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INTEGRAL BASES FOR AN INFINITE FAMILY OF CYCLIC QUINTIC FIELDS*

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Abstract. An explicit integral basis is given for infinitely many cyclic quintic fields.

Key words. Integral basis, family of quintic fields.

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1. Introduction. We denote the set of integers by $\mathbb{Z}$ and the set of positive integers by $\mathbb{N}$. Let $n \in \mathbb{Z}$. The Lehmer quintic $f_n(x) \in \mathbb{Z}[x]$ is defined by

$$\begin{aligned} f_n(x) = x^5 + n^2x^4 &- (2n^3 + 6n^2 + 10n + 10)x^3 \\ &+(n^4 + 5n^3 + 11n^2 + 15n + 5)x^2 + (n^3 + 4n^2 + 10n + 10)x + 1, \end{aligned}$$

see [5, p. 539]. Schoof and Washington [6, p. 548] have shown that $f_n(x)$ is irreducible for all $n \in \mathbb{Z}$. Let $\theta \in \mathbb{C}$ be a root of $f_n(x) = 0$. Set $K = \mathbb{Q}(\theta)$ so that $[K : \mathbb{Q}] = 5$. It is known that K is a cyclic field [6, p. 548]. We denote the ring of integers of K by O_K. The discriminant $d(K)$ of K has been determined by Jeannin [4, p. 76], see also Spearman and Williams [7, p. 215], namely $d(K) = f(K)^4$, where the conductor $f(K)$ of K is given by

$$f(K) = 5^b \prod_{\substack{p \equiv 1 \pmod 5 \\ v_p(n^4+5n^3+15n^2+25n+25) \not\equiv 0 \pmod 5}} p, \tag{1.1}$$

where $v_p(k)$ denotes the exponent of the largest power of the prime p dividing the nonzero integer k and

$$b = \begin{cases} 0, & \text{if } 5 \nmid n, \\ 2, & \text{if } 5 \mid n. \end{cases} \tag{1.2}$$

Set

$$m = n^4 + 5n^3 + 15n^2 + 25n + 25 \in \mathbb{Z}, \tag{1.3}$$

$$d = n^3 + 5n^2 + 10n + 7 \in \mathbb{Z}, \tag{1.4}$$

$$a = m^3 - 10m^2 + 5m \in \mathbb{Z}. \tag{1.5}$$

From (1.3) we have

$$m = (n+2)(n+1)\left((n+1)^2 + 6\right) + 11$$

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and, as $(n+2)(n+1) \geq 0$ for all $n \in \mathbb{Z}$, we deduce that $m \geq 11$ so that

$$m \in \mathbb{N}. \tag{1.6}$$

Then, from (1.5), we obtain $a = m^2(m-10) + 5m \geq 176$ so that

$$a \in \mathbb{N}. \tag{1.7}$$

As $x^3 + 5x^2 + 10x + 7$ is irreducible in $\mathbb{Z}[x]$, we deduce from (1.4) that

$$d \neq 0. \tag{1.8}$$

A MAPLE calculation gives

$$\begin{aligned} a = \; & (n^3 + 5n^2 + 10n + 7)(n^9 + 10n^8 + 60n^7 + 243n^6 + 730n^5 \\ & + 1650n^4 + 2824n^3 + 3520n^2 + 2990n + 1357) + 1. \end{aligned} \tag{1.9}$$

From (1.2) and (1.3) we observe that

$$5^b \parallel m. \tag{1.10}$$

From (1.4) and (1.9) we see that

$$a = 1 + dk, \tag{1.11}$$

where

$$\begin{aligned} k = \; & n^9 + 10n^8 + 60n^7 + 243n^6 + 730n^5 + 1650n^4 \\ & + 2824n^3 + 3520n^2 + 2990n + 1357 \in \mathbb{Z} \setminus \{0\}. \end{aligned} \tag{1.12}$$

Gaál and Pohst [2, p. 1690] have shown that under the condition

$$p^2 \nmid m \text{ for any prime } p \neq 5 \tag{1.13}$$

an integral basis for K is given by

$$\{1, \theta, \theta^2, \theta^3, \omega_5\}, \tag{1.14}$$

where

$$\omega_5 = \frac{1}{d}\left((n+2) + (2n^2 + 9n + 9)\theta + (2n^2 + 4n - 1)\theta^2 + (-3n-4)\theta^3 + \theta^4\right). \tag{1.15}$$

Although it is very likely that there are infinitely many $n \in \mathbb{Z}$ such that (1.13) holds this has not yet been proved. Gaál and Pohst used their integral basis in a search for cyclic quintic fields with a power basis. They proved under the condition that m is squarefree that the field K admits a power basis if and only if $n = -1$ or $n = -2$ [2, Theorem, p. 1695], and noted that these values of n give the same field K [2, p. 1689]. They also observed [2, Remark, p. 1695] that their result is a special case of a theorem of Gras [3], which asserts that there is only one cyclic quintic field with a power basis, namely, the maximal real subfield of the cyclotomic field of 11-th roots of unity.

In this work we give an integral basis for K under the weaker condition

$$m \text{ is cubefree.} \tag{1.16}$$

From now on we assume that (1.16) holds except in Lemma 2.2. In view of (1.6), (1.10) and (1.16), we have

$$m = 5^b PQ^2, \tag{1.17}$$

where b is given by (1.2) and $P, Q \in \mathbb{N}$ are such that

$$5 \nmid P, \quad 5 \nmid Q, \quad (P,Q) = 1, \quad P, Q \text{ squarefree.} \tag{1.18}$$

By [4, Lemme 2.1.1] every prime factor ($\neq 5$) of m is $\equiv 1 \pmod 5$. Hence, by (1.1), we have

$$f(K) = 5^b PQ \tag{1.19}$$

and

$$p \text{ (prime)} \mid PQ \Longrightarrow p \equiv 1 \pmod{5}. \tag{1.20}$$

By (1.17) we have $Q \mid m$. By (1.5) we have $m \mid a$. Hence $Q \mid a$. Then, by (1.11), we have $Q \mid 1 + dk$ from which we deduce

$$(d, Q) = 1. \tag{1.21}$$

We define

$$v_4 = \frac{1}{Q}\left(\theta - \frac{n^2}{5}(Q-1)\right)^3 \in K \tag{1.22}$$

and

$$v_5 = \frac{ad\omega_5 + (1-a)Qv_4\theta}{dQ} \in K. \tag{1.23}$$

We note that (1.8) ensures that v_5 is well-defined. We prove

THEOREM. *Under the assumption* (1.16)

$$\{1, \theta, \theta^2, v_4, v_5\}$$

is an integral basis for K.

We note that if (1.13) holds then

$$Q = 1, \quad v_4 = \theta^3, \quad v_5 = \frac{ad\omega_5 + (1-a)\theta^4}{d}.$$

Appealing to (1.11) we deduce

$$v_5 = \omega_5 + k(d\omega_5 - \theta^4).$$

As $d\omega_5 - \theta^4$ is a cubic polynomial in θ with coefficients in $\mathbb{Z}$, we deduce from the theorem that $\{1, \theta, \theta^2, \theta^3, \omega_5\}$ is an integral basis for K showing that our theorem includes that of Gaál and Pohst [2, p. 1690].

By a theorem of Erdös [1] there exists an infinite set S of integers n such that $m = n^4 + 5n^3 + 15n^2 + 25n + 25$ is cubefree. For $n \in S$ the integer m has the form (1.17). Clearly S contains an infinite subset S_1 such that the values of $5^b PQ$ are distinct for $n \in S_1$. Thus, by (1.19), the conductors $f(K)$ are distinct for $n \in S_1$ thus ensuring that the cyclic quintic fields K are distinct for $n \in S_1$. Thus our theorem gives an integral basis for infinitely many cyclic quintic fields.

2. Proof of Theorem. We require a number of lemmas.

LEMMA 2.1. *Under the assumption* (1.16), *we have* $v_4 \in O_K$.

Proof. The asserted result is immediate if $Q = 1$. Hence we may assume that $Q > 1$. By (1.19) we see that $Q \mid f(K)$. Hence all the prime divisors q of Q ramify in O_K. Moreover, as K is a cyclic quintic field, each prime factor q ramifies totally. Hence there is a prime ideal $\wp$ of O_K such that $< q >= \wp^5$ and $N(\wp) = q$. Let $g_n(x) \in \mathbb{Z}[x]$ be the minimal polynomial of $5\theta + n^2$. Using MAPLE we find

$$g_n(0) = m(4n^6 + 30n^5 + 65n^4 - 200n^2 - 125n + 125). \tag{2.1}$$

From (1.17) and (2.1) we deduce that

$$Q^2 \mid g_n(0) = \pm N(5\theta + n^2). \tag{2.2}$$

Let

$$< 5\theta + n^2 >= P_1^{a_1} \cdots P_r^{a_r} \tag{2.3}$$

be the prime ideal decomposition of $< 5\theta + n^2 >$ into distinct prime ideals of O_K so

$$|N(5\theta + n^2)| = N(< 5\theta + n^2 >) = N(P_1)^{a_1} \cdots N(P_r)^{a_r}. \tag{2.4}$$

From (2.2) and (2.4) we see that

$$q^2 \mid N(P_1)^{a_1} \cdots N(P_r)^{a_r}. \tag{2.5}$$

Thus $P_i = \wp$ and $a_i \geq 2$ for some $i \in \{1, 2, \ldots, r\}$. Hence by (2.3) we have

$$\wp^2 \mid < 5\theta + n^2 > . \tag{2.6}$$

Since $\wp^5 \mid Q$ we deduce from (2.6) that

$$\wp^2 \mid < 5\theta + n^2 - n^2 Q > . \tag{2.7}$$

As $5 \nmid Q$ we have $\wp \nmid < 5 >$. Also by (1.20) we have $Q \equiv 1 \pmod 5$. Thus

$$\wp^2 \mid < \theta - n^2 \left(\frac{Q-1}{5} \right) > .$$

Hence

$$\wp^5 \mid < \theta - n^2 \left(\frac{Q-1}{5} \right) >^3 . \tag{2.8}$$

As (2.8) is true for each prime divisor q of Q we have

$$Q \mid < \theta - n^2 \left(\frac{Q-1}{5} \right) >^3 .$$

This proves that

$$v_4 = \frac{1}{Q} \left(\theta - \frac{n^2}{5}(Q-1) \right)^3 \in O_K$$

as asserted. □

LEMMA 2.2. *For all $n \in \mathbb{Z}$ we have $\omega_5 \in O_K$.*

Proof. The proof is given in [2, pp. 1690-1691], where the case $n = -2$ should be dealt with separately. □

LEMMA 2.3. *Under the assumption* (1.16), *we have* $\upsilon_5 \in O_K$.

Proof. Let

$$\alpha = ad\omega_5 + (1-a)Q\upsilon_4\theta. \tag{2.9}$$

By Lemmas 2.1 and 2.2 we have $\upsilon_4 \in O_K$ and $\omega_5 \in O_K$ so

$$\alpha \in O_K.$$

From (1.5) and (1.17) we have $Q \mid a$. Hence

$$\alpha \equiv 0 \pmod{Q}$$

in O_K. From (1.11) we have $d \mid 1-a$. Hence

$$\alpha \equiv 0 \pmod{d}$$

in O_K. Then, by (1.21), we deduce that

$$\alpha \equiv 0 \pmod{dQ}$$

in O_K so that by (1.23) and (2.9)

$$\upsilon_5 = \frac{\alpha}{dQ} \in O_K$$

as claimed. □

Proof of Theorem. We have

$$\begin{aligned}\alpha &= dQ\upsilon_5 = ad\omega_5 + (1-a)Q\upsilon_4\theta \\ &= a\left(\theta^4 + c(\theta)\right) + (1-a)\theta\left(\theta - \frac{n^2}{5}(Q-1)\right)^3,\end{aligned}$$

where

$$c(\theta) \in \mathbb{Z}[\theta], \quad \deg c(\theta) = 3.$$

Thus

$$\alpha = \theta^4 + d(\theta),$$

where

$$d(\theta) \in \mathbb{Z}[\theta], \quad \deg d(\theta) \le 3.$$

Similarly

$$Qv_4 = \theta^3 + e(\theta),$$

where

$$e(\theta) \in \mathbb{Z}[\theta], \ \deg e(\theta) \leq 2.$$

Thus

$$\mathrm{disc}(1, \theta, \theta^2, Qv_4, \alpha) = \mathrm{disc}(1, \theta, \theta^2, \theta^3, \alpha) = \mathrm{disc}(1, \theta, \theta^2, \theta^3, \theta^4) = m^4 d^2,$$

by [2, p. 1691]. Therefore

$$\mathrm{disc}(1, \theta, \theta^2, v_4, v_5) = \frac{\mathrm{disc}(1, \theta, \theta^2, Qv_4, \alpha)}{Q^2(dQ)^2} = \frac{m^4}{Q^4} = 5^{4b}P^4Q^4 = f(K)^4 = d(K).$$

As $v_4 \in O_K$ and $v_5 \in O_K$ by Lemmas 2.1 and 2.3 respectively, we deduce that $\{1, \theta, \theta^2, v_4, v_5\}$ is an integral basis for K. □

We conclude with an example.

EXAMPLE. Let $n = 14$ so that

$$K = \mathbb{Q}(\theta), \ \ \theta^5 + 196\theta^4 - 6814\theta^3 + 545070\theta^2 + 3678\theta + 1 = 0.$$

We use the theorem to determine an integral basis for K. Here

$$\begin{aligned}
&m = 11 \times 71^2, \ b = 0, \ P = 11, \ Q = 71, \\
&d = 7^2 \times 79, \\
&a = 2^4 \times 11 \times 71^2 \times 192141181, \\
&k = 5 \times 8807580989, \\
&v_4 = \frac{1}{71}(\theta - 2744)^3, \ \ v_4 \equiv \frac{5 + 29\theta + 4\theta^2 + \theta^3}{71} (\mathrm{mod}\ 1), \\
&\omega_5 = \frac{16 + 527\theta + 447\theta^2 - 46\theta^3 + \theta^4}{3871},
\end{aligned}$$

and

$$v_5 = \frac{r + s\theta - t\theta^2 + u\theta^3 + \theta^4}{274841}$$

with

$$r = 2727531680673536, \ \ s = 352210381854043381655 7072,$$

$$t = 38506202959783786 36848, \ \ u = 1395473396124589624,$$

so that

$$v_5 \equiv \frac{50339 + 276240\theta + 1127060\theta^2 + 2206010\theta^3 + \theta^4}{274841} (\mathrm{mod}\ 1).$$

Thus by the theorem

$$\left\{1, \theta, \theta^2, \frac{5 + 29\theta + 4\theta^2 + \theta^3}{71}, \frac{50339 + 276240\theta + 1127060\theta^2 + 2206010\theta^3 + \theta^4}{274841}\right\}$$

is an integral basis for K. As

$$\begin{aligned}\frac{65823+624639\theta+701259\theta^2+38259\theta^3+\theta^4}{274841}\end{aligned}$$

$$\begin{aligned}&\frac{65823+624630\theta+701250\theta^2+38250\theta^3+\theta^4}{274841}\\&=\frac{50339+276240\theta+1127060\theta^2+2206010\theta^3+\theta^4}{274841}\\&\quad-56\left(\frac{5+29\theta+4\theta^2+\theta^3}{71}\right)+\left(4+23\theta+3\theta^2\right),\end{aligned}$$

we see that

$$\left\{1,\theta,\theta^2,\frac{5+29\theta+4\theta^2+\theta^3}{71},\frac{65823+624630\theta+701250\theta^2+38250\theta^3+\theta^4}{274841}\right\}$$

is also an integral basis for K in agreement with MAPLE.

We close by remarking that when m is not cubefree the cyclic quintic field K may not have an integral basis of the type given in our theorem. To see this take $n=44$ so that $m=41^3\times 61$. In this case $(18+20\theta+\theta^2)/41$ is an integer of K and so θ^2 is not a minimal integer of degree 2. Hence K cannot have an integral basis of the type $\{1,\theta,\theta^2,*,*\}$.

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