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SUMS OF SIX SQUARES

ERIN MCAFEE and KENNETH S. WILLIAMS

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Abstract

A new elementary arithmetic proof of Jacobi's six squares formula is presented.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of natural numbers, integers, real numbers and complex numbers, respectively. Let $k \in \mathbb{N}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The number $r_k(n)$ of representations of n as the sum of k squares is defined by

$$r_k(n) = \text{card}\{(x_1, \ldots, x_k) \in \mathbb{Z}^k : x_1^2 + \cdots + x_k^2 = n\}. \tag{1.1}$$

When $k = 2$ we have the well known formula

$$r_2(n) = 4\sum_{d \mid n}\left(\frac{-4}{d}\right), \quad n \in \mathbb{N}, \tag{1.2}$$

where the sum is taken over all $d \in \mathbb{N}$ dividing n and the Kronecker symbol $\left(\frac{-4}{n}\right) = 1, -1$ or 0 according as $n \equiv 1 \pmod 4$, $3 \pmod 4$ or

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0 (mod 2). We note that $r_2(n) = 0$ for $n \equiv 3 \pmod 4$. When $k = 4$ we have

$$r_4(n) = 8\sum_{\substack{d \mid n \\ 4 \nmid d}} d, \quad n \in \mathbb{N}. \tag{1.3}$$

Many elementary arithmetic proofs of (1.2) and (1.3) are known, see for example [2, p. 80], [8, pp. 427-435], [9]. When $k = 6$ Jacobi's formula for $r_6(n)$ asserts that

$$r_6(n) = 16\sum_{d \mid n}\left(\frac{-4}{n/d}\right)d^2 - 4\sum_{d \mid n}\left(\frac{-4}{d}\right)d^2, \quad n \in \mathbb{N}. \tag{1.4}$$

Although many analytic proofs of Jacobi's formula have appeared in the literature, see for example [1], only one truly arithmetic proof of Jacobi's formula appears to be known. A presentation of this proof has been given by Nathanson in his beautiful book on elementary methods in number theory [8, pp. 436-439]. The proof makes use of an elementary formula due to Liouville [5] to show that the function Φ, defined by $\Phi(0) = 1$ and

$$\Phi(n) = 16\sum_{d \mid n}\left(\frac{-4}{n/d}\right)d^2 - 4\sum_{d \mid n}\left(\frac{-4}{d}\right)d^2$$
$$= 4\sum_{d \mid n,\, n/d \text{ odd}} (-1)^{(n/d-1)/2}(4d^2 - (n/d)^2)$$

for $n \in \mathbb{N}$, satisfies the recursion formula

$$\sum_{|x| \le \sqrt{n}} (n - 7x^2)\Phi(n - x^2) = 0$$

for $n \in \mathbb{N}$. Since it is easily shown that $r_6(n)$ satisfies this recursion [8, Theorem 14.1, p. 424] and $r_6(0) = 1 = \Phi(0)$, it follows that $r_6(n) = \Phi(n)$ for all $n \in \mathbb{N}_0$, which proves Jacobi's formula (1.4).

It is the purpose of this paper to present an entirely different elementary arithmetic proof of Jacobi's formula. Our starting point is the formula (1.2) for the number $r_2(n)$ of representations of $n \in \mathbb{N}$ as the sum of two squares. Clearly

$$r_4(n) = \sum_{\substack{(n_1, n_2) \in \mathbb{N}_0^2 \\ n_1 + n_2 = n}} r_2(n_1) r_2(n_2), \quad n \in \mathbb{N}.$$

Taking into account terms with n_1 or $n_2 = 0$, we obtain (as $r_2(0) = 1$)

$$r_4(n) = 2r_2(n) + \sum_{\substack{(n_1, n_2) \in \mathbb{N}^2 \\ n_1 + n_2 = n}} r_2(n_1) r_2(n_2), \quad n \in \mathbb{N}.$$

Then for $n \in \mathbb{N}$ we have

$$r_6(n) = \sum_{\substack{(n_1, n_2, n_3) \in \mathbb{N}_0^3 \\ n_1 + n_2 + n_3 = n}} r_2(n_1) r_2(n_2) r_2(n_3)$$

$$= 3r_2(n) + 3 \sum_{\substack{(n_1, n_2) \in \mathbb{N}^2 \\ n_1 + n_2 = n}} r_2(n_1) r_2(n_2) + \sum_{\substack{(n_1, n_2, n_3) \in \mathbb{N}^3 \\ n_1 + n_2 + n_3 = n}} r_2(n_1) r_2(n_2) r_2(n_3)$$

$$= 3r_4(n) - 3r_2(n) + \sum_{\substack{(n_1, n_2, n_3) \in \mathbb{N}^3 \\ n_1 + n_2 + n_3 = n}} r_2(n_1) r_2(n_2) r_2(n_3).$$

Appealing to (1.2), we obtain as $\left(\frac{-4}{n}\right)$ is a multiplicative function of $n \in \mathbb{N}$

$$r_6(n) - 3r_4(n) + 3r_2(n) = 64 \sum_{\substack{(n_1, n_2, n_3) \in \mathbb{N}^3 \\ n_1 + n_2 + n_3 = n}} \sum_{a | n_1, b | n_2, c | n_3} \left(\frac{-4}{abc}\right).$$

Thus

$$r_6(n) - 3r_4(n) + 3r_2(n) = 64 \sum_{\substack{(a, b, c, x, y, z) \in \mathbb{N}^6 \\ ax + by + cz = n}} \left(\frac{-4}{abc}\right). \tag{1.5}$$

A simple case by case examination of a, b and $c \in \mathbb{Z}$ modulo 4 shows that

$$\left(\frac{-4}{abc}\right) = F(a + b + c) + F(a - b - c) - F(a + b - c) - F(a - b + c) \tag{1.6}$$

with

$$F(m) = -\frac{1}{4}\left(\frac{-4}{m}\right), \quad m \in \mathbb{Z}. \tag{1.7}$$

Hence for $n \in \mathbb{N}$ we have

$$r_6(n) - 3r_4(n) + 3r_2(n)$$
$$= 64 \sum_{\substack{(a,b,c,x,y,z)\in\mathbb{N}^6 \\ ax+by+cz=n}} (F(a+b+c)+F(a-b-c)-F(a+b-c)-F(a-b+c)). \tag{1.8}$$

Recently the authors have proved in an entirely elementary manner the following formula [7], see also [6], which is similar to one stated by Liouville [4, p. 331]. We set $\sigma(n) = \sum_{d|n} d$, $n \in \mathbb{N}$.

Theorem 1.1. *Let* $n \in \mathbb{N}$ *and let* $F : \mathbb{Z} \to \mathbb{C}$ *be an odd function. Then*

$$\sum_{ax+by+cz=n} (F(a+b+c)+F(a-b-c)-F(a+b-c)-F(a-b+c))$$
$$= \sum_{d|n}\left(\frac{d^2-3d+2}{2} + 3\left(\frac{n}{d}-1\right)\left(\frac{n}{d}-d\right)\right)F(d)$$
$$+ 3\sum_{d|n}\sum_{1\le k<d}\left(2d - \frac{2n}{d} - k\right)F(k) - 6\sum_{\substack{n_1,n_2\in\mathbb{N} \\ n=n_1+n_2}} \sigma(n_1)\sum_{d|n_2} F(d),$$

where the sum on the left-hand side is taken over all $(a, b, c, x, y, z) \in \mathbb{N}^6$ *such that* $ax + by + cz = n$.

Using Theorem 1.1 with F as in (1.7) enables us to turn the sum on the right-hand side of (1.8) into sums over divisors of n from which we obtain Jacobi's formula (1.4) for $r_6(n)$ in a completely arithmetic manner. The details are given in Section 2.

2. Arithmetic Proof of Jacobi's Six Squares Theorem

We begin our evaluation of $r_6(n)$ by considering the case when n is odd. The following two arithmetic identities stated by Liouville [3], [4] are proved in an elementary way in [6].

Lemma 2.1. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Then*

$$\sum_{ax+by=n} (F(a-b) - F(a+b))$$

$$= F(0)(\sigma(n) - d(n)) + \sum_{d \mid n}\left(1 - d + \frac{2n}{d}\right)F(d) - 2\sum_{d \mid n}\sum_{1 \le k \le d} F(k),$$

where the summation on the left-hand side is over all $(a, b, x, y) \in \mathbb{N}^4$ such that $ax + by = n$.

Lemma 2.2. *Let n be an odd positive integer. Let $F : \mathbb{Z} \to \mathbb{C}$ be an even function. Then*

$$2\sum_{\substack{ax+by=n \\ a,x,y \text{ odd}}} (F(a-b) - F(a+b)) = \sum_{d \mid n}\left(\frac{n}{d} - d\right)F(d),$$

where the summation on the left-hand side is over all $(a, b, x, y) \in \mathbb{N}^4$ with a, x, y odd, such that $ax + by = n$.

The following four elementary lemmas evaluate sums that arise in the proof of Jacobi's formula (1.4).

Lemma 2.3. *Let n be an odd positive integer. Then*

$$\sum_{d \mid n}\sum_{1 \le k < d}\left(\frac{-4}{k}\right)k = \frac{1}{2}\sum_{d \mid n}\left(\frac{-4}{d}\right)(1 - d).$$

If $n \equiv 3 \pmod 4$, then

$$\sum_{d \mid n}\sum_{1 \le k < d}\left(\frac{-4}{k}\right)k = -\frac{1}{2}\sum_{d \mid n}\left(\frac{-4}{d}\right)d.$$

Proof. We have

$$\sum_{d \mid n}\sum_{1 \le k < d}\left(\frac{-4}{k}\right)k = \sum_{d \mid n}(-1)^{(d-1)/2}\frac{(1-d)}{2} = \frac{1}{2}\sum_{d \mid n}\left(\frac{-4}{d}\right)(1 - d),$$

as claimed. When $n \equiv 3 \pmod 4$ we have $r_2(n) = 0$ so by (1.2)

$$\sum_{d \mid n} \left(\frac{-4}{d} \right) = 0,$$

giving the desired result.

Lemma 2.4. *Let n be an odd positive integer. Then*

$$2 \sum_{\substack{n = n_1 + 2n_2 \\ n_1, n_2 \text{ odd}}} r_2(n_1)\sigma(n_2) - \sum_{\substack{n = n_1 + 2^s n_2 \\ n_1, n_2 \text{ odd}, s \geq 2}} 2^s r_2(n_1)\sigma(n_2) = \frac{n}{4} r_2(n) - \sum_{d \mid n} \left(\frac{-4}{d} \right) d^2.$$

Proof. We note that the first sum vanishes if $n \equiv 1 \pmod 4$ and the second sum vanishes if $n \equiv 3 \pmod 4$. In order to prove Lemma 2.4, we apply Lemma 2.2 with the even function $F(x) = \left(\frac{-4}{x} \right) x$. We begin by calculating the left-hand side of Lemma 2.2. The left-hand side is

$$2 \sum_{\substack{n = n_1 + 2^s n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1 \mid n_1} \sum_{d_2 \mid n_2} \left(\left(\frac{-4}{d_1 - 2^s d_2} \right) (d_1 - 2^s d_2) - \left(\frac{-4}{d_1 + 2^s d_2} \right) (d_1 + 2^s d_2) \right).$$

As n is odd, we have $s \geq 1$. Thus $2^s \equiv -2^s \pmod 4$, and so $d_1 + 2^s d_2 \equiv d_1 - 2^s d_2 \pmod 4$. Therefore the left-hand side is

$$2 \sum_{\substack{n = n_1 + 2^s n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1 \mid n_1} \sum_{d_2 \mid n_2} \left(\left(\frac{-4}{d_1 + 2^s d_2} \right) (d_1 - 2^s d_2) - \left(\frac{-4}{d_1 + 2^s d_2} \right) (d_1 + 2^s d_2) \right)$$

$$= -4 \sum_{\substack{n = n_1 + 2^s n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1 \mid n_1} \sum_{d_2 \mid n_2} 2^s d_2 \left(\frac{-4}{d_1 + 2^s d_2} \right)$$

$$= -4 \sum_{\substack{n = n_1 + 2^s n_2 \\ n_1, n_2 \text{ odd}, s \geq 2}} \sum_{d_1 \mid n_1} \sum_{d_2 \mid n_2} 2^s d_2 \left(\frac{-4}{d_1} \right) - 4 \sum_{\substack{n = n_1 + 2n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1 \mid n_1} \sum_{d_2 \mid n_2} 2d_2 \left(\frac{-4}{d_1 + 2d_2} \right)$$

$$= -4 \sum_{\substack{n = n_1 + 2^s n_2 \\ n_1, n_2 \text{ odd}, s \geq 2}} \sum_{d_1 \mid n_1} \sum_{d_2 \mid n_2} 2^s d_2 \left(\frac{-4}{d_1} \right) + 8 \sum_{\substack{n = n_1 + 2n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1 \mid n_1} \sum_{d_2 \mid n_2} d_2 \left(\frac{-4}{d_1} \right)$$

$$= -\sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}, s\geq 2}} 2^s \cdot 4 \sum_{d_1 \mid n_1} \left(\frac{-4}{d_1}\right) \sum_{d_2 \mid n_2} d_2 + 2 \sum_{\substack{n=n_1+2n_2 \\ n_1, n_2 \text{ odd}}} 4 \sum_{d_1 \mid n_1} \left(\frac{-4}{d_1}\right) \sum_{d_2 \mid n_2} d_2$$

$$= -\sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}, s\geq 2}} 2^s r_2(n_1)\sigma(n_2) + 2 \sum_{\substack{n=n_1+2n_2 \\ n_1, n_2 \text{ odd}}} r_2(n_1)\sigma(n_2).$$

The right-hand side of Lemma 2.2 is

$$\sum_{d \mid n} \left(\frac{n}{d} - d\right)\left(\frac{-4}{d}\right) d = n \sum_{d \mid n} \left(\frac{-4}{d}\right) - \sum_{d \mid n} \left(\frac{-4}{d}\right) d^2 = \frac{n}{4} r_2(n) - \sum_{d \mid n} \left(\frac{-4}{d}\right) d^2.$$

Equating the left-hand and right-hand sides, we get the desired result.

Lemma 2.5. *Let n be an odd positive integer. Then*

$$\sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}, s\geq 1}} r_2(n_2)\sigma(n_1) = \begin{cases} \dfrac{r_6(n)}{12} - \dfrac{r_4(n)}{8}, & \text{if } n \equiv 1 \pmod 4, \\ \dfrac{r_6(n)}{20} - \dfrac{r_4(n)}{8}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. From (1.3) we have

$$r_4(m) = 8\sigma(m), \quad \text{if } m \ (\in \mathbb{N}) \text{ is odd.}$$

Let $n \equiv 1 \pmod 4$. If $n = x_1^2 + \cdots + x_6^2$, then exactly one of $x_1, \ldots, x_6$ is even or exactly one of $x_1, \ldots, x_6$ is odd. Set

$$t_0(n) = \text{card}\{(x_1, \ldots, x_6) \in \mathbb{Z}^6 \mid n = x_1^2 + \cdots + x_6^2,\ x_1 \text{ even},\ x_2, \ldots, x_6 \text{ odd}\},$$

$$t_1(n) = \text{card}\{(x_1, \ldots, x_6) \in \mathbb{Z}^6 \mid n = x_1^2 + \cdots + x_6^2,\ x_1 \text{ odd},\ x_2, \ldots, x_6 \text{ even}\},$$

so that

$$r_6(n) = 6t_0(n) + 6t_1(n).$$

Next

$$t_0(n) = \sum_{\substack{1 \leq n_1 < n \\ n_1 \equiv 3 \pmod 4}} \sum_{\substack{x_1, x_2, x_3, x_4 \in \mathbb{Z} \\ x_1 \text{ even}, x_2, x_3, x_4 \text{ odd} \\ n_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2}} 1 \sum_{\substack{x_5, x_6 \in \mathbb{Z} \\ x_5, x_6 \text{ odd} \\ n - n_1 = x_5^2 + x_6^2}} 1$$

and

$$t_1(n) = \sum_{\substack{1 \le n_1 \le n \\ n_1 \equiv 1 \pmod 4}} \sum_{\substack{x_1, x_2, x_3, x_4 \in \mathbf{Z} \\ x_1 \text{ odd}, x_2, x_3, x_4 \text{ even} \\ n_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2}} 1 \sum_{\substack{x_5, x_6 \in \mathbf{Z} \\ x_5, x_6 \text{ even} \\ n - n_1 = x_5^2 + x_6^2}} 1.$$

Hence

$$t_0(n) = \sum_{\substack{1 \le n_1 < n \\ n_1 \equiv 3 \pmod 4}} \frac{1}{4} r_4(n_1) r_2(n - n_1) = \frac{1}{4} \sum_{\substack{1 \le n_1 < n \\ n_1 \equiv 3 \pmod 4}} r_4(n_1) r_2(n - n_1)$$

and

$$t_1(n) = \sum_{\substack{1 \le n_1 \le n \\ n_1 \equiv 1 \pmod 4}} \frac{1}{4} r_4(n_1) r_2(n - n_1) = \frac{1}{4} \sum_{\substack{1 \le n_1 < n \\ n_1 \equiv 1 \pmod 4}} r_4(n_1) r_2(n - n_1) + \frac{1}{4} r_4(n).$$

Thus

$$\begin{aligned} \frac{r_6(n)}{6} = t_0(n) + t_1(n) &= \frac{1}{4} \sum_{\substack{1 \le n_1 < n \\ n_1 \equiv 1 \pmod 2}} r_4(n_1) r_2(n - n_1) + \frac{1}{4} r_4(n) \\ &= 2 \sum_{\substack{1 \le n_1 < n \\ n_1 \equiv 1 \pmod 2}} \sigma(n_1) r_2(n - n_1) + \frac{1}{4} r_4(n) \\ &= 2 \sum_{\substack{n_1 + 2^s n_2 = n \\ n_1, n_2 \text{ odd}, s \ge 1}} \sigma(n_1) r_2(2^s n_2) + \frac{1}{4} r_4(n) \\ &= 2 \sum_{\substack{n_1 + 2^s n_2 = n \\ n_1, n_2 \text{ odd}, s \ge 1}} \sigma(n_1) r_2(n_2) + \frac{1}{4} r_4(n). \end{aligned}$$

Therefore when $n \equiv 1 \pmod 4$ we have

$$\sum_{\substack{n_1 + 2^s n_2 = n \\ n_1, n_2 \text{ odd}, s \ge 1}} \sigma(n_1) r_2(n_2) = \frac{r_6(n)}{12} - \frac{r_4(n)}{8},$$

as claimed. The proof for the case when $n \equiv 3 \pmod 4$ follows a similar method using the fact that if $n = x_1^2 + \cdots + x_6^2$, then exactly three of $x_1, \ldots, x_6$ are even and exactly three of $x_1, \ldots, x_6$ are odd.

The following lemma only applies to the case when $n \equiv 1 \pmod 4$. A similar result is not needed when $n \equiv 3 \pmod 4$.

Lemma 2.6. *Let* $n \in \mathbb{N}$ *satisfy* $n \equiv 1 \pmod 4$. *Then*

$$\sum_{\substack{n=2^s n_1+n_2 \\ n_1, n_2 \text{ odd}, s\geq 1}} r_2(n_2)\sigma(n_1) = \frac{1}{6}\left(\sum_{d\mid n}\left(\frac{-4}{d}\right)d + \sum_{d\mid n}\left(\frac{-4}{d}\right)d^2 - 2\sum_{d\mid n}\sum_{k=1}^{d}\left(\frac{-4}{k}\right)k\right).$$

Proof. We apply Lemma 2.1 with the even function $F(x) = \left(\frac{-4}{x}\right)x$. The left-hand side of Lemma 2.1 is

$$\sum_{n=n_1+n_2}\sum_{d_1\mid n_1}\sum_{d_2\mid n_2}\left((d_1-d_2)\left(\frac{-4}{d_1-d_2}\right)-(d_1+d_2)\left(\frac{-4}{d_1+d_2}\right)\right)$$

$$= \sum_{n=n_1+n_2}\left(\sum_{\substack{d_1\mid n_1, d_2\mid n_2 \\ 2\nmid d_1, 2\mid d_2}}\left((d_1-d_2)\left(\frac{-4}{d_1-d_2}\right)-(d_1+d_2)\left(\frac{-4}{d_1+d_2}\right)\right)\right.$$

$$\left. + \sum_{\substack{d_1\mid n_1, d_2\mid n_2 \\ 2\mid d_1, 2\nmid d_2}}\left((d_1-d_2)\left(\frac{-4}{d_1-d_2}\right)-(d_1+d_2)\left(\frac{-4}{d_1+d_2}\right)\right)\right)$$

$$= \sum_{n=n_1+n_2}\left(\sum_{\substack{d_1\mid n_1, d_2\mid n_2 \\ 2\nmid d_1, 2\mid d_2}}\left((d_1-d_2)\left(\frac{-4}{d_1-d_2}\right)-(d_1+d_2)\left(\frac{-4}{d_1-d_2}\right)\right)\right.$$

$$\left. + \sum_{\substack{d_1\mid n_1, d_2\mid n_2 \\ 2\mid d_1, 2\nmid d_2}}\left((d_1-d_2)\left(\frac{-4}{d_1-d_2}\right)+(d_1+d_2)\left(\frac{-4}{d_1-d_2}\right)\right)\right)$$

$$= \sum_{n=n_1+n_2} \left(\sum_{\substack{d_1 \mid n_1, d_2 \mid n_2 \\ 2 \nmid d_1, 2 \mid d_2}} (-2d_2)\left(\frac{-4}{d_1-d_2}\right) + \sum_{\substack{d_1 \mid n_1, d_2 \mid n_2 \\ 2 \mid d_1, 2 \nmid d_2}} (2d_1)\left(\frac{-4}{d_1-d_2}\right) \right)$$

$$= \sum_{n=n_1+n_2} \left(\sum_{\substack{d_1 \mid n_1, d_2 \mid n_2 \\ 2 \nmid d_1, 2 \mid d_2}} (-2d_2)\left(\frac{-4}{d_1-d_2}\right) + \sum_{\substack{d_1 \mid n_1, d_2 \mid n_2 \\ 2 \nmid d_1, 2 \mid d_2}} (2d_2)\left(\frac{-4}{d_2-d_1}\right) \right)$$

$$= \sum_{n=n_1+n_2} \sum_{\substack{d_1 \mid n_1, d_2 \mid n_2 \\ 2 \nmid d_1, 2 \mid d_2}} (-4d_2)\left(\frac{-4}{d_1-d_2}\right)$$

$$= \sum_{n=n_1+n_2} \sum_{\substack{d_1 \mid n_1, d_2 \mid n_2 \\ 2 \nmid d_1, 2 \| d_2}} (-4d_2)\left(\frac{-4}{d_1-d_2}\right)$$

$$+ \sum_{n=n_1+n_2} \sum_{\substack{d_1 \mid n_1, d_2 \mid n_2 \\ 2 \nmid d_1, 4 \mid d_2}} (-4d_2)\left(\frac{-4}{d_1-d_2}\right)$$

$$= \sum_{n=n_1+n_2} \sum_{\substack{d_1 \mid n_1, d_2 \mid n_2 \\ 2 \nmid d_1, 2 \| d_2}} (4d_2)\left(\frac{-4}{d_1}\right) + \sum_{n=n_1+n_2} \sum_{\substack{d_1 \mid n_1, d_2 \mid n_2 \\ 2 \nmid d_1, 4 \mid d_2}} (-4d_2)\left(\frac{-4}{d_1}\right)$$

$$= 4 \sum_{n=n_1+n_2} \sum_{\substack{d_1 \mid n_1 \\ 2 \nmid d_1}} \left(\frac{-4}{d_1}\right) \sum_{\substack{d_2 \mid n_2 \\ 2 \| d_2}} d_2 - 4 \sum_{n=n_1+n_2} \sum_{\substack{d_1 \mid n_1 \\ 2 \nmid d_1}} \left(\frac{-4}{d_1}\right) \sum_{\substack{d_2 \mid n_2 \\ 4 \mid d_2}} d_2$$

$$= \sum_{n=n_1+n_2} r_2(n_1) \sum_{\substack{d_2 \mid n_2 \\ 2 \| d_2}} d_2 - \sum_{n=n_1+n_2} r_2(n_1) \sum_{\substack{d_2 \mid n_2 \\ 4 \mid d_2}} d_2$$

$$= \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 1}} r_2(n_1) \sum_{\substack{d_2 \mid 2^s N_2 \\ 2 \| d_2}} d_2 - \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 2}} r_2(n_1) \sum_{\substack{d_2 \mid 2^s N_2 \\ 4 \mid d_2}} d_2$$

$$= \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 1}} r_2(n_1) \sum_{e_2 \mid N_2} 2e_2 - \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 2}} r_2(n_1) \sum_{e_2 \mid 2^{s-2} N_2} 4e_2$$

$$= 2 \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s\ge 1}} r_2(n_1)\sigma(N_2) - 4 \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s\ge 2}} r_2(n_1)\sigma(2^{s-2}N_2)$$

$$= 2 \sum_{\substack{n=n_1+2N_2 \\ n_1, N_2 \text{ odd}}} r_2(n_1)\sigma(N_2) + 2 \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s\ge 2}} r_2(n_1)(1-2\sigma(2^{s-2}))\sigma(N_2)$$

$$= 2 \sum_{\substack{n=n_1+2N_2 \\ n_1 \equiv 3 \pmod 4}} r_2(n_1)\sigma(N_2) + 2 \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s\ge 2}} r_2(n_1)(1-2(2^{s-1}-1))\sigma(N_2)$$

$$= 2 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}, s\ge 2}} r_2(n_1)(3-2^s)\sigma(n_2)$$

$$= 6 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}, s\ge 2}} r_2(n_1)\sigma(n_2) - 2 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}, s\ge 2}} 2^s r_2(n_1)\sigma(n_2)$$

$$= 6 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}, s\ge 2}} r_2(n_1)\sigma(n_2) - 2\left(\sum_{d\mid n}\left(\frac{-4}{d}\right)d^2 - \frac{n}{4}r_2(n)\right),$$

by Lemma 2.4. Next the right-hand side of Lemma 2.1 is

$$\sum_{d\mid n}\left(1-d+\frac{2n}{d}\right)\left(\frac{-4}{d}\right)d - 2\sum_{d\mid n}\sum_{1\le k\le d}\left(\frac{-4}{k}\right)k$$

$$= \sum_{d\mid n}\left(\frac{-4}{d}\right)d - \sum_{d\mid n}\left(\frac{-4}{d}\right)d^2 + 2n\sum_{d\mid n}\left(\frac{-4}{d}\right) - 2\sum_{d\mid n}\sum_{1\le k\le d}\left(\frac{-4}{k}\right)k$$

$$= \sum_{d\mid n}\left(\frac{-4}{d}\right)d - \sum_{d\mid n}\left(\frac{-4}{d}\right)d^2 + \frac{n}{2}r_2(n) - 2\sum_{d\mid n}\sum_{1\le k\le d}\left(\frac{-4}{k}\right)k.$$

Equating both sides we obtain the desired result.

Proof of Jacobi's formula (1.4) when n is odd. Let n be an odd positive integer. Then by (1.8) the left-hand side of Theorem 1.1 with the odd function $F(x) = -\frac{1}{4}\left(\frac{-4}{x}\right)$ is $\dfrac{r_6(n)-3r_4(n)+3r_2(n)}{64}$

The right-hand side of Theorem 1.1 with $F(x) = -\frac{1}{4}\left(\frac{-4}{x}\right)$ is

$$\sum_{d\mid n}\left(\frac{d^2-3d+2}{2}+3\left(\frac{n}{d}-1\right)\left(\frac{n}{d}-d\right)\right)\left(-\frac{1}{4}\right)\left(\frac{-4}{d}\right)$$

$$+3\sum_{d\mid n}\sum_{1\le k<d}\left(2d-\frac{2n}{d}-k\right)\left(-\frac{1}{4}\right)\left(\frac{-4}{k}\right)-6\sum_{\substack{n=n_1+n_2\\ n_1,n_2\in\mathbb{N}}}\sigma(n_1)\sum_{d\mid n_2}\left(-\frac{1}{4}\right)\left(\frac{-4}{d}\right).$$

The first sum is

$$-\frac{1}{4}\sum_{d\mid n}\left(\frac{d^2-3d+2}{2}+3\left(\frac{n}{d}-1\right)\left(\frac{n}{d}-d\right)\right)\left(\frac{-4}{d}\right)$$

$$=-\frac{1}{8}\sum_{d\mid n}\left(\frac{-4}{d}\right)d^2+\frac{3}{8}\sum_{d\mid n}\left(\frac{-4}{d}\right)d-\frac{1}{4}\sum_{d\mid n}\left(\frac{-4}{d}\right)-\frac{3}{4}\sum_{d\mid n}\left(\frac{-4}{d}\right)\left(\frac{n}{d}\right)^2$$

$$+\frac{3}{4}n\sum_{d\mid n}\left(\frac{-4}{d}\right)+\frac{3}{4}\sum_{d\mid n}\left(\frac{-4}{d}\right)\frac{n}{d}-\frac{3}{4}\sum_{d\mid n}\left(\frac{-4}{d}\right)d$$

$$=-\frac{1}{8}\sum_{d\mid n}\left(\frac{-4}{d}\right)d^2+\frac{3}{8}\sum_{d\mid n}\left(\frac{-4}{d}\right)d-\frac{1}{16}r_2(n)-\frac{3}{4}\sum_{d\mid n}\left(\frac{-4}{n/d}\right)d^2$$

$$+\frac{3}{16}nr_2(n)+\frac{3}{4}\sum_{d\mid n}\left(\frac{-4}{n/d}\right)d-\frac{3}{4}\sum_{d\mid n}\left(\frac{-4}{d}\right)d$$

$$=\begin{cases}-\frac{7}{8}\sum_{d\mid n}\left(\frac{-4}{d}\right)d^2-\frac{r_2(n)}{16}(1-3n)+\frac{3}{8}\sum_{d\mid n}\left(\frac{-4}{d}\right)d, & \text{if } n\equiv 1 \pmod 4,\\ \frac{5}{8}\sum_{d\mid n}\left(\frac{-4}{d}\right)d^2-\frac{9}{8}\sum_{d\mid n}\left(\frac{-4}{d}\right)d, & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

The second sum is

$$3\sum_{d\mid n}\sum_{1\le k<d}\left(2d-\frac{2n}{d}-k\right)\left(-\frac{1}{4}\right)\left(\frac{-4}{k}\right)$$

$$=\frac{3}{2}\sum_{d\mid n}\left(\frac{n}{d}-d\right)\sum_{1\le k<d}\left(\frac{-4}{k}\right)+\frac{3}{4}\sum_{d\mid n}\sum_{1\le k<d}k\left(\frac{-4}{k}\right).$$

As d is odd, $\sum_{1 \le k < d} \left(\frac{-4}{k}\right)$ is 0 if $d \equiv 1 \pmod 4$ and 1 if $d \equiv 3 \pmod 4$.

Therefore

$$\sum_{d \mid n} \left(\frac{n}{d} - d\right) \sum_{1 \le k < d} \left(\frac{-4}{k}\right) = \begin{cases} 0, & \text{if } n \equiv 1 \pmod 4, \\ \sum_{d \mid n} \left(\frac{-4}{d}\right) d, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Therefore the second sum is

$$\frac{3}{4} \sum_{d \mid n} \sum_{1 \le k < d} k \left(\frac{-4}{k}\right) + \begin{cases} 0, & \text{if } n \equiv 1 \pmod 4, \\ \sum_{d \mid n} \left(\frac{-4}{d}\right) d, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Finally the third sum is

$$\frac{3}{2} \sum_{\substack{n = 2^{s_1} n_1 + 2^{s_2} n_2 \\ n_1, n_2 \text{ odd}}} \sigma(2^{s_1} n_1) \sum_{d \mid n_2} \left(\frac{-4}{d}\right)$$

$$= \frac{3}{2} \sum_{\substack{n = 2^{s_1} n_1 + 2^{s_2} n_2 \\ n_1, n_2 \text{ odd}}} \sigma(2^{s_1}) \sigma(n_1) \frac{r_2(n_2)}{4}$$

$$= \frac{3}{8} \sum_{\substack{n = 2^{s_1} n_1 + 2^{s_2} n_2 \\ n_1, n_2 \text{ odd}}} r_2(n_2)(2^{s_1+1} - 1)\sigma(n_1)$$

$$= \frac{3}{4} \sum_{\substack{n = 2^{s_1} n_1 + 2^{s_2} n_2 \\ n_1, n_2 \text{ odd}}} 2^{s_1} r_2(n_2)\sigma(n_1) - \frac{3}{8} \sum_{\substack{n = 2^{s_1} n_1 + 2^{s_2} n_2 \\ n_1, n_2 \text{ odd}}} r_2(n_2)\sigma(n_1)$$

$$= \frac{3}{4} \sum_{\substack{n = 2^{s_1} n_1 + n_2 \\ n_1, n_2 \text{ odd}, s_1 \ge 1}} 2^{s_1} r_2(n_2)\sigma(n_1) + \frac{3}{4} \sum_{\substack{n = n_1 + 2^{s_2} n_2 \\ n_1, n_2 \text{ odd}, s_2 \ge 1}} r_2(n_2)\sigma(n_1)$$

$$- \frac{3}{8} \sum_{\substack{n = 2^{s_1} n_1 + n_2 \\ n_1, n_2 \text{ odd}, s_1 \ge 1}} r_2(n_2)\sigma(n_1) - \frac{3}{8} \sum_{\substack{n = n_1 + 2^{s_2} n_2 \\ n_1, n_2 \text{ odd}, s_2 \ge 1}} r_2(n_2)\sigma(n_1)$$

$$= \frac{3}{4} \sum_{\substack{n=2^{s_1} n_1+n_2 \\ n_1, n_2 \text{ odd}, s_1 \geq 1}} 2^{s_1} r_2(n_2)\sigma(n_1) + \frac{3}{8} \sum_{\substack{n=n_1+2^{s_2} n_2 \\ n_1, n_2 \text{ odd}, s_2 \geq 1}} r_2(n_2)\sigma(n_1)$$

$$- \frac{3}{8} \sum_{\substack{n=2^{s_1} n_1+n_2 \\ n_1, n_2 \text{ odd}, s_1 \geq 1}} r_2(n_2)\sigma(n_1).$$

This expression can be simplified further when $n \equiv 3 \pmod 4$ as

$$\sum_{\substack{n=2^{s_1} n_1+n_2 \\ n_1, n_2 \text{ odd}, s_1 \geq 1}} 2^{s_1} r_2(n_2)\sigma(n_1) = 2 \sum_{\substack{n=2n_1+n_2 \\ n_1, n_2 \text{ odd}}} r_2(n_2)\sigma(n_1)$$

$$= 2 \sum_{\substack{n=2^{s_1} n_1+n_2 \\ n_1, n_2 \text{ odd}, s_1 \geq 1}} r_2(n_2)\sigma(n_1).$$

Appealing to Lemmas 2.4, 2.5 and 2.6 we simplify the third sum for the more complicated case when $n \equiv 1 \pmod 4$. The case when $n \equiv 3 \pmod 4$ can be proven similarly by applying Lemmas 2.4 and 2.5. We have

$$\frac{3}{2} \sum_{\substack{n=2^{s_1} n_1+2^{s_2} n_2 \\ n_1, n_2 \text{ odd}}} \sigma(2^{s_1} n_1) \sum_{d \mid n_2} \left(\frac{-4}{d}\right)$$

$$= \frac{3}{4}\left(\sum_{d \mid n}\left(\frac{-4}{d}\right)d^2 - \frac{n}{4} r_2(n)\right) + \frac{3}{8}\left(\frac{r_6(n)}{12} - \frac{r_4(n)}{8}\right)$$

$$- \frac{3}{8} \cdot \frac{1}{6}\left(\sum_{d \mid n}\left(\frac{-4}{d}\right)d + \sum_{d \mid n}\left(\frac{-4}{d}\right)d^2 - 2\sum_{d \mid n} \sum_{k=1}^{d}\left(\frac{-4}{k}\right)k\right)$$

$$= \frac{1}{32} r_6(n) - \frac{3}{64} r_4(n) - \frac{3n}{16} r_2(n) - \frac{1}{16} \sum_{d \mid n}\left(\frac{-4}{d}\right)d + \frac{11}{16} \sum_{d \mid n}\left(\frac{-4}{d}\right)d^2$$

$$+ \frac{1}{8} \sum_{d \mid n} \sum_{k=1}^{d}\left(\frac{-4}{k}\right)k$$

$$= \frac{1}{32} r_6(n) - \frac{3}{64} r_4(n) - \frac{3n}{16} r_2(n) + \frac{1}{16} \sum_{d|n} \left(\frac{-4}{d}\right) d + \frac{11}{16} \sum_{d|n} \left(\frac{-4}{d}\right) d^2$$

$$+ \frac{1}{8} \sum_{d|n} \sum_{k=1}^{d-1} \left(\frac{-4}{k}\right) k.$$

Therefore when $n \equiv 1 \pmod 4$ the right-hand side is

$$\left(-\frac{7}{8} \sum_{d|n} \left(\frac{-4}{d}\right) d^2 - \frac{r_2(n)}{16} (1 - 3n) + \frac{3}{8} \sum_{d|n} \left(\frac{-4}{d}\right) d\right)$$

$$+ \left(\frac{3}{4} \sum_{d|n} \sum_{1 \le k < d} \left(\frac{-4}{k}\right) k\right)$$

$$+ \left(\frac{1}{32} r_6(n) - \frac{3}{64} r_4(n) - \frac{3n}{16} r_2(n)\right.$$

$$\left. + \frac{1}{16} \sum_{d|n} \left(\frac{-4}{d}\right) d + \frac{11}{16} \sum_{d|n} \left(\frac{-4}{d}\right) d^2 + \frac{1}{8} \sum_{d|n} \sum_{k=1}^{d-1} \left(\frac{-4}{k}\right) k\right)$$

$$= \frac{1}{32} r_6(n) - \frac{3}{64} r_4(n) - \frac{1}{16} r_2(n) + \frac{7}{16} \sum_{d|n} \left(\frac{-4}{d}\right) d - \frac{3}{16} \sum_{d|n} \left(\frac{-4}{d}\right) d^2$$

$$+ \frac{7}{8} \sum_{d|n} \sum_{1 \le k < d} \left(\frac{-4}{k}\right) k.$$

Equating both sides we obtain

$$r_6(n) - 3r_4(n) + 3r_2(n) = 2r_6(n) - 3r_4(n) - 4r_2(n) + 28 \sum_{d|n} \left(\frac{-4}{d}\right) d$$

$$- 12 \sum_{d|n} \left(\frac{-4}{d}\right) d^2 + 56 \sum_{d|n} \sum_{1 \le k < d} \left(\frac{-4}{k}\right) k,$$

so that

$$r_6(n) = 12 \sum_{d|n} \left(\frac{-4}{d}\right) d^2 + 7r_2(n) - 28 \sum_{d|n} \left(\frac{-4}{d}\right) d - 56 \sum_{d|n} \sum_{1 \le k < d} \left(\frac{-4}{k}\right) k$$

$$= 12\sum_{d \mid n}\left(\frac{-4}{d}\right)d^2 + 28\sum_{d \mid n}\left(\frac{-4}{d}\right) - 28\sum_{d \mid n}\left(\frac{-4}{d}\right)d - 56\sum_{d \mid n}\sum_{1 \le k < d}\left(\frac{-4}{k}\right)k$$

$$= 12\sum_{d \mid n}\left(\frac{-4}{d}\right)d^2 + 28\left(\sum_{d \mid n}\left(\frac{-4}{d}\right)(1-d) - 2\sum_{d \mid n}\sum_{1 \le k < d}\left(\frac{-4}{k}\right)k\right)$$

$$= 12\sum_{d \mid n}\left(\frac{-4}{d}\right)d^2,$$

by Lemma 2.3. This gives the desired result for $n \equiv 1 \pmod 4$. The result follows similarly for $n \equiv 3 \pmod 4$.

This completes our arithmetic proof of Jacobi's formula (1.4) when n is odd.

Proof of Jacobi's formula (1.4) when n is even. Let $n \in \mathbb{N}$ be even. We use standard arguments to relate $r_6(n)$ for n even to $r_6(n)$ for n odd, so that we can apply (1.4) with n odd. We write N for the odd part of n. For $a \in \{0, 1, 2, 3, 4, 5, 6\}$, we define $r_6^{a, 6-a}(n)$ to be the number of representations $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}^6$ with $n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$ and a of $x_1, x_2, x_3, x_4, x_5, x_6$ even and $6-a$ of $x_1, x_2, x_3, x_4, x_5, x_6$ odd. For $n \in \mathbb{N}$ it is easy to see that

$$r_6^{6,0}(n) = 0, \text{ if } n \not\equiv 0 \pmod 4,$$

$$r_6^{5,1}(n) = 0, \text{ if } n \not\equiv 1 \pmod 4,$$

$$r_6^{4,2}(n) = 0, \text{ if } n \not\equiv 2 \pmod 4,$$

$$r_6^{3,3}(n) = 0, \text{ if } n \not\equiv 3 \pmod 4,$$

$$r_6^{2,4}(n) = 0, \text{ if } n \not\equiv 0 \pmod 4,$$

$$r_6^{1,5}(n) = 0, \text{ if } n \not\equiv 1 \pmod 4,$$

$$r_6^{0,6}(n) = 0, \text{ if } n \not\equiv 6 \pmod 8. \tag{2.1}$$

From (2.1) and the relation $r_6(n) = \sum_{a=0}^{6} r_6^{a,6-a}(n)$, we obtain

$$r_6(n) = \begin{cases} r_6^{2,4}(n) + r_6^{6,0}(n), & \text{if } n \equiv 0 \pmod 4, \\ r_6^{1,5}(n) + r_6^{5,1}(n), & \text{if } n \equiv 1 \pmod 4, \\ r_6^{0,6}(n) + r_6^{4,2}(n), & \text{if } n \equiv 2 \pmod 4, \\ r_6^{3,3}(n), & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{2.2}$$

Lemma 2.7. *Let* $n \in \mathbb{N}$ *satisfy* $n \equiv 6 \pmod 8$. *Then*

$$r_6^{0,6}(n) = \frac{2}{5} r_6\left(\frac{n}{2}\right).$$

Proof. Suppose $n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$, where x_1, x_2, x_3, x_4, x_5 and x_6 are odd. For $i, j \in \{1, 2, 3, 4, 5, 6\}$, we observe that either $x_i \equiv x_j \pmod 4$ or $x_i \equiv -x_j \pmod 4$. We define A to be the set of solutions $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{N}^6$ to $n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$ with $x_1, x_2, x_3, x_4, x_5, x_6$ odd and $x_1 \equiv x_2 \pmod 4$, $x_3 \equiv x_4 \pmod 4$ and $x_5 \equiv x_6 \pmod 4$. Then, writing $|A|$ for card A, we have

$$r_6^{0,6}(n) = 8|A|.$$

Next we define B to be the set of solutions $(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{N}^6$ to $n/2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2$ with y_1, y_3, y_5 odd, and y_2, y_4, y_6 even. Then

$$r_6^{3,3}(n/2) = \binom{6}{3}|B| = 20|B|.$$

It is easily checked that $\Phi : A \to B$ defined by

$$\Phi(x_1, x_2, x_3, x_4, x_5, x_6)$$
$$= \left(\frac{x_1 + x_2}{2}, \frac{x_1 - x_2}{2}, \frac{x_3 + x_4}{2}, \frac{x_3 - x_4}{2}, \frac{x_5 + x_6}{2}, \frac{x_5 - x_6}{2}\right)$$

is a bijection. Therefore by (2.2) we have

$x_5 \not\equiv x_6 \pmod 2$, we replace x_5 by $-x_5 - 1$. Then $-x_5 - 1 \equiv x_6 \pmod 2$, d the sum is unaffected since $(2(-x_5 - 1) + 1)^2 = (2x_5 + 1)^2$. Next we ine

$$B = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in A \mid x_5 \equiv x_6 \pmod 2\},$$

that $|A| = 2|B|$. In order to calculate the cardinality of B, we define

$$B_1 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \equiv x_2 \pmod 2,\ x_3 \equiv x_4 \pmod 2\},$$

$$B_2 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \not\equiv x_2 \pmod 2,\ x_3 \equiv x_4 \pmod 2\},$$

$$B_3 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \equiv x_2 \pmod 2,\ x_3 \not\equiv x_4 \pmod 2\},$$

$$B_4 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \not\equiv x_2 \pmod 2,\ x_3 \not\equiv x_4 \pmod 2\}.$$

en clearly $|B| = |B_1| + |B_2| + |B_3| + |B_4|$. Let C be the set of solutions $, y_2, y_3, y_4, y_5, y_6) \in \mathbb{N}^6$ such that $n/2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2$, d define

$$C_1 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_1, y_2, y_3, y_4, y_6 \text{ even}, y_5 \text{ odd}\},$$

$$C_2 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_3, y_4, y_6 \text{ even}, y_1, y_2, y_5 \text{ odd}\},$$

$$C_3 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_1, y_2, y_6 \text{ even}, y_3, y_4, y_5 \text{ odd}\},$$

$$C_4 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_6 \text{ even}, y_1, y_2, y_3, y_4, y_5 \text{ odd}\}.$$

en

$$|C_1| = \frac{r_6^{5,1}\left(\frac{n}{2}\right)}{6},\ |C_2| = \frac{r_6^{3,3}\left(\frac{n}{2}\right)}{20},\ |C_3| = \frac{r_6^{3,3}\left(\frac{n}{2}\right)}{20},\ |C_4| = \frac{r_6^{1,5}\left(\frac{n}{2}\right)}{6}$$

is easily checked that $\Phi_i : B_i \to C_i$ defined by

$$\Phi_i(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$= (x_1 + x_2, x_1 - x_2, x_3 + x_4, x_3 - x_4, x_5 + x_6 + 1, x_5 - x_6)$$

a bijection. It follows that

$$r_6^{0,6}(n) = 8|A| = 8|B| = \frac{8}{20} r_6^{3,3}\left(\frac{n}{2}\right) = \frac{2}{5} r_6\left(\frac{n}{2}\right),$$

as required.

Lemma 2.8. *Let* $n \in \mathbb{N}$ *satisfy* $n \equiv 2 \pmod 4$. *Then*

$$r_6^{0,6}(n) = \left(4\left(\frac{-4}{N}\right) - 4\right) \sum_{d \mid N} \left(\frac{-4}{d}\right) d^2.$$

Proof. If $n \equiv 2 \pmod 8$, then we have $N \equiv 1 \pmod 4$, so that $\left(\frac{-4}{N}\right) = 1$ and by (2.1) we see that

$$r_6^{0,6}(n) = 0 = \left(4\left(\frac{-4}{N}\right) - 4\right) \sum_{d \mid N} \left(\frac{-4}{d}\right) d^2,$$

as required. When $n \equiv 6 \pmod 8$, we have $N = \frac{n}{2} \equiv 3 \pmod 4$, so that $\left(\frac{-4}{N}\right) = -1$ and

$$\left(4\left(\frac{-4}{N}\right) - 4\right) \sum_{d \mid N} \left(\frac{-4}{d}\right) d^2 = -8 \sum_{d \mid N} \left(\frac{-4}{d}\right) d^2 = \frac{2}{5} r_6\left(\frac{n}{2}\right) = r_6^{0,6}(n),$$

by (1.4) (for $n/2 \equiv 1 \pmod 2$) and Lemma 2.7.

Lemma 2.9. *Let* $n \in \mathbb{N}$ *satisfy* $n \equiv 2 \pmod 4$. *Then*

$$r_6^{4,2}(n) = \left(4 + (-1)^{\frac{n-2}{4}}\right) r_6\left(\frac{n}{2}\right).$$

Proof. Let A be the set of solutions $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{N}^6$ to

$$n = (2x_1)^2 + (2x_2)^2 + (2x_3)^2 + (2x_4)^2 + (2x_5 + 1)^2 + (2x_6 + 1)^2.$$

Then

$$r_6^{4,2}(n) = \binom{6}{2} |A| = 15|A|.$$

If $x_5 \not\equiv x_6 \pmod 2$, we replace x_5 by $-x_5 - 1$. Then $-x_5 - 1 \equiv x_6 \pmod 2$, and the sum is unaffected since $(2(-x_5 - 1) + 1)^2 = (2x_5 + 1)^2$. Next we define

$$B = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in A \mid x_5 \equiv x_6 \pmod 2\},$$

so that $|A| = 2|B|$. In order to calculate the cardinality of B, we define

$$B_1 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \equiv x_2 \pmod 2,\ x_3 \equiv x_4 \pmod 2\},$$

$$B_2 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \not\equiv x_2 \pmod 2,\ x_3 \equiv x_4 \pmod 2\},$$

$$B_3 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \equiv x_2 \pmod 2,\ x_3 \not\equiv x_4 \pmod 2\},$$

$$B_4 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \not\equiv x_2 \pmod 2,\ x_3 \not\equiv x_4 \pmod 2\}.$$

Then clearly $|B| = |B_1| + |B_2| + |B_3| + |B_4|$. Let C be the set of solutions $(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{N}^6$ such that $n/2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2$, and define

$$C_1 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_1, y_2, y_3, y_4, y_6 \text{ even}, y_5 \text{ odd}\},$$

$$C_2 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_3, y_4, y_6 \text{ even}, y_1, y_2, y_5 \text{ odd}\},$$

$$C_3 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_1, y_2, y_6 \text{ even}, y_3, y_4, y_5 \text{ odd}\},$$

$$C_4 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_6 \text{ even}, y_1, y_2, y_3, y_4, y_5 \text{ odd}\}.$$

Then

$$|C_1| = \frac{r_6^{5,1}\left(\frac{n}{2}\right)}{6}, \ |C_2| = \frac{r_6^{3,3}\left(\frac{n}{2}\right)}{20}, \ |C_3| = \frac{r_6^{3,3}\left(\frac{n}{2}\right)}{20}, \ |C_4| = \frac{r_6^{1,5}\left(\frac{n}{2}\right)}{6}.$$

It is easily checked that $\Phi_i : B_i \to C_i$ defined by

$$\Phi_i(x_1, x_2, x_3, x_4, x_5, x_6)$$
$$= (x_1 + x_2, x_1 - x_2, x_3 + x_4, x_3 - x_4, x_5 + x_6 + 1, x_5 - x_6)$$

is a bijection. It follows that

$$r_6^{4,2}(n) = 15|A| = 30|B| = 30(|B_1| + |B_2| + |B_3| + |B_4|)$$

$$= 30(|C_1| + |C_2| + |C_3| + |C_4|)$$

$$= 30\left(\frac{r_6^{5,1}\left(\frac{n}{2}\right)}{6} + \frac{r_6^{3,3}\left(\frac{n}{2}\right)}{20} + \frac{r_6^{3,3}\left(\frac{n}{2}\right)}{20} + \frac{r_6^{1,5}\left(\frac{n}{2}\right)}{6}\right)$$

$$= 5r_6^{5,1}\left(\frac{n}{2}\right) + 3r_6^{3,3}\left(\frac{n}{2}\right) + 5r_6^{1,5}\left(\frac{n}{2}\right)$$

$$= \begin{cases} 5\left(r_6^{5,1}\left(\frac{n}{2}\right) + r_6^{1,5}\left(\frac{n}{2}\right)\right), & \text{if } \frac{n}{2} \equiv 1 \pmod 4, \\ 3r_6^{3,3}\left(\frac{n}{2}\right), & \text{if } \frac{n}{2} \equiv 3 \pmod 4, \end{cases}$$

$$= (4 + (-1)^{\frac{n-2}{4}})r_6\left(\frac{n}{2}\right),$$

by (2.1) and (2.2), as required.

Lemma 2.10. *Let* $n \in \mathbb{N}$ *satisfy* $n \equiv 2 \pmod 4$. *Then*

$$r_6^{4,2}(n) = 60\left(\frac{-4}{N}\right)\sum_{d \mid N}\left(\frac{-4}{d}\right)d^2.$$

Proof. Appealing to Lemma 2.9 and (1.4) (for $N \equiv 1 \pmod 2$), we obtain

$$r_6^{4,2}(n) = (4 + (-1)^{(N-1)/2})r_6(N) = 60\left(\frac{-4}{N}\right)\sum_{d \mid N}\left(\frac{-4}{d}\right)d^2,$$

as claimed.

Lemma 2.11. *Let* $n \in \mathbb{N}$ *be even. Set* $n/N = 2^k$ *with* $k \in \mathbb{N}$. *Suppose* $m = 2^l N$ *for a nonnegative integer* $l \le k$. *Then*

$$\sum_{d \mid m}\left(\frac{-4}{m/d}\right)d^2 = 2^{2(l-k)}\sum_{d \mid n}\left(\frac{-4}{n/d}\right)d^2.$$

Proof. This follows by changing the summation variable in the sum on the left-hand sum from d to $c = 2^{k-l}d$.

We are now ready to prove the formula for $r_6(n)$ when $n \equiv 2 \pmod 4$. Appealing to (2.2), Lemmas 2.8, 2.10 and 2.11, we obtain

$$\begin{aligned} r_6(n) &= r_6^{0,6}(n) + r_6^{4,2}(n) \\ &= \left(64\left(\frac{-4}{N}\right) - 4\right)\sum_{d \mid N}\left(\frac{-4}{d}\right)d^2 \\ &= 64\sum_{d \mid N}\left(\frac{-4}{N/d}\right)d^2 - 4\sum_{d \mid n}\left(\frac{-4}{d}\right)d^2 \\ &= 16\sum_{d \mid n}\left(\frac{-4}{n/d}\right)d^2 - 4\sum_{d \mid n}\left(\frac{-4}{d}\right)d^2, \end{aligned}$$

which is (1.4) when $n \equiv 2 \pmod 4$.

The following two lemmas address the case when $n \equiv 0 \pmod 4$.

Lemma 2.12. *Let $n \in \mathbb{N}$ satisfy $n \equiv 0 \pmod 4$. Then*

$$r_6^{6,0}(n) = r_6\left(\frac{n}{4}\right).$$

Proof. Clearly

$$n = (2x_1)^2 + \cdots + (2x_6)^2 \Leftrightarrow \frac{n}{4} = x_1^2 + \cdots + x_6^2$$

and the result follows.

Lemma 2.13. *Let $n \in \mathbb{N}$ satisfy $n \equiv 0 \pmod 4$. Let $n = 2^k N$, where $k \geq 2$. Then*

$$r_6^{2,4}(n) = 15 \cdot 2^{2k}\left(\frac{-4}{N}\right)\sum_{d \mid N}\left(\frac{-4}{d}\right)d^2.$$

Proof. Let A be the set of solutions $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{N}^6$ to

$$n = (2x_1)^2 + (2x_2)^2 + (2x_3 + 1)^2 + (2x_4 + 1)^2 + (2x_5 + 1)^2 + (2x_6 + 1)^2,$$

so that

$$r_6^{2,4}(n) = \binom{6}{2}|A| = 15|A|.$$

Similarly to the proof of Lemma 2.9, we can choose x_3, x_4, x_5, x_6 so that $x_3 \equiv x_4 \pmod 2$ and $x_5 \equiv x_6 \pmod 2$. We define

$$B = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in A \mid x_3 \equiv x_4 \pmod 2,\ x_5 \equiv x_6 \pmod 2\},$$

so that $|A| = 4|B|$. In order to calculate the cardinality of B, we define

$$B_1 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \equiv x_2 \pmod 2\},$$

$$B_2 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \not\equiv x_2 \pmod 2\}.$$

Then clearly $|B| = |B_1| + |B_2|$. Let C be the set of solutions $(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{N}^6$ to $n/2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2$, and define

$$C_1 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_1, y_2, y_4, y_6 \text{ even}, y_3, y_5 \text{ odd}\},$$

$$C_2 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_4, y_6 \text{ even}, y_1, y_2, y_3, y_5 \text{ odd}\}.$$

Then clearly

$$|C_1| = \frac{r_6^{4,2}\left(\frac{n}{2}\right)}{15} \quad \text{and} \quad |C_2| = \frac{r_6^{2,4}\left(\frac{n}{2}\right)}{15}.$$

It is easily checked that $\Phi_i : B_i \to C_i$ $(i = 1, 2)$ defined by

$$\Phi_i(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$= (x_1 + x_2, x_1 - x_2, x_3 + x_4 + 1, x_3 - x_4, x_5 + x_6 + 1, x_5 - x_6),$$

is a bijection. It follows that

$$r_6^{2,4}(n) = 15|A| = 60|B| = 60(|B_1| + |B_2|) = 60(|C_1| + |C_2|)$$

$$= 60\left(\frac{r_6^{4,2}\left(\frac{n}{2}\right)}{15} + \frac{r_6^{2,4}\left(\frac{n}{2}\right)}{15}\right) = 4\begin{cases} r_6^{2,4}\left(\frac{n}{2}\right), & \text{if } n \equiv 0 \pmod 8, \\ r_6^{4,2}\left(\frac{n}{2}\right), & \text{if } n \equiv 4 \pmod 8, \end{cases}$$

by (2.1). Finally, making use of Lemma 2.10, we obtain

$$r_6^{2,4}(n) = r_6^{2,4}(2^k N) = 2^2 r_6^{2,4}(2^{k-1} N) = \cdots = 2^{2(k-2)} r_6^{2,4}(4N)$$

$$= 2^{2(k-1)} r_6^{4,2}(2N) = 15 \cdot 2^{2k} \left(\frac{-4}{N}\right) \sum_{d \mid N} \left(\frac{-4}{d}\right) d^2,$$

as required.

We can now prove formula (1.4) for $n \equiv 4 \pmod 8$. In this case we have $n = 4N$. Then

$$r_6(n) = r_6^{2,4}(4N) + r_6^{6,0}(4N) \quad \text{(by (2.2))}$$

$$= 15 \cdot 2^4 \sum_{d \mid N} \left(\frac{-4}{N/d}\right) d^2 + r_6(N) \quad \text{(by Lemmas 2.12 and 2.13)}$$

$$= 15 \sum_{d \mid n} \left(\frac{-4}{n/d}\right) d^2 + r_6(N) \quad \text{(by Lemma 2.11)}$$

$$= 15 \sum_{d \mid n} \left(\frac{-4}{n/d}\right) d^2 + 16 \sum_{d \mid N} \left(\frac{-4}{N/d}\right) d^2 - 4 \sum_{d \mid N} \left(\frac{-4}{d}\right) d^2$$

$$= 16 \sum_{d \mid n} \left(\frac{-4}{n/d}\right) d^2 - 4 \sum_{d \mid n} \left(\frac{-4}{d}\right) d^2 \quad \text{(by Lemma 2.11)}$$

which is (1.4) for $n \equiv 4 \pmod 8$.

Finally we prove (1.4) for $n \equiv 0 \pmod 8$. In this case we have $n = 2^k N$ with $k \geq 3$. We have

$$r_6(n) = r_6^{2,4}(2^k N) + r_6^{6,0}(2^k N) \quad \text{(by (2.2))}$$

$$= r_6^{2,4}(2^k N) + r_6(2^{k-2} N) \quad \text{(by Lemma 2.12)}$$

$$= r_6^{2,4}(2^k N) + r_6^{2,4}(2^{k-2} N) + r_6^{6,0}(2^{k-2} N) \quad \text{(by (2.2))}$$

$$= r_6^{2,4}(2^k N) + r_6^{2,4}(2^{k-2} N) + r_6(2^{k-4} N) \quad \text{(by Lemma 2.12)}$$

$$= \begin{cases} r_6^{2,4}(2^k N) + r_6^{2,4}(2^{k-2} N) + \cdots + r_6^{2,4}(4N) + r_6^{6,0}(4N), & \text{if } k \equiv 0 \pmod 2, \\ r_6^{2,4}(2^k N) + r_6^{2,4}(2^{k-2} N) + \cdots + r_6^{2,4}(8N) + r_6^{6,0}(8N), & \text{if } k \equiv 1 \pmod 2, \end{cases}$$

$$= \begin{cases} r_6^{2,4}(2^k N) + r_6^{2,4}(2^{k-2}N) + \cdots + r_6^{2,4}(4N) + r_6(N), & \text{if } k \equiv 0 \pmod 2, \\ r_6^{2,4}(2^k N) + r_6^{2,4}(2^{k-2}N) + \cdots + r_6^{2,4}(8N) + r_6(2N), & \text{if } k \equiv 1 \pmod 2. \end{cases}$$

Appealing to Lemmas 2.13 and 2.11, we obtain

$$r_6^{2,4}(2^l N) = 15 \cdot 2^{2l} \sum_{d \mid N} \left(\frac{-4}{N/d} \right) d^2 = 15 \cdot 2^{2(l-k)} \sum_{d \mid n} \left(\frac{-4}{n/d} \right) d^2,$$

for $l \in \mathbb{N}$, $1 \le l \le k$. Therefore

$$r_6(n) = \begin{cases} 15 \sum_{d \mid n} \left(\frac{-4}{n/d} \right) d^2 \sum_{i=0}^{(k-2)/2} 2^{-4i} + r_6(N), & \text{if } k \equiv 0 \pmod 2, \\ 15 \sum_{d \mid n} \left(\frac{-4}{n/d} \right) d^2 \sum_{i=0}^{(k-3)/2} 2^{-4i} + r_6(2N), & \text{if } k \equiv 1 \pmod 2, \end{cases}$$

$$= \begin{cases} 16 \sum_{d \mid n} \left(\frac{-4}{n/d} \right) d^2 (1 - 2^{-2k}) + r_6(N), & \text{if } k \equiv 0 \pmod 2, \\ 16 \sum_{d \mid n} \left(\frac{-4}{n/d} \right) d^2 (1 - 2^{-2k+2}) + r_6(2N), & \text{if } k \equiv 1 \pmod 2. \end{cases}$$

By (1.4) for $N \equiv 1 \pmod 2$ and Lemma 2.11, we have

$$r_6(N) = 16 \sum_{d \mid N} \left(\frac{-4}{N/d} \right) d^2 - 4 \sum_{d \mid N} \left(\frac{-4}{d} \right) d^2$$

$$= 16 \cdot 2^{-2k} \sum_{d \mid n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d \mid n} \left(\frac{-4}{d} \right) d^2,$$

and by (1.4) for $2N \equiv 2 \pmod 4$ and Lemma 2.11, we have

$$r_6(2N) = 16 \sum_{d \mid 2N} \left(\frac{-4}{2N/d} \right) d^2 - 4 \sum_{d \mid 2N} \left(\frac{-4}{d} \right) d^2$$

$$= 16 \cdot 2^{-2k+2} \sum_{d \mid n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d \mid n} \left(\frac{-4}{d} \right) d^2,$$

giving (1.4) for $n \equiv 0 \pmod 8$.

This completes the proof of Jacobi's formula (1.4) for all $n \in \mathbb{N}$.

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School of Mathematics and Statistics
Carleton University
Ottawa, Ontario, Canada K1S 5B6
e-mail: williams@math.carleton.ca