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THE CONVOLUTION SUM $\sum_{m<n/9} \sigma(m)\sigma(n-9m)$

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The evaluation of the sum $\sum_{m<n/9} \sigma(m)\sigma(n-9m)$ is carried out for all positive integers n. This evaluation is used to detemine the number of solutions to

$$n = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 3(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)$$

in integers $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$.

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1. Introduction

Let $\mathbb{N}$ denote the set of natural numbers. For $n \in \mathbb{N}$ we set

$$\sigma(n) = \sum_{d|n} d,$$

where d runs through the positive divisors of n. If $n \notin \mathbb{N}$ we set $\sigma(n) = 0$. For $k \in \mathbb{N}$ we define

$$W_k(n) := \sum_{m<n/k} \sigma(m)\sigma(n-km), \tag{1.1}$$

where m runs through the positive integers $< n/k$. The evaluation of $W_1(n)$ goes back to the middle of the nineteenth century, when Besge [4] gave the following formula in a letter to Liouville

$$W_1(n) = \frac{5}{12}\sigma_3(n) + \frac{(1-6n)}{12}\sigma(n), \tag{1.2}$$

where

$$\sigma_3(n) = \sum_{d|n} d^3 \ \ (n \in \mathbb{N}), \quad \sigma_3(n) = 0 \ \ (n \notin \mathbb{N}).$$

Lützen [11, p. 81] has observed in his study of Liouville that Besge is a pseudonym for Liouville. Formula (1.2) was first proved by Glaisher [6; 7, Paper 20]. In 1998 Melfi [12, Theorem 12; 13] evaluated $W_k(n)$ for $k = 2, 3, 4$ and $\gcd(n, k) = 1$ using the theory of modular forms. In 2000 Huard *et al.* [8] gave completely elementary arithmetic proofs of the evaluations

$$W_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) + \frac{(1-3n)}{24}\sigma(n) + \frac{(1-6n)}{24}\sigma\left(\frac{n}{2}\right), \tag{1.3}$$

$$W_3(n) = \frac{1}{24}\sigma_3(n) + \frac{3}{8}\sigma_3\left(\frac{n}{3}\right) + \frac{(1-2n)}{24}\sigma(n) + \frac{(1-6n)}{24}\sigma\left(\frac{n}{3}\right), \tag{1.4}$$

$$W_4(n) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) + \frac{(2-3n)}{48}\sigma(n) + \frac{(1-6n)}{24}\sigma\left(\frac{n}{4}\right), \tag{1.5}$$

which are valid for all $n \in \mathbb{N}$. Melfi [12, Theorem 12; 13] also gave the evaluations

$$W_5(n) = \frac{5}{312}\sigma_3(n) + \frac{(5-6n)}{120}\sigma(n), \tag{1.6}$$

if $n \equiv 8 \pmod{16}$ and $n \not\equiv 0 \pmod 5$, and

$$W_9(n) = \frac{1}{216}\sigma_3(n) + \frac{(3-2n)}{72}\sigma(n), \tag{1.7}$$

if $n \equiv 1 \pmod 3$ and there exists a prime $p \equiv 2 \pmod 3$ with $p \| n$, or $n \equiv 2 \pmod 3$. No elementary proofs of (1.6) and (1.7) are known. For $k \neq 1, 2, 3, 4, 5, 9$ the value of $W_k(n)$ is unknown. Williams [17] has recently given a simple analytic proof of

$$W_9(n) = \begin{cases} \frac{1}{6}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{4}\sigma_3\left(\frac{n}{9}\right) + \frac{(3-2n)}{18}\sigma\left(\frac{n}{3}\right) \\ \quad - \frac{(1+2n)}{12}\sigma\left(\frac{n}{9}\right), & \text{if } n \equiv 0 \pmod 3, \\ \frac{1}{216}\sigma_3(n) + \frac{(3-2n)}{72}\sigma(n), & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{1.8}$$

It is the purpose of this work to give an analytic proof of a general formula for $W_9(n)$, which includes (1.7) and (1.8) as special cases. Our formula involves integers $a(n)$ $(n \in \mathbb{N})$, which are related to Ramanujan's tau function $\tau(n)$ $(n \in \mathbb{N})$ [15].

Ramanujan's tau function $\tau(n)$ $(n \in \mathbb{N})$ is defined by [15, Eq. (92); 16, p. 151]

$$\Delta(q) := q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n, \quad q \in \mathbb{C}, \ |q| < 1. \tag{1.9}$$

Clearly

$$\Delta(q^3) = q^3\prod_{n=1}^{\infty}(1-q^{3n})^{24} = \sum_{n=1}^{\infty}\tau(n)q^{3n} = \sum_{n=1}^{\infty}\tau\left(\frac{n}{3}\right)q^n \tag{1.10}$$

so that

$$\Delta(q^3)^{\frac{1}{3}} = q\prod_{n=1}^{\infty}(1-q^{3n})^8 = \sum_{n=1}^{\infty} a(n)q^n \tag{1.11}$$

for integers $a(n)$ ($n \in \mathbb{N}$). From (1.11) we see that

$$a(n) = 0, \quad n \equiv 0, 2 (\text{mod } 3). \tag{1.12}$$

From (1.10)–(1.12), we deduce that

$$\sum_{\substack{r,s,t \in \mathbb{N} \\ r+s+t=3n \\ r\equiv s\equiv t\equiv 1 \ (\text{mod } 3)}} a(r)a(s)a(t) = \tau(n), \quad n \in \mathbb{N}. \tag{1.13}$$

Taking $n = 1, 2, 3, 4, 5$ in (1.13), we obtain

$$\begin{cases} a(1)^3 = \tau(1), \quad 3a(1)^2a(4) = \tau(2), \\ 3a(1)^2a(7) + 3a(1)a(4)^2 = \tau(3), \\ 3a(1)^2a(10) + 6a(1)a(4)a(7) + a(4)^3 = \tau(4), \\ 3a(1)^2a(13) + 6a(1)a(4)a(10) + 3a(1)a(7)^2 + 3a(4)^2a(7) = \tau(5). \end{cases} \tag{1.14}$$

As $\tau(1) = 1$, $\tau(2) = -24$, $\tau(3) = 252$, $\tau(4) = -1472$, $\tau(5) = 4830$ [15, Table V; 16, p. 153], we deduce from (1.14)

$$a(1) = 1, \quad a(4) = -8, \quad a(7) = 20, \quad a(10) = 0, \quad a(13) = -70. \tag{1.15}$$

From the work of Mordell [14, p. 121] we know that

$$a(mn) = a(m)a(n), \quad \gcd(m, n) = 1. \tag{1.16}$$

In Sec. 2 we prove

Theorem 1.1. *For all $n \in \mathbb{N}$ we have*

$$\begin{aligned} W_9(n) &= \sum_{m<n/9} \sigma(m)\sigma(n-9m) \\ &= \frac{1}{216}\sigma_3(n) + \frac{1}{27}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{8}\sigma_3\left(\frac{n}{9}\right) + \frac{(3-2n)}{72}\sigma(n) \\ &\quad + \frac{(1-6n)}{24}\sigma\left(\frac{n}{9}\right) - \frac{1}{54}a(n). \end{aligned}$$

Before giving the proof of this theorem, we show that it is in agreement with (1.8) when $n \equiv 0 (\text{mod } 3)$ and $n \equiv 2 (\text{mod } 3)$. In these cases by (1.12) we have $a(n) = 0$.

When $n \equiv 2 (\text{mod } 3)$, the right-hand side of the theorem reduces to $\frac{1}{216}\sigma_3(n) + \frac{(3-2n)}{72}\sigma(n)$, in agreement with (1.8).

When $n \equiv 0 (\text{mod } 3)$ we have the elementary identities

$$\sigma(n) - 4\sigma\left(\frac{n}{3}\right) + 3\sigma\left(\frac{n}{9}\right) = 0$$

and

$$\sigma_3(n) - 28\sigma_3\left(\frac{n}{3}\right) + 27\sigma_3\left(\frac{n}{9}\right) = 0,$$

so that the right-hand side of the theorem is

$$\begin{aligned}
&\frac{1}{216}\sigma_3(n) + \frac{1}{27}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{8}\sigma_3\left(\frac{n}{9}\right) + \frac{(3-2n)}{72}\sigma(n) + \frac{(1-6n)}{2}\sigma\left(\frac{n}{9}\right)\\
&= \frac{1}{216}\left(28\sigma_3\left(\frac{n}{3}\right) - 27\sigma_3\left(\frac{n}{9}\right)\right) + \frac{1}{27}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{8}\sigma_3\left(\frac{n}{9}\right)\\
&\quad + \frac{(3-2n)}{72}\left(4\sigma\left(\frac{n}{3}\right) - 3\sigma\left(\frac{n}{9}\right)\right) + \frac{(1-6n)}{24}\sigma\left(\frac{n}{9}\right)\\
&= \frac{1}{6}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{4}\sigma_3\left(\frac{n}{9}\right) + \frac{(3-2n)}{18}\sigma\left(\frac{n}{3}\right) - \frac{(1+2n)}{12}\sigma\left(\frac{n}{9}\right)
\end{aligned}$$

in agreement with (1.8).

Now let $n \in \mathbb{N}$ be such that

$$n \equiv 1 (\text{mod } 3), \quad n = rs (r, s \in \mathbb{N}), \quad r \equiv 2 (\text{mod } 3), \quad \gcd(r, s) = 1.$$

Then, by (1.12) and (1.16), we have

$$a(n) = a(rs) = a(r)a(s) = 0a(s) = 0,$$

so that the right-hand side of Theorem 1.1 is $\frac{1}{216}\sigma_3(n) + \frac{(3-2n)}{72}\sigma(n)$, showing that Melfi's evaluation (1.7) is a special case of Theorem 1.1.

In Sec. 3 we apply Theorem 1.1 to evaluate the sum

$$S(a, 3) = \sum_{\substack{m=1\\ m \equiv a (\text{mod } 3)}}^{n-1} \sigma(m)\sigma(n-m)$$

for $a = 0, 1, 2$ and all $n \in \mathbb{N}$, completing the partial evaluation of this sum given in [8, Theorem 8, p. 256].

Theorem 1.2. *For all $n \in \mathbb{N}$ we have*

$$\begin{aligned}
S(0, 3) = &\frac{11}{72}\sigma_3(n) + \frac{25}{18}\sigma_3\left(\frac{n}{3}\right) - \frac{9}{8}\sigma_3\left(\frac{n}{9}\right) + \frac{(1-6n)}{24}\sigma(n)\\
&+ \frac{(1-6n)}{6}\sigma\left(\frac{n}{3}\right) - \frac{(1-6n)}{8}\sigma\left(\frac{n}{9}\right) + \frac{1}{18}a(n).
\end{aligned}$$

The values of $S(1, 3)$ and $S(2, 3)$ follow from Theorem 1.2 and the relations

$$S(0, 3) + S(1, 3) + S(2, 3) = \frac{5}{12}\sigma_3(n) + \frac{(1-6n)}{12}\sigma(n)$$

and

$$\begin{cases} S(1, 3) = S(2, 3), & \text{if } n \equiv 0 (\text{mod } 3),\\ S(0, 3) = S(1, 3), & \text{if } n \equiv 1 (\text{mod } 3),\\ S(0, 3) = S(2, 3), & \text{if } n \equiv 2 (\text{mod } 3), \end{cases}$$

proved in [8, p. 257].

In Sec. 4 we apply Theorem 1.1 to determine the number $N(n)$ $(n \in \mathbb{N})$ of solutions of

$$n = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 3\left(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2\right)$$

in integers $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$.

Theorem 1.3. *For all $n \in \mathbb{N}$ we have*

$$N(n) = 4\sigma_3(n) - 88\sigma_3\left(\frac{n}{3}\right) + 324\sigma_3\left(\frac{n}{9}\right) + 8a(n).$$

2. Proof of Theorem 1.1

Let $q \in \mathbb{C}$ be such that $|q| < 1$. The Eisenstein series $L(q)$, $M(q)$ and $N(q)$ are defined by

$$L(q) := 1 - 24\sum_{n=1}^{\infty} \sigma(n)q^n, \tag{2.1}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n, \tag{2.2}$$

$$N(q) := 1 - 504\sum_{n=1}^{\infty} \sigma_5(n)q^n, \tag{2.3}$$

see for example [15, Eq. (25); 16, p. 140]. It was shown by Ramanujan [15, Eq. (44); 16, p. 144] that

$$\Delta(q) = \frac{1}{1728}\left(M(q)^3 - N(q)^2\right). \tag{2.4}$$

First we determine the generating function of $W_9(n)$ $(n \in \mathbb{N})$.

Lemma 2.1. $\sum_{n=1}^{\infty} W_9(n)q^n = \frac{1}{576} - \frac{1}{576}L(q) - \frac{1}{576}L(q^9) + \frac{1}{576}L(q)L(q^9)$.

Proof. By (2.1) we have

$$\begin{aligned}
(1 - L(q))(1 - L(q^9)) &= \left(24\sum_{l=1}^{\infty}\sigma(l)q^l\right)\left(24\sum_{m=1}^{\infty}\sigma(m)q^{9m}\right)\\
&= 576\sum_{l,m=1}^{\infty}\sigma(l)\sigma(m)q^{l+9m}\\
&= 576\sum_{n=1}^{\infty}q^n\left(\sum_{\substack{l,m=1\\ l+9m=n}}^{\infty}\sigma(l)\sigma(m)\right)\\
&= 576\sum_{n=1}^{\infty}q^n\sum_{1\le m<n/9}\sigma(m)\sigma(n-9m)\\
&= 576\sum_{n=1}^{\infty}W_9(n)q^n,
\end{aligned}$$

from which the asserted result follows. □

Next we modify the result of Lemma 2.1 to determine the function whose generating function is $L(q)L(q^9)$.

Lemma 2.2. $1+\sum_{n=1}^{\infty}\big(576W_9(n)-24\sigma(n)-24\sigma\left(\frac{n}{9}\right)\big)q^n = L(q)L(q^9)$.

Proof. Letting $q \to q^9$ in (2.1) yields

$$L(q^9) = 1-24\sum_{n=1}^{\infty}\sigma(n)q^{9n} = 1-24\sum_{n=1}^{\infty}\sigma\left(\frac{n}{9}\right)q^n. \tag{2.5}$$

Putting (2.1) and (2.5) into Lemma 2.1, we obtain the result. □

In order to determine $W_9(n)$, we determine the power series expansion of $L(q)L(q^9)$ in a different way, see Lemma 2.14. The assertion of Theorem 1.1 then follows by equating the coefficients of q^n in Lemmas 2.2 and 2.14. We obtain the power series expansion of $L(q)L(q^9)$ in powers of q from those of $L(q)^2$, $L(q^9)^2$ and $(L(q)-9L(q^9))^2$ by means of the elementary identity

$$L(q)L(q^9) = \frac{1}{18}L(q)^2 + \frac{9}{2}L(q^9)^2 - \frac{1}{18}(L(q)-9L(q^9))^2.$$

First we consider $L(q)^2$.

Lemma 2.3. $L(q)^2 = 1+\sum_{n=1}^{\infty}(240\sigma_3(n)-288n\sigma(n))q^n$.

Proof. This result is well known, see for example Lahiri [9, p. 197]. It follows by squaring (2.1) and applying the result (1.2). □

Next we consider $L(q^9)^2$. Letting $q \to q^9$ in Lemma 2.3, we obtain

Lemma 2.4. $L(q^9)^2 = 1+\sum_{n=1}^{\infty}\big(240\sigma_3\left(\frac{n}{9}\right)-32n\sigma\left(\frac{n}{9}\right)\big)q^n$.

We now turn to the consideration of $L(q)-9L(q^9)$, see Lemma 2.10. For $z \in \mathbb{C}$ with $|z|<1$ we set

$$w(z) := {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;z\right) = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(1)_n}\frac{z^n}{n!}, \tag{2.6}$$

where ${}_2F_1$ is the Gaussian hypergeometric function and $(a)_n$ is the Pochhammer symbol, see for example [5, p. 247]. Clearly $w(0)=1$. The infinite series (2.6) diverges at $z=1$ [5, p. 249] so that $w(1)=+\infty$. For $x \in \mathbb{R}$ with $0 \le x < 1$ we have

$$w(x) = 1+\sum_{n=1}^{\infty}\frac{3n!}{n!^3 3^{3n}}x^n \ge 1$$

so that

$$w(x) \neq 0, \quad 0 \le x < 1. \tag{2.7}$$

The derivative of the function

$$y(x) := \frac{2\pi}{\sqrt{3}} \frac{w(1-x)}{w(x)}, \quad 0 < x < 1, \tag{2.8}$$

is [1, p. 87]

$$y'(x) = \frac{-1}{x(1-x)w(x)^2}, \quad 0 < x < 1. \tag{2.9}$$

Thus, by (2.7) and (2.9), we have

$$y'(x) < 0, \quad 0 < x < 1. \tag{2.10}$$

Hence, as x increases from 0 to 1, $y(x)$ strictly decreases from $y(0) = \frac{2\pi}{\sqrt{3}} \frac{w(1)}{w(0)} = +\infty$ to $y(1) = \frac{2\pi}{\sqrt{3}} \frac{w(0)}{w(1)} = 0$. Now restrict q to satisfy $q \in \mathbb{R}$, $0 < q < 1$. Then $0 < -\log q < +\infty$ and there is a unique real number x with $0 < x < 1$ such that

$$\frac{2\pi}{\sqrt{3}} \frac{w(1-x)}{w(x)} = -\log q. \tag{2.11}$$

We set

$$w := w(x). \tag{2.12}$$

The values of $L(q)$, $M(q)$ and $N(q)$ can be given explicitly in terms of x and w, see for example [3, Lemma 4.1, p. 4178; Theorem 4.2, p. 4179; Theorem 4.3, p. 4179].

Lemma 2.5.

(a) $L(q) = (1-4x)w^2 + 12x(1-x)w\frac{dw}{dx}$.
(b) $M(q) = (1+8x)w^4$.
(c) $N(q) = (1-20x-8x^2)w^6$.

The principle of triplication [2, p. 101; 3, p. 4174] asserts that if $q \to q^3$ then

$$x \to \left(\frac{1-(1-x)^{\frac{1}{3}}}{1+2(1-x)^{\frac{1}{3}}} \right)^3$$

and

$$w \to \frac{1}{3}(1+2(1-x)^{\frac{1}{3}})w.$$

Applying the principle of triplication to Lemma 2.5, we obtain

Lemma 2.6.

(a) $L(q^3) = \left(1 - \frac{4}{3}x\right)w^2 + 4x(1-x)w\frac{dw}{dx}$.
(b) $M(q^3) = \left(1 - \frac{8}{9}x\right)w^4$.
(c) $N(q^3) = \left(1 - \frac{4}{3}x + \frac{8}{27}x^2\right)w^6$.

Proof. See [2, Eq. (13.17), p. 178; Theorems 4.4 and 4.5, p. 107)], [3, Theorem 4.4, p. 4179; Theorem 4.5, p. 4180]. □

Applying triplication again, this time to Lemma 2.6, we obtain

Lemma 2.7.

(a) $L(q^9) = (11 - 12x + 8(1-x)^{\frac{1}{3}} + 8(1-x)^{\frac{2}{3}})\frac{w^2}{27} + \frac{4}{3}x(1-x)w\frac{dw}{dx}$.
(b) $M(q^9) = (249 - 248x + 240(1-x)^{\frac{1}{3}} + 240(1-x)^{\frac{2}{3}} - 160x(1-x)^{\frac{1}{3}})\frac{w^4}{729}$.
(c) $N(q^9) = (6579 - 8604x + 2024x^2 + 6552(1-x)^{\frac{1}{3}} + 6552(1-x)^{\frac{2}{3}} - 6384x(1-x)^{\frac{1}{3}} - 4704x(1-x)^{\frac{2}{3}})\frac{w^6}{19683}$.

Proof. See [17, Eqs. (5.4)–(5.6)]. □

From (2.4) and Lemma 2.5(b)(c), we obtain

Lemma 2.8. $\Delta(q) = \frac{x(1-x)^3w^{12}}{27}$.

Applying the process of triplication to Lemma 2.8, we deduce

Lemma 2.9. $\Delta(q^3) = \frac{x^3(1-x)w^{12}}{19683}$.

From Lemma 2.5(a) and Lemma 2.7(a), we obtain $L(q) - 9L(q^9)$.

Lemma 2.10. $L(q) - 9L(q^9) = -\frac{8}{3}\left(1 + (1-x)^{\frac{1}{3}} + (1-x)^{\frac{2}{3}}\right)w^2$.

In the next lemma we determine $w^4, xw^4, x(1-x)^{\frac{1}{3}}w^4$ and $((1-x)^{\frac{1}{3}} + (1-x)^{\frac{2}{3}})w^4$ in terms of M and Δ. These quantities occur in the determination of $(L(q) - 9L(q^9))^2$ in Lemma 2.12.

Lemma 2.11.

(a) $w^4 = \frac{1}{10}M(q) + \frac{9}{10}M(q^3)$.
(b) $xw^4 = \frac{9}{80}M(q) - \frac{9}{80}M(q^3)$.
(c) $x(1-x)^{\frac{1}{3}}w^4 = 27\Delta(q^3)^{\frac{1}{3}}$
(d) $\left((1-x)^{\frac{1}{3}} + (1-x)^{\frac{2}{3}}\right)w^4 = \frac{1}{80}M(q) - \frac{21}{20}M(q^3) + \frac{243}{80}M(q^9) + 18\Delta(q^3)^{\frac{1}{3}}$.

Proof. Parts (a) and (b) follow from Lemmas 2.5(b) and 2.6(b). Part (c) follows by taking cube roots in Lemma 2.9. Part (d) follows from Lemmas 2.5(b), 2.6(b), 2.7(b) and part (c). □

We are now ready to determine the value of $(L(q) - 9L(q^9))^2$.

Lemma 2.12. $(L(q) - 9L(q^9))^2 = \frac{4}{5}M(q) - \frac{8}{5}M(q^3) + \frac{324}{5}M(q^9) + 192\Delta(q^3)^{\frac{1}{3}}$.

Proof. By Lemmas 2.10 and 2.11, we have

$$\begin{aligned}
&(L(q)-9L(q^9))^2\\
&=\frac{64}{9}\Big(3-2x+3\Big((1-x)^{\frac{1}{3}}+(1-x)^{\frac{2}{3}}\Big)-x(1-x)^{\frac{1}{3}}\Big)w^4\\
&=\frac{64}{9}\Bigg(\Big(\frac{3}{10}M(q)+\frac{27}{10}M(q^3)\Big)-\Big(\frac{9}{40}M(q)-\frac{9}{40}M(q^3)\Big)\\
&\quad+\Big(\frac{3}{80}M(q)-\frac{63}{20}M(q^3)+\frac{729}{80}M(q^9)+54\Delta(q^3)^{\frac{1}{3}}\Big)-27\Delta(q^3)^{\frac{1}{3}}\Bigg)\\
&=\frac{4}{5}M(q)-\frac{8}{5}M(q^3)+\frac{324}{5}M(q^9)+192\Delta(q^3)^{\frac{1}{3}},
\end{aligned}$$

as asserted. □

We are now in a position to determine the power series expansion of $(L(q)-9L(q^9))^2$ in powers of q.

Lemma 2.13.

$$(L(q)-9L(q^9))^2=64+\sum_{n=1}^{\infty}\Big(192\sigma_3(n)-384\sigma_3\left(\frac{n}{3}\right)+15552\sigma_3\left(\frac{n}{9}\right)+192a(n)\Big)q^n.$$

Proof. Appealing to Lemma 2.12, (2.2) and (1.11), we obtain

$$\begin{aligned}
&(L(q)-9L(q^9))^2\\
&=\frac{4}{5}\Big(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\Big)-\frac{8}{5}\Big(1+240\sum_{n=1}^{\infty}\sigma_3\left(\frac{n}{3}\right)q^n\Big)\\
&\quad+\frac{324}{5}\Big(1+240\sum_{n=1}^{\infty}\sigma_3\left(\frac{n}{9}\right)q^n\Big)+192\sum_{n=1}^{\infty}a(n)q^n\\
&=64+192\sum_{n=1}^{\infty}\sigma_3(n)q^n-384\sum_{n=1}^{\infty}\sigma_3\left(\frac{n}{3}\right)q^n+15552\sum_{n=1}^{\infty}\sigma_3\left(\frac{n}{9}\right)q^n\\
&\quad+192\sum_{n=1}^{\infty}a(n)q^n,
\end{aligned}$$

which gives the asserted result. □

We now deduce from Lemma 2.13 the power series expansion of $L(q)L(q^9)$ in powers of q.

Lemma 2.14.

$$\begin{aligned}
L(q)L(q^9)=1+\sum_{n=1}^{\infty}(&\frac{8}{3}\sigma_3(n)+\frac{64}{3}\sigma_3\Big(\frac{n}{3}\Big)+216\sigma_3\Big(\frac{n}{9}\Big)\\
&-16n\sigma(n)-144n\sigma\Big(\frac{n}{9}\Big)-\frac{32}{3}a(n))q^n.
\end{aligned}$$

Proof. Appealing to Lemmas 2.3, 2.4 and 2.13, we obtain

$$\begin{aligned}
&L(q)L(q^9)\\
&\quad= \frac{1}{18}\left(L(q)^2 + 81L(q^9)^2 - (L(q) - 9L(q^9))^2\right)\\
&\quad= \frac{1}{18}\Bigg(1 + \sum_{n=1}^{\infty}(240\sigma_3(n) - 288n\sigma(n))q^n + 81 + \sum_{n=1}^{\infty}\Bigg(19440\sigma_3\left(\frac{n}{9}\right)\\
&\qquad - 2592n\sigma\left(\frac{n}{9}\right)\Bigg)q^n - 64 - \sum_{n=1}^{\infty}\Bigg(192\sigma_3(n) - 384\sigma_3\left(\frac{n}{3}\right)\\
&\qquad + 15552\sigma_3\left(\frac{n}{9}\right) + 192a(n)\Bigg)q^n\Bigg)\\
&\quad= 1 + \sum_{n=1}^{\infty}\Bigg(\frac{8}{3}\sigma_3(n) + \frac{64}{3}\sigma_3\left(\frac{n}{3}\right) + 216\sigma_3\left(\frac{n}{9}\right)\\
&\qquad - 16n\sigma(n) - 144n\sigma\left(\frac{n}{9}\right) - \frac{32}{3}a(n)\Bigg)q^n,
\end{aligned}$$

as asserted. □

Proof of Theorem 1.1. Equating coefficients of q^n in Lemmas 2.2 and 2.14, we obtain

$$\begin{aligned}
&576W_9(n) - 24\sigma(n) - 24\sigma\left(\frac{n}{9}\right)\\
&\quad= \frac{8}{3}\sigma_3(n) + \frac{64}{3}\sigma_3\left(\frac{n}{3}\right) + 216\sigma_3\left(\frac{n}{9}\right) - 16n\sigma(n) - 144n\sigma\left(\frac{n}{9}\right) - \frac{32}{3}a(n),
\end{aligned}$$

from which we deduce

$$\begin{aligned}
W_9(n) = {}& \frac{1}{216}\sigma_3(n) + \frac{1}{27}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{8}\sigma_3\left(\frac{n}{9}\right) + \frac{1}{24}\sigma(n) - \frac{1}{36}n\sigma(n)\\
&+ \frac{1}{24}\sigma\left(\frac{n}{9}\right) - \frac{1}{4}n\sigma\left(\frac{n}{9}\right) - \frac{1}{54}a(n),
\end{aligned}$$

which is the asserted formula. □

3. First Application: Proof of Theorem 1.2

By (1.4) and Theorem 1.1 we have

$$\begin{aligned}
S(0,3) &= \sum_{\substack{m=1\\ m\equiv 0 \pmod 3}}^{n-1} \sigma(m)\sigma(n-m)\\
&= \sum_{k<n/3} \sigma(3k)\sigma(n-3k)\\
&= \sum_{k<n/3} \left(4\sigma(k) - 3\sigma\left(\frac{k}{3}\right)\right)\sigma(n-3k)
\end{aligned}$$

$$
\begin{aligned}
&= 4\sum_{k<n/3}\sigma(k)\sigma(n-3k) - 3\sum_{k<n/9}\sigma(k)\sigma(n-9k)\\
&= 4W_3(n) - 3W_9(n)\\
&= 4\Big(\frac{1}{24}\sigma_3(n) + \frac{3}{8}\sigma_3\left(\frac{n}{3}\right) + \frac{(1-2n)}{24}\sigma(n) + \frac{(1-6n)}{24}\sigma\left(\frac{n}{3}\right)\Big)\\
&\quad - 3\Big(\frac{1}{216}\sigma_3(n) + \frac{1}{27}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{8}\sigma_3\left(\frac{n}{9}\right) + \frac{(3-2n)}{72}\sigma(n)\\
&\quad + \frac{(1-6n)}{24}\sigma\left(\frac{n}{9}\right) - \frac{1}{54}a(n)\Big)\\
&= \frac{11}{72}\sigma_3(n) + \frac{25}{18}\sigma_3\left(\frac{n}{3}\right) - \frac{9}{8}\sigma_3\left(\frac{n}{9}\right) + \frac{(1-6n)}{24}\sigma(n) + \frac{(1-6n)}{6}\sigma\left(\frac{n}{3}\right)\\
&\quad - \frac{(1-6n)}{8}\sigma\left(\frac{n}{9}\right) + \frac{1}{18}a(n),
\end{aligned}
$$

as asserted. □

It is easy to check that Theorem 1.2 is in agreement with the partial evaluation of $S(0,3)$ given in [8, Theorem 8, p. 256].

4. Second Application: Proof of Theorem 1.3

Let $n \in \mathbb{N}$. We use Theorem 1.1 to determine the number $N(n)$ of solutions of

$$n = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 3\big(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2\big)$$

in integers $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$. We have

$$
\begin{aligned}
N(n) &= \sum_{\substack{(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\,\in\,\mathbb{Z}^8\\ n=x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2+3(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2)}} 1\\
&= \sum_{\substack{l,m\in\mathbb{Z}\\ l,m\geq 0\\ l+3m=n}} M(l)M(m),
\end{aligned}
$$

where

$$M(k) = \sum_{\substack{(x_1,x_2,x_3,x_4)\,\in\,\mathbb{Z}^4\\ k=x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2}} 1, \quad k \in \mathbb{N}\cup\{0\}.$$

Clearly $M(0) = 1$. If $k \notin \mathbb{N} \cup \{0\}$ we set $M(k) = 0$. Thus

$$\begin{aligned} N(n) &= M(n) + M\left(\frac{n}{3}\right) + \sum_{\substack{l,m \in \mathbb{N} \\ l+3m=n}} M(l)M(m) \\ &= M(n) + M\left(\frac{n}{3}\right) + \sum_{m<n/3} M(m)M(n-3m). \end{aligned}$$

Now

$$M(k) = 12\sigma(k) - 36\sigma\left(\frac{k}{3}\right), \quad k \in \mathbb{N},$$

see for example [8, Theorem 13, p. 266], so that

$$\begin{aligned} N(n) &= 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right) + 12\sigma\left(\frac{n}{3}\right) - 36\sigma\left(\frac{n}{9}\right) \\ &\quad + \sum_{m<n/3} \left(12\sigma(m) - 36\sigma\left(\frac{m}{3}\right)\right)\left(12\sigma(n-3m) - 36\sigma\left(\frac{n}{3} - m\right)\right) \\ &= 12\sigma(n) - 24\sigma\left(\frac{n}{3}\right) - 36\sigma\left(\frac{n}{9}\right) \\ &\quad + 144 \sum_{m<n/3} \sigma(m)\sigma(n-3m) - 432 \sum_{m<n/3} \sigma(m)\sigma\left(\frac{n}{3} - m\right) \\ &\quad - 432 \sum_{m<n/3} \sigma(n-3m)\sigma\left(\frac{m}{3}\right) + 1296 \sum_{m<n/3} \sigma\left(\frac{m}{3}\right)\sigma\left(\frac{n}{3} - m\right) \\ &= 12\sigma(n) - 24\sigma\left(\frac{n}{3}\right) - 36\sigma\left(\frac{n}{9}\right) + 144W_3(n) - 432W_1\left(\frac{n}{3}\right) \\ &\quad - 432W_9(n) + 1296W_3\left(\frac{n}{3}\right). \end{aligned}$$

Appealing to (1.2), (1.4) and Theorem 1.1, we obtain the asserted formula for $N(n)$. □

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n	$N(n)$	$\sigma_3(n)$	$a(n)$	$4\sigma_3(n) - 88\sigma_3(n/3) + 324\sigma_3(n/9) + 8a(n)$
1	12	1	1	12
2	36	9	0	36
3	24	28	0	24
4	228	73	−8	228
5	504	126	0	504
6	216	252	0	216
7	1536	344	20	1536
8	2340	585	0	2340
9	888	757	0	888
10	4536	1134	0	4536

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