\ ps-qr=1\},$$

where

$$a(p,r)=ap^2+bpr+cr^2,\ b(p,q,r,s)=2apq+bps+bqr+2crs,\ c(q,s)=aq^2+bqs+cs^2.$$

The group of classes of positive-definite, primitive, integral, binary quadratic forms of discriminant d under Gaussian composition is denoted by $H(d)$. $H(d)$ is a finite abelian group. We denote its order by $h(d)$. The identity I of the group $H(d)$ is the principal class

$$(3.6)\qquad I=\begin{cases}[1,0,-d/4], & \text{if } d\equiv 0 \pmod 4,\\ [1,1,(1-d)/4], & \text{if } d\equiv 1 \pmod 4.\end{cases}$$

The inverse of the class $K=[a,b,c]\in H(d)$ is the class $K^{-1}=[a,-b,c]\in H(d)$. If p is a prime with $\left(\frac{d}{p}\right)=1$, we let h_1 and h_2 be the solutions of $h^2\equiv d \pmod{4p}$, $0\le h<2p$, with $h_1<h_2$. The class K_p of $H(d)$ is defined by

$$K_p=\left[p,h_1,\frac{h_1^2-d}{4p}\right].$$

Then

$$K_p^{-1}=\left[p,-h_1,\frac{h_1^2-d}{4p}\right]=\left[p,h_2,\frac{h_2^2-d}{4p}\right],$$

as $h_1+h_2=2p$. If p is a prime with $\left(\frac{d}{p}\right)=0$, $p\nmid f$, the class K_p of $H(d)$ is defined by

$$K_p=\begin{cases}[p,0,-d/4p], & \text{if } p>2,\ d\equiv 0 \pmod 4,\\ [p,p,(p^2-d)/4p], & \text{if } p>2,\ d\equiv 1 \pmod 4,\\ [2,0,-d/8], & \text{if } p=2,\ d\equiv 8 \pmod{16},\\ [2,2,(4-d)/8], & \text{if } p=2,\ d\equiv 12 \pmod{16},\end{cases}$$

so that $K_p=K_p^{-1}$. We do not define K_p for any other primes p.

As $H(d)$ is a finite abelian group, there exist positive integers $h_1,h_2,\ldots,h_\nu$ and generators $A_1,A_2,\ldots,A_\nu\in H(d)$ such that

$$h_1h_2\cdots h_\nu=h(d),\quad 1<h_1\mid h_2\mid\ldots\mid h_\nu,\quad \operatorname{ord}(A_i)=h_i\ (i=1,\ldots,\nu),$$

and, for each $K \in H(d)$, there exist unique integers $k_1, \ldots, k_\nu$ with

$$K = A_1^{k_1} \cdots A_\nu^{k_\nu} \ (0 \le k_j < h_j,\ j = 1, \ldots, \nu).$$

We fix once and for all the generators $A_1, \ldots, A_\nu$ of the group $H(d)$. For $j = 1, \ldots, \nu$ we set

$$\mathrm{ind}_{A_j}(K) := k_j,$$

and for $K,\ L \in H(d)$, we define $\chi : H(d) \times H(d) \longrightarrow \Omega_{h_\nu}$ (group of h_ν th roots of unity) by

$$\chi(K, L) = e^{2\pi i \sum_{j=1}^{\nu} \frac{\mathrm{ind}_{A_j}(K)\, \mathrm{ind}_{A_j}(L)}{h_j}}$$

The function χ has the properties

$$\begin{aligned}
\chi(K, L) &= \chi(L, K), \text{ for all } K, L \in H(d),\\
\chi(K, I) &= 1, \text{ for all } K \in H(d),\\
\chi(KL, M) &= \chi(K, M)\chi(L, M), \text{ for all } K, L, M \in H(d),\\
\chi(K^r, L^s) &= \chi(K, L)^{rs}, \text{ for all } K, L \in H(d) \text{ and all } r, s \in \mathbb{Z},
\end{aligned}$$

see [6, Lemma 6.2]. It is known that for $K(\neq I) \in H(d)$ the limit

$$j(K, d) = \lim_{s \to 1^+} \prod_{\substack{p \\ \left(\frac{d}{p}\right)=1}} \left(1 - \frac{\chi(K, K_p)}{p^s}\right)\left(1 - \frac{\chi(K^{-1}, K_p)}{p^s}\right) \tag{3.7}$$

exists and is a nonzero real number such that $j(K, d) = j(K^{-1}, d)$, see [6, Lemma 7.6]. For $n \in \mathbb{N}$ and $L \in H(d)$ we define

$$H_L(n) := \mathrm{card}\{h \mid 0 \le h < 2n, \quad h^2 \equiv d \pmod{4n}, \quad \left[n, h, \frac{h^2 - d}{4n}\right] = L\}.$$

The properties of $H_L(n)$ are developed in [6, Section 5]. Then, for $n \in \mathbb{N}$ and $K \in H(d)$, we set

$$Y_K(n) := \sum_{L \in H(d)} \chi(K, L) H_L(n).$$

Properties of $Y_K(n)$ are given in [6, Section 7]. Further, for a prime p and a class $K(\neq I) \in H(d)$, we set

$$A(K, d, p) = \sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^j}. \tag{3.8}$$

Next, for $K(\neq I) \in H(d)$, we set

$$l(K,d) = \prod_{\substack{p \mid d \\ p \nmid f}} \left(1 + \frac{\chi(K, K_p)}{p}\right) \prod_{p \mid f} A(K,d,p), \tag{3.9}$$

where the products are over all primes p satisfying the stated conditions. Finally, for $K \in H(d)$, we define

$$E(K,d) = \frac{\pi\sqrt{|d|}w(d)}{48h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L,K)^{-1} \frac{t_1(d)}{j(L,d)} l(L,d), \tag{3.10}$$

see [6, Section 9], where

$$w(d) = 6, 4 \text{ or } 2 \text{ according as } d = -3, d = -4 \text{ or } d < -4, \tag{3.11}$$

and

$$t_1(d) := \prod_{\substack{p \\ \left(\frac{d}{p}\right) = 1}} \left(1 - \frac{1}{p^2}\right). \tag{3.12}$$

The following evaluation of $\eta(\sqrt{-\lambda})$ follows immediately from [6, Theorem 1], as $\eta(\sqrt{-\lambda})$ is real and positive for $\lambda \in \mathbb{N}$.

Proposition 1. *Let $\lambda \in \mathbb{N}$. Let $d = -4\lambda = \Delta f^2$, where Δ and f are defined in (3.2) and (3.3). Let $K = [1, 0, \lambda] \in H(d)$. Then*

$$\eta(\sqrt{-\lambda}) = 2^{-3/4}\pi^{-1/4}\lambda^{-1/4} \prod_{p \mid f} p^{\alpha_p(\Delta,f)/4} \left(\prod_{m=1}^{|\Delta|} \Gamma\left(\frac{m}{|\Delta|}\right)^{\left(\frac{\Delta}{m}\right)}\right)^{\frac{w(\Delta)}{8h(\Delta)}} e^{-E(K,d)},$$

where $\alpha_p(\Delta, f)$ is defined in (3.4) and $\left(\frac{\Delta}{m}\right)$ is the Kronecker symbol.

The following result is Theorem 3 of [6].

Proposition 2. *Let $\lambda \in \mathbb{N}$. Let $d = -4\lambda$. Let $K = [1, 0, \lambda] \in H(d)$.*
(a) $\lambda \equiv 0 \pmod 4$. *Set*

$$\begin{aligned} M_0 &= [4, 4, \lambda + 1] \in H(4d), \\ M_1 &= \left[1, 0, \frac{\lambda}{4}\right] \in H\left(\frac{d}{4}\right), \\ M_2 &= [1, 0, 4\lambda] \in H(4d). \end{aligned}$$

Let $\lambda = 4^\alpha \mu$, where α is a positive integer and $\mu \equiv 1, 2$ or $3 \pmod 4$.

(i) $\mu \equiv 1$ *or* $2 \pmod 4$ (*so that* Δ *is even and* $v_2(f) = \alpha$). *We have*

$$\begin{aligned} f_0(\sqrt{-\lambda}) &= 2^{\frac{1}{2^{\alpha+3}}} e^{E(K,d)-E(M_0,4d)}, \\ f_1(\sqrt{-\lambda}) &= 2^{\frac{2^{\alpha+1}-1}{2^{\alpha+2}}} e^{E(K,d)-E(M_1,d/4)}, \\ f_2(\sqrt{-\lambda}) &= 2^{\frac{1}{2^{\alpha+3}}} e^{E(K,d)-E(M_2,4d)}. \end{aligned}$$

(ii) $\mu \equiv 3 \pmod 4$ (*so that* $\Delta \equiv -\mu \pmod 8$ *and* $v_2(f) = \alpha + 1$). *If* $\mu \equiv 3 \pmod 8$, *we have*

$$\begin{aligned} f_0(\sqrt{-\lambda}) &= 2^{\frac{1}{3\cdot 2^{\alpha+2}}} e^{E(K,d)-E(M_0,4d)}, \\ f_1(\sqrt{-\lambda}) &= 2^{\frac{3\cdot 2^{\alpha}-1}{3\cdot 2^{\alpha+1}}} e^{E(K,d)-E(M_1,d/4)}, \\ f_2(\sqrt{-\lambda}) &= 2^{\frac{1}{3\cdot 2^{\alpha+2}}} e^{E(K,d)-E(M_2,4d)}. \end{aligned}$$

If $\mu \equiv 7 \pmod 8$, *we have*

$$\begin{aligned} f_0(\sqrt{-\lambda}) &= e^{E(K,d)-E(M_0,4d)}, \\ f_1(\sqrt{-\lambda}) &= \sqrt{2} e^{E(K,d)-E(M_1,d/4)}, \\ f_2(\sqrt{-\lambda}) &= e^{E(K,d)-E(M_2,4d)}. \end{aligned}$$

(b) $\lambda \equiv 1 \pmod 4$ (*so that* Δ *is even and* f *is odd*). *Set*

$$\begin{aligned} M_0 &= \left[2, 2, \frac{\lambda+1}{2}\right] \in H(d), \\ M_1 &= [4, 0, \lambda] \in H(4d), \\ M_2 &= [1, 0, 4\lambda] \in H(4d). \end{aligned}$$

Then

$$\begin{aligned} f_0(\sqrt{-\lambda}) &= 2^{1/4} e^{E(K,d)-E(M_0,d)}, \\ f_1(\sqrt{-\lambda}) &= 2^{1/8} e^{E(K,d)-E(M_1,4d)}, \\ f_2(\sqrt{-\lambda}) &= 2^{1/8} e^{E(K,d)-E(M_2,4d)}. \end{aligned}$$

(c) $\lambda \equiv 2 \pmod 4$ (*so that* Δ *is even and* f *is odd*). *Set*

$$\begin{aligned} M_0 &= [4, 4, \lambda+1] \in H(4d), \\ M_1 &= \left[2, 0, \frac{\lambda}{2}\right] \in H(d), \\ M_2 &= [1, 0, 4\lambda] \in H(4d). \end{aligned}$$

Then

$$\begin{aligned} f_0(\sqrt{-\lambda}) &= 2^{1/8} e^{E(K,d)-E(M_0,4d)}, \\ f_1(\sqrt{-\lambda}) &= 2^{1/4} e^{E(K,d)-E(M_1,d)}, \\ f_2(\sqrt{-\lambda}) &= 2^{1/8} e^{E(K,d)-E(M_2,4d)}. \end{aligned}$$

(d) $\lambda \equiv 3 \pmod 4$ *(so that* $\lambda \equiv -\Delta \pmod 8$ *and* $f \equiv 2 \pmod 4$*). Set*

$$\begin{aligned} M_0 &= \left[1, 1, \frac{\lambda+1}{4}\right] \in H\left(\frac{d}{4}\right), \\ M_1 &= [4, 0, \lambda] \in H(4d), \\ M_2 &= [1, 0, 4\lambda] \in H(4d). \end{aligned}$$

Then, for $\lambda \equiv 3 \pmod 8$*, we have*

$$\begin{aligned} f_0(\sqrt{-\lambda}) &= 2^{1/3} e^{E(K,d)-E(M_0,d/4)}, \\ f_1(\sqrt{-\lambda}) &= 2^{1/12} e^{E(K,d)-E(M_1,4d)}, \\ f_2(\sqrt{-\lambda}) &= 2^{1/12} e^{E(K,d)-E(M_2,4d)}, \end{aligned}$$

and, for $\lambda \equiv 7 \pmod 8$*, we have*

$$\begin{aligned} f_0(\sqrt{-\lambda}) &= \sqrt{2} e^{E(K,d)-E(M_0,d/4)}, \\ f_1(\sqrt{-\lambda}) &= e^{E(K,d)-E(M_1,4d)}, \\ f_2(\sqrt{-\lambda}) &= e^{E(K,d)-E(M_2,4d)}. \end{aligned}$$

4. Formulae for $k(\lambda)$ and $K[\sqrt{\lambda}]$

From (2.25), (2.26), Proposition 1 and Proposition 2, we obtain the main result of this paper, namely, the formulae for the singular modulus $k(\lambda)$ and the complete elliptic integral of the first kind $K[\sqrt{\lambda}]$ at the singular modulus valid for every $\lambda \in \mathbb{N}$.

Theorem 1. *Let* $\lambda \in \mathbb{N}$*. Let* $d = -4\lambda$*. Let* $K = [1, 0, \lambda] \in H(d)$*.*
(a) $\lambda \equiv 0 \pmod 4$*. Set*

$$M_0 = [4, 4, \lambda + 1] \in H(4d), \quad M_2 = [1, 0, 4\lambda] \in H(4d).$$

Then

$$k(\lambda) = e^{4(E(M_0,4d)-E(M_2,4d))}.$$

Let $\lambda = 4^\alpha \mu$*, where* α *is a positive integer and* $\mu \equiv 1, 2$ *or* $3 \pmod 4$*. Then*

$$K[\sqrt{\lambda}] = 2^{\beta}\pi^{1/2}\lambda^{-1/2}\prod_{p|f} p^{\alpha_p(\Delta,f)/2}\left(\prod_{m=1}^{|\Delta|}\Gamma\left(\frac{m}{|\Delta|}\right)^{\left(\frac{\Delta}{m}\right)}\right)^{\frac{w(\Delta)}{4h(\Delta)}} e^{2E(K,d)-4E(M_0,4d)},$$

where

$$\beta = \begin{cases} \dfrac{1}{2^{\alpha+1}} - \dfrac{5}{2}, & \text{if } \mu \equiv 1 \text{ or } 2 \pmod 4, \\ \dfrac{1}{3\cdot 2^{\alpha}} - \dfrac{5}{2}, & \text{if } \mu \equiv 3 \pmod 8, \\ -\dfrac{5}{2}, & \text{if } \mu \equiv 7 \pmod 8, \end{cases}$$

(b) $\lambda \equiv 1 \pmod 4$. *Set*

$$M_0 = \left[2, 2, \frac{\lambda+1}{2}\right] \in H(d), \quad M_2 = [1, 0, 4\lambda] \in H(4d).$$

Then

$$k(\lambda) = 2^{-1/2} e^{4(E(M_0,d)-E(M_2,4d))}$$

and

$$K[\sqrt{\lambda}] = 2^{-3/2}\pi^{1/2}\lambda^{-1/2}\prod_{p|f} p^{\alpha_p(\Delta,f)/2}\left(\prod_{m=1}^{|\Delta|}\Gamma\left(\frac{m}{|\Delta|}\right)^{\left(\frac{\Delta}{m}\right)}\right)^{\frac{w(\Delta)}{4h(\Delta)}} e^{2E(K,d)-4E(M_0,d)}.$$

(c) $\lambda \equiv 2 \pmod 4$. *Set*

$$M_0 = [4, 4, \lambda+1] \in H(4d), \quad M_2 = [1, 0, 4\lambda] \in H(4d).$$

Then

$$k(\lambda) = e^{4(E(M_0,4d)-E(M_2,4d))}$$

and

$$K[\sqrt{\lambda}] = 2^{-2}\pi^{1/2}\lambda^{-1/2}\prod_{p|f} p^{\alpha_p(\Delta,f)/2}\left(\prod_{m=1}^{|\Delta|}\Gamma\left(\frac{m}{|\Delta|}\right)^{\left(\frac{\Delta}{m}\right)}\right)^{\frac{w(\Delta)}{4h(\Delta)}} e^{2E(K,d)-4E(M_0,4d)}.$$

(d) $\lambda \equiv 3 \pmod 4$. *Set*

$$M_0 = \left[1, 1, \frac{\lambda+1}{4}\right] \in H\left(\frac{d}{4}\right), \quad M_2 = [1, 0, 4\lambda] \in H(4d).$$

Then, for $\lambda \equiv 3 \pmod 8$, *we have*

$$k(\lambda) = 2^{-1} e^{4(E(M_0,d/4)-E(M_2,4d))}$$

and

$$K[\sqrt{\lambda}] = 2^{-7/6}\pi^{1/2}\lambda^{-1/2}\prod_{p|f} p^{\alpha_p(\Delta,f)/2}\left(\prod_{m=1}^{|\Delta|}\Gamma\left(\frac{m}{|\Delta|}\right)^{\left(\frac{\Delta}{m}\right)}\right)^{\frac{w(\Delta)}{4h(\Delta)}} e^{2E(K,d)-4E(M_0,d/4)},$$

and, for $\lambda \equiv 7 \pmod 8$, *we have*

$$k(\lambda) = 2^{-2}e^{4(E(M_0,d/4)-E(M_2,4d))}$$

and

$$K[\sqrt{\lambda}] = 2^{-1/2}\pi^{1/2}\lambda^{-1/2}\prod_{p|f} p^{\alpha_p(\Delta,f)/2}\left(\prod_{m=1}^{|\Delta|}\Gamma\left(\frac{m}{|\Delta|}\right)^{\left(\frac{\Delta}{m}\right)}\right)^{\frac{w(\Delta)}{4h(\Delta)}} e^{2E(K,d)-4E(M_0,d/4)}.$$

5. Evaluation of $K[\sqrt{17}]$

In this section we use Theorem 1 to evaluate the complete elliptic integral of the first kind $K[\sqrt{17}]$. Thus $\lambda = 17$, $d = -4\lambda = -68$, $\Delta = -68$ and $f = 1$ in the notation of Sections 3 and 4. The group $H(-68)$ of classes of positive-definite, primitive, integral binary quadratic forms of discriminant -68 under composition is

$$H(-68) = \{I, A, A^2, A^3\}, \quad A^4 = I,$$

where

$$I = [1, 0, 17], \;\; A = [3, -2, 6], \;\; A^2 = [2, 2, 9], \;\; A^3 = [3, 2, 6].$$

In order to determine $K[\sqrt{17}]$ explicitly using Theorem 1, we must determine $E(I, -68)$ and $E(A^2, -68)$ (see Lemma 14). This requires finding $j(A^m, -68)$ ($m = 1, 2, 3$) (see Lemma 13). To compute $j(A^m, -68)$ ($m = 1, 2, 3$) from (3.7) we must determine those primes p satisfying $\left(\frac{-1}{p}\right) = \left(\frac{p}{17}\right) = 1$ for which $K_p = I$ and those for which $K_p = A^2$. This depends upon whether p is of the form x^2+17y^2 for integers x and y or of the form $2x^2 + 2xy + 9y^2$ for integers x and y. By class field theory the former occurs if and only if the quartic polynomial x^4+x^2+2x+1 is the product of four linear factors (mod p). This leads us to consider the arithmetic of the field $K = \mathbb{Q}(\theta)$, where θ is a root of $x^4 + x^2 + 2x + 1$.

Let $f(x)$ be the irreducible quartic polynomial given by

$$f(x) = x^4 + x^2 + 2x + 1 \in \mathbb{Z}[x]. \tag{5.1}$$

The discriminant of $f(x)$ is $272 = 2^4 \cdot 17$ and its Galois group is D_8 (the dihedral group of order 8) [8, p. 441]. The four roots of $f(x)$ are

$$\frac{1}{2}(i + (-1+4i)^{\frac{1}{2}}), \quad \frac{1}{2}(i - (-1+4i)^{\frac{1}{2}}),$$
$$\frac{1}{2}(-i + (-1-4i)^{\frac{1}{2}}), \quad \frac{1}{2}(-i - (-1-4i)^{\frac{1}{2}}),$$

where $z^{\frac{1}{2}}$ denotes the principal value of the square root of the complex number z. Let

$$\theta = \frac{1}{2}(i + (-1+4i)^{\frac{1}{2}})$$

and set

$$K = Q(\theta) \tag{5.2}$$

so that K is the totally complex quartic field $Q((-1+4i)^{\frac{1}{2}})$. Thus the number of real embeddings of K is $r_1 = 0$ and the number of imaginary embeddings is $2r_2 = 4$. The ring of integers of K is

$$O_K = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2 + \mathbb{Z}\theta^3, \tag{5.3}$$

see [8, p. 441]. As K is monogenic, its discriminant $d(K) = \mathrm{disc}(f(x)) = 272$. It is known that O_K has classnumber $h_K = 1$ [8, p. 435] so that it is a principal ideal domain. As $r_1 + r_2 - 1 = 0 + 2 - 1 = 1$ we know by Dirichlet's unit theorem that O_K has a single fundamental unit. This unit can be taken to be θ [8, p. 441]. The regulator

$$R(K) = 2\log|\theta| = \log\left|\frac{i + (-1+4i)^{\frac{1}{2}}}{2}\right|^2 = \log\left(\frac{1 + \sqrt{2 + 2\sqrt{17}} + \sqrt{17}}{4}\right) \approx 0.732,$$

see [8, p. 441]. The quartic field K contains a unique subfield ($\neq \mathbb{Q}, K$), namely, $Q(i)$. The only roots of unity in O_K are ± 1 and $\pm i$. Thus the number of roots of unity in O_K is $w(K) = 4$.

We now give the factorization of $f(x)$ modulo a prime p. We use the notation (m) to denote a monic irreducible polynomial of degree m with integer coefficients. Thus $g(x) \equiv (2)(2) \pmod p$ means that $g(x)$ is the product of two distinct monic irreducible quadratic polynomials modulo p and $h(x) \equiv (2)^2$ means that $h(x)$ is the square of a monic irreducible quadratic polynomial modulo p. From class field theory or indeed by elementary arguments one can show that the factorization of $f(x) \pmod p$, where p is a prime $\neq 2, 17$, is given as follows:

If

$$\left(\frac{-1}{p}\right) = \left(\frac{p}{17}\right) = 1 \text{ and } p = u^2 + 17v^2 \text{ for some integers } u \text{ and } v$$

then

$$f(x) \equiv (1)(1)(1)(1) \pmod{p}.$$

If

$$\left(\frac{-1}{p}\right) = \left(\frac{p}{17}\right) = 1 \text{ and } p = 2u^2 + 2uv + 9v^2 \text{ for some integers } u \text{ and } v$$

then

$$f(x) \equiv (2)(2) \pmod{p}.$$

If

$$\left(\frac{-1}{p}\right) = -1, \left(\frac{p}{17}\right) = 1$$

then

$$f(x) \equiv (2)(2) \pmod{p}.$$

If

$$\left(\frac{-1}{p}\right) = 1, \left(\frac{p}{17}\right) = -1$$

then

$$f(x) \equiv (1)(1)(2) \pmod{p}.$$

If

$$\left(\frac{-1}{p}\right) = \left(\frac{p}{17}\right) = -1$$

then

$$f(x) \equiv (4) \pmod{p}.$$

For $p = 2$

$$f(x) \equiv (2)^2 \pmod{2}$$

and for $p = 17$

$$f(x) \equiv (1)(1)(1)^2 \pmod{17}.$$

Using these results, a standard algebraic number theoretic argument gives the factorization of the principal ideal pO_K into prime ideals in O_K, where p is a prime.

Lemma 1. *Let p be a prime $\neq 2, 17$.*

(i) *If*

$$\left(\frac{-1}{p}\right) = \left(\frac{17}{p}\right) = 1 \text{ and } p = x^2 + 17y^2 \text{ for some integers } x \text{ and } y$$

then

$$pO_K = PQRS, \quad N(P) = N(Q) = N(R) = N(S) = p,$$

where P, Q, R, S *are distinct prime ideals of* O_K.

(ii) *If*

$$\left(\frac{-1}{p}\right) = \left(\frac{17}{p}\right) = 1 \text{ and } p = 2x^2 + 2xy + 9y^2 \text{ for some integers } x \text{ and } y$$

then

$$pO_K = PQ, \quad N(P) = N(Q) = p^2,$$

where P *and* Q *are distinct prime ideals of* O_K.

(iii) *If*

$$\left(\frac{-1}{p}\right) = -1, \left(\frac{17}{p}\right) = 1$$

then

$$pO_K = PQ, \quad N(P) = N(Q) = p^2,$$

where P *and* Q *are distinct prime ideals of* O_K.

(iv) *If*

$$\left(\frac{-1}{p}\right) = 1, \left(\frac{17}{p}\right) = -1$$

then

$$pO_K = PQR, \quad N(P) = N(Q) = p, \quad N(R) = p^2,$$

where P, Q *and* R *are distinct prime ideals of* O_K.

(v) *If*

$$\left(\frac{-1}{p}\right) = \left(\frac{17}{p}\right) = -1$$

then

$$pO_K = P, \quad N(P) = p^4,$$

where P *is a prime ideal.*

(vi) $2O_K = P^2$, $N(P) = 2^2$, *where* P *is a prime ideal.*

(vii) $17O_K = PQR^2$, $N(P) = N(Q) = N(R) = 17$, *where* P, Q *and* R *are distinct prime ideals.*

The next lemma determines the class K_p of $H(-68)$ when p is a prime such that $\left(\frac{-68}{p}\right) = 1$.

Lemma 2. *Let p be a prime such that $\left(\frac{-68}{p}\right) = 1$. Then*

$$K_p = I \iff p = x^2 + 17y^2 \text{ for some integers } x \text{ and } y,$$
$$K_p = A^2 \iff p = 2x^2 + 2xy + 9y^2 \text{ for some integers } x \text{ and } y,$$
$$K_p = A \text{ or } A^3 \iff p = 3x^2 \pm 2xy + 6y^2 \text{ for some integers } x \text{ and } y.$$

Proof. As $\left(\frac{-68}{p}\right) = 1$ there exist integers x and y such that

$$p = x^2 + 17y^2 \text{ or } 2x^2 + 2xy + 9y^2, \text{ if } \left(\frac{-1}{p}\right) = \left(\frac{17}{p}\right) = 1,$$

and such that

$$p = 3x^2 \pm 2xy + 6y^2, \text{ if } \left(\frac{-1}{p}\right) = \left(\frac{17}{p}\right) = -1.$$

We recall that as p is a prime the only classes representing p are K_p and $K_p{}^{-1}$. Hence

$$p = x^2 + 17y^2 \implies [1, 0, 17] \text{ represents } p \implies I = K_p \text{ or } K_p^{-1} \implies K_p = I,$$
$$p = 2x^2 + 2xy + 9y^2 \implies [2, 2, 9] \text{ represents } p \implies A^2 = K_p \text{ or } K_p^{-1} \implies K_p = A^2,$$
$$p = 3x^2 \pm 2xy + 6y^2 \implies [3, 2, 6] \text{ represents } p \implies A^3 = K_p \text{ or } K_p^{-1} \implies K_p = A \text{ or } A^3.$$

This completes the proof of Lemma 2. ■

Definition 1. For $s > 1$ and $\epsilon, \eta \in \{-1, +1\}$ we define

$$A_{\epsilon,\eta}(s) := \prod_{\substack{p \neq 2, 17 \\ \left(\frac{-1}{p}\right) = \epsilon,\ \left(\frac{17}{p}\right) = \eta}} \left(1 + \frac{1}{p^s}\right)^{-1}$$

and

$$B_{\epsilon,\eta}(s) := \prod_{\substack{p \neq 2, 17 \\ \left(\frac{-1}{p}\right) = \epsilon,\ \left(\frac{17}{p}\right) = \eta}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

For brevity we just write $A_{+1,+1}(s), A_{+1,-1}(s), \ldots$ as $A_{++}(s), A_{+-}(s), \ldots$ respectively. In view of Lemmas 1 and 2 we can split each of $A_{++}(s)$ and $B_{++}(s)$ into two products as

$$A_{++}(s) = A'_{++}(s)A''_{++}(s), \quad B_{++} = B'_{++}(s)B''_{++}(s),$$

where

$$A'_{++}(s) := \prod_{\substack{p \neq 2,17 \\ K_p = I}} \left(1 + \frac{1}{p^s}\right)^{-1}, \quad A''_{++}(s) := \prod_{\substack{p \neq 2,17 \\ K_p = A^2}} \left(1 + \frac{1}{p^s}\right)^{-1}$$

and

$$B'_{++}(s) := \prod_{\substack{p \neq 2,17 \\ K_p = I}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad B''_{++}(s) := \prod_{\substack{p \neq 2,17 \\ K_p = A^2}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Lemma 3. *For $s > 1$ we have*

$$A_{\epsilon,\eta}(s) = \frac{B_{\epsilon,\eta}(2s)}{B_{\epsilon,\eta}(s)}, \quad \textit{where } \epsilon, \eta \in \{-1, +1\},$$

and

$$A'_{++}(s) = \frac{B'_{++}(2s)}{B'_{++}(s)}, \quad A''_{++}(s) = \frac{B''_{++}(2s)}{B''_{++}(s)}.$$

Proof. We just prove the first equality as the other two equalities can be proved similarly. We have

$$\begin{aligned} A_{\epsilon,\eta}(s)B_{\epsilon,\eta}(s) &= \prod_{\substack{p \neq 2,17 \\ \left(\frac{-1}{p}\right) = \epsilon,\ \left(\frac{17}{p}\right) = \eta}} \left(1 + \frac{1}{p^s}\right)^{-1} \prod_{\substack{p \neq 2,17 \\ \left(\frac{-1}{p}\right) = \epsilon,\ \left(\frac{17}{p}\right) = \eta}} \left(1 - \frac{1}{p^s}\right)^{-1} \\ &= \prod_{\substack{p \neq 2,17 \\ \left(\frac{-1}{p}\right) = \epsilon,\ \left(\frac{17}{p}\right) = \eta}} \left(1 - \frac{1}{p^{2s}}\right)^{-1} \\ &= B_{\epsilon,\eta}(2s), \end{aligned}$$

from which the asserted result follows. ∎

For $s > 1$ the Riemann zeta function is given by

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product is taken over all primes p. If D is an integer with $D \equiv 0$ or $1 \pmod 4$ the Dirichlet L-series $L(s, D)$ $(s > 1)$ is given by

$$L(s, D) = \prod_p \left(1 - \frac{\left(\frac{D}{p}\right)}{p^s}\right)^{-1}.$$

We prove

Lemma 4. *For $s > 1$ we have*

$$(i)\ \zeta(s) = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{17^s}\right)^{-1} B_{--}(s) B_{-+}(s) B_{+-}(s) B_{++}(s),$$

$$(ii)\ L(s, -4) = \left(1 - \frac{1}{17^s}\right)^{-1} \frac{B_{--}(2s)}{B_{--}(s)} \frac{B_{-+}(2s)}{B_{-+}(s)} B_{+-}(s) B_{++}(s),$$

$$(iii)\ L(s, 17) = \left(1 - \frac{1}{2^s}\right)^{-1} \frac{B_{--}(2s)}{B_{--}(s)} B_{-+}(s) \frac{B_{+-}(2s)}{B_{+-}(s)} B_{++}(s),$$

$$(iv)\ L(s, -68) = B_{--}(s) \frac{B_{-+}(2s)}{B_{-+}(s)} \frac{B_{+-}(2s)}{B_{+-}(s)} B_{++}(s).$$

Proof. We just give the proofs of (i) and (ii). Equations (iii) and (iv) can be proved similarly. Let

$$X = \{(-1, -1), (-1, +1), (+1, -1), (+1, +1)\}.$$

First we prove (i). We have

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{17^s}\right)\zeta(s) = \prod_{p \neq 2, 17} \left(1 - \frac{1}{p^s}\right)^{-1}$$
$$= \prod_{(\epsilon,\eta) \in X} \prod_{\substack{p \neq 2, 17 \\ \left(\frac{-1}{p}\right) = \epsilon,\ \left(\frac{17}{p}\right) = \eta}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

from which (i) now follows by Definition 1.

Next we prove (ii). We have

$$\begin{aligned} L(s,-4) &= \prod_p \left(1 - \frac{\left(\frac{-4}{p}\right)}{p^s}\right)^{-1} \\ &= \left(1 - \frac{1}{17^s}\right)^{-1} \prod_{p\neq 2,17} \left(1 - \frac{\left(\frac{-4}{p}\right)}{p^s}\right)^{-1} \\ &= \left(1 - \frac{1}{17^s}\right)^{-1} \prod_{(\epsilon,\eta)\in X} \prod_{\substack{p\neq 2,17 \\ \left(\frac{-1}{p}\right)=\epsilon,\ \left(\frac{17}{p}\right)=\eta}} \left(1 - \frac{\epsilon}{p^s}\right)^{-1} \\ &= \left(1 - \frac{1}{17^s}\right)^{-1} A_{--}(s)A_{-+}(s)B_{+-}(s)B_{++}(s), \end{aligned}$$

and (ii) follows using Lemma 3. ■

Lemma 5. *For $s > 1$ we have*

$$\begin{aligned} B_{--}(s)^4 &= L(s,-4)^{-1}L(s,17)^{-1}L(s,-68)B_{--}(2s)^2\zeta(s), \\ B_{-+}(s)^4 &= \left(1 - \frac{1}{2^s}\right)^2 L(s,-4)^{-1}L(s,17)L(s,-68)^{-1}B_{-+}(2s)^2\zeta(s), \\ B_{+-}(s)^4 &= \left(1 - \frac{1}{17^s}\right)^2 L(s,-4)L(s,17)^{-1}L(s,-68)^{-1}B_{+-}(2s)^2\zeta(s), \\ B_{++}(s)^4 &= \left(1 - \frac{1}{2^s}\right)^2 \left(1 - \frac{1}{17^s}\right)^2 L(s,-4)L(s,17)L(s,-68) \\ &\quad \times B_{--}(2s)^{-2}B_{-+}(2s)^{-2}B_{+-}(2s)^{-2}\zeta(s), \end{aligned}$$

Proof. We obtain the asserted equalities by solving the equations (i)-(iv) in Lemma 4 for $B_{--}(s)$, $B_{-+}(s)$, $B_{+-}(s)$ and $B_{++}(s)$. ■

The Dedekind zeta function for the field K is given by

$$\zeta_K(s) = \prod_P \left(1 - \frac{1}{N(P)^s}\right)^{-1},$$

where the product is taken over all prime ideals of O_K.

Lemma 6. *For $s > 1$ we have*

$$\zeta_K(s) = \left(1 - \frac{1}{2^{2s}}\right)^{-1} \left(1 - \frac{1}{17^s}\right)^{-3}$$

$$B_{--}(4s)B_{-+}(2s)^2B_{+-}(2s)B''_{++}(2s)^2B_{+-}(s)^2B'_{++}(s)^4.$$

Proof. We split $\zeta_K(s)$ into seven products and make use of Lemma 1 to recognize each of these products in terms of the $B_{\epsilon,\eta}$. We have

$$\zeta_K(s) = \Pi_1\Pi_2\Pi_3\Pi_4\Pi_5\Pi_6\Pi_7,$$

where

$$\Pi_1 := \prod_{P|2O_K}\left(1-\frac{1}{N(P)^s}\right)^{-1} = \left(1-\frac{1}{4^s}\right)^{-1} = \left(1-\frac{1}{2^{2s}}\right)^{-1},$$

$$\Pi_2 := \prod_{P|17O_K}\left(1-\frac{1}{N(P)^s}\right)^{-1} = \left(1-\frac{1}{17^s}\right)^{-3},$$

$$\Pi_3 := \prod_{\substack{p\neq 2,\,17\\ \left(\frac{-1}{p}\right)=\left(\frac{17}{p}\right)=-1}}\ \prod_{P|pO_K}\left(1-\frac{1}{N(P)^s}\right)^{-1}$$

$$= \prod_{\substack{p\neq 2,\,17\\ \left(\frac{-1}{p}\right)=\left(\frac{17}{p}\right)=-1}}\left(1-\frac{1}{p^{4s}}\right)^{-1} = B_{--}(4s),$$

$$\Pi_4 := \prod_{\substack{p\neq 2,\,17\\ \left(\frac{-1}{p}\right)=-1,\ \left(\frac{17}{p}\right)=1}}\ \prod_{P|pO_K}\left(1-\frac{1}{N(P)^s}\right)^{-1}$$

$$= \prod_{\substack{p\neq 2,\,17\\ \left(\frac{-1}{p}\right)=-1,\ \left(\frac{17}{p}\right)=1}}\left(1-\frac{1}{p^{2s}}\right)^{-2} = B_{-+}(2s)^2,$$

$$\Pi_5 := \prod_{\substack{p\neq 2,\,17\\ \left(\frac{-1}{p}\right)=1,\ \left(\frac{17}{p}\right)=-1}}\ \prod_{P|pO_K}\left(1-\frac{1}{N(P)^s}\right)^{-1}$$

$$= \prod_{\substack{p\neq 2,\,17\\ \left(\frac{-1}{p}\right)=1,\ \left(\frac{17}{p}\right)=-1}}\left(1-\frac{1}{p^{s}}\right)^{-2}\left(1-\frac{1}{p^{2s}}\right)^{-1} = B_{+-}(s)^2B_{+-}(2s),$$

$$\Pi_6 := \prod_{\substack{p\neq 2,\,17\\ \left(\frac{-1}{p}\right)=\left(\frac{17}{p}\right)=1\\ p=x^2+17y^2}}\ \prod_{P|pO_K}\left(1-\frac{1}{N(P)^s}\right)^{-1}$$

$$= \prod_{\substack{p\neq 2,\,17\\ \left(\frac{-1}{p}\right)=\left(\frac{17}{p}\right)=1\\ K_p=I}}\left(1-\frac{1}{p^{s}}\right)^{-4} = B'_{++}(s)^4,$$

$$\begin{aligned}\Pi_7 := & \prod_{\substack{p\neq 2,\,17\\ \left(\frac{-1}{p}\right)=\left(\frac{17}{p}\right)=1\\ p=2x^2+2xy+9y^2}} \prod_{P|pO_K}\left(1-\frac{1}{N(P)^s}\right)^{-1}\\ = & \prod_{\substack{p\neq 2,\,17\\ \left(\frac{-1}{p}\right)=\left(\frac{17}{p}\right)=1\\ K_p=A^2}}\left(1-\frac{1}{p^{2s}}\right)^{-2} = B''_{++}(2s)^2.\end{aligned}$$

Multipying $\Pi_1, \Pi_2, \ldots, \Pi_7$ together, we obtain the asserted equality. ■

Lemma 7. *For $s>1$ we have*

$$\begin{aligned}B'_{++}(s)^8 = & \left(1-\frac{1}{2^s}\right)^2\left(1+\frac{1}{2^s}\right)^2\left(1-\frac{1}{17^s}\right)^4 L(s,-4)^{-1}L(s,17)L(s,-68)\\ & \times B_{--}(4s)^{-2}B_{-+}(2s)^{-4}B_{+-}(2s)^{-4}B''_{++}(2s)^{-4}\zeta_K(s)^2\zeta(s)^{-1},\\ B''_{++}(s)^8 = & \left(1-\frac{1}{2^s}\right)^2\left(1+\frac{1}{2^s}\right)^{-2} L(s,-4)^3L(s,17)L(s,-68)\\ & \times B_{--}(4s)^2B_{--}(2s)^{-4}B''_{++}(2s)^4\zeta_K(s)^{-2}\zeta(s)^3.\end{aligned}$$

Proof. The first equality follows by replacing $B_{+-}(s)^4$ in the square of the equality in Lemma 6 by its value given in Lemma 5. The second equality then follows from $B'_{++}(s)^8B''_{++}(s)^8 = B_{++}(s)^8$ and the value of $B_{++}(s)^8$ given by Lemma 5. ■

Lemma 8.

(*i*) $B_{--}(2)B_{-+}(2)B_{+-}(2)B_{++}(2) = \dfrac{36\pi^2}{289}$,

(*ii*) $t_1(-68) = \dfrac{289}{36\pi^2}B_{-+}(2)B_{+-}(2)$.

Proof. By Lemma 4(i) we have (as $\zeta(2)=\pi^2/6$)

$$B_{--}(2)B_{-+}(2)B_{+-}(2)B_{++}(2) = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{17^2}\right)\zeta(2) = \frac{36}{289}\pi^2,$$

which is (i). Then

$$t_1(-68) = \frac{1}{B_{--}(2)B_{++}(2)} = \frac{289}{36\pi^2}B_{-+}(2)B_{+-}(2).$$

by (3.12), Definition 1 and (i). ■

Lemma 9.

$$\lim_{s\to 1^+}\left(\frac{\zeta_K(s)}{\zeta(s)}\right)=\frac{\pi^2}{4\sqrt{17}}\log\left(\frac{1+\sqrt{2+2\sqrt{17}}+\sqrt{17}}{4}\right).$$

Proof. By [7, Theorem 7.1, p. 326] we have

$$\lim_{s\to 1^+}(s-1)\zeta_K(s)=\frac{2^{r_1+r_2}\pi^{r_2}R(K)h(K)}{w(K)|d(K)|^{1/2}}=\frac{\pi^2}{4\sqrt{17}}\log\left(\frac{1+\sqrt{2+2\sqrt{17}}+\sqrt{17}}{4}\right).$$

As

$$\lim_{s\to 1^+}(s-1)\zeta(s)=1$$

the asserted result follows. ■

Lemma 10.

$$L(1,-4)=\frac{\pi}{4},\quad L(1,17)=\frac{2}{\sqrt{17}}\log(4+\sqrt{17}),\quad L(1,-68)=\frac{2\pi}{\sqrt{17}}.$$

Proof. Dirichlet's class number formula [7, Theorem 7.1, p. 326] for the quadratic field $\mathbb{Q}(\sqrt{d})$ of discriminant d asserts that

$$L(1,d)=\frac{2h(d)\log\eta(d)}{\sqrt{d}},\quad \text{if } d>0,$$

and

$$L(1,d)=\frac{2\pi h(d)}{w(d)\sqrt{|d|}},\quad \text{if } d<0,$$

where $h(d)$ is the class number of $\mathbb{Q}(\sqrt{d})$, $\eta(d)$ is the fundamental unit >1 of $\mathbb{Q}(\sqrt{d})$ when $d>0$, and $w(d)=2,4$ or 6 according as $d<-4$, $d=-4$ or $d=-3$ when $d<0$. As

$$h(-4)=1,\quad h(17)=1,\quad h(-68)=4,\quad \eta(17)=4+\sqrt{17}$$

the asserted result follows. ■

Lemma 11.

$$\lim_{s\to 1^+}\left(\frac{B_{--}(s)}{B_{++}(s)}\right)^2=\frac{17\sqrt{17}B_{--}(2)^2B_{-+}(2)B_{+-}(2)}{4\pi\log(4+\sqrt{17})}.$$

Proof. By Lemma 5 we have

$$\left(\frac{B_{--}(s)}{B_{++}(s)}\right)^2 = \left(1-\frac{1}{2^s}\right)^{-1}\left(1-\frac{1}{17^s}\right)^{-1} L(s,-4)^{-1}L(s,17)^{-1}$$
$$\times B_{--}(2s)^2 B_{-+}(2s)B_{+-}(2s).$$

Letting $s \to 1^+$ and appealing to Lemma 10, we obtain the asserted limit. ∎

Lemma 12.

$$\lim_{s\to 1^+}\left(\frac{B'_{++}(s)}{B''_{++}(s)}\right)^2 = \frac{24\pi}{17\sqrt{17}}\log\left(\frac{1+\sqrt{2+2\sqrt{17}}+\sqrt{17}}{4}\right)$$
$$\times B_{--}(4)^{-1}B_{--}(2)B_{-+}(2)^{-1}B_{+-}(2)^{-1}B''_{++}(2)^{-2}.$$

Proof. By Lemma 7 we have

$$\left(\frac{B'_{++}(s)}{B''_{++}(s)}\right)^2 = \left(1+\frac{1}{2^s}\right)\left(1-\frac{1}{17^s}\right)L(s,-4)^{-1}B_{--}(4s)^{-1}B_{--}(2s)B_{-+}(2s)^{-1}$$
$$\times B_{+-}(2s)^{-1}B''_{++}(2s)^{-2}\left(\frac{\zeta_K(s)}{\zeta(s)}\right).$$

Letting $s \to 1+$ and appealing to Lemmas 9 and 10, we obtain the asserted limit. ∎

We note (in the notation of Section 3) that

$$K_2 = [2,2,9] = A^2,$$
$$K_{17} = [17,0,1] = [1,0,17] = I,$$
$$\chi(A^j,A^k) = i^{jk},$$
$$l(A^j,-68) = \left(1+\frac{\chi(A^j,A^2)}{2}\right)\left(1+\frac{\chi(A^j,I)}{17}\right) = \left(1+\frac{(-1)^j}{2}\right)\left(1+\frac{1}{17}\right)$$
$$= \begin{cases} \dfrac{9}{17}, & \text{if } j=1,3,\\[2mm] \dfrac{27}{17}, & \text{if } j=2.\end{cases}$$

Lemma 13.

$$j(A,-68) = j(A^3,-68) = \frac{17\sqrt{17}B_{-+}(2)B_{+-}(2)}{24\pi\log\left(\dfrac{1+\sqrt{2+2\sqrt{17}}+\sqrt{17}}{4}\right)},$$

$$j(A^2,-68) = \frac{17\sqrt{17}B_{-+}(2)B_{+-}(2)}{4\pi\log(4+\sqrt{17})}.$$

Proof. For $r = 1, 2, 3$ we have by (3.7)

$$j(A^r,-68) = \lim_{s\to 1^+} \prod_{\substack{p\neq 2,17\\ \left(\frac{-1}{p}\right)=\left(\frac{17}{p}\right)}} \left(1-\frac{\chi(A^r,K_p)}{p^s}\right)\left(1-\frac{\chi(A^{-r},K_p)}{p^s}\right).$$

Thus, by Lemmas 1 and 2, we have

$$j(A^r,-68) = \lim_{s\to 1^+} \prod_{\substack{\left(\frac{-1}{p}\right)=\left(\frac{17}{p}\right)=1\\ K_p=I}} \left(1-\frac{1}{p^s}\right)^2 \prod_{\substack{\left(\frac{-1}{p}\right)=\left(\frac{17}{p}\right)=1\\ K_p=A^2}} \left(1-\frac{(-1)^r}{p^s}\right)^2$$
$$\times \prod_{\left(\frac{-1}{p}\right)=\left(\frac{17}{p}\right)=-1} \left(1-\frac{i^r}{p^s}\right)\left(1-\frac{i^{-r}}{p^s}\right).$$

Hence

$$\begin{aligned} j(A^2,-68) &= \lim_{s\to 1^+} B_{++}(s)^{-2}A_{--}(s)^{-2} \\ &= \lim_{s\to 1^+}\frac{1}{B_{--}(2s)^2}\left(\frac{B_{--}(s)}{B_{++}(s)}\right)^2 \quad \text{(by Lemma 3)} \\ &= \frac{1}{B_{--}(2)^2}\lim_{s\to 1^+}\left(\frac{B_{--}(s)}{B_{++}(s)}\right)^2. \end{aligned}$$

The determination of $j(A^2,-68)$ now follows by Lemma 11.

Finally

$$\begin{aligned} j(A,-68) = j(A^3,-68) &= \lim_{s\to 1^+} B'_{++}(s)^{-2}A''_{++}(s)^{-2}A_{--}(2s)^{-1} \\ &= \lim_{s\to 1^+}\frac{B_{--}(2s)}{B''_{++}(2s)^2B_{--}(4s)}\left(\frac{B'_{++}(s)}{B''_{++}(s)}\right)^{-2} \quad \text{(by Lemma 3)} \\ &= \frac{B_{--}(2)}{B''_{++}(2)^2B_{--}(4)}\lim_{s\to 1^+}\left(\frac{B'_{++}(s)}{B''_{++}(s)}\right)^{-2}. \end{aligned}$$

The determination of $j(A,-68)$ now follows by Lemma 12. ∎

Lemma 14.

$$E(I,-68) = \frac{1}{4}\log\left(\frac{1+\sqrt{2+2\sqrt{17}}+\sqrt{17}}{4}\right)+\frac{1}{16}\log(4+\sqrt{17}),$$

$$E(A^2,-68) = -\frac{1}{4}\log\left(\frac{1+\sqrt{2+2\sqrt{17}}+\sqrt{17}}{4}\right)+\frac{1}{16}\log(4+\sqrt{17}).$$

Proof. From (3.10) we have for $r = 0, 1, 2, 3$

$$E(A^r,-68) = \frac{\pi\sqrt{68}w(-68)}{48h(-68)}\sum_{m=1}^{3}\chi(A^m,A^r)^{-1}\frac{t_1(-68)}{j(A^m,-68)}l(A^m,-68)$$

$$= \frac{289\sqrt{17}}{1728\pi}B_{-+}(2)B_{+-}(2)\sum_{m=1}^{3} i^{-mr}\frac{l(A^m,-68)}{j(A^m,-68)} \quad \text{(by Lemma 8(ii))}$$

$$= \frac{17\sqrt{17}}{192\pi}B_{-+}(2)B_{+-}(2)\left(\frac{i^{-r}}{j(A,-68)} + 3\frac{i^{-2r}}{j(A^2,-68)} + \frac{i^{-3r}}{j(A^3,-68)}\right).$$

The asserted results now follow by taking $r = 0$ and $r = 2$ and appealing to Lemma 13. ■

From Proposition 2(b) and Lemma 14 we obtain

$$f_0(\sqrt{-17}) = 2^{1/4}\left(\frac{1+\sqrt{2+2\sqrt{17}}+\sqrt{17}}{4}\right)^{1/2}$$

in agreement with [9, p. 721].

Theorem 2.

$$K[\sqrt{17}] = 2^{-9/2}17^{-1/2}\pi^{1/2}(\sqrt{17}-4)^{1/8}$$
$$\times(1+\sqrt{2+2\sqrt{17}}+\sqrt{17})^{3/2}\left\{\prod_{m=1}^{68}\Gamma\left(\tfrac{m}{68}\right)^{\left(\frac{-68}{m}\right)}\right\}^{1/8}.$$

Proof. We apply Theorem 1(b) with $\lambda = 17$ so that $K = [1,0,17] = I$ and $M_0 = [2,2,9] = A^2$. We obtain

$$K[\sqrt{17}] = 2^{-3/2}\pi^{1/2}17^{-1/2}\left\{\prod_{m=1}^{68}\Gamma\left(\frac{m}{68}\right)^{\left(\frac{-68}{m}\right)}\right\}^{1/8} e^{2E(I,-68)-4E(A^2,-68)}.$$

By Lemma 14 we have

$$2E(I,-68)-4E(A^2,-68) = \frac{3}{2}\log\left(\frac{1+\sqrt{2+2\sqrt{17}}+\sqrt{17}}{4}\right) - \frac{1}{8}\log(4+\sqrt{17}),$$

so that

$$e^{2E(I,-68)-4E(A^2,-68)} = \frac{\left(\frac{1+\sqrt{2+2\sqrt{17}}+\sqrt{17}}{4}\right)^{3/2}}{(4+\sqrt{17})^{1/8}}$$

$$= 2^{-3}(\sqrt{17}-4)^{1/8}\left(1+\sqrt{2+2\sqrt{17}}+\sqrt{17}\right)^{3/2},$$

and Theorem 2 follows. ∎

In a similar manner it can be shown that the singular modulus $k(17)$ is given by

$$k(17) = \frac{1}{2}(\sqrt{U}-\sqrt{V}) = 0.006156\ldots,$$

where

$$U = 21 + 5\sqrt{17} - 8\sqrt{2+2\sqrt{17}} - 6\sqrt{2\sqrt{17}-2}$$

and

$$V = -19 - 5\sqrt{17} + 8\sqrt{2+2\sqrt{17}} + 6\sqrt{2\sqrt{17}-2}.$$

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Habib Muzaffar
Department of Mathematics,
University of Toledo,
Toledo, Ohio 43606, USA
E-mail: hmuzaff@utnet.utoledo.edu

Kenneth S. Williams
Centre for Research in Algebra and Number Theory,
School of Mathematics and Statistics,
Carleton University, Ottawa,
Ontario K1S 5B6, Canada
E-mail: kwilliam@connect.carleton.ca