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TWO TERM CLASS NUMBER FORMULAE OF DIRICHLET TYPE

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Abstract

Let D be the discriminant of a quadratic field. Let q_1 and q_2 be rational numbers such that $0 \le q_1 < q_2 \le 1$. A sum of the form $\sum_{q_1|D|<n<q_2|D|} \left(\frac{D}{n}\right)$, where n runs through the positive integers in the specified range and $\left(\frac{D}{n}\right)$ is the Kronecker symbol, is called a short character sum. A generalization of a theorem of Johnson and Mitchell is proved and used to find new formulae expressing the class number $h(-e|D|)$ as a linear combination of two short character sums for certain positive integers e and all discriminants D in certain arithmetic progressions.

1. Introduction

Let K be a quadratic field. Let D be the discriminant of K so that $K = \mathbb{Q}(\sqrt{D})$. A quadratic field is uniquely determined by its discriminant. An integer which is the discriminant of a quadratic field is called a *fundamental discriminant*. We denote the class number of K by $h(D)$.

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In 1840 Dirichlet [2, 3] proved several class number formulae, such as

$$h(-p) = \sum_{0<n<p/2} \left(\frac{n}{p}\right), \quad p \equiv 7 \pmod 8, \tag{1}$$

where p is a prime. We note that $\left(\frac{n}{p}\right) = \left(\frac{-p}{n}\right)$ by the law of quadratic reciprocity, where $-p$ is the discriminant of $\mathbb{Q}(\sqrt{-p})$. In 1982 Hudson and Williams [13] defined a class number formula of Dirichlet type to be a formula of the form

$$h(-ep) = c \sum_{(k-1)p/le<n<kp/le} \left(\frac{n}{p}\right), \quad p \equiv r \pmod m, \tag{2}$$

where

(1) p is a prime greater than 3,

(2) e is a positive integer such that $-ep$ is the discriminant of an imaginary quadratic field, that is, a negative fundamental discriminant,

(3) l is a positive integer,

(4) k is a positive integer satisfying $1 \le k \le \frac{1}{2}(le+1)$,

(5) $m = 4\prod_{q \mid le} q$, where q runs through the distinct primes dividing le,

(6) r is an integer satisfying $1 \le r \le m$, $(r, 2le) = 1$,

(7) c is a rational number.

Hudson and Williams [13] undertook a computer search to find all such formulae with e and l in the ranges $1 \le e \le 28$ and $1 \le le \le 30$. Within these limits 127 formulae were found. Of the 127 formulae, 84 were found in the literature. For example

$$h(-p) = \sum_{0<n<p/6} \left(\frac{n}{p}\right), \quad p \equiv 7 \pmod{24}, \tag{3}$$

$$h(-p) = \frac{1}{2} \sum_{3p/10<n<p/3} \left(\frac{n}{p}\right), \quad p \equiv 43 \pmod{120}, \tag{4}$$

$$h(-5p) = 2 \sum_{p/5<n<2p/5} \left(\frac{n}{p}\right), \quad p \equiv 19 \pmod{20}, \tag{5}$$

$$h(-8p) = -4 \sum_{3p/8<n<p/2} \left(\frac{n}{p}\right), \quad p \equiv 7 \pmod{8}, \tag{6}$$

$$h(-15p) = -4 \sum_{11p/30<n<2p/5} \left(\frac{n}{p}\right), \quad p \equiv 13 \pmod{120}, \tag{7}$$

can be found in Holden [7], Johnson and Mitchell [14], Lerch [16], Dirichlet [3] and Karpinski [15] respectively. Others can be found in [1], [2], [4], [6], [8]-[12], [17]. The remaining 43 formulae were new. Examples of these are

$$h(-p) = \sum_{5p/14<n<3p/7} \left(\frac{n}{p}\right), \quad p \equiv 43 \pmod{56}, \tag{8}$$

$$h(-3p) = -2 \sum_{p/3<n<p/2} \left(\frac{n}{p}\right), \quad p \equiv 17 \pmod{24}, \tag{9}$$

$$h(-5p) = 2 \sum_{p/4<n<3p/10} \left(\frac{n}{p}\right), \quad p \equiv 39 \pmod{40}. \tag{10}$$

In 1999 Schinzel,Urbanowicz and Van Wamelen [19] extended the idea of a class number formula from that defined by Hudson and Williams to

$$h(-e|D|) = c \sum_{q_1|D|<n<q_2|D|} \left(\frac{D}{n}\right), \quad |D| \equiv s_0 \pmod{m}, \tag{11}$$

where

(1) q_1 and q_2 are rational numbers with $0 \le q_1 < q_2$, $q_1 + q_2 \le 1$, and $q_2 \le \frac{1}{2}$ if $q_1 = 0$,

(2) r is the least common denominator of q_1 and q_2,

(3) $D(\neq 1)$ is a fundamental discriminant with $(D, r) = 1$,

(4) e is a positive integer such that $(e, D) = 1$, $-e|D|$ is a fundamental discriminant and $e|D| > 4$,

(5) $D < 0$ is excluded if $q_2 > \frac{1}{2}$.

Using an identity due to Zagier, the authors derived 222 formulae in the ranges $1 \le e \le 30$ and $1 \le r \le 60$. These formulae are listed in [19] and [20]. They include the 127 formulae found by Hudson and Williams.

We call the class number formulae of the forms (2) and (11) *single term class number formulae of Dirichlet type*. The sums appearing in (2) and (11) are called *short character sums*.

2. Two Term Class Number Formulae of Dirichlet Type

In this paper we determine class number formulae involving two short character sums, namely, formulae of the type

$$h(-e|D|) = c_1 \sum_{q_1|D|<n<q_2|D|} \left(\frac{D}{n}\right) + c_2 \sum_{q_3|D|<n<q_4|D|} \left(\frac{D}{n}\right),$$

$$|D| \equiv s_0 \pmod{m}, \qquad (12)$$

where

(1) e is a positive integer ,

(2) D is an odd fundamental discriminant with $(D, e) = 1$ such that $-e|D| < -4$ is a fundamental discriminant,

(3) $c_1, c_2 \in \mathbb{Q}$,

(4) $q_1, q_2, q_3, q_4 \in \mathbb{Q}$ with least common denominator r such that $e \mid r$,

(5) $0 \le q_1 < q_2 \le q_3 < q_4 \le \frac{1}{2}$,

(6) $q_2 = q_3$ is allowed only if $c_1 \ne c_2$,

(7) $m = 4\prod_{p \mid r} p$, where p runs through the distinct primes dividing r,

(8) $s_0 \in \{1, 2, ..., m-1\}$ and $(s_0, m) = 1$,

(9) neither of the short character sums is itself a rational multiple of the class number, that is, there are no class number formulae of the forms

$$h(-e|D|) = C_1 \sum_{q_1|D|<n<q_2|D|} \left(\frac{D}{n}\right), \quad |D| \equiv s_0 \pmod{m}, \tag{13}$$

or

$$h(-e|D|) = C_2 \sum_{q_3|D|<n<q_4|D|} \left(\frac{D}{n}\right), \quad |D| \equiv s_0 \pmod{m}. \tag{14}$$

We call class number formulae of the form (12) *two term class number formulae of Dirichlet type.*

A few formulae of the type (12) have appeared in the literature. For example Dirichlet [3] has shown that if d is a positive fundamental discriminant with $d \equiv 1 \pmod 4$, then

$$h(-8d) = 2 \sum_{0<n<\frac{d}{8}} \left(\frac{d}{n}\right) - 2 \sum_{\frac{3}{8}d<n<\frac{d}{2}} \left(\frac{d}{n}\right). \tag{15}$$

In order to determine possible class number formulae of the type (12), a computer search was carried out within certain limits. For each $e \in \{1, 2, \ldots, 30\}$ and each $(q_1, q_2, q_3, q_4) \in \mathbb{Q}^4$ satisfying $e|r$, $r \le 30$ and $0 \le q_1 < q_2 \le q_3 < q_4 \le \frac{1}{2}$, where r is the least common denominator of q_1, q_2, q_3, q_4, a sequence of 10 fundamental discriminants D was found for each $s_0 \in \{1, 2, \ldots, m-1\}$ with $(s_0, m) = 1$ satisfying $(D, e) = 1$, $e|D| > 4$ and $|D| \equiv s_0 \pmod{m}$, where $m = 4\prod_{p|r} p$. For each such D,

$$\alpha = h(-e|D|), \quad \beta = \sum_{q_1|D|<n<q_2|D|} \left(\frac{D}{n}\right), \quad \gamma = \sum_{q_3|D|<n<q_4|D|} \left(\frac{D}{n}\right)$$

were calculated. If the ratio $\alpha : \beta : \gamma$ was the same for all the D in the sequence, the computer checked to see if there were one term formulae for each of the sums. If not, the system of equations given by $\alpha = c_1\beta + c_2\gamma$ for the D in the sequence was checked for a unique solution (c_1, c_2). If there was a unique solution to the system of equations, then

$$h(-e|D|) = c_1 \sum_{q_1|D|<n<q_2|D|} \left(\frac{d}{n}\right) + c_2 \sum_{q_3|D|<n<q_4|D|} \left(\frac{d}{n}\right)$$

became a conjectured formula for $|D| \equiv s_0 \pmod{m}$. When the programs were run over 8316 conjectured formulae were found. These formulae can be found in [5] and on the website http://math.carleton.ca, and include the following three formulae

$$h(-|D|) = \frac{2}{3} \sum_{\frac{1}{10}|D|<n<\frac{1}{5}|D|} \left(\frac{D}{n}\right) + \frac{1}{3} \sum_{\frac{1}{5}|D|<n<\frac{2}{5}|D|} \left(\frac{D}{n}\right),$$

$$|D| \equiv 3 \pmod{40}, \qquad (16)$$

$$h(-3|D|) = 2 \sum_{\frac{1}{12}|D|<n<\frac{1}{3}|D|} \left(\frac{D}{n}\right) - 2 \sum_{\frac{5}{12}|D|<n<\frac{1}{2}|D|} \left(\frac{D}{n}\right),$$

$$|D| \equiv 13 \pmod{24}, \qquad (17)$$

$$h(-5|D|) = -2 \sum_{0<n<\frac{1}{5}|D|} \left(\frac{D}{n}\right) + 2 \sum_{\frac{3}{10}|D|<n<\frac{2}{5}|D|} \left(\frac{D}{n}\right),$$

$$|D| \equiv 3 \pmod{40}. \qquad (18)$$

How the conjectured formulae were proved is described in the next section.

3. Extension of a Theorem of Johnson and Mitchell

Hudson and Williams [13] used symmetries of the Legendre symbol $\left(\frac{n}{p}\right)$ as well as a theorem of Johnson and Mitchell [14] relating short character sums to prove their 127 single term class number formulae of Dirichlet type. Our conjectured two term class number formulae involve the Kronecker symbol $\left(\frac{D}{n}\right)$ rather than the Legendre symbol $\left(\frac{n}{p}\right)$ so in order to prove them we must extend Johnson and Mitchell's result to short character sums involving the Kronecker symbol. We prove the following theorem [5] using the ideas of [14].

Theorem 3.1. *Let D be a nonsquare integer $\equiv 0$ or $1 \pmod 4$. Let n and q be positive integers such that $(qn, D) = 1$. Set*

$$S(r, n, D) = \sum_{\frac{(r-1)|D|}{n} \le a < \frac{r|D|}{n}} \left(\frac{D}{a}\right), \quad r = 1, 2, \dots, n. \tag{19}$$

Then

$$\left(\frac{D}{q}\right) S(r, n, D) = \sum_{j=0}^{[(q-1)/2]} S(jn + r, nq, D) + \operatorname{sgn}(D) \sum_{j=1}^{[q/2]} S(jn - r + 1, nq, D). \tag{20}$$

Proof. The set

$$\{j \in \mathbb{Z} \mid -[(q-1)/2] \le j \le [q/2]\}$$

is a complete residue system modulo q. As $(q, D) = 1$ the set

$$\{j|D| \mid j \in \mathbb{Z}, -[(q-1)/2] \le j \le [q/2]\}$$

is also a complete residue system modulo q. Hence

$$S(r, n, D) = \sum_{\frac{(r-1)|D|}{n} \le a < \frac{r|D|}{n}} \left(\frac{D}{a}\right)$$

$$= \sum_{j=-[(q-1)/2]}^{[q/2]} \sum_{\substack{\frac{(r-1)|D|}{n} \le a < \frac{r|D|}{n} \\ a \equiv j|D| \pmod q}} \left(\frac{D}{a}\right).$$

As $(n, |D|) = 1$ the rational number $\frac{(r-1)|D|}{n}$ is an integer if and only if $n \mid r-1$, that is, if and only if $r = 1$, as $0 \le r - 1 < n$. When $r = 1$ the term $a = \frac{(r-1)|D|}{n} = 0$ contributes $\left(\frac{D}{a}\right) = \left(\frac{D}{0}\right) = 0$ to the inner sum. Hence

$$S(r, n, D) = \sum_{j=-[(q-1)/2]}^{[q/2]} \sum_{\substack{\frac{(r-1)|D|}{n} < a < \frac{r|D|}{n} \\ a \equiv j|D| \pmod q}} \left(\frac{D}{a}\right).$$

Set

$$S_1 = \sum_{j=1}^{[q/2]} \sum_{\substack{\frac{(r-1)|D|}{n} < a < \frac{r|D|}{n} \\ a \equiv j|D| \pmod q}} \left(\frac{D}{a}\right) \tag{21}$$

and

$$S_2 = \sum_{j=-[(q-1)/2]}^{0} \sum_{\substack{\frac{(r-1)|D|}{n} < a < \frac{r|D|}{n} \\ a \equiv j|D| \pmod q}} \left(\frac{D}{a}\right) \tag{22}$$

so that

$$S(r, n, D) = S_1 + S_2. \tag{23}$$

In (21), by changing the summation variable a in the inner sum to $b = \frac{j|D| - a}{q}$, we obtain

$$S_1 = \sum_{j=1}^{[q/2]} \sum_{\frac{(jn-r)|D|}{qn} < b < \frac{(jn-r+1)|D|}{qn}} \left(\frac{D}{j|D| - bq}\right)$$

$$= \sum_{j=1}^{[q/2]} \sum_{\frac{(jn-r)|D|}{qn} < b < \frac{(jn-r+1)|D|}{qn}} \operatorname{sgn}(D)\left(\frac{D}{bq}\right). \tag{24}$$

We show next that $\frac{(jn-r)|D|}{qn}$ is not an integer unless $j = 1$ and $r = n$. Suppose $\frac{(jn-r)|D|}{qn} \in \mathbb{Z}$. Then $qn \mid (jn-r)|D|$. As $(qn, |D|) = 1$ we must have $qn \mid jn - r$. Thus $n \mid r$. But $1 \le r \le n$ so $r = n$. Hence $q \mid j - 1$.

But $1 \le j \le \left[\frac{q}{2}\right] < q$ so $0 \le j-1 < q$. Thus $j = 1$. When $(j, r) = (1, n)$ we have $\frac{(jn-r)|D|}{qn} = 0$. As $\left(\frac{D}{b}\right) = 0$ for $b = 0$, we deduce from (24) that

$$S_1 = \operatorname{sgn}(D)\left(\frac{D}{q}\right)\sum_{j=1}^{[q/2]}\sum_{\frac{(jn-r)|D|}{qn} \le b < \frac{(jn-r+1)|D|}{qn}}\left(\frac{D}{b}\right)$$

$$= \operatorname{sgn}(D)\left(\frac{D}{q}\right)\sum_{j=1}^{[q/2]} S(jn-r+1, qn, D). \tag{25}$$

Next, changing the summation variable j in the outer sum in (22) to $-j$, and then changing a to $c = \frac{a + j|D|}{q}$ in the inner sum, we obtain

$$S_2 = \sum_{j=0}^{[(q-1)/2]}\sum_{\substack{\frac{(r-1)|D|}{n} < a < \frac{r|D|}{n} \\ a \equiv -j|D| \pmod q}}\left(\frac{D}{a}\right)$$

$$= \sum_{j=0}^{[(q-1)/2]}\sum_{\frac{(jn+r-1)|D|}{qn} < c < \frac{(jn+r)|D|}{qn}}\left(\frac{D}{cq - j|D|}\right)$$

$$= \sum_{j=0}^{[(q-1)/2]}\sum_{\frac{(jn+r-1)|D|}{qn} < c < \frac{(jn+r)|D|}{qn}}\left(\frac{D}{cq}\right)$$

$$= \left(\frac{D}{q}\right)\sum_{j=0}^{[(q-1)/2]}\sum_{\frac{(jn+r-1)|D|}{qn} < c < \frac{(jn+r)|D|}{qn}}\left(\frac{D}{c}\right). \tag{26}$$

We now show that $\frac{(jn+r-1)|D|}{qn}$ is not an integer unless $j = 0$ and $r = 1$. Suppose $\frac{(jn+r-1)|D|}{qn} \in \mathbb{Z}$. Hence $qn \mid (jn+r-1)|D|$. As $(qn, |D|) = 1$

we have $qn | jn + r - 1$. Thus $n | r - 1$. But $1 \le r \le n$ so $0 \le r - 1 < n$. Hence $r = 1$. Thus $q | j$. But $0 \le j \le \left[\frac{q-1}{2}\right] < q$ so $j = 0$. When $(j, r) = (0, 1)$ we have $\frac{(jn + r - 1)|D|}{qn} = 0$. As $\left(\frac{D}{c}\right) = 0$ for $c = 0$, we deduce from (26) that

$$S_2 = \left(\frac{D}{q}\right) \sum_{j=0}^{[(q-1)/2]} \sum_{\frac{(jn+r-1)|D|}{qn} \le c < \frac{(jn+r)|D|}{qn}} \left(\frac{D}{c}\right)$$

$$= \left(\frac{D}{q}\right) S(jn + r, qn, D). \tag{27}$$

Thus from (23), (25) and (27) we obtain

$$S(r, n, D) = \operatorname{sgn}(D)\left(\frac{D}{q}\right) \sum_{j=1}^{[q/2]} S(jn - r + 1, qn, D) + \left(\frac{D}{q}\right) \sum_{j=0}^{[(q-1)/2]} S(jn + r, qn, D).$$

Multiplying both sides of this equation by $\left(\frac{D}{q}\right) (= \pm 1)$, we obtain (20).

We note that by (19)

$$\sum_{r=1}^{n} S(r, n, D) = \sum_{r=1}^{n} \sum_{\frac{(r-1)|D|}{n} \le a < \frac{r|D|}{n}} \left(\frac{D}{a}\right) = \sum_{a=0}^{|D|-1} \left(\frac{D}{a}\right) = 0. \tag{28}$$

Theorem 3.2. *Let D be a nonsquare integer $\equiv 0$ or $1 \pmod 4$. Let n and q be positive integers such that $(qn, D) = 1$. For $k = 1, 2, \ldots, qn$ set*

$$S_k = S(k, qn, D). \tag{29}$$

Then $S_1, S_2, \ldots, S_{qn}$ satisfy the linear equations

$$\sum_{j=1}^{qn} S_j = 0 \tag{30}$$

and

$$\sum_{j=0}^{[(q-1)/2]} S_{jn+r} + \operatorname{sgn}(D) \sum_{j=1}^{[q/2]} S_{jn-r+1} - \left(\frac{D}{q}\right) \sum_{j=1}^{q} S_{rq-q+j} = 0, \; r = 1, 2, \ldots, n. \tag{31}$$

Proof. Equation (30) follows immediately from (28) and (29) (with qn replacing n). Equation (31) is a simple consequence of Theorem 3.1. For $r = 1, 2, \ldots, n$ we have

$$\begin{aligned} S(r, n, D) &= \sum_{\frac{(r-1)|D|}{n} \le a < \frac{r|D|}{n}} \left(\frac{D}{a}\right) \\ &= \sum_{j=1}^{q} \sum_{\frac{(rq-q+j-1)|D|}{qn} \le a < \frac{(rq-q+j)|D|}{qn}} \left(\frac{D}{a}\right) \\ &= \sum_{j=1}^{q} S(rq-q+j, qn, D) \\ &= \sum_{j=1}^{q} S_{rq-q+j}. \end{aligned}$$

The asserted linear equations now follow from Theorem 3.1.

We remark that when $q = 1$ formula (31) yields only the trivial assertion

$$S_r + \operatorname{sgn}(D) \times 0 - \left(\frac{D}{1}\right) S_r = 0.$$

We also require class number formulae in terms of the Kronecker symbol. These are provided by the following theorem of Lerch [16, 17] and Mordell [18]. A proof is given in [20].

Theorem 3.3. *Let $d_1 < 0$ and $d_2 > 0$ be coprime fundamental discriminants. Then*

$$h(d_1 d_2) = 2 \sum_{1 \le a \le |d_1|/2} \left(\frac{d_1}{a}\right) \sum_{1 \le n \le a d_2/|d_1|} \left(\frac{d_2}{n}\right) \tag{32}$$

$$= -2 \sum_{1 \le a \le d_2/2} \left(\frac{d_2}{a}\right) \sum_{1 \le n \le a|d_1|/d_2} \left(\frac{d_1}{n}\right). \tag{33}$$

For our purposes we need the following consequence of Theorem 3.3.

Theorem 3.4. (i) *Let* $D > 0$ *be an odd fundamental discriminant. Let* e *be a positive integer such that* $-e$ *is a fundamental discriminant* (*so that* $e \equiv 3 \pmod 4$ *or* $e \equiv 4, 8 \pmod{16}$) *coprime with* D. *Then* $-e|D|$ *is a negative fundamental discriminant and*

$$h(-e|D|) = 2 \sum_{1 \le a \le e/2} \left(\sum_{a \le k \le e/2} \left(\frac{-e}{k} \right) \right) S(a, e, D). \tag{34}$$

(ii) *Let* $D < 0$ *be an odd fundamental discriminant. Let* e *be a positive fundamental discriminant* (*so that* $e \equiv 1 \pmod 4$ *or* $e \equiv 0 \pmod 4$ *and* $\frac{e}{4} \equiv 2, 3 \pmod 4$) *coprime with* D. *Then* $-e|D|$ *is a negative fundamental discriminant and*

$$h(-e|D|) = -2 \sum_{1 \le a \le e/2} \left(\sum_{a \le k \le e/2} \left(\frac{e}{k} \right) \right) S(a, e, D). \tag{35}$$

Proof. We prove (i). From (32) we obtain

$$\begin{aligned} h(-e|D|) &= 2 \sum_{1 \le a \le e/2} \left(\frac{-e}{a} \right) \sum_{1 \le n \le a|D|/e} \left(\frac{D}{n} \right) \\ &= 2 \sum_{1 \le a \le e/2} \left(\frac{-e}{a} \right) \sum_{1 \le n \le a} S(n, e, D) \\ &= 2 \sum_{1 \le a \le e/2} \sum_{1 \le n \le a} \left(\frac{-e}{a} \right) S(n, e, D) \\ &= 2 \sum_{1 \le n \le e/2} \sum_{n \le a \le e/2} \left(\frac{-e}{a} \right) S(n, e, D) \\ &= 2 \sum_{1 \le a \le e/2} \sum_{a \le n \le e/2} \left(\frac{-e}{n} \right) S(a, e, D). \end{aligned}$$

The proof of (ii) is essentially the same as that of (i) using (33) in place of (32).

We now have the necessary mathematical machinery to describe the computer implementation of the proofs of the conjectured class number formulae of the form (12). For each conjectured formula, we solve the homogeneous system of linear equations resulting from Theorem 3.2 by taking all pairs of positive integers $q(>1)$ and n with $qn = r$ (r = lcm of the denominators q_1, q_2, q_3, q_4) for $S_1, S_2, \dots, S_r$, where $S_j = S(j, r, D)$, in terms of some integral parameters $k_1, \dots, k_{r-t}$, where t is the rank of the coefficient matrix of the system. Then, by means of Theorem 3.4, as $e \mid r$, $h(-e|D|)$ can be expressed in terms of $S_1, \dots, S_r$ and thus in terms of $k_1, \dots, k_{r-t}$. Also, as r is the least common denominator of q_1, q_2, q_3, q_4, the quantity

$$c_1 \sum_{q_1|D|<n<q_2|D|} \left(\frac{D}{n}\right) + c_2 \sum_{q_3|D|<n<q_4|D|} \left(\frac{D}{n}\right)$$

can be expressed in terms of $S_1, \dots, S_r$ and thus in terms of $k_1, \dots, k_{r-t}$. If the two expressions in terms of $k_1, \dots, k_{r-t}$ agree, then the formula is a valid one. The congruence class to which $|D|$ belongs determines the quantities $\operatorname{sgn}(D)$ and $\left(\frac{D}{q}\right)$ occurring in Theorem 3.2. All conjectured formulae were found to be valid by these means. In the next section we give the details of the proof of one of these formulae.

4. Proof of One Conjectured Formula

We give the details of the proof of one of the conjectured formulae, namely (17) based on the method described at the end of Section 3.

Let D be a fundamental discriminant with $|D| \equiv 13 \pmod{24}$, so that $D > 0$, $\operatorname{sgn}(D) = +1$, $\left(\frac{D}{2}\right) = -1$, $\left(\frac{D}{3}\right) = +1$. Taking $(q, n) = (2, 6)$, $(3, 4)$, $(4, 3)$, $(6, 2)$, and $(12, 1)$ in Theorem 3.2 (these are all pairs of positive integers (q, n) with $q > 1$ such that $qn = 12$), we obtain

$$S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 + S_8 + S_9 + S_{10} + S_{11} + S_{12} = 0,$$

$$S_r + S_{7-r} + S_{2r-1} + S_{2r} = 0, \quad r = 1, 2, 3, 4, 5, 6,$$

$$S_r + S_{r+4} + S_{5-r} - (S_{3r-2} + S_{3r-1} + S_{3r}) = 0, \quad r = 1, 2, 3, 4,$$

$$S_r + S_{r+3} + S_{4-r} + S_{7-r} - (S_{4r-3} + S_{4r-2} + S_{4r-1} + S_{4r}) = 0, \quad r = 1, 2, 3,$$

$$S_r + S_{r+2} + S_{r+4} + S_{3-r} + S_{5-r} + S_{7-r} - (S_{6r-5} + S_{6r-4} + S_{6r-3} + S_{6r-2} + S_{6r-1} + S_{6r}) = 0, \quad r = 1, 2,$$

$$S_1 + S_2 + S_3 + S_4 + S_5 + S_6 - (S_7 + S_8 + S_9 + S_{10} + S_{11} + S_{12}) = 0.$$

Solving these $1 + 6 + 4 + 3 + 2 + 1 = 17$ linear equations for the 12 unknowns $S_1, \ldots, S_{12}$, we obtain

$$S_1 = S_3 = S_{10} = S_{12} = k_1,$$
$$S_2 = S_6 = S_7 = S_{11} = -k_1,$$
$$S_4 = S_9 = k_2,$$
$$S_5 = S_8 = -k_2,$$

for some $k_1, k_2 \in \mathbb{Z}$.

By Theorem 3.4 (i) we have

$$\begin{aligned} h(-3|D|) &= 2 \sum_{1 \le a \le 3/2} \left(\sum_{a \le k \le 3/2} \left(\frac{-3}{k} \right) \right) S(a, 3, D) \\ &= 2S(1, 3, D) \\ &= 2S(1, 12, D) + 2S(2, 12, D) + 2S(3, 12, D) + 2S(4, 12, D) \\ &= 2S_1 + 2S_2 + 2S_3 + 2S_4. \end{aligned}$$

Using the parametric equations for the S_k, we obtain

$$h(-3|D|) = 2(k_1) + 2(-k_1) + 2(k_1) + 2(k_2) = 2k_1 + 2k_2.$$

Similarly, using the parametric equations for the S_k in the right hand side of (17), we obtain

$$\begin{aligned} &2 \sum_{\frac{1}{12}|D| < n < \frac{1}{3}|D|} \left(\frac{D}{n} \right) - 2 \sum_{\frac{5}{12}|D| < n < \frac{1}{2}|D|} \left(\frac{D}{n} \right) \\ &= 2(S(2, 12, D) + S(3, 12, D) \\ &\quad + S(4, 12, D)) - 2S(6, 12, D) \end{aligned}$$

$$= 2S_2 + 2S_3 + 2S_4 - 2S_6$$

$$= 2(-k_1) + 2(k_1) + 2(k_2) - 2(-k_1)$$

$$= 2k_1 + 2k_2.$$

This completes the proof of (17).

References

[1] B. C. Berndt, Classical theorems on quadratic residues, Enseignement Math. 22 (1976), 261-304.

[2] G. L. Dirichlet, Recherches sur diverses applications de l'analyse infitesimale à la theorie des nombres, J. Reine Angew. Math. 19 (1839), 324-369.

[3] G. L. Dirichlet, Recherches sur diverses applications de l'analyse infitesimale à la theorie des nombres, Second Partie, J. Reine Angew. Math. 21 (1840), 134-155.

[4] J. W. L. Glaisher, On the expression for the number of classes of a negative determinant, and on the number of positives in the octants of P, Quart. J. Math. 34 (1903), 178-204.

[5] S. M. F. Godin, Two term class number formulae of Dirichlet type, M.Sc. thesis, Carleton University, 2002.

[6] H. Holden, On various expressions for h, the number of properly primitive classes for a determinant $-p$, where p is a prime of the form $4n+3$ (first paper), Messenger Math. 35 (1905/1906), 73-80.

[7] H. Holden, On various expressions for h, the number of properly primitive classes for a determinant $-p$, where p is of the form $4n+3$ and is a prime or the product of different primes (second paper), Messenger Math. 35 (1905/1906), 102-110.

[8] H. Holden, On various expressions for h, the number of properly primitive classes for any negative determinant, not involving a square factor (third paper), Messenger Math. 35 (1905/1906), 110-117.

[9] H. Holden, On various expressions for h, the number of properly primitive classes for a negative determinant (fourth paper), Messenger Math. 36 (1906/1907), 69-75.

[10] H. Holden, On various expressions for h, the number of properly primitive classes for a determinant $-p$, where p is of the form $4n+3$, and is a prime or the product of different primes (addition to the second paper), Messenger Math. 36 (1906/1907), 75-77.

[11] H. Holden, On various expressions for h, the number of properly primitive classes for a negative determinant, not involving a square factor (fifth paper), Messenger Math. 36 (1906/1907), 126-134.

[12] H. Holden, On various expressions for h, the number of properly primitive classes for any negative determinant, not involving a square factor (sixth paper), Messenger Math. 37 (1907/1908), 13-16.

[13] R. H. Hudson and K. S. Williams, Class number formulae of Dirichlet type, Math. Comp. 39 (1982), 725-732.

[14] W. Johnson and K. J. Mitchell, Symmetries for sums of the Legendre symbol, Pacific J. Math. 69 (1977), 117-124.

[15] L. C. Karpinski, Über die Verteilung der quadratischen Reste, J. Reine Angew. Math. 127 (1904), 1-19.

[16] M. Lerch, Essais sur le calcul du nombre des classes de formes quadratiques binaires aux coefficients entiers, Acta Math. 29 (1905), 333-424.

[17] M. Lerch, Essais sur le calcul du nombre des classes de formes quadratiques binaires aux coefficients entiers, Acta Math. 30 (1906), 203-293.

[18] L. J. Mordell, On Lerch's class number for binary quadratic forms, Ark. Mat. 5 (1963), 97-100.

[19] A. Schinzel, J. Urbanowicz and P. Van Wamelen, Class numbers and short sums of Kronecker symbols, J. Number Theory 78 (1999), 64-84.

[20] J. Urbanowicz and K. S. Williams, Congruences for L-functions, Kluwer Academic Publishers, 2000.

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