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# A cubic transformation formula for ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; z)$ and some arithmetic convolution formulae

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*Abstract*

A cubic transformation formula for the hypergeometric function ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; z)$ is proved. As an application of this formula a number of arithmetic convolution sums are evaluated. For example, Melfi's formula,

$$\sum_{\substack{k=1 \\ k\equiv 1 \pmod 3}}^{n-1} \sigma(k)\sigma(n-k) = \frac{1}{9}\sigma_3(n), \quad n \equiv 2 \pmod 3,$$

is proved without the use of modular forms.

## 1. *Introduction*: *preliminaries and statement of main results*

As usual we let $\mathbb{C}$ denote the complex plane and $\mathbb{R}$ the real number line in $\mathbb{C}$. We recall that the function $z^{\frac{1}{3}}$ is defined for all $z \in \mathbb{C}$ by

$$z^{\frac{1}{3}} = \begin{cases} |z|^{\frac{1}{3}} e^{\frac{1}{3} i \arg z}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0, \end{cases}$$

where the argument $\arg z$ of the complex number $z$ is chosen to satisfy

$$-\pi < \arg z \leqslant \pi, \quad z \in \mathbb{C}\backslash\{0\}.$$

In particular if $a \in \mathbb{R}$ is such that $a < 0$ then

$$a^{\frac{1}{3}} = -|a|^{\frac{1}{3}}\omega^2,$$

where $\omega$ is the complex cube root of unity given by

$$\omega = e^{\frac{2\pi i}{3}} = \tfrac{1}{2}(-1 + i\sqrt{3}),$$

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so that for example we have

$$\left(-\tfrac{1}{8}\right)^{\frac{1}{3}} = -\tfrac{1}{2}\omega^2. \tag{1·1}$$

The function $z^{\frac{1}{3}}$ is regular [**8**, p. 36] in $\mathbb{C}\backslash(-\infty, 0]$, see [**8**, p. 49]. Let $D$ denote the domain [**8**, p. 15] in the complex plane given by

$$D = \mathbb{C}\backslash[1, \infty). \tag{1·2}$$

Hence, the function $(1-z)^{\frac{1}{3}}$ is defined for all $z \in \mathbb{C}$ and is regular in $D$. (1·3)

We define the function $h$ by

$$h(z) = 9z(1-z)^{\frac{1}{3}} + ((1-z)^{\frac{1}{3}} - \omega)^4, \quad z \in \mathbb{C}. \tag{1·4}$$

We show that

$$h(z) \neq 0 \text{ for all } z \in \mathbb{C}. \tag{1·5}$$

To see this we suppose that $z \in \mathbb{C}$ is such that $h(z) = 0$. From (1·1) and (1·4) we have $z \neq 0,\ \frac{9}{8}$. Set

$$z_1 = \omega^2(1-z)^{\frac{1}{3}} \in \mathbb{C}$$

so that $z_1 \neq 1,\ -\frac{1}{2}$. Then $h(z) = 0$ gives

$$9\left(1 - z_1^3\right)\omega z_1 + (\omega z_1 - \omega)^4 = 0$$

so that

$$(1 - z_1)(1 + 2z_1)^3 = 0,$$

a contradiction.

In view of (1·3), (1·4) and (1·5), we can define functions $f$ and $g$ by

$$f(z) = \frac{9z(1-z)^{\frac{1}{3}}}{9z(1-z)^{\frac{1}{3}} + ((1-z)^{\frac{1}{3}} - \omega)^4}, \quad z \in \mathbb{C}, \tag{1·6}$$

and

$$g(z) = \frac{1 + 2\omega^2(1-z)^{\frac{1}{3}}}{1 + 2\omega^2}, \quad z \in \mathbb{C}. \tag{1·7}$$

From (1·3), (1·4), (1·5), (1·6) and (1·7), we see that

$$f(z) \text{ and } g(z) \text{ are defined for all } z \in \mathbb{C} \text{ and are regular in } D. \tag{1·8}$$

For $r \in \mathbb{R}^+$ we set $C(r) = \{z \in \mathbb{C} \mid |z| < r\}$. We note that

$$f(0) = 0, \quad f(1) = 0, \quad f\left(\tfrac{9}{8}\right) = \tfrac{9}{8}, \tag{1·9}$$

$$g(0) = 1, \quad g(1) = \tfrac{1}{3}(1 + 2\omega), \quad g\left(\tfrac{9}{8}\right) = 1 + \omega, \tag{1·10}$$

$$f(z) = 0 \iff z = 0,\ 1, \tag{1·11}$$

$$g(z) \neq 0 \text{ for } z \in \mathbb{C}, \tag{1·12}$$

$$f(z) \in C\left(\tfrac{1}{2}\right) \text{ for } z \in C\left(\tfrac{1}{8}\right), \tag{1·13}$$

$$f(x) \notin \mathbb{R} \text{ for } x \in \left(0, \tfrac{1}{8}\right), \tag{1·14}$$

$$f'(0) = \lim_{z\to 0} \frac{f(z)}{z} = \frac{9}{(1-\omega)^4} = \omega, \tag{1·15}$$

$$g'(0) = \frac{-2\omega^2}{3(1+2\omega^2)} = -\tfrac{2}{9}(1-\omega). \tag{1·16}$$

We just prove (1·13) and (1·14) as the others are clear.

*Proof of* (1·13). For $z \in C\left(\frac{1}{8}\right)$ we have

$$|1-z|^{\frac{1}{3}} < 2, \quad 9|z||1-z|^{\frac{1}{3}} < \tfrac{9}{4}.$$

By the generalized binomial theorem, we see that for $a = \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$ and $z \in C\left(\frac{1}{8}\right)$ we have

$$|(1-z)^a - 1| \leqslant \sum_{n=1}^{\infty} \left|\binom{a}{n}\right| |z|^n \leqslant \sum_{n=1}^{\infty} a|z|^n = \frac{a|z|}{1-|z|} < \frac{a}{7}.$$

Thus

$$\begin{aligned}
&|((1-z)^{\frac{1}{3}} - \omega)^4 - (1-\omega)^4| \\
&\quad = |((1-z)^{\frac{4}{3}} - 1) + 4\omega z + 6\omega^2((1-z)^{\frac{2}{3}} - 1) - 4((1-z)^{\frac{1}{3}} - 1)| \\
&\quad \leqslant |(1-z)^{\frac{4}{3}} - 1| + 4|z| + 6|(1-z)^{\frac{2}{3}} - 1| + 4|(1-z)^{\frac{1}{3}} - 1| \\
&\quad < \tfrac{4}{21} + \tfrac{1}{2} + \tfrac{4}{7} + \tfrac{4}{21} \\
&\quad = \tfrac{61}{42} \\
&\quad < \tfrac{3}{2}.
\end{aligned}$$

Hence

$$\begin{aligned}
&|9z(1-z)^{\frac{1}{3}} + ((1-z)^{\frac{1}{3}} - \omega)^4)| \\
&\quad = |(1-\omega)^4 + 9z(1-z)^{\frac{1}{3}} + (((1-z)^{\frac{1}{3}} - \omega)^4 - (1-\omega)^4)| \\
&\quad \geqslant |(1-\omega)^4| - 9|z||1-z|^{\frac{1}{3}} - |((1-z)^{\frac{1}{3}} - \omega)^4 - (1-\omega)^4)| \\
&\quad > 9 - \tfrac{9}{4} - \tfrac{3}{2} \\
&\quad = \tfrac{21}{4}.
\end{aligned}$$

Then, for $z \in C\left(\frac{1}{8}\right)$, we have

$$|f(z)| = \frac{9|z||1-z|^{\frac{1}{3}}}{|9z(1-z)^{\frac{1}{3}} + ((1-z)^{\frac{1}{3}} - \omega)^4|} < \frac{\frac{9}{4}}{\frac{21}{4}} = \tfrac{9}{21} < \tfrac{1}{2}.$$

*Proof of* (1·14). Suppose that $f(x) \in \mathbb{R}$ for $x \in \left(0, \frac{1}{8}\right)$. Then $f(x) = \overline{f(x)}$. As $1 - x > 0$ we have $(1-x)^{\frac{1}{3}} \in \mathbb{R}$ so that by (1·6)

$$((1-x)^{\frac{1}{3}} - \omega)^4 = ((1-x)^{\frac{1}{3}} - \omega^2)^4.$$

Expanding the fourth powers we obtain, after cancelling the factor $\omega - \omega^2$,

$$6(1-x)^{\frac{2}{3}} = -3 + 4x.$$

Cubing both sides, we deduce that

$$64x^3 - 360x^2 + 540x - 243 = 0.$$

Hence

$$(8x-9)(8x^2 - 36x + 27) = 0$$

so that

$$x = \tfrac{9}{8}(\approx 1.1), \quad \tfrac{9-3\sqrt{3}}{4}(\approx 0.9) \text{ or } \tfrac{9+3\sqrt{3}}{4}(\approx 3.5).$$

This is a contradiction as none of these values lies in the interval $\left(0, \frac{1}{8}\right)$.

The integral

$$\int_0^1 t^{-\frac{1}{3}}(1-t)^{-\frac{2}{3}}(1-zt)^{-\frac{1}{3}}\,dt$$

is uniformly convergent in any closed domain of $D$ and so represents an analytic function of $z$, which is regular in $D$ [8, p. 249]. We set

$$w(z) := \frac{\sqrt{3}}{2\pi}\int_0^1 t^{-\frac{1}{3}}(1-t)^{-\frac{2}{3}}(1-zt)^{-\frac{1}{3}}\,dt, \quad z \in D. \tag{1·17}$$

Hence

$$w(z) \text{ is defined and regular in } D. \tag{1·18}$$

For $z \in C(1)$ it is known that [8, p. 249]

$$\frac{\sqrt{3}}{2\pi}\int_0^1 t^{-\frac{1}{3}}(1-t)^{-\frac{2}{3}}(1-zt)^{-\frac{1}{3}}\,dt = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n} \frac{z^n}{n!},$$

where for $a \in \mathbb{R}^+$ and $n \in \{0, 1, 2, \ldots\}$ the Pochhammer symbol $(a)_n$ is defined by

$$(a)_n = a(a+1)\cdots(a+n-1),\ n \in \mathbb{N}; \quad (a)_0 = 1. \tag{1·19}$$

Thus $w(z)$ is the analytic continuation of the hypergeometric function ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right)$ from $C(1)$ to $D$, so that

$$w(z) = {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; z\right), \quad z \in C(1).$$

The function $w(z)$ is the unique solution of the hypergeometric differential equation

$$z(1-z)w'' + (1-2z)w' - \tfrac{2}{9}w = 0, \tag{1·20}$$

which is regular in $D$ and satisfies

$$w(0) = 1, \quad w'(0) = \tfrac{2}{9}, \tag{1·21}$$

see [**8**, pp. 246–248]. We now show that

$$w(z) \neq 0 \text{ for } z \in C\left(\tfrac{1}{2}\right). \tag{1·22}$$

As

$$\left(\tfrac{1}{3}\right)_n = \frac{1\cdot 4\cdot 7\cdots(3n-2)}{3^n}, \quad \left(\tfrac{2}{3}\right)_n = \frac{2\cdot 5\cdot 8\cdots(3n-1)}{3^n}, \quad (1)_n = n!,$$

by (1·19), we have

$$\frac{\left(\tfrac{1}{3}\right)_n\left(\tfrac{2}{3}\right)_n}{(1)_n} = \frac{3n!}{n!^2 3^{3n}}$$

so that

$$|w(z)-1| = \left|\sum_{n=1}^{\infty} \frac{3n!}{n!^3 3^{3n}} z^n\right| \leqslant \sum_{n=1}^{\infty} \frac{3n!}{n!^3 3^{3n}} |z|^n, \quad |z|<1.$$

The multinomial coefficient

$$\binom{3n}{n,\ n,\ n} = \frac{3n!}{n!^3}$$

is one of the terms in the multinomial expansion of $(1+1+1)^{3n}$ so that

$$\frac{3n!}{n!^3} \leqslant 3^{3n}.$$

Hence

$$|w(z)-1| \leqslant \sum_{n=1}^{\infty} |z|^n = \frac{|z|}{1-|z|}.$$

If $z \in C\left(\tfrac{1}{2}\right)$ then $\dfrac{|z|}{1-|z|} < 1$, so that $|w(z)-1|<1$, proving (1·22).

For $x \in \mathbb{R}$ with $0 \leqslant x < 1$ we have

$$w(x) = 1 + \sum_{n=1}^{\infty} \frac{3n!}{n!^3 3^{3n}} x^n$$

so that

$$w(x) \geqslant 1, \quad 0 \leqslant x < 1,$$

and thus in particular

$$w(x) \neq 0, \quad 0 \leqslant x < 1. \tag{1·23}$$

It is convenient to define the subset $D^*$ of $D$ by

$$D^* = \{z \in D \mid f(z) \in D\}. \tag{1·24}$$

By (1·8), (1·18) and (1·24) we see that

$$w(f(z)) \text{ is defined and regular in } D^* \tag{1·25}$$

and

$$g(z)w(z) \text{ is defined and regular in } D. \tag{1·26}$$

Let $z \in C\left(\frac{1}{8}\right)$. By (1·2) we have $z \in D$. By (1·13) we have $f(z) \in C\left(\frac{1}{2}\right)$ so that $f(z) \in D$. Hence, by (1·24), we have

$$C\left(\tfrac{1}{8}\right) \subseteq D^* \subseteq D. \tag{1·27}$$

The main result of this paper is the following cubic transformation formula for $w(z)$.

THEOREM 1·1. *For $z \in D^*$ we have*

$$w(f(z)) = g(z)w(z). \tag{1·28}$$

The proof of Theorem 1·1 is given in Section 2. Theorem 1·1 is similar to the cubic transformation formula of Ramanujan [**13**, second notebook, p. 258]

$$w\left(1 - \left(\frac{1-x}{1+2x}\right)^3\right) = (1+2x)w(x^3), \tag{1·29}$$

which is valid for $x \in \mathbb{R}$ with $|x|$ sufficiently small, see [**4**, corollary 2·4, p. 97]. Proofs of (1·29) have been given by Berndt, Bhargava and Garvan [**5**, corollary 2·4, p. 4170], Borwein and Borwein [**6**, p. 694] and Chan [**7**, sections 5 and 6, pp. 201–203]. We have not been able to deduce Theorem 1·1 from (1·29).

We define the subset $D^{**}$ of $D^*$ by

$$D^{**} = \{z \in D^* \mid w(z) \neq 0,\ w(f(z)) \neq 0,\ 1 - z \in D,\ 1 - f(z) \in D\}. \tag{1·30}$$

It is clear from (1·30) that

$$\frac{2\pi}{\sqrt{3}}\frac{w(1-z)}{w(z)} - \frac{2\pi}{\sqrt{3}}\frac{w(1-f(z))}{w(f(z))} \text{ is defined and regular in } D^{**}. \tag{1·31}$$

For $x \in \left(0, \frac{1}{8}\right)$ we have

$$x \in D^*, \quad \text{by} \quad (1·27), \tag{1·32}$$

$$w(x) \neq 0, \quad \text{by} \quad (1·22), \tag{1·33}$$

$$w(f(x)) \neq 0, \quad \text{by} \quad (1·13) \text{ and } (1·22), \tag{1·34}$$

$$1 - x \in D, \quad \text{by} \quad (1·2), \tag{1·35}$$

$$1 - f(x) \in D, \quad \text{by} \quad (1·2) \text{ and } (1·14). \tag{1·36}$$

From (1·30) and (1·32) – (1·36) we see that

$$\left(0, \tfrac{1}{8}\right) \subseteq D^{**}. \tag{1·37}$$

We use Theorem 1·1 to prove the following important identity satisfied by $w(z)$.

THEOREM 1·2. *For $z \in D^{**}$ we have*

$$e^{-\frac{2\pi}{\sqrt{3}}\frac{w(1-f(z))}{w(f(z))}} = \omega e^{-\frac{2\pi}{\sqrt{3}}\frac{w(1-z)}{w(z)}}. \tag{1·38}$$

The proof of Theorem 1·2 is given in Section 3. Theorem 1·2 is the analogue of Jacobi's "change of sign" formula [**3**, p. 126] for a "change of cube root of unity."

We now recall the Eisenstein series

$$L(q) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n, \qquad q \in C(1), \tag{1·39}$$

$$M(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \quad q \in C(1), \tag{1·40}$$

$$N(q) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \quad q \in C(1), \tag{1·41}$$

treated by Ramanujan in [**12**], [**14**, pp. 136–162]. (Ramanujan actually used $P$, $Q$, $R$ in place of $L$, $M$, $N$, see [**12**, equation (25), p. 140].) Now let $q \in \mathbb{R}$ be such that $0 < q < 1$ so that $0 < -\log q < +\infty$. The derivative of the function

$$y(x) := \frac{2\pi}{\sqrt{3}}\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-x\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right)} = \frac{2\pi}{\sqrt{3}}\frac{w(1-x)}{w(x)}, \quad 0 < x < 1,$$

is, [**2**, p. 87],

$$y'(x) = \frac{-1}{x(1-x)\{w(x)\}^2}, \quad 0 < x < 1. \tag{1·42}$$

By (1·23), we have $w(x) \neq 0$ for $0 < x < 1$, so that

$$y'(x) < 0, \quad 0 < x < 1.$$

Hence, as $x$ increases from 0 to 1, $y(x)$ decreases from $y(0) = +\infty$ to $y(1) = 0$, so that there is a unique real number $x$ with $0 < x < 1$ such that

$$\frac{2\pi}{\sqrt{3}}\frac{w(1-x)}{w(x)} = -\log q,$$

equivalently

$$q = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{w(1-x)}{w(x)}\right) = e^{-y(x)}, \tag{1·43}$$

see [**4**, equation (1·7), p. 91]. It is also convenient to set

$$w := w(x) = {}_2F_1\left(\tfrac{1}{3},\tfrac{2}{3};1;x\right).$$

It is known that

$$L(q) = (1-4x)w^2 + 12x(1-x)w\frac{dw}{dx}, \tag{1·44}$$

$$M(q) = (1+8x)w^4, \tag{1·45}$$

$$N(q) = (1-20x-8x^2)w^6, \tag{1·46}$$

see [**4**, lemma 4·1, theorem 4·2, theorem 4·3, pp. 105–106]. In Section 4, we apply Theorem 1·2 to (1·44), (1·45) and (1·46) to obtain $L(\omega q)$, $M(\omega q)$ and $N(\omega q)$.

THEOREM 1·3.

$$L(\omega q) = \left(11 - 12x + 8\omega^2(1-x)^{\frac{1}{3}} + 8\omega(1-x)^{\frac{2}{3}}\right)\frac{w^2}{3} + 12x(1-x)w\frac{dw}{dx}, \tag{1·47}$$

$$M(\omega q) = \left(249 - 248x - 120(1-x)^{\frac{1}{3}} - 120(1-x)^{\frac{2}{3}} + 80x(1-x)^{\frac{1}{3}} + (\omega-\omega^2)\left(80x(1-x)^{\frac{1}{3}} - 120(1-x)^{\frac{1}{3}} + 120(1-x)^{\frac{2}{3}}\right)\right)\frac{w^4}{9}, \tag{1·48}$$

$$N(\omega q) = \left(6579 - 8604x + 2024x^2 - 3276(1-x)^{\frac{1}{3}} - 3276(1-x)^{\frac{2}{3}} + 3192x(1-x)^{\frac{1}{3}} + 2352x(1-x)^{\frac{2}{3}} - (\omega-\omega^2)\left(3276(1-x)^{\frac{1}{3}} - 3276(1-x)^{\frac{2}{3}} - 3192x(1-x)^{\frac{1}{3}} + 2352x(1-x)^{\frac{2}{3}}\right)\right)\frac{w^6}{27}. \tag{1·49}$$

For $j = 1, 2, 3$ we introduce the series

$$L_j(q) = \sum_{\substack{n=1 \\ n\equiv j \pmod 3}}^{\infty} \sigma(n)q^n, \tag{1·50}$$

$$M_j(q) = \sum_{\substack{n=1 \\ n\equiv j \pmod 3}}^{\infty} \sigma_3(n)q^n, \tag{1·51}$$

$$N_j(q) = \sum_{\substack{n=1 \\ n\equiv j \pmod 3}}^{\infty} \sigma_5(n)q^n. \tag{1·52}$$

In Section 5 we use Theorem 1·3 and the method of triplication [**4**, pp. 101–102] to evaluate $L_1(q), L_2(q), \ldots, N_3(q)$.

THEOREM 1·4.

$$L_1(q) = \left(1 + (1-x)^{\frac{1}{3}} - 2(1-x)^{\frac{2}{3}}\right)\frac{w^2}{27}, \tag{1·53}$$

$$L_2(q) = \left(1 - 2(1-x)^{\frac{1}{3}} + (1-x)^{\frac{2}{3}}\right)\frac{w^2}{27}, \tag{1·54}$$

$$L_3(q) = \frac{1}{24} + \left(-25 + 36x + 8(1-x)^{\frac{1}{3}} + 8(1-x)^{\frac{2}{3}}\right)\frac{w^2}{216} - \frac{1}{2}x(1-x)w\frac{dw}{dx}, \tag{1·55}$$

$$M_1(q) = \left(-3 + 4x - 3(1-x)^{\frac{1}{3}} + 6(1-x)^{\frac{2}{3}} + 2x(1-x)^{\frac{1}{3}}\right)\frac{w^4}{81}, \tag{1·56}$$

$$M_2(q) = \left(-3 + 4x + 6(1-x)^{\frac{1}{3}} - 3(1-x)^{\frac{2}{3}} - 4x(1-x)^{\frac{1}{3}}\right)\frac{w^4}{81}, \tag{1·57}$$

$$M_3(q) = -\frac{1}{240} + \left(507 - 424x - 240(1-x)^{\frac{1}{3}} - 240(1-x)^{\frac{2}{3}} + 160x(1-x)^{\frac{1}{3}}\right)\frac{w^4}{6480}, \tag{1·58}$$

$$N_1(q) = \Big(117 - 144x + 40x^2 + 117(1-x)^{\frac{1}{3}} - 234(1-x)^{\frac{2}{3}} - 114x(1-x)^{\frac{1}{3}} + 168x(1-x)^{\frac{2}{3}}\Big)\frac{w^6}{729}, \tag{1·59}$$

$$N_2(q) = \Big(117 - 144x + 40x^2 - 234(1-x)^{\frac{1}{3}} + 117(1-x)^{\frac{2}{3}} + 228x(1-x)^{\frac{1}{3}} - 848x(1-x)^{\frac{2}{3}}\Big)\frac{w^6}{729}, \tag{1·60}$$

$$N_3(q) = \frac{1}{504} + \Big(-355995 + 479196x - 103464x^2 + 176904(1-x)^{\frac{1}{3}} + 176904(1-x)^{\frac{2}{3}} - 172368x(1-x)^{\frac{1}{3}} - 127008x(1-x)^{\frac{2}{3}}\Big)\frac{w^6}{1102248}. \tag{1·61}$$

The relationships in Theorems 1·5, 1·6 and 1·7 below follow easily from Theorem 1·4, for proofs see Sections 6, 7 and 8.

THEOREM 1·5.

$$L_1(q)^2 = \frac{1}{9}M_2(q),$$

$$L_1(q)L_2(q) = \frac{1}{80}M(q^3) - \frac{1}{80}M(q^9),$$

$$L_2(q)L_3(q) = \frac{1}{24}L_2(q) + \frac{11}{72}M_2(q) - \frac{1}{4}q\frac{dL_2(q)}{dq},$$

$$L_3(q)^2 = -\frac{1}{40}M(q^3) + \frac{1}{40}M(q^9) + \frac{1}{12}L_3(q) + \frac{5}{12}M_3(q) - \frac{1}{2}q\frac{dL_3(q)}{dq}.$$

THEOREM 1·6.

$$L_1(q)L(q^3) = -M_1(q) + 2q\frac{dL_1(q)}{dq},$$

$$L_2(q)L(q^3) = -M_2(q) + 2q\frac{dL_2(q)}{dq},$$

$$L_3(q)L(q^3) = \frac{1}{24}L(q^3) - \frac{3}{80}M(q^3) - M_3(q) - \frac{3}{16}q\frac{dL(q^9)}{dq} + \frac{7}{2}q\frac{dL_3(q)}{dq} - \frac{1}{240},$$

$$L(q)L(q^3) = \frac{1}{10}M(q) + \frac{9}{10}M(q^3) + 2q\frac{dL(q)}{dq} + \frac{9}{2}q\frac{dL(q^9)}{dq} - 36q\frac{dL_3(q)}{dq}.$$

THEOREM 1·7.

$$L_2(q)L(q^9) = -\frac{1}{9}M_2(q) + \frac{2}{3}q\frac{dL_2(q)}{dq},$$

$$L_3(q)L(q^9) = \frac{1}{24}L(q^9) - \frac{23}{560}M(q^9) - \frac{1}{7}M_3(q) - \frac{1}{4}q\frac{dL(q^9)}{dq} + \frac{2}{3}q\frac{dL_3(q)}{dq} - \frac{1}{1680}.$$

We set (as in [**9**, p. 255])

$$S(a,3) = \sum_{\substack{m=1 \\ m\equiv a \pmod 3}}^{n-1} \sigma(m)\sigma(n-m), \quad a = 0,1,2.$$

By equating coefficients of $q^n$ on both sides of the four identities in Theorem 1·5, we obtain the following arithmetic convolution identities, see [**9**, theorem 8, p. 256].

THEOREM 1·8.

$$S(1,3) = \frac{1}{9}\sigma_3(n), \qquad \text{if } n \equiv 2 \pmod 3,$$
$$S(1,3) = S(2,3) = \frac{1}{9}(\sigma_3(n) - \sigma_3(n/3)), \qquad \text{if } n \equiv 0 \pmod 3,$$
$$S(0,3) = S(2,3) = \frac{1}{72}(11\sigma_3(n) + (3-18n)\sigma(n)), \qquad \text{if } n \equiv 2 \pmod 3,$$
$$S(0,3) = \frac{1}{36}(7\sigma_3(n) + (3-18n)\sigma(n) + 8\sigma_3(n/3)), \qquad \text{if } n \equiv 0 \pmod 3.$$

The first identity in Theorem 1·8 is due to Melfi [**10**, **11**], who proved it using modular forms. Our proof is the first proof without the use of modular forms, see Section 9.

Similarly, equating coefficients of $q^n$ in the four identities in Theorem 1.6, we obtain the following result.

THEOREM 1·9.

$$\sum_{m<n/3} \sigma(m)\sigma(n-3m) = \frac{1}{24}(\sigma_3(n) + (1-2n)\sigma(n) + 9\sigma_3(n/3) + (1-6n)\sigma(n/3)).$$

The first identity in Theorem 1·6 gives the case $n \equiv 1 \pmod 3$ of Theorem 1·9, the second the case $n \equiv 2 \pmod 3$, and the third $n \equiv 0 \pmod 3$. The fourth identity gives the result for all $n$. For $n \not\equiv 0 \pmod 3$ Theorem 1·9 is due to Melfi [**10**, **11**]. His proof used modular forms. For general $n$ the theorem is due to Huard, Ou, Spearman and Williams [**9**]. Their proof is completely elementary. The proof of Theorem 1·9 is given in Section 10.

Finally, by equating coefficients of $q^n$ in the two identities in Theorem 1·7, we obtain the following result, see Section 11.

THEOREM 1·10.

$$\sum_{k<n/9} \sigma(k)\sigma(n-9k) = \frac{1}{216}\left(\sigma_3(n) + (9-6n)\sigma(n)\right), \qquad \text{if } n \equiv 2 \pmod 3.$$
$$\sum_{k<n/9} \sigma(k)\sigma(n-9k) = \frac{1}{36}\big(6\sigma_3(n/3) + (6-4n)\sigma(n/3) + 9\sigma_3(n/9) - (3+6n)\sigma(n/9)\big), \qquad \text{if } n \equiv 0 \pmod 3.$$

The first identity in Theorem 1·10 was proved by Melfi [**10**, **11**] using modular forms. Our proof makes no use of modular forms. The second identity is due to Huard, Ou, Spearman and Williams [**9**]. Their proof is elementary. Melfi has also given an evaluation of the sum in Theorem 1·10 for certain $n \equiv 1 \pmod 3$. No other proof of his evaluation is known.

Further arithmetic identities can be obtained by equating coefficients of $q^n$ in the identities resulting from differentiating those in Theorems 1·5, 1·6 and 1·7. We just give one example. Differentiating the first identity in Theorem 1·5, we obtain

$$L_1(q)\frac{dL_1(q)}{dq} = \frac{1}{18}\frac{dM_2(q)}{dq}. \tag{1·62}$$

Equating coefficients of $q^n$ ($n \equiv 2 \pmod 3$) in (1·62), we obtain the following identity, see Section 12.

THEOREM 1·11.

$$\sum_{\substack{k=1 \\ k\equiv 1 \pmod 3}}^{n-1} k\sigma(k)\sigma(n-k) = \frac{1}{18}n\sigma_3(n), \quad \textit{if } n \equiv 2 \pmod 3.$$

This identity can also be proved by changing the variable $k$ to $n-k$ in the sum and appealing to the first identity in Theorem 1·8.

## 2. *Proof of Theorem* 1·1

We set

$$a(z) = w(f(z)), \quad z \in D^*, \tag{2·1}$$

$$b(z) = g(z)w(z), \quad z \in D^*. \tag{2·2}$$

Let $z \in C\left(\frac{1}{8}\right) \subseteq D^*$. By (1·24) – (1·26), we know that $a(z)$ and $b(z)$ are defined and regular in $D^*$. Differentiating (2·1) twice, and appealing to (1·20), we find that $a(z)$ satisfies the differential equation

$$a'' + p(z)a' + q(z)a = 0,$$

where

$$p(z) = \frac{(1-2f(z))f'(z)}{f(z)(1-f(z))} - \frac{f''(z)}{f'(z)}, \quad q(z) = \frac{-2f'(z)^2}{9f(z)(1-f(z))}. \tag{2·3}$$

Similarly, differentiating (2·2) twice, and appealing to (1·20), we find that $b(z)$ satisfies the differential equation

$$b'' + r(z)b' + s(z)b = 0,$$

where

$$r(z) = \frac{1-2z}{z(1-z)} - \frac{2g'(z)}{g(z)}, \tag{2·4}$$

$$s(z) = \frac{2g'(z)^2}{g(z)^2} - \frac{g''(z)}{g(z)} - \frac{2}{9z(1-z)} - \frac{(1-2z)g'(z)}{z(1-z)g(z)}. \tag{2·5}$$

A simple MAPLE calculation shows that

$$p(z) = r(z), \quad q(z) = s(z), \tag{2·6}$$

so that $a(z)$ and $b(z)$ satisfy the same second order homogeneous linear differential equation

$$y'' + r(z)y' + q(z)y = 0. \tag{2·7}$$

From (1·10), (1·16) and (2·4), we deduce that

$$\lim_{z\to 0} zr(z) = \lim_{z\to 0}\left(\frac{1-2z}{1-z} - \frac{2zg'(z)}{g(z)}\right) = 1 \tag{2·8}$$

and, from (1·9), (1·15) and (2·3), that

$$\lim_{z\to 0} z^2 q(z) = \lim_{z\to 0}\left(\frac{-2zf'(z)^2}{9\dfrac{f(z)}{z}(1-f(z))}\right) = \frac{-2\cdot 0\cdot \omega^2}{9\cdot\omega\cdot 1} = 0. \tag{2·9}$$

We deduce from (2·8) and (2·9) that $z = 0$ is a regular singular point of (2·7) [**8**, p. 237]. The indicial equation is $\alpha(\alpha-1)+1\alpha+0=0$ [**8**, p. 238], which has the double root $\alpha = 0$. Thus the general solution of (2·7) in a neighbourhood of $z = 0$ is of the form

$$y = Ay_1(z) + By_2(z), \quad A, B \in \mathbb{C},$$

where $y_1(z)$ is a solution of (2·7), which is regular in a neighbourhood of $z = 0$, and $y_2(z)$ is a solution of (2·7) possessing a logarithmic branch point at $z = 0$ [**8**, p. 242]. Hence (2·7) has a unique solution $y(z)$ regular in a neighbourhood of $z = 0$ with a prescribed value for $y(0)$. Since (by (2·1), (1·9), (1·10) and (2·2))

$$a(0) = w(f(0)) = w(0) = g(0)w(0) = b(0),$$

this proves that

$$w(f(z)) = a(z) = b(z) = g(z)w(z) \text{ for } z \in C\left(\tfrac{1}{8}\right).$$

Thus as $w(f(z))$ and $g(z)w(z)$ are defined and regular in $D^*$, and equal in the subset $C\left(\frac{1}{8}\right)$ of $D^*$, we have

$$w(f(z)) = g(z)w(z) \text{ for } z \in D^*,$$

which is (1·28). This completes the proof of Theorem 1·1.

## 3. *Proof of Theorem* 1·2

In this section we use Theorem 1·1 to prove Theorem 1·2. By (1·31) and (1·37) the function

$$\frac{2\pi}{\sqrt{3}}\frac{w(1-z)}{w(z)} - \frac{2\pi}{\sqrt{3}}\frac{w(1-f(z))}{w(f(z))}$$

is defined and regular in $D^{**} \supseteq \left(0, \frac{1}{8}\right)$. For $x \in \left(0, \frac{1}{8}\right)$ we have by [**2**, p. 87]

$$\frac{d}{dx}\left(\frac{2\pi}{\sqrt{3}}\frac{w(1-x)}{w(x)}\right) = \frac{-1}{x(1-x)w(x)^2}.$$

Thus

$$\frac{d}{dx}\left(\frac{2\pi}{\sqrt{3}}\frac{w(1-f(x))}{w(f(x))}\right) = \frac{-f'(x)}{f(x)(1-f(x))w(f(x))^2}.$$

By Theorem 1·1 we have $w(f(x)) = g(x)w(x)$ so that

$$\frac{d}{dx}\left(\frac{2\pi}{\sqrt{3}}\frac{w(1-x)}{w(x)} - \frac{2\pi}{\sqrt{3}}\frac{w(1-f(x))}{w(f(x))}\right)$$

$$= \left(\frac{f'(x)}{f(x)(1-f(x))g(x)^2} - \frac{1}{x(1-x)}\right)\frac{1}{w(x)^2}.$$

A MAPLE calculation shows that

$$\frac{f'(x)}{f(x)(1-f(x))g(x)^2} = \frac{1}{x(1-x)}$$

so that there is a constant $K \in \mathbb{C}$ such that

$$\frac{2\pi}{\sqrt{3}}\frac{w(1-x)}{w(x)} - \frac{2\pi}{\sqrt{3}}\frac{w(1-f(x))}{w(f(x))} = K, \text{ for } 0 < x < \frac{1}{8}.$$

Thus

$$\begin{aligned}
&\lim_{x\to 0^+}(g(x)w(1-x) - w(1-f(x))) \\
&\qquad = \lim_{x\to 0^+}\left(g(x)w(x)\left(\frac{w(1-x)}{w(x)} - \frac{w(1-f(x))}{w(f(x))}\right)\right) \\
&\qquad = g(0)w(0)\frac{\sqrt{3}}{2\pi}K \\
&\qquad = \frac{\sqrt{3}}{2\pi}K,
\end{aligned}$$

by (1·10) and (1·21). On the other hand by [**1**, formula (15.3.10), p. 559], we have

$$w(1-x) = -\frac{\sqrt{3}}{2\pi}w(x)\log x + A(x), \quad |x| < 1,$$

where

$$A(x) = \frac{\sqrt{3}}{2\pi}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(n!)^2}\left(2\psi(n+1) - \psi\left(n+\tfrac{1}{3}\right) - \psi\left(n+\tfrac{2}{3}\right)\right)x^n,$$

and

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0,$$

is the digamma function. Hence

$$\begin{aligned}
&\lim_{x\to 0^+}(g(x)w(1-x) - w(1-f(x))) \\
&\qquad = \lim_{x\to 0^+}(g(x)A(x) - A(f(x)) + \frac{\sqrt{3}}{2\pi}(\log f(x) - \log x)w(f(x)) \\
&\qquad = g(0)A(0) - A(0) + \frac{\sqrt{3}}{2\pi}\lim_{x\to 0^+}\log\frac{f(x)}{x} \\
&\qquad = \frac{\sqrt{3}}{2\pi}\log\omega \qquad \text{(by (1·15))} \\
&\qquad = \frac{i}{\sqrt{3}}.
\end{aligned}$$

Hence

$$K = \frac{2\pi i}{3}.$$

Thus

$$\frac{2\pi}{\sqrt{3}}\frac{w(1-x)}{w(x)} - \frac{2\pi}{\sqrt{3}}\frac{w(1-f(x))}{w(f(x))} - \frac{2\pi i}{3} = 0 \text{ for } x \in \left(0, \tfrac{1}{8}\right). \tag{3·1}$$

As the left–hand side of (3·1) is defined and regular in $D^{**}$ and $D^{**} \supseteq \left(0, \frac{1}{8}\right)$, we have

$$\frac{2\pi}{\sqrt{3}} \frac{w(1-z)}{w(z)} - \frac{2\pi}{\sqrt{3}} \frac{w(1-f(z))}{w(f(z))} - \frac{2\pi i}{3} = 0 \text{ for } z \in D^{**},$$

which is (1·38). This completes the proof of Theorem 1·2.

## 4. *Proof of Theorem* 1·3

We just give the details for (1·48) as (1·47) and (1·49) can be proved in a similar manner. The function $M(q)$ is analytic in $C(1)$. For $0 < x < 1$ we have by (1·43) and (1·45)

$$M(e^{-\frac{2\pi}{\sqrt{3}} \frac{w(1-x)}{w(x)}}) = (1+8x)w(x)^4.$$

By the principle of analytic continuation, $M$ has an analytic continuation $M^*$ such that

$$M^*(e^{-\frac{2\pi}{\sqrt{3}} \frac{w(1-z)}{w(z)}}) = (1+8z)w(z)^4, \quad z \in D, \quad w(z) \neq 0,$$

and

$$M^*(q) = M(q), \quad q \in C(1).$$

For $z \in D^{**}$, we have $f(z) \in D$, $1 - f(z) \in D$ and $w(f(z)) \neq 0$, so that

$$M^*(e^{-\frac{2\pi}{\sqrt{3}} \frac{w(1-f(z))}{w(f(z))}}) = (1+8f(z))w(f(z))^4.$$

Then, by Theorems 1·1 and 1·2, we obtain

$$M^*(\omega e^{-\frac{2\pi}{\sqrt{3}} \frac{w(1-z)}{w(z)}}) = (1+8f(z))(g(z)w(z))^4.$$

Now take $z = x \in \left(0, \frac{1}{8}\right) \subseteq D^{**}$. Then

$$M^*(\omega q) = (1+8f(x))(g(x)w(x))^4.$$

As $|\omega q| = |q| < 1$ we have $M^*(\omega q) = M(\omega q)$ so that

$$M(\omega q) = (1+8f(x))(g(x)w(x))^4. \tag{4·1}$$

Replacing $f(x)$ and $g(x)$ in (4·1) by the expressions in (1·6) and (1·7) respectively, we obtain (1·48).

## 5. *Proof of Theorem* 1·4

We begin by applying the principle of triplication to (1·44), (1·45) and (1·46). As triplication sends

$$q \longrightarrow q^3, \quad x \longrightarrow \left(\frac{1-(1-x)^{\frac{1}{3}}}{1+2(1-x)^{\frac{1}{3}}}\right)^3, \quad w \longrightarrow \frac{1}{3}\left(1+2(1-x)^{\frac{1}{3}}\right) w,$$

see [**4**, theorem 3·1, p. 101], we obtain

$$L(q^3) = \left(1 - \frac{4}{3}x\right) w^2 + 4x(1-x)w\frac{dw}{dx}, \tag{5·1}$$

$$M(q^3) = \left(1 - \frac{8}{9}x\right) w^4, \tag{5·2}$$

$$N(q^3) = \left(1 - \frac{4}{3}x + \frac{8}{27}x^2\right) w^6, \tag{5·3}$$

see [**4**, equation (13·17), p. 178, theorems 4·4 and 4·5, p. 107]. Applying the principle of triplication to (5·1), (5·2) and (5·3), we obtain

$$L(q^9) = \left(11 - 12x + 8(1-x)^{\frac{1}{3}} + 8(1-x)^{\frac{2}{3}}\right) \frac{w^2}{27} + \frac{4}{3}x(1-x)w\frac{dw}{dx}, \tag{5·4}$$

$$M(q^9) = \left(249 - 248x + 240(1-x)^{\frac{1}{3}} + 240(1-x)^{\frac{2}{3}} - 160x(1-x)^{\frac{1}{3}}\right) \frac{w^4}{729}, \tag{5·5}$$

$$N(q^9) = \Big(6579 - 8604x + 2024x^2 + 6552(1-x)^{\frac{1}{3}} + 6552(1-x)^{\frac{2}{3}}$$
$$- 6384x(1-x)^{\frac{1}{3}} - 4704x(1-x)^{\frac{2}{3}}\Big) \frac{w^6}{19683}. \tag{5·6}$$

Clearly, from (1·50) – (1·52) and (1·39) – (1·41), we obtain

$$L_1(q) + L_2(q) + L_3(q) = \sum_{n=1}^{\infty} \sigma(n)q^n = \frac{1 - L(q)}{24}, \tag{5·7}$$

$$M_1(q) + M_2(q) + M_3(q) = \sum_{n=1}^{\infty} \sigma_3(n)q^n = \frac{M(q) - 1}{240}, \tag{5·8}$$

$$N_1(q) + N_2(q) + N_3(q) = \sum_{n=1}^{\infty} \sigma_5(n)q^n = \frac{1 - N(q)}{504}. \tag{5·9}$$

Now for all $k \in \mathbb{N}$ we have

$$\sigma(3k) = 4\sigma(k) - 3\sigma(k/3)$$

so that

$$\begin{aligned} L_3(q) &= \sum_{k=1}^{\infty} \sigma(3k)q^{3k} \\ &= 4\sum_{k=1}^{\infty} \sigma(k)q^{3k} - 3\sum_{k=1}^{\infty} \sigma(k/3)q^{3k} \\ &= 4\sum_{k=1}^{\infty} \sigma(k)q^{3k} - 3\sum_{k=1}^{\infty} \sigma(k)q^{9k} \\ &= 4\left(\frac{1 - L(q^3)}{24}\right) - 3\left(\frac{1 - L(q^9)}{24}\right), \end{aligned}$$

that is

$$L_3(q) = \frac{1 - 4L(q^3) + 3L(q^9)}{24}. \tag{5·10}$$

Then, from (5·7) and (5·10), we deduce

$$L_1(q) + L_2(q) = \frac{-L(q) + 4L(q^3) - 3L(q^9)}{24}. \tag{5·11}$$

Similarly, using

$$\sigma_3(3k) = 28\sigma_3(k) - 27\sigma_3(k/3),$$
$$\sigma_5(3k) = 244\sigma_5(k) - 243\sigma_5(k/3),$$

we obtain

$$M_3(q) = \frac{-1 + 28M(q^3) - 27M(q^9)}{240}, \tag{5·12}$$

$$M_1(q) + M_2(q) = \frac{M(q) - 28M(q^3) + 27M(q^9)}{240}, \tag{5·13}$$

$$N_3(q) = \frac{1 - 244N(q^3) + 243N(q^9)}{504}, \tag{5·14}$$

$$N_1(q) + N_2(q) = \frac{-N(q) + 244N(q^3) - 243N(q^9)}{504}. \tag{5·15}$$

From (5·1)–(5·6), (5·10), (5·12) and (5·14), we obtain

$$L_3(q) = \frac{1}{24} + \left(-25 + 36x + 8(1-x)^{\frac{1}{3}} + 8(1-x)^{\frac{2}{3}}\right)\frac{w^2}{216} - \frac{1}{2}x(1-x)w\frac{dw}{dx}, \tag{5·16}$$

$$M_3(q) = -\frac{1}{240} + \left(507 - 424x - 240(1-x)^{\frac{1}{3}} - 240(1-x)^{\frac{2}{3}} + 160x(1-x)^{\frac{1}{3}}\right)\frac{w^4}{6480}, \tag{5·17}$$

$$N_3(q) = \frac{1}{504} + \left(-355995 + 479196x - 103464x^2 + 176904(1-x)^{\frac{1}{3}} + 176904(1-x)^{\frac{2}{3}} - 172368x(1-x)^{\frac{1}{3}} - 127008x(1-x)^{\frac{2}{3}}\right)\frac{w^6}{1102248}. \tag{5·18}$$

From (1·44) – (1·46), (5·1) – (5·6), (5·11), (5·13) and (5·15), we obtain

$$L_1(q) + L_2(q) = \left(2 - (1-x)^{\frac{1}{3}} - (1-x)^{\frac{2}{3}}\right)\frac{w^2}{27}, \tag{5·19}$$

$$M_1(q) + M_2(q) = \left(-6 + 8x + 3(1-x)^{\frac{1}{3}} + 3(1-x)^{\frac{2}{3}} - 2x(1-x)^{\frac{1}{3}}\right)\frac{w^4}{81}, \tag{5·20}$$

$$N_1(q) + N_2(q) = \left(234 - 288x + 80x^2 - 117(1-x)^{\frac{1}{3}} - 117(1-x)^{\frac{2}{3}} + 114x(1-x)^{\frac{1}{3}} + 84x(1-x)^{\frac{2}{3}}\right)\frac{w^6}{729}. \tag{5·21}$$

The formulae (1·39) – (1·41) are valid for all $q \in \mathbb{C}$ with $|q| < 1$. Replacing $q$ by $\omega q$ in each of (1·39), (1·40) and (1·41), we obtain respectively

$$L(\omega q) = 1 - 24\omega L_1(q) - 24\omega^2 L_2(q) - 24L_3(q), \tag{5·22}$$

$$M(\omega q) = 1 + 240\omega M_1(q) + 240\omega^2 M_2(q) + 240 M_3(q), \tag{5·23}$$

$$N(\omega q) = 1 - 504\omega N_1(q) - 504\omega^2 N_2(q) - 504 N_3(q). \tag{5·24}$$

From (1·47), (5·16) and (5·22), we obtain

$$\omega L_1(q) + \omega^2 L_2(q) = \left(-1 + (\omega - 2\omega^2)(1-x)^{\frac{1}{3}} + (-2\omega + \omega^2)(1-x)^{\frac{2}{3}}\right)\frac{w^2}{27}. \tag{5·25}$$

Solving (5·19) and (5·25) for $L_1(q)$ and $L_2(q)$, we obtain

$$L_1(q) = \left(1 + (1-x)^{\frac{1}{3}} - 2(1-x)^{\frac{2}{3}}\right)\frac{w^2}{27}, \tag{5·26}$$

$$L_2(q) = \left(1 - 2(1-x)^{\frac{1}{3}} + (1-x)^{\frac{2}{3}}\right)\frac{w^2}{27}. \tag{5·27}$$

From (1·48), (5·17) and (5·23), we obtain

$$\begin{aligned}\omega M_1(q) + \omega^2 M_2(q) = &\Big(6 - 8x - 3(1-x)^{\frac{1}{3}} - 3(1-x)^{\frac{2}{3}} + 2x(1-x)^{\frac{1}{3}} \\ &+ (\omega - \omega^2)(-9(1-x)^{\frac{1}{3}} + 9(1-x)^{\frac{2}{3}} \\ &+ 6x(1-x)^{\frac{1}{3}})\Big)\frac{w^4}{162}.\end{aligned} \tag{5·28}$$

Solving (5·20) and (5·28) for $M_1(q)$ and $M_2(q)$, we obtain

$$M_1(q) = \left(-3 + 4x - 3(1-x)^{\frac{1}{3}} + 6(1-x)^{\frac{2}{3}} + 2x(1-x)^{\frac{1}{3}}\right)\frac{w^4}{81}, \tag{5·29}$$

$$M_2(q) = \left(-3 + 4x + 6(1-x)^{\frac{1}{3}} - 3(1-x)^{\frac{2}{3}} - 4x(1-x)^{\frac{1}{3}}\right)\frac{w^4}{81}. \tag{5·30}$$

From (1·49), (5·18) and (5·24), we obtain

$$\omega N_1(q) + \omega^2 N_2(q) = \left(A + (\omega - \omega^2)B\right)\frac{z^6}{1458}, \tag{5·31}$$

where

$$A = -234 + 288x - 89x^2 + 117(1-x)^{\frac{1}{3}} + 117(1-x)^{\frac{2}{3}} - 114x(1-x)^{\frac{1}{3}} - 84x(1-x)^{\frac{2}{3}}$$

and

$$B = 351(1-x)^{\frac{1}{3}} - 351(1-x)^{\frac{2}{3}} - 342x(1-x)^{\frac{1}{3}} + 252x(1-x)^{\frac{2}{3}}.$$

Solving (5·21) and (5·31) for $N_1(q)$ and $N_2(q)$, we obtain

$$\begin{aligned}N_1(q) = &\Big(117 - 144x + 40x^2 + 117(1-x)^{\frac{1}{3}} - 234(1-x)^{\frac{2}{3}} \\ &-114x(1-x)^{\frac{1}{3}} + 168x(1-x)^{\frac{2}{3}}\Big)\frac{w^6}{729},\end{aligned} \tag{5·32}$$

$$\begin{aligned}N_2(q) = &\Big(117 - 144x + 40x^2 - 234(1-x)^{\frac{1}{3}} + 117(1-x)^{\frac{2}{3}} \\ &+228x(1-x)^{\frac{1}{3}} - 848x(1-x)^{\frac{2}{3}}\Big)\frac{w^6}{729}.\end{aligned} \tag{5·33}$$

This completes the proof of Theorem 1·4.

6. *Proof of Theorem* 1·5

We just give the proof of the first identity in Theorem 1·5 as the remaining three identities can be proved similarly. From (1·53) and (1·57), we obtain

$$\begin{aligned} L_1(q)^2 &= \Big(1 + (1-x)^{\frac{2}{3}} + 4(1-x)(1-x)^{\frac{1}{3}} + 2(1-x)^{\frac{1}{3}} \\ &- 4(1-x)^{\frac{2}{3}} - 4(1-x)\Big)\frac{w^4}{729} \\ &= \left(-3 + 4x + 6(1-x)^{\frac{1}{3}} - 3(1-x)^{\frac{2}{3}} - 4x(1-x)^{\frac{1}{3}}\right)\frac{w^4}{729} \\ &= \frac{1}{9}M_2(q) \end{aligned}$$

as asserted.

7. *Proof of Theorem* 1·6

We just give the proof of the second identity in Theorem 1·6 as the remaining three identities can be proved similarly. By Theorem 1·4 we have

$$L_2(q) = \left(1 - 2(1-x)^{\frac{1}{3}} + (1-x)^{\frac{2}{3}}\right)\frac{w^2}{27}$$

so that

$$\frac{dL_2}{dx} = \left(\frac{2}{3}(1-x)^{-\frac{2}{3}} - \frac{2}{3}(1-x)^{-\frac{1}{3}}\right)\frac{w^2}{27} + \left(1 - 2(1-x)^{\frac{1}{3}} + (1-x)^{\frac{2}{3}}\right)\frac{2w}{27}\frac{dw}{dx}.$$

By (1·43) and (1·42), we have

$$\frac{dq}{dx} = -e^{-y}\frac{dy}{dx} = \frac{q}{x(1-x)w^2}$$

so that

$$\begin{aligned} q\frac{dL_2}{dq} &= x(1-x)w^2\frac{dL_2}{dx} \\ &= x\big((1-x)^{\frac{1}{3}} - (1-x)^{\frac{2}{3}}\big)\frac{2}{81}w^4 \\ &\quad + x(1-x)\left(1 - 2(1-x)^{\frac{1}{3}} + (1-x)^{\frac{2}{3}}\right)\frac{2w^3}{27}\frac{dw}{dx}. \end{aligned}$$

From (1·54) and (5·1), we obtain

$$\begin{aligned} L_2(q)L(q^3) &= \left((3-4x) - 2(3-4x)(1-x)^{\frac{1}{3}} + (3-4x)(1-x)^{\frac{2}{3}}\right)\frac{w^4}{81} \\ &\quad + 4\left(1 - 2(1-x)^{\frac{1}{3}} + (1-x)^{\frac{2}{3}}\right)x(1-x)\frac{w^3}{27}\frac{dw}{dx}. \end{aligned}$$

Hence, appealing to (1·57), we deduce

$$\begin{aligned} L_2(q)L(q^3) + M_2(q) &= 4x\left((1-x)^{\frac{1}{3}} - (1-x)^{\frac{2}{3}}\right)\frac{w^4}{81} \\ &\quad + 4\left(1 - 2(1-x)^{\frac{1}{3}} + (1-x)^{\frac{2}{3}}\right)x(1-x)\frac{w^3}{27}\frac{dw}{dx} \\ &= 2q\frac{dL_2}{dq}, \end{aligned}$$

completing the proof of the identity.

## 8. *Proof of Theorem* 1·7

We just give the proof of the first identity in Theorem 1·7. The second identity can be proved similarly. By (1·54) and (5·4), we obtain

$$9L_2(q)L(q^9) = \left(3 - 4x - 6(1-x)^{\frac{1}{3}} - 9(1-x)^{\frac{2}{3}} + 16x(1-x)^{\frac{1}{3}}\right)\frac{w^4}{81} + \frac{4}{9}\left(1 - 2(1-x)^{\frac{1}{3}} + (1-x)^{\frac{2}{3}}\right)x(1-x)w^3\frac{dw}{dx}.$$

Hence, appealing to (1·57), we have

$$\begin{aligned} &9L_2(q)L(q^9) + M_2(q) \\ &= 4x\left((1-x)^{\frac{1}{3}} - (1-x)^{\frac{2}{3}}\right)\frac{w^4}{27} + \frac{4}{9}\left(1 - 2(1-x)^{\frac{1}{3}} + (1-x)^{\frac{2}{3}}\right)x(1-x)w^3\frac{dw}{dx} \\ &= 6q\frac{dL_2}{dq}, \end{aligned}$$

completing the proof of the identity.

## 9. *Proof of Theorem* 1·8

We just give the proof of the first identity. The other identities can be proved similarly. We have

$$\begin{aligned} \sum_{\substack{n=1 \\ n\equiv 2 \,(\mathrm{mod}\, 3)}}^{\infty} S(1,3)q^n &= \sum_{\substack{n=1 \\ n\equiv 2 \,(\mathrm{mod}\, 3)}}^{\infty} \sum_{\substack{k=1 \\ k\equiv 1 \,(\mathrm{mod}\, 3)}}^{n-1} \sigma(k)\sigma(n-k)q^n \\ &= \sum_{N=0}^{\infty} \sum_{\substack{k=1 \\ k\equiv 1 \,(\mathrm{mod}\, 3)}}^{3N+1} \sigma(k)\sigma(3N+2-k)q^{3N+2} \\ &= \sum_{N=0}^{\infty}\sum_{m=0}^{N} \sigma(3m+1)\sigma(3(N-m)+1)q^{3N+2} \\ &= \sum_{m,n=0}^{\infty} \sigma(3m+1)\sigma(3n+1)q^{3(m+n)+2} \\ &= \left(\sum_{n=0}^{\infty} \sigma(3n+1)q^{3n+1}\right)^2 \\ &= L_1(q)^2 \\ &= \frac{1}{9}M_2(q) \\ &= \frac{1}{9}\sum_{\substack{n=1 \\ n\equiv 2 \,(\mathrm{mod}\, 3)}}^{\infty} \sigma_3(n)q^n, \end{aligned}$$

by the first identity in Theorem 1·5, and the result follows on equating coefficients of $q^n$ with $n \equiv 2 \,(\mathrm{mod}\, 3)$.

## 10. *Proof of Theorem* 1·9

Appealing to the fourth identity in Theorem 1·6, we obtain

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m<n/3} \sigma(m)\sigma(n-3m)q^n \\
&= \sum_{l=1}^{\infty} \sigma(l)q^l \sum_{m=1}^{\infty} \sigma(m)q^{3m} \\
&= \left(\frac{1-L(q)}{24}\right)\left(\frac{1-L(q^3)}{24}\right) \\
&= \frac{1}{576}\left(1-L(q)-L(q^3)+L(q)L(q^3)\right) \\
&= \frac{1}{576}\left(1-L(q)-L(q^3)+\frac{1}{10}M(q)+\frac{9}{10}M(q^3)+2q\frac{dL(q)}{dq}+\frac{9}{2}q\frac{dL(q^9)}{dq}-36q\frac{dL_3(q)}{dq}\right) \\
&= \frac{1}{576}\left(24\sum_{n=1}^{\infty}\sigma(n)q^n+24\sum_{n=1}^{\infty}\sigma(n/3)q^n+24\sum_{n=1}^{\infty}\sigma_3(n)q^n+216\sum_{n=1}^{\infty}\sigma_3(n/3)q^n\right. \\
&\qquad \left. -48\sum_{n=1}^{\infty}n\sigma(n)q^n-108\sum_{n=1}^{\infty}n\sigma(n/9)q^n-36\sum_{n\geqslant 1,\,3\mid n}n\sigma(n)q^n\right).
\end{aligned}
$$

Equating coefficients of $q^n$, and appealing to the elementary result

$$
4\sigma(n/3)-3\sigma(n/9)=\begin{cases}\sigma(n), & \text{if } 3\mid n,\\ 0, & \text{if } 3\nmid n.\end{cases}
$$

we obtain the assertion of Theorem 1·9.

## 11. *Proof of Theorem* 1·10

We just give the proof of the first identity. The second identity can be proved similarly. We have by (1·50) and (1·39)

$$
\begin{aligned}
9L_2(q)L(q^9) &= 9\left(\sum_{\substack{m=1\\ m\equiv 2 \pmod 3}}^{\infty}\sigma(m)q^m\right)\left(1-24\sum_{l=1}^{\infty}\sigma(l)q^{9l}\right) \\
&= 9\sum_{\substack{n=1\\ n\equiv 2 \pmod 3}}^{\infty}\sigma(n)q^n-216\sum_{n=1}^{\infty}q^n\sum_{\substack{l,m\geqslant 1\\ 9l+m=n\\ m\equiv 2 \pmod 3}}\sigma(l)\sigma(m) \\
&= 9\sum_{\substack{n=1\\ n\equiv 2 \pmod 3}}^{\infty}\sigma(n)q^n-216\sum_{\substack{n=1\\ n\equiv 2 \pmod 3}}^{\infty}q^n\sum_{l<n/9}\sigma(l)\sigma(n-9l).
\end{aligned}
$$

On the other hand, by the first identity in Theorem 1·7, we have

$$
9L_2(q)L(q^9)=6q\frac{dL_2}{dq}-M_2(q)=6\sum_{\substack{n=1\\ n\equiv 2 \pmod 3}}^{\infty}n\sigma(n)q^n-\sum_{\substack{n=1\\ n\equiv 2 \pmod 3}}^{\infty}\sigma_3(n)q^n.
$$

Equating coefficients of $q^n$ ($n \equiv 2 \pmod 3$), we obtain the asserted identity.

## 12. *Proof of Theorem* 1·11

We have by (1·62)

$$\sum_{\substack{n=1 \\ n\equiv 2 \ (\mathrm{mod}\ 3)}}^{\infty} \sum_{\substack{k=1 \\ k\equiv 1 \ (\mathrm{mod}\ 3)}}^{n-1} k\sigma(k)\sigma(n-k)q^n = \sum_{\substack{l=1 \\ l\equiv 1 \ (\mathrm{mod}\ 3)}}^{\infty} \sigma(l)q^l \sum_{\substack{k=1 \\ k\equiv 1 \ (\mathrm{mod}\ 3)}}^{\infty} k\sigma(k)q^k$$

$$= L_1(q)q\frac{dL_1(q)}{dq} = \frac{1}{18}q\frac{dM_2(q)}{dq}$$

$$= \frac{1}{18} \sum_{\substack{n=1 \\ n\equiv 2 \ (\mathrm{mod}\ 3)}}^{\infty} n\sigma_3(n)q^n.$$

Equating coefficients of $q^n$ ($n \equiv 2 \pmod 3$), we obtain the asserted result.

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