Math. J. Okayama Univ. **48** (2006), 43–46

THE PRIME IDEAL FACTORIZATION OF 2 IN PURE QUARTIC FIELDS WITH INDEX 2

Blair K. SPEARMAN and Kenneth S. WILLIAMS

Abstract. The prime ideal decomposition of 2 in a pure quartic field with field index 2 is determined explicitly.

1. Introduction

Let K be an algebraic number field and O_K its ring of integers. When determining generators of the ideals in the prime ideal factorization of a (rational) prime p in O_K, the most difficult case occurs when p divides the field index $i(K)$ of K. In this paper we examine the case when K is a pure quartic field. Here $i(K) = 1$ or 2, and we determine explicit generators of the prime ideals in the decomposition of 2 when $i(K) = 2$.

Let K be a pure quartic field. Then there exists a fourth power free integer m such that $K = \mathbb{Q}(m^{1/4})$. It follows from the work of Funakura [1, p. 36] that the field index $i(K)$ of K is given by

$$i(K) = \begin{cases} 2, & \text{if } m \equiv 1 \pmod{16}, \\ 1, & \text{if } m \not\equiv 1 \pmod{16}. \end{cases}$$

From now on we assume that $i(K) = 2$ so that $m \equiv 1 \pmod{16}$, say $m = 16k + 1$. In this case the prime ideal factorization of $< 2 >$ in O_K is

$$< 2 >= P_1^2 P_2 P_3,$$

where P_1, P_2, P_3 are distinct prime ideals, see [1, p. 36]. In this paper we determine explicit generators of P_1, P_2 and P_3.

Theorem. *Let m be a fourth power free integer such that $K = \mathbb{Q}(m^{1/4})$ is a pure quartic field with $i(K) = 2$. Then $< 2 >= P_1^2 P_2 P_3$, where the*

Mathematics Subject Classification. 11R16.

Key words and phrases. pure quartic field, discriminant, prime decomposition.

Both authors were supported by research grants from the Natural Sciences and Engineering Research Council of Canada.

distinct prime ideals P_1, P_2, P_3 of O_K are given by

$$P_1 = \ < 2, \frac{3}{2} + m^{1/4} + \frac{1}{2}m^{1/2} >,$$

$$P_2 = \begin{cases} < 2, \frac{5}{4} + \frac{1}{4}m^{1/4} + \frac{1}{4}m^{1/2} + \frac{1}{4}m^{3/4} >, & \text{if } m \equiv 1 \,(\mathrm{mod}\ 32), \\ < 2, \frac{3}{4} + \frac{5}{4}m^{1/4} + \frac{3}{4}m^{1/2} + \frac{1}{4}m^{3/4} >, & \text{if } m \equiv 17 \,(\mathrm{mod}\ 32), \end{cases}$$

$$P_3 = \begin{cases} < 2, \frac{5}{4} - \frac{1}{4}m^{1/4} + \frac{1}{4}m^{1/2} - \frac{1}{4}m^{3/4} >, & \text{if } m \equiv 1 \,(\mathrm{mod}\ 32), \\ < 2, \frac{3}{4} - \frac{5}{4}m^{1/4} + \frac{3}{4}m^{1/2} - \frac{1}{4}m^{3/4} >, & \text{if } m \equiv 17 \,(\mathrm{mod}\ 32). \end{cases}$$

2. Proof of Theorem

Let $L = \mathbb{Q}(m^{1/2})$ so that $\mathbb{Q} \subset L \subset K$ and $[L : \mathbb{Q}] = 2$. Set

$$Q_1 =< 2, \frac{1 + m^{1/2}}{2} >, \quad Q_2 =< 2, \frac{1 - m^{1/2}}{2} > .$$

Q_1 and Q_2 are distinct prime ideals of O_L such that $< 2 >= Q_1 Q_2$. Let m_2 be the largest integer such that $m_2^2 \mid m$. Set $m_1 = m/m_2^2$ so that m_1 is a squarefree integer having the same sign as m. Clearly $m^{1/2} = m_2 m_1^{1/2}$. Then

$$Q_1 = \begin{cases} < 2, \frac{1 + m_1^{1/2}}{2} >, & \text{if } m_2 \equiv 1 \,(\mathrm{mod}\ 4), \\ < 2, \frac{1 - m_1^{1/2}}{2} >, & \text{if } m_2 \equiv 3 \,(\mathrm{mod}\ 4). \end{cases}$$

Next, by [2, Table D, cases D1, D2, p. 92], we see that

$$Q_1 = P_1^2$$

for some prime ideal P_1 of O_K. We claim that

$$P_1 =< 2, \frac{3}{2} + m^{1/4} + \frac{1}{2}m^{1/2} > .$$

First we show that P_1 is a prime ideal of O_K. The minimal polynomial of $\theta = \frac{3}{2} + m^{1/4} + \frac{1}{2}m^{1/2}$ over $\mathbb{Q}$ is

$$g(x) = x^4 - 6x^3 + (13 - 8k)x^2 + (-14 - 8k)x + (6 + 16k + 16k^2).$$

Hence $N(\theta) = \pm(6 + 16k + 16k^2) \equiv 2 \,(\mathrm{mod}\ 4)$. Let $< \theta >= S_1 S_2 \cdots S_r$ be the prime ideal factorization of $< \theta >$ in O_K. Hence $N(< \theta >) =$

$N(S_1)N(S_2)\cdots N(S_r)$. As $2 \parallel N(<\theta>)$ there exists a unique $S = S_i$ such that $2 \parallel N(S)$, that is $N(S) = 2$. Thus $<\theta>$ has exactly one prime ideal to exponent 1 in its prime factorization lying above 2. As $P_1 =< 2, \theta >$ we deduce that $P_1 = S$ so that P_1 is a prime ideal of O_K. Next we show that $P_1 \mid Q_1$. We set $\phi = \frac{3}{2} - m^{1/4} + \frac{1}{2}m^{1/2}$. An easy calculation shows that

$$\frac{1 + m^{1/2}}{2} = \theta\phi - (2k + 1)2.$$

Hence, as $2 \in P_1$ and $\theta \in P_1$, we deduce that $\frac{1 + m^{1/2}}{2} \in P_1$. Thus we have $Q_1 =< 2, \frac{1 + m^{1/2}}{2} >\subseteq P_1$, and so $P_1 \mid Q_1$. As Q_1 is the square of a prime ideal in O_K, we deduce that $Q_1 = P_1^2$ as asserted.

Let

$$k = \begin{cases} 2g, & \text{if } m \equiv 1 \pmod{32}, \\ 2g + 1, & \text{if } m \equiv 17 \pmod{32}. \end{cases}$$

For $\epsilon = \pm 1$, the minimal polynomial of

$$\alpha(\epsilon) = \begin{cases} \frac{5}{4} + \frac{\epsilon}{4}m^{1/4} + \frac{1}{4}m^{1/2} + \frac{\epsilon}{4}m^{3/4}, & \text{if } m \equiv 1 \pmod{32}, \\ \\ \frac{3}{4} + \frac{5\epsilon}{4}m^{1/4} + \frac{3}{4}m^{1/2} + \frac{\epsilon}{4}m^{3/4}, & \text{if } m \equiv 17 \pmod{32}, \end{cases}$$

is

$$x^4 - 5x^3 + (9 - 12g)x^2 + (-7 + 24g - 64g^2)x + (2 - 12g + 64g^2 - 128g^3),$$

if $m \equiv 1 \pmod{32}$, and

$$x^4 - 3x^3 + (-37 - 76g)x^2 + (-75 - 240g - 192g^2)x$$
$$+(-38 - 172g - 256g^2 - 128g^3),$$

if $m \equiv 17 \pmod{32}$. Clearly $N(\alpha(\epsilon)) \equiv 2 \pmod 4$ in both cases, and similarly to the argument above, we deduce that $I_+ =< 2, \alpha(1) >$ and $I_- =< 2, \alpha(-1) >$ are conjugate prime ideals of O_K lying above 2. If $m \equiv 1 \pmod{32}$ we have

$$\frac{1 - m^{1/2}}{2} = 2(1 - g - gm^{1/2}) - \alpha(1)\alpha(-1) \in I_+ \cap I_-$$

and if $m \equiv 17 \pmod{32}$

$$\frac{1 - m^{1/2}}{2} = 2(-g - (1 + g)m^{1/2}) - \alpha(1)\alpha(-1) \in I_+ \cap I_-.$$

Hence $\dfrac{1-m^{1/2}}{2} \in I_+ \cap I_-$. Thus I_+ and I_- are conjugate prime ideals of O_K lying above the prime ideal Q_2 of O_L. As $< 2 >= P_1^2 P_2 P_3 = Q_1 Q_2$ and $Q_1 = P_1^2$, we see that $Q_2 = P_2 P_3$ and that we can take

$$P_2 = I_+ =< 2, \alpha(1) >$$

and

$$P_3 = I_- =< 2, \alpha(-1) > .$$

This completes the proof.

References

[1] T. FUNAKURA, *On integral bases of pure quartic fields*, Math. J. Okayama Univ. **26** (1984), 27-41.

[2] J. G. HUARD, B. K. SPEARMAN and K. S. WILLIAMS, *Integral bases for quartic fields with quadratic subfields,* J. Number Theory **51** (1995), 87-102.

Blair K. Spearman
Department of Mathematics and Statistics
University of British Columbia Okanagan
Kelowna, B.C. Canada V1V 1V7
e-mail address: blair.spearman@ubc.ca

Kenneth S. Williams
School of Mathematics and Statistics
Carleton University
Ottawa, Ontario, Canada K1S 5B6
e-mail address: kwilliam@connect.carleton.ca

(*Received May 20, 2005*)