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EVALUATION OF WEBER'S FUNCTIONS AT QUADRATIC IRRATIONALITIES

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Abstract

Let $\eta(z)$ denote the Dedekind eta function. Let $ax^2 + bxy + cy^2$ be a positive-definite, primitive, integral, binary quadratic form of discriminant $d(= b^2 - 4ac < 0)$. The value of $|\eta((b + \sqrt{d})/2a)|$ is determined for an arbitrary discriminant d. This result generalizes the corresponding result when d is fundamental, which was obtained by van der Poorten and Williams [Canad. J. Math. 51 (1999), 176-224. Corrigendum, Canad. J. Math. 53 (2001), 434-448]. As a consequence of our evaluation of $|\eta(z)|$ for $z = ((b + \sqrt{d})/2a)$, formulae are obtained for the moduli of Weber's functions $f(z)$, $f_1(z)$ and $f_2(z)$ [Lehrbuch der Algebra, Vol. III, Chelsea Publishing Co., New York, 1961, p. 114]. From

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these formulae the values of $f(\sqrt{-m})$, $f_1(\sqrt{-m})$ and $f_2(\sqrt{-m})$ are determined for an arbitrary positive integer m.

1. Introduction

The Dedekind eta function $\eta(z)$ is defined for all $z = x + iy \in \mathbb{C}$ with $y > 0$ by

$$\eta(z) = e^{\pi iz/12}\prod_{m=1}^{\infty}(1 - e^{2\pi imz}). \tag{1}$$

The fundamental transformation formulae of $\eta(z)$ are [18, pp. 17-18], [20, p. 113]

$$\eta(z+1) = e^{\pi i/12}\eta(z), \quad \eta\left(-\frac{1}{z}\right) = \sqrt{-iz}\,\eta(z). \tag{2}$$

We also note that

$$\eta(iy) \in \mathbb{R}^+, \quad e^{-\pi i/24}\eta\left(\frac{1+iy}{2}\right) \in \mathbb{R}^+, \tag{3}$$

for $y > 0$. Weber's three functions $f(z)$, $f_1(z)$ and $f_2(z)$ are defined in terms of the Dedekind eta function as follows:

$$f(z) = \frac{e^{-\pi i/24}\eta\left(\frac{z+1}{2}\right)}{\eta(z)}, \tag{4}$$

$$f_1(z) = \frac{\eta\left(\frac{z}{2}\right)}{\eta(z)}, \tag{5}$$

and

$$f_2(z) = \sqrt{2}\,\frac{\eta(2z)}{\eta(z)}, \tag{6}$$

see [20, p. 114]. It is convenient to set

$$f_0(z) = f(z)$$

so that $f_i(z)$ is defined for $i = 0, 1, 2$. Weber's functions satisfy the relations [20, p. 114]

$$f_0(z)^8 = f_1(z)^8 + f_2(z)^8 \tag{7}$$

and

$$f_0(z)f_1(z)f_2(z) = \sqrt{2}. \tag{8}$$

For n a positive integer, Ramanujan's class invariants G_n and g_n are defined by

$$G_n = 2^{-1/4} f_0(\sqrt{-n}) \tag{9}$$

and

$$g_n = 2^{-1/4} f_1(\sqrt{-n}), \tag{10}$$

see [2, p. 183]. From (3)-(6) we see that $f_i(\sqrt{-n}) \in \mathbb{R}^+$ for $i = 0, 1, 2$ so that G_n, $g_n \in \mathbb{R}^+$. The values of G_n and g_n have been determined for many values of n. Traditionally G_n is determined for odd values of n and g_n for even values of n [2, p. 184]. Berndt gives a wealth of values of n for which G_n and g_n have been determined by various authors using such techniques as modular equations, Kronecker's limit formula or class field theory [2, Chapter 34]. In this paper we use Kronecker's limit formula together with some new results on binary quadratic forms to determine the value of

$$\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|,$$

where a, b, c, d are integers such that $ax^2 + bxy + cy^2$ is a positive-definite, primitive, integral, binary quadratic form of discriminant $d(= b^2 - 4ac < 0)$, see Theorem 1 in Section 9. The Chowla-Selberg formula [5, 17] gives the value of

$$\prod_{K=[a,b,c]\in H(d)} \left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|$$

for a fundamental discriminant d, that is, a discriminant d for which $d/m^2 \equiv 0$ or $1 \pmod 4 \Rightarrow |m| = 1$. This formula was extended to an arbitrary discriminant d by Kaneko [10], Nakkajima and Taguchi [13], and Kaplan and Williams [11]. We show that this result is a simple consequence of Theorem 1, see Corollary 1 in Section 9. Theorem 1 extends a recent result of van der Poorten and Williams [16] giving the value of $|\eta((b+\sqrt{d})/2a)|$ in the case when d is fundamental, see Corollary 2 in Section 9. Appealing to (4)-(6) and Theorem 1, we obtain the values of

$$\left| f_i\left(\frac{b+\sqrt{d}}{2a}\right)\right|, \quad i = 0, 1, 2,$$

see Theorem 2 in Section 10. From Theorem 2 we deduce the values of

$$f_i(\sqrt{-m}), \quad i = 0, 1, 2,$$

for an arbitrary positive integer m, see Theorem 3 in Section 10. We illustrate Theorem 3 by deducing from it the known result (Weber [20])

$$f(\sqrt{-19}) = \theta, \tag{11}$$

where θ is the unique real root of the cubic equation $x^3 - 2x - 2 = 0$, see Theorem 4 in Section 11. In a future paper we plan to obtain further explicit results of this type from our formulae.

Throughout this paper n denotes a positive integer, p denotes a prime and d is an integer satisfying

$$d < 0, \; d \equiv 0 \text{ or } 1 \pmod 4. \tag{12}$$

We denote the Kronecker symbol, see for example [9, p. 278], by $\left(\frac{d}{m}\right)$ or (d/m) as convenient. The unique integer v such that $p^v \mid n$ and $p^{v+1} \nmid n$ is denoted by $v_p(n)$. The largest positive integer f such that

$$f^2 \mid d, \; d/f^2 \equiv 0 \text{ or } 1 \pmod 4) \tag{13}$$

is called the *conductor* of d. The integer $\Delta = d/f^2$ is called the *fundamental discriminant* associated with d. If $ax^2 + bxy + cy^2$ is a positive-definite, primitive, integral, binary quadratic form of discriminant d, the class of this form under the action of the modular group is denoted by $[a, b, c]$. We describe briefly how two classes A_1 and A_2 of forms of the same discriminant d can be multiplied (composed). This method of composing two classes is due to Dirichlet. Representatives of the classes A_1 and A_2 can be chosen so that

$$A_1 = [a, B, cC] \text{ and } A_2 = [c, B, aC].$$

Then the product (composition) of A_1 and A_2 is given by

$$A_1A_2 = [ac, B, C].$$

For more details the reader should consult [7, Chapter IX]. With respect to this multiplication, the classes of forms of discriminant d form a finite abelian group $H(d)$ called the *form class group* of discriminant d. The order of $H(d)$ is called the *class number* of discriminant d and is denoted by $h(d)$. The identity I of the group $H(d)$ is the principal class

$$I = \begin{cases} [1, 0, -d/4], & \text{if } d \equiv 0 \pmod 4, \\ [1, 1, (1-d)/4], & \text{if } d \equiv 1 \pmod 4. \end{cases} \tag{14}$$

The inverse of the class $K = [a, b, c] \in H(d)$ is the class $K^{-1} = [a, -b, c]$. The genus group $G(d)$ is defined to be

$$G(d) = H(d)/H^2(d). \tag{15}$$

Its order is

$$|G(d)| = 2^{t(d)}, \tag{16}$$

where $t(d)$ is a nonnegative integer. The reader will find the exact value of $t(d)$ in [9, p. 277]. An element of $G(d)$ is called a *genus*. The reader will find discussions of the theory of binary quadratic forms in [3], [7], [8] and [15].

If x and y are integers such that

$$n = ax^2 + bxy + cy^2,$$

then (x, y) is called a *representation* of the positive integer n by the form $ax^2 + bxy + cy^2$. As $ax^2 + bxy + cy^2$ is a positive-definite form the number $R_{(a,b,c)}(n, d)$ of representations of n by the form $ax^2 + bxy + cy^2$ is finite. It is well-known that if $Ax^2 + Bxy + Cy^2$ is a form equivalent to the form $ax^2 + bxy + cy^2$, then $R_{(A,B,C)}(n, d) = R_{(a,b,c)}(n, d)$. Hence we can define the number of representations of the positive integer n by the class $K \in H(d)$ by

$$R_K(n, d) = R_{(a,b,c)}(n, d) \text{ for any form } ax^2 + bxy + cy^2 \in K. \tag{17}$$

If $G \in G(d)$ the number of representations of n by the classes in the genus G is denoted by $R_G(n, d)$ and is given by

$$R_G(n, d) = \sum_{K \in G} R_K(n, d). \tag{18}$$

The total number of representations of n by the classes in $H(d)$ is

$$N(n, d) = \sum_{K \in H(d)} R_K(n, d) = \sum_{G \in G(d)} R_G(n, d). \tag{19}$$

The number $w(d)$ of automorphs of a primitive, positive-definite, binary quadratic form of discriminant d is given by (see for example [15, pp. 172-176])

$$w(d) = \begin{cases} 6, & \text{if } d = -3, \\ 4, & \text{if } d = -4, \\ 2, & \text{if } d < -4. \end{cases} \tag{20}$$

In Sections 2-8 we develop the results on binary quadratic forms that we need in order to prove our main results, namely Theorems 1, 2, 3 in Sections 9 and 10.

2. Three Lemmas

Our first lemma is well-known.

Lemma 2.1. *Let n_1 and n_2 be relatively prime positive integers.*

(i) *Let B be an integer such that $0 \le B < 2n_1n_2$, $B^2 \equiv d \pmod{4n_1n_2}$. Then there exist unique integers b_1 and b_2 such that*

$$0 \le b_1 < 2n_1, \; b_1^2 \equiv d \pmod{4n_1}, \; b_1 \equiv B \pmod{2n_1},$$

$$0 \le b_2 < 2n_2, \; b_2^2 \equiv d \pmod{4n_2}, \; b_2 \equiv B \pmod{2n_2}.$$

(ii) *Let* b_1 *and* b_2 *be integers such that* $0 \le b_1 < 2n_1$, $b_1^2 \equiv d \pmod{4n_1}$, $0 \le b_2 < 2n_2$, $b_2^2 \equiv d \pmod{4n_2}$. *Then there exists a unique integer* B *with*

$$0 \le B < 2n_1n_2, \quad B^2 \equiv d \pmod{4n_1n_2},$$

$$B \equiv b_1 \pmod{2n_1}, \quad B \equiv b_2 \pmod{2n_2}.$$

Proof. This is essentially [7, Lemma 2, p. 134].

Lemma 2.2. *Let* p *be a prime which does not divide* d. *Let* h_1 *be an integer such that*

$$0 \le h_1 < 2p, \quad h_1^2 \equiv d \pmod{4p}.$$

Then, for each positive integer n, *there is a unique integer* h_n *such that*

$$0 \le h_n < 2p^n, \quad h_n \equiv h_1 \pmod{2p}, \quad h_n^2 \equiv d \pmod{4p^n}.$$

Proof. We note that $p \nmid h_1$ since $p \nmid d$. We use induction on n. The result is clearly true for $n = 1$. We assume that the result is true for $n = N$. Thus there is a unique integer h_N with

$$0 \le h_N < 2p^N, \quad h_N \equiv h_1 \pmod{2p}, \quad h_N^2 \equiv d \pmod{4p^N}.$$

Note that $p \nmid h_N$. Hence there is a unique integer λ_N satisfying

$$0 \le \lambda_N < p, \quad \lambda_N h_N \equiv -\frac{(h_N^2 - d)}{4p^N} \pmod{p}.$$

We set $h_{N+1} = h_N + 2\lambda_N p^N$. Then we have $0 \le h_{N+1} \le 2p^N - 1 + 2(p-1)p^N < 2p^{N+1}$ and $h_{N+1} \equiv h_N \equiv h_1 \pmod{2p}$. Also

$$h_{N+1}^2 = h_N^2 + 4\lambda_N h_N p^N + 4\lambda_N^2 p^{2N} \equiv h_N^2 - (h_N^2 - d) \equiv d \pmod{4p^{N+1}}.$$

To show that h_{N+1} is unique, let $h'_{N+1} \ne h_{N+1}$ be another integer such that

$$0 \le h'_{N+1} < 2p^{N+1}, \quad h'_{N+1} \equiv h_1 \pmod{2p}, \quad h'^2_{N+1} \equiv d \pmod{4p^{N+1}}.$$

We define nonzero integers u and v by

$$u = \frac{1}{2}(h'_{N+1} - h_{N+1}), \quad v = \frac{1}{2}(h'_{N+1} + h_{N+1}).$$

Then $u \equiv 0 \pmod{p}$ and $uv \equiv 0 \pmod{p^{N+1}}$. Let $p^{\alpha} \parallel u$ and $p^{\beta} \parallel v$, so that $\alpha \geq 1$, $\alpha + \beta \geq N+1$. If $\beta \geq 1$, then $p \mid u$ and $p \mid v$ so that $p \mid v - u = h_{N+1}$, a contradiction. Hence $\beta = 0$ and $\alpha \geq N+1$. Thus

$$h'_{N+1} \equiv h_{N+1} \pmod{2p^{N+1}}, \quad 0 \leq h_{N+1}, h'_{N+1} < 2p^{N+1}.$$

This gives $h'_{N+1} = h_{N+1}$, a contradiction. Hence h_{N+1} is unique and the induction is complete.

Lemma 2.3. *Let a, b, c, n be integers with $a > 0$, $n > 0$, $[a^n, b, c] \in H(d)$ and $(a, b) = 1$. Then*

$$[a^n, b, c] = [a, b, a^{n-1}c]^n.$$

Proof. Note that $[a^{n-i}, b, a^i c] \in H(d)$ for $0 \leq i \leq n$. We have $[a, b, a^{n-1}c][a^{n-i}, b, a^i c] = [a^{n-(i-1)}, b, a^{i-1}c]$ for $1 \leq i \leq n$. Thus

$$\begin{aligned}
[a^n, b, c] &= [a, b, a^{n-1}c][a^{n-1}, b, ac] \\
&= [a, b, a^{n-1}c]^2[a^{n-2}, b, a^2c] \\
&= [a, b, a^{n-1}c]^3[a^{n-3}, b, a^3c] \\
&= \cdots \\
&= [a, b, a^{n-1}c]^{n-1}[a, b, a^{n-1}c] \\
&= [a, b, a^{n-1}c]^n,
\end{aligned}$$

which is the asserted result.

3. The Integer $k(n, d)$

In Definition 3.1 we define the first of three integers that we shall need which count the number of solutions of the congruence $h^2 \equiv d \pmod{4n}$ having certain properties.

Definition 3.1. We define $k(n, d)$ to be the number of integers h satisfying

$$0 \leq h < 2n, \ h^2 \equiv d \pmod{4n}.$$

When $(n, d) = 1$ the value of $k(n, d)$ is well-known, see for example [7, p. 78].

Lemma 3.1. *If* $(n, d) = 1$, *then*

$$k(n, d) = \prod_{p \mid n} \left(1 + \left(\frac{d}{p}\right)\right).$$

Next we generalize Lemma 3.1.

Lemma 3.2. *If* $(n, f) = 1$, *then*

$$k(n, d) = \begin{cases} 0, & \text{if there exists a prime } p \text{ with } p^2 \mid n \text{ and } p \mid d, \\ \prod_{p \mid n} \left(1 + \left(\frac{d}{p}\right)\right), & \text{otherwise.} \end{cases}$$

Proof. We first assume that there exists a prime p with $p^2 \mid n$ and $p \mid d$ and that there exists an integer h with $0 \leq h < 2n$, $h^2 \equiv d \pmod{4n}$. Then $p \mid h$ and so $p^2 \mid d$. Since $d = \Delta f^2$ and $p \nmid f$, we have $p^2 \mid \Delta$. If $p > 2$, then this is a contradiction since Δ is fundamental. If $p = 2$, then we have

$$\frac{\Delta}{4} \equiv \frac{\Delta f^2}{4} \equiv \frac{d}{4} \equiv \left(\frac{h}{2}\right)^2 \equiv 0 \text{ or } 1 \pmod{4},$$

which is again a contradiction as Δ is fundamental. Hence $k(n, d) = 0$.

Next, we assume that there is no prime p with $p^2 \mid n$ and $p \mid d$. Hence if p is a prime with $p \mid n$, then $p \parallel n$ or $p^2 \mid n$, and $p \nmid d$. Thus, if $p > 2$, then the number of solutions of the congruence $h^2 \equiv d \pmod{p^{v_p(n)}}$ is

$1+\left(\frac{d}{p}\right)$. If $p = 2$, then the number of solutions of the congruence $h^2 \equiv d \pmod{2^{2+v_2(n)}}$ is

$$\begin{cases} 2, & \text{if } 2 \nmid n, \\ 2\left(1+\left(\frac{d}{2}\right)\right), & \text{if } 2 \parallel n, \\ 2\left(1+\left(\frac{d}{2}\right)\right), & \text{if } 2^2 \mid n,\ 2 \nmid d. \end{cases}$$

Hence the total number of solutions of the congruence $h^2 \equiv d \pmod{4n}$ is

$$2\prod_{p \mid n}\left(1+\left(\frac{d}{p}\right)\right).$$

The result follows as $k(n, d)$ is half this number.

4. The Integer $H(n, d)$

In Definition 4.1 we define the second of the three integers related to the congruence $h^2 \equiv d \pmod{4n}$. This integer is denoted by $H(n, d)$. We give a comprehensive treatment of the evaluation of $H(p^j, d)$, where p is a prime and j is a nonnegative integer, even though not all of these properties will be used in later sections.

Definition 4.1. We define $H(n, d)$ to be the number of integers h satisfying

$$0 \le h < 2n, \quad h^2 \equiv d \pmod{4n}, \quad \left(n, h, \frac{h^2-d}{4n}\right) = 1.$$

Clearly $H(n, d) \le k(n, d)$. We determine for use later $H(p^j, d)$ when $p \mid f$ and $j \ge 1$. In this case we have

$$H(p^j, d) = \operatorname{card}\left\{h : 0 \le h < 2p^j,\ h^2 \equiv d \pmod{4p^j},\ p \nmid \frac{h^2-d}{4p^j}\right\}.$$

Lemma 4.1. *Let* $\Delta = p^v s$, $f = p^u t$, $p \nmid st$, $u \geq 1$, p *odd, so that* $v = 0$ *or* 1. *If* $v = 0$, *then*

$$H(p^j, d) = \begin{cases} 0, & \text{if } j = 2l + 1,\ 0 \leq l < u, \\ p^l - p^{l-1}, & \text{if } j = 2l,\ 1 \leq l < u, \\ p^u - \left(1 + \left(\frac{s}{p}\right)\right) p^{u-1}, & \text{if } j = 2u, \\ \left(1 + \left(\frac{s}{p}\right)\right)(p^u - p^{u-1}), & \text{if } j > 2u. \end{cases}$$

If $v = 1$, *then*

$$H(p^j, d) = \begin{cases} 0, & \text{if } j = 2l + 1,\ 0 \leq l < u, \\ p^u, & \text{if } j = 2u + 1, \\ p^l - p^{l-1}, & \text{if } j = 2l,\ 1 \leq l \leq u, \\ 0, & \text{if } j > 2u + 1. \end{cases}$$

Proof. Since $p \mid f$ and p is odd, we have

$$H(p^j, d) = \text{card}\left\{h : 0 \leq h < 2p^j,\ h^2 \equiv d \pmod{4p^j},\ p \nmid \frac{h^2 - d}{p^j}\right\}$$

for $j \geq 1$. Also

$$h^2 \equiv d \pmod{4p^j} \Leftrightarrow h \equiv d \pmod 2 \text{ and } h^2 \equiv d \pmod{p^j}.$$

For $1 \leq j \leq 2u + v$, as $d = p^{2u+v} st^2 \equiv 0 \pmod{p^j}$, we have

$$h^2 \equiv d \pmod{4p^j} \Leftrightarrow h \equiv d \pmod 2 \text{ and } h^2 \equiv 0 \pmod{p^j}. \tag{21}$$

(a) Let $j = 2l + 1$, $0 \leq l < u$. Then, by (21), we have

$$\begin{aligned} h^2 \equiv d \pmod{4p^j} &\Leftrightarrow h \equiv d \pmod 2 \text{ and } h \equiv 0 \pmod{p^{l+1}} \\ &\Leftrightarrow h = \lambda p^{l+1} \text{ and } \lambda \equiv d \pmod 2. \end{aligned} \tag{22}$$

For h satisfying (22), we have

$$\frac{h^2 - d}{p^j} = p\lambda^2 - st^2 p^{2(u-l)+v-1} \equiv 0 \pmod p.$$

Thus $H(p^j, d) = 0$.

(b) Next let $j = 2u + 1$, $v = 1$. Then, by (21), we have

$$h^2 \equiv d \,(\mathrm{mod}\; 4p^j) \Leftrightarrow h \equiv d \,(\mathrm{mod}\; 2) \text{ and } h \equiv 0 \,(\mathrm{mod}\; p^{u+1})$$

$$\Leftrightarrow h = \lambda p^{u+1} \text{ and } \lambda \equiv d \,(\mathrm{mod}\; 2). \tag{23}$$

For h satisfying (23), we have

$$0 \le h < 2p^j \Leftrightarrow 0 \le \lambda < 2p^u$$

and

$$\frac{h^2 - d}{p^j} = \lambda^2 p - st^2 \not\equiv 0 \,(\mathrm{mod}\; p).$$

Hence $H(p^j, d) = p^u$.

(c) Next let $j = 2l$, $1 \le l \le u$. Then, by (21), we have

$$h^2 \equiv d \,(\mathrm{mod}\; 4p^j) \Leftrightarrow h \equiv d \,(\mathrm{mod}\; 2) \text{ and } h \equiv 0 \,(\mathrm{mod}\; p^l)$$

$$\Leftrightarrow h = \lambda p^l \text{ and } \lambda \equiv d \,(\mathrm{mod}\; 2). \tag{24}$$

For h satisfying (24), we have

$$0 \le h < 2p^j \Leftrightarrow 0 \le \lambda < 2p^l,$$

and

$$\frac{h^2 - d}{p^j} = \lambda^2 - p^{2(u-l)+v} st^2$$

$$\equiv \begin{cases} \lambda^2 \,(\mathrm{mod}\; p), & \text{if } 1 \le l < u \text{ or } l = u,\ v = 1, \\ \lambda^2 - st^2 \,(\mathrm{mod}\; p), & \text{if } l = u,\ v = 0. \end{cases}$$

Thus, for $1 \le l < u$ or $l = u$, $v = 1$, we have

$$H(p^j, d) = \mathrm{card}\{\lambda : 0 \le \lambda < 2p^l,\ \lambda \equiv d \,(\mathrm{mod}\; 2),\ \lambda \not\equiv 0 \,(\mathrm{mod}\; p)\}$$

$$= p^l - p^{l-1},$$

and for $l = u$, $v = 0$, we have

$$H(p^j, d) = \text{card}\{\lambda : 0 \le \lambda < 2p^u,\ \lambda \equiv d \pmod 2,\ \lambda^2 \not\equiv st^2 \pmod p\}$$

$$= \begin{cases} p^u, & \text{if } \left(\frac{s}{p}\right) = -1, \\ p^u - 2p^{u-1}, & \text{if } \left(\frac{s}{p}\right) = 1, \end{cases}$$

$$= p^u - \left(1 + \left(\frac{s}{p}\right)\right) p^{u-1}.$$

(d) Next let $j > 2u + 1$, $v = 1$. Then the congruence

$$h^2 \equiv d \equiv p^{2u+1} st^2 \pmod{4p^j}$$

has no solutions so that $H(p^j, d) = 0$.

(e) Finally let $j > 2u$, $v = 0$. Let h satisfy

$$h^2 \equiv d \equiv p^{2u} st^2 \pmod{4p^j},\ 0 \le h < 2p^j,\ p \nmid \frac{h^2 - d}{p^j}.$$

Then $p^u \mid h$. Let $h = p^u h_1$ so that

$$h_1^2 \equiv st^2 \pmod{4p^{j-2u}},\ 0 \le h_1 < 2p^{j-u},\ p \nmid \frac{h_1^2 - st^2}{p^{j-2u}},$$

as $(h^2 - d)/p^j = (h_1^2 - st^2)/p^{j-2u}$. Thus

$$H(p^j, d) = \text{card}\left\{h_1 : 0 \le h_1 < 2p^{j-u},\ h_1^2 \equiv st^2 \pmod{4p^{j-2u}},\ p \nmid \frac{h_1^2 - st^2}{p^{j-2u}}\right\}.$$

If $\left(\frac{s}{p}\right) = -1$, then it is clear that

$$H(p^j, d) = 0 = \left(1 + \left(\frac{s}{p}\right)\right)(p^u - p^{u-1}).$$

If $\left(\frac{s}{p}\right) = 1$, then

$$h_1^2 \equiv st^2 \pmod{4p^{j-2u}} \Leftrightarrow h_1^2 \equiv st^2 \equiv d \pmod 4 \text{ and } h_1^2 \equiv st^2 \pmod{p^{j-2u}}$$

$$\Leftrightarrow h_1 \equiv d \pmod 2 \text{ and } h_1^2 \equiv st^2 \pmod{p^{j-2u}}.$$

The congruence

$$h_1^2 \equiv st^2 \pmod{p^{j-2u}} \tag{25}$$

has two solutions satisfying $0 < h_1 < p^{j-2u}$. If x is one such solution, then the other is $p^{j-2u} - x$. As these two solutions are of opposite parity, we may assume that $x \equiv d \pmod 2$. Thus the solutions to (25) satisfying

$$0 \le h_1 < 2p^{j-u}, \quad h_1 \equiv d \pmod 2,$$

are

$$h_1 = x + 2mp^{j-2u} \text{ and } h_1 = p^{j-2u} - x + (2m+1)p^{j-2u},$$

for $0 \le m \le p^u - 1$. For $h_1 = x + 2mp^{j-2u}$, we have

$$\frac{h_1^2 - st^2}{p^{j-2u}} \equiv 4mx + \frac{x^2 - st^2}{p^{j-2u}} \pmod p,$$

so that

$$\frac{h_1^2 - st^2}{p^{j-2u}} \equiv 0 \pmod p \Leftrightarrow m \equiv -(4x)^{-1} \frac{x^2 - st^2}{p^{j-2u}} \pmod p.$$

Similarly we find for $h_1 = p^{j-2u} - x + (2m+1)p^{j-2u}$ that

$$\frac{h_1^2 - st^2}{p^{j-2u}} \equiv 0 \pmod p \Leftrightarrow m \equiv (4x)^{-1} \frac{x^2 - st^2}{p^{j-2u}} - 1 \pmod p.$$

Thus

$$H(p^j, d) = 2(p^u - p^{u-1}) = \left(1 + \left(\frac{s}{p}\right)\right)(p^u - p^{u-1}),$$

completing the proof of Lemma 4.1.

Lemma 4.2. *Let* $n \equiv 1 \pmod 8$ *and let* l *and* m *be integers with* $l \geq 3$ *and* $m > 0$. *Then*

$$\operatorname{card}\left\{x : x^2 \equiv n \pmod{2^l},\ 0 \leq x < 2^l m,\ \frac{x^2 - n}{2^l}\ \text{odd}\right\} = 2m.$$

Proof. The congruence $x^2 \equiv n \pmod{2^l}$ has four solutions satisfying $0 < x < 2^l$. Let x_0 be the least one of these. Then $0 < x_0 < 2^{l-1}$ (otherwise $2^l - x_0$ would be a smaller solution). The other three solutions are given by $x_1 = 2^l - x_0$, $x_2 = x_0 + 2^{l-1}$, $x_3 = 2^{l-1} - x_0$. Then we have

$$\frac{x_1^2 - n}{2^l} \equiv \frac{x_0^2 - n}{2^l} \pmod 2$$

and

$$\frac{x_2^2 - n}{2^l} \equiv \frac{x_3^2 - n}{2^l} \equiv 1 + \frac{x_0^2 - n}{2^l} \pmod 2.$$

The solutions to $x^2 \equiv n \pmod{2^l}$ satisfying $0 \leq x < 2^l m$ are $x_i + 2^l r$ for $0 \leq r < m$, $0 \leq i \leq 3$. Also

$$\frac{(x_i + 2^l r)^2 - n}{2^l} \equiv \frac{x_i^2 - n}{2^l} \pmod 2.$$

Thus the required number is $2m$.

Lemma 4.3. *Let* $\Delta = 2^v s$, $f = 2^u t$, $2 \nmid st$, $u \geq 1$, *so that* $v = 0, 2, 3$, *and*

$$s \equiv \begin{cases} 1 \pmod 4), & \text{if } v = 0, \\ 3 \pmod 4), & \text{if } v = 2. \end{cases}$$

If $v = 2v_1$, *where* $v_1 = 0$ *or* 1, *then*

$$H(2^j, d) = \begin{cases} 0, & \textit{if } j = 2l+1, 0 \le l \le u + v_1 - 2, \\ 2^{l-1}, & \textit{if } j = 2l, 1 \le l \le u + v_1 - 1, \\ 0, & \textit{if } j = 2u + 2v_1 - 1, s \equiv 1 \pmod 4, \\ 2^{u+v_1-1}, & \textit{if } j = 2u + 2v_1 - 1, s \equiv 3 \pmod 4, \\ 0, & \textit{if } j = 2u + 2v_1, s \not\equiv 5 \pmod 8, \\ 2^{u+v_1}, & \textit{if } j = 2u + 2v_1, s \equiv 5 \pmod 8, \\ 0, & \textit{if } j > 2u + 2v_1, s \not\equiv 1 \pmod 8, \\ 2^{u+v_1}, & \textit{if } j > 2u + 2v_1, s \equiv 1 \pmod 8. \end{cases}$$

If $v = 3$, *then*

$$H(2^j, d) = \begin{cases} 0, & \textit{if } j = 2l + 1, 0 \le l < u, \\ 2^u, & \textit{if } j = 2u + 1, \\ 2^{l-1}, & \textit{if } j = 2l, 1 \le l \le u, \\ 0, & \textit{if } j > 2u + 1. \end{cases}$$

Proof. For $j \ge 1$, we have

$$H(2^j, d) = \text{card}\left\{h : 0 \le h < 2^{j+1}, h^2 \equiv d \pmod{2^{j+2}}, \frac{h^2 - d}{2^{j+2}} \text{ odd}\right\}.$$

If $j + 2 \le 2u + v$, then we have

$$h^2 \equiv d \pmod{2^{j+2}} \Leftrightarrow h^2 \equiv 0 \pmod{2^{j+2}}.$$

(a) Let $j = 2l + 1$, $0 \le l \le u + (v - 3)/2$ (so that $0 \le l \le u$ if $v = 3$ and $0 \le l \le u + v_1 - 2$ if $v = 2v_1$ where $v_1 = 0$ or 1). Then

$$h^2 \equiv d \pmod{2^{j+2}} \Leftrightarrow h^2 \equiv 0 \pmod{2^{2l+3}} \Leftrightarrow h = \lambda 2^{l+2} \text{ for some integer } \lambda.$$

For any such h, we have

$$0 \le h < 2^{j+1} = 2^{2l+2} \Leftrightarrow 0 \le \lambda < 2^l,$$

and

$$\frac{h^2 - d}{2^{j+2}} = 2\lambda^2 - st^2 2^{2u-2l+v-3}$$

$$\equiv \begin{cases} 0 \pmod 2, & \text{if } v = 3, l < u \text{ or } v = 2v_1, v_1 = 0 \text{ or } 1, \\ 1 \pmod 2, & \text{if } v = 3, l = u. \end{cases}$$

Thus

$$H(2^j, d) = \begin{cases} 0, & \text{if } v = 3,\ j = 2l+1,\ 0 \le l < u, \\ 0, & \text{if } v = 2v_1,\ v_1 = 0 \text{ or } 1,\ j = 2l+1,\ 0 \le l \le u + v_1 - 2, \\ 2^u, & \text{if } v = 3,\ j = 2u+1. \end{cases}$$

(b) Next let $j = 2l$, $1 \le l \le u + v/2 - 1$ (so that $1 \le l \le u$ if $v = 3$ and $1 \le l \le u + v_1 - 1$ if $v = 2v_1$, $v_1 = 0$ or 1). Then

$$h^2 \equiv d \pmod{2^{j+2}} \Leftrightarrow h^2 \equiv 0 \pmod{2^{2l+2}} \Leftrightarrow h = \lambda 2^{l+1} \text{ for some integer } \lambda.$$

For such h we have

$$0 \le h < 2^{j+1} = 2^{2l+1} \Leftrightarrow 0 \le \lambda < 2^l,$$

and

$$\frac{h^2 - d}{2^{j+2}} = \lambda^2 - 2^{2u-2l+v-2} st^2$$

$$\equiv \begin{cases} \lambda \pmod 2, & \text{if } v = 3 \text{ or } v = 2v_1,\ v_1 = 0 \text{ or } 1,\ l < u + v_1 - 1, \\ \lambda + 1 \pmod 2, & \text{if } v = 2v_1,\ v_1 = 0 \text{ or } 1,\ l = u + v_1 - 1. \end{cases}$$

Thus, in all cases under consideration, we have $H(2^j, d) = 2^{l-1}$.

(c) Next let $v = 3$ and $j > 2u+1$. Then the congruence $h^2 \equiv d \equiv 2^{2u+3} st^2 \pmod{2^{j+2}}$ has no solutions so that $H(2^j, d) = 0$.

If $v = 2v_1$, $v_1 = 0$ or 1 and $j + 2 > 2u + v = 2u + 2v_1$, it is easily shown that

$$H(2^j, d) = \operatorname{card}\left\{ h_1 : 0 \le h_1 < 2^{j+1-u-v_1},\right.$$
$$\left. h_1^2 \equiv st^2 \pmod{2^{j+2-2u-2v_1}},\ \frac{h_1^2 - st^2}{2^{j+2-2u-2v_1}} \text{ odd} \right\}.$$

(d) Let $j = 2u + 2v_1 - 1$, $v = 2v_1$, $v_1 = 0$ or 1. Then

$$H(2^j, d) = \operatorname{card}\left\{ h_1 : 0 \le h_1 < 2^{u+v_1},\ h_1^2 \equiv st^2 \pmod 2,\ \frac{h_1^2 - st^2}{2} \text{ is odd} \right\}.$$

We have

$$h_1^2 \equiv st^2 \pmod 2 \Leftrightarrow h_1 \text{ is odd}.$$

If h_1 is odd, then $h_1^2 - st^2 \equiv 1 - s \pmod 4$. Thus, for the cases under consideration, we have

$$H(2^j, d) = \begin{cases} 0, & \text{if } s \equiv 1 \pmod 4, \\ 2^{u+v_1-1}, & \text{if } s \equiv 3 \pmod 4. \end{cases}$$

(e) Next let $j = 2u + 2v_1$, $v = 2v_1$, $v_1 = 0$ or 1. Then

$$H(2^j, d) = \text{card}\left\{h_1 : 0 \le h_1 < 2^{u+v_1+1},\ h_1^2 \equiv st^2 \pmod 4,\ \frac{h_1^2 - st^2}{4} \text{ odd}\right\}.$$

If $s \equiv 3 \pmod 4$, then the congruence $h_1^2 \equiv st^2 \equiv 3 \pmod 4$ has no solutions so that $H(2^j, d) = 0$. If $s \equiv 1 \pmod 4$, then $h_1^2 \equiv st^2 \pmod 4 \Leftrightarrow h_1$ is odd. If h_1 is odd, then $h_1^2 - st^2 \equiv 1 - s \pmod 8$. Thus

$$H(2^j, d) = \begin{cases} 0, & \text{if } s \equiv 1 \pmod 8, \\ 2^{u+v_1}, & \text{if } s \equiv 5 \pmod 8. \end{cases}$$

(f) Finally let $j > 2u + 2v_1$, $v = 2v_1$, $v_1 = 0$ or 1. Then the congruence

$$h_1^2 \equiv st^2 \pmod{2^{j+2-2u-2v_1}}$$

has no solutions if $s \not\equiv 1 \pmod 8$ so that $H(2^j, d) = 0$. If $s \equiv 1 \pmod 8$, then $H(2^j, d) = 2(2^{u+v_1-1}) = 2^{u+v_1}$ by Lemma 4.2.

From Lemmas 4.1 and 4.3 we deduce for $p \mid f$ that

$$0 \le H(p^j, d) \le 2p^u \le 2f. \tag{26}$$

5. The Integer $H_K(n)$

We now define the third of our three integers connected to the congruence $h^2 \equiv d \pmod{4n}$.

Definition 5.1. For $K \in H(d)$, we define $H_K(n)$ to be the number of integers h satisfying

$$0 \le h < 2n, \quad h^2 \equiv d \,(\mathrm{mod}\ 4n), \quad \left[n,\ h,\ \frac{h^2 - d}{4n}\right] = K.$$

It is clear from Definitions 3.1, 4.1 and 5.1 that

$$0 \le H_K(n) \le H(n, d) \le k(n, d) \tag{27}$$

and

$$\sum_{K \in H(d)} H_K(n) = H(n, d). \tag{28}$$

It is a classical result of elementary number theory [15, p. 174] that for $K \in H(d)$,

$$R_K(n, d) = w(d) \sum_{m^2 \mid n} H_K(n/m^2). \tag{29}$$

By (27) and Lemma 3.2, we have

Lemma 5.1. *Let* $K \in H(d)$. *If* $(n, f) = 1$, *then*

$$H_K(n) = 0, \text{ if there is a prime } p \text{ with } p^2 \mid n,\ p \mid d,$$

and

$$0 \le H_K(n) \le \prod_{p \mid n} \left(1 + \left(\frac{d}{p}\right)\right), \text{ otherwise.}$$

Lemma 5.2. *Let* $K \in H(d)$. *Then*

$$H_K(1) = \begin{cases} 1, & \text{if } K = I, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We have

$$H_K(1) = \sum_{\substack{h=0 \\ h^2 \equiv d \,(\mathrm{mod}\ 4) \\ [1,\ h,\ (h^2-d)/4] = K}}^{1} 1.$$

The only possibility for h in the sum is $h = 0$ if $d \equiv 0 \pmod 4$ and $h = 1$ if $d \equiv 1 \pmod 4$. Hence

$$H_K(1) = \begin{cases} 0, & \text{if } K \neq \begin{cases} [1, 0, -d/4], & \text{if } d \equiv 0 \pmod 4, \\ [1, 1, (1-d)/4], & \text{if } d \equiv 1 \pmod 4, \end{cases} \\ 1, & \text{if } K = \begin{cases} [1, 0, -d/4], & \text{if } d \equiv 0 \pmod 4, \\ [1, 1, (1-d)/4], & \text{if } d \equiv 1 \pmod 4, \end{cases} \end{cases}$$

and the result follows.

If p is a prime with $\left(\frac{d}{p}\right) = 1$, by Lemma 3.1 we have $k(p, d) = 2$. Let h_1 and h_2 be the two solutions of $h^2 \equiv d \pmod{4p}$, $0 \leq h < 2p$, chosen so that $h_1 < h_2$. Since $p \nmid d$, we have $p \nmid h_1$, $p \nmid h_2$ and $h_2 = 2p - h_1$. We have

$$\left[p, h_i, \frac{h_i^2 - d}{4p}\right] \in H(d), \ i = 1, 2.$$

Also

$$\left[p, h_2, \frac{h_2^2 - d}{4p}\right] = \left[p, 2p - h_1, \frac{(2p - h_1)^2 - d}{4p}\right]$$

$$= \left[p, -h_1, \frac{h_1^2 - d}{4p}\right] = \left[p, h_1, \frac{h_1^2 - d}{4p}\right]^{-1}.$$

Motivated by these observations, we make the following definition.

Definition 5.2. If p is a prime with $\left(\frac{d}{p}\right) = 1$, we let h_1 and h_2 be the solutions of $h^2 \equiv d \pmod{4p}$, $0 \leq h < 2p$, with $h_1 < h_2$. We define $K_p \in H(d)$ by

$$K_p = \left[p, h_1, \frac{h_1^2 - d}{4p}\right],$$

so that

$$K_p^{-1} = \left[p, h_2, \frac{h_2^2 - d}{4p}\right].$$

If p is a prime with $\left(\frac{d}{p}\right) = 0$, $p \nmid f$, then we define $K_p \in H(d)$ by

$$K_p = \begin{cases} [p, 0, -d/4p], & \text{if } p > 2,\ d \equiv 0 \pmod 4, \\ [p, p, (p^2 - d)/4p], & \text{if } p > 2,\ d \equiv 1 \pmod 4, \\ [2, 0, -d/8], & \text{if } p = 2,\ d \equiv 8 \pmod{16}, \\ [2, 2, (4-d)/8], & \text{if } p = 2,\ d \equiv 12 \pmod{16}. \end{cases}$$

Clearly $K_p = K_p^{-1}$.

Lemma 5.3. *Let p be a prime and let $K \in H(d)$.*

(a) *If $\left(\frac{d}{p}\right) = -1$, then $H_K(p) = 0$.*

(b) *If $\left(\frac{d}{p}\right) = 1$ and $K_p \neq K_p^{-1}$, then p is represented by exactly two classes of $H(d)$, namely K_p and K_p^{-1}. If $\left(\frac{d}{p}\right) = 1$ and $K_p = K_p^{-1}$, then p is represented by exactly one class of $H(d)$ namely K_p. Moreover*

$$H_K(p) = \begin{cases} 0, & \text{if } K \neq K_p, K_p^{-1}, \\ 1, & \text{if } K = K_p \neq K_p^{-1} \text{ or } K = K_p^{-1} \neq K_p, \\ 2, & \text{if } K = K_p = K_p^{-1}. \end{cases}$$

(c) *If $\left(\frac{d}{p}\right) = 0$ and $p \mid f$, then $H_K(p) = 0$.*

(d) *If $\left(\frac{d}{p}\right) = 0$ and $p \nmid f$, then p is represented by exactly one class in $H(d)$, namely K_p and*

$$H_K(p) = \begin{cases} 0, & \text{if } K \neq K_p, \\ 1, & \text{if } K = K_p. \end{cases}$$

Proof. (a) This follows immediately from Lemma 5.1.

(b) Let L be a class of $H(d)$ which represents p. Then there exist integers r and s such that $L = [p, r, s]$. Note that $r^2 - 4ps = d$ and $(p, r, s) = 1$. Let $r = 2pt + h$ with $0 \leq h < 2p$. We have $h^2 \equiv r^2 \equiv d \pmod{4p}$.

Hence $h = h_1$ or h_2. Thus

$$L = [p,\ 2pt + h,\ s] = \left[p,\ h,\ \frac{h^2 - d}{4p}\right] = K_p \text{ or } K_p^{-1}.$$

Also

$$H_K(p) = \sum_{\substack{0 \le h < 2p \\ h^2 \equiv d \pmod{4p} \\ [p,\, h,\, (h^2-d)/4p] = K}} 1 = \sum_{\substack{h = h_1, h_2 \\ [p,\, h,\, (h^2-d)/4p] = K}} 1.$$

The required result follows from this and the definition of K_p (Definition 5.2).

(c) This follows from Lemma 4.1, Lemma 4.3 and (27).

(d) Let L be a class of $H(d)$ that represents p. Then there exist integers r, s with $(p, r, s) = 1$, $r^2 - 4ps = d$ and $L = [p, r, s]$. Let $r = 2pt + h$, $0 \le h < 2p$. Then $h^2 \equiv r^2 \equiv d \pmod{4p}$. By Lemma 3.2 we have $k(p, d) = 1$. The unique solution h of $h^2 \equiv d \pmod{4p}$, $0 \le h < 2p$ is

$$h = \begin{cases} 0, & \text{if } p > 2,\ d \equiv 0 \pmod 4, \\ p, & \text{if } p > 2,\ d \equiv 1 \pmod 4, \\ 0, & \text{if } p = 2,\ d \equiv 8 \pmod{16}, \\ 2, & \text{if } p = 2,\ d \equiv 12 \pmod{16}. \end{cases}$$

Hence $L = [p, 2pt + h, s] = [p, h, (h^2 - d)/4p] = K_p$. The result for $H_K(p)$ follows at once.

Lemma 5.4. *Let $K \in H(d)$ and let k be an integer with $k \ge 2$. Then*

$$H_K(p^k) = \begin{cases} \displaystyle\sum_{L^k = K} H_L(p), & \text{if } p \nmid d, \\ 0, & \text{if } p \mid d,\ p \nmid f, \end{cases}$$

where the sum is taken over all $L \in H(d)$ with $L^k = K$.

Proof. First we consider the case $p \nmid d$. We have

$$H_K(p^k) = \sum_{\substack{0 \le h < 2p^k \\ h^2 \equiv d \pmod{4p^k} \\ [p^k,\, h,\, (h^2-d)/4p^k] = K}} 1.$$

For each h occurring in the sum we have $p \nmid h$. Thus, by Lemma 2.3, we have

$$\left[p^k,\, h,\, \frac{h^2-d}{4p^k}\right] = \left[p,\, h,\, \frac{h^2-d}{4p}\right]^k.$$

Hence

$$\begin{aligned} H_K(p^k) &= \sum_{\substack{0 \le h < 2p^k \\ h^2 \equiv d \pmod{4p^k} \\ [p,\, h,\, (h^2-d)/4p]^k = K}} 1 \\ &= \sum_{0 \le h_1 < 2p} \sum_{\substack{0 \le h < 2p^k \\ h \equiv h_1 \pmod{2p} \\ h^2 \equiv d \pmod{4p^k} \\ [p,\, h,\, (h^2-d)/4p]^k = K}} 1 \\ &= \sum_{\substack{0 \le h_1 < 2p \\ h_1^2 \equiv d \pmod{4p}}} \sum_{\substack{0 \le h < 2p^k \\ h \equiv h_1 \pmod{2p} \\ h^2 \equiv d \pmod{4p^k} \\ [p,\, h,\, (h^2-d)/4p]^k = K}} 1. \end{aligned}$$

For $h \equiv h_1 \pmod{2p}$, we have $[p, h, (h^2-d)/4p] = [p, h_1, (h_1^2-d)/4p]$. Thus

$$H_K(p^k) = \sum_{\substack{0 \le h_1 < 2p \\ h_1^2 \equiv d \pmod{4p} \\ [p,\, h_1,\, (h_1^2-d)/4p]^k = K}} \sum_{\substack{0 \le h < 2p^k \\ h \equiv h_1 \pmod{2p} \\ h^2 \equiv d \pmod{4p^k}}} 1.$$

By Lemma 2.2, corresponding to each h_1, there is a unique h satisfying the conditions of the inner sum. Thus

$$H_K(p^k) = \sum_{\substack{0 \le h_1 < 2p \\ h_1^2 \equiv d \pmod{4p} \\ [p,\, h_1,\, (h_1^2-d)/4p]^k = K}} 1 = \sum_{L \in H(d)} \sum_{\substack{0 \le h_1 < 2p \\ h_1^2 \equiv d \pmod{4p} \\ [p,\, h_1,\, (h_1^2-d)/4p]^k = K \\ [p,\, h_1,\, (h_1^2-d)/4p] = L}} 1$$

$$= \sum_{\substack{L \in H(d) \\ L^k = K}} \sum_{\substack{0 \le h_1 < 2p \\ h_1^2 \equiv d \pmod{4p} \\ [p,\, h_1,\, (h_1^2-d)/4p] = L}} 1 = \sum_{\substack{L \in H(d) \\ L^k = K}} H_L(p).$$

In the case $p \mid d$, $p \nmid f$, the result follows from Lemma 5.1.

Lemma 5.5. *Let n_1 and n_2 be relatively prime positive integers. Let $K \in H(d)$. Then*

$$H_K(n_1 n_2) = \sum_{K_1 K_2 = K} H_{K_1}(n_1) H_{K_2}(n_2),$$

where K_1, K_2 run through all the classes in $H(d)$ whose product is K.

Proof. We have

$$H_K(n_1 n_2) = \sum_{\substack{0 \le B < 2n_1 n_2 \\ B^2 \equiv d \pmod{4n_1 n_2} \\ [n_1 n_2,\, B,\, (B^2-d)/4n_1 n_2] = K}} 1.$$

If $B\,(0 \le B < 2n_1 n_2,\ B^2 \equiv d \pmod{4n_1 n_2})$ is such that $[n_1 n_2, B, (B^2 - d)/4n_1 n_2] = K$, then as $(n_1, n_2) = 1$, we have

$$\left[n_1,\, B,\, \frac{B^2-d}{4n_1}\right] \in H(d) \text{ and } \left[n_2,\, B,\, \frac{B^2-d}{4n_2}\right] \in H(d).$$

Hence

$$\left[n_1 n_2,\, B,\, \frac{B^2-d}{4n_1 n_2}\right] = \left[n_1,\, B,\, \frac{B^2-d}{4n_1}\right]\left[n_2,\, B,\, \frac{B^2-d}{4n_2}\right].$$

Conversely, if $B\,(0 \le B < 2n_1n_2,\ B^2 \equiv d \pmod{4n_1n_2})$ is such that

$$\left[n_1,\ B,\ \frac{B^2-d}{4n_1}\right],\ \left[n_2,\ B,\ \frac{B^2-d}{4n_2}\right] \in H(d)$$

and

$$\left[n_1,\ B,\ \frac{B^2-d}{4n_1}\right]\left[n_2,\ B,\ \frac{B^2-d}{4n_2}\right] = K,$$

then

$$\left[n_1n_2,\ B,\ \frac{B^2-d}{4n_1n_2}\right] = K.$$

Thus

$$H_K(n_1n_2) = \sum_{\substack{0\le B<2n_1n_2\\ B^2\equiv d \,(\mathrm{mod}\ 4n_1n_2)\\ [n_1,\ B,\ (B^2-d)/4n_1]\in H(d)\\ [n_2,\ B,\ (B^2-d)/4n_2]\in H(d)\\ [n_1,\ B,\ (B^2-d)/4n_1][n_2,\ B,\ (B^2-d)/4n_2]=K}} 1$$

$$= \sum_{K_1K_2=K} \sum_{\substack{0\le B<2n_1n_2\\ B^2\equiv d \,(\mathrm{mod}\ 4n_1n_2)\\ [n_1,\ B,\ (B^2-d)/4n_1]=K_1\\ [n_2,\ B,\ (B^2-d)/4n_2]=K_2}} 1$$

$$= \sum_{K_1K_2=K} \sum_{\substack{0\le b_1<2n_1\\ 0\le b_2<2n_2}} \sum_{\substack{0\le B<2n_1n_2\\ B^2\equiv d \,(\mathrm{mod}\ 4n_1n_2)\\ B\equiv b_1 \,(\mathrm{mod}\ 2n_1)\\ B\equiv b_2 \,(\mathrm{mod}\ 2n_2)\\ [n_1,\ B,\ (B^2-d)/4n_1]=K_1\\ [n_2,\ B,\ (B^2-d)/4n_2]=K_2}} 1$$

$$= \sum_{K_1K_2=K} \sum_{\substack{0\le b_1<2n_1 \\ b_1^2\equiv d \pmod{4n_1}}} \sum_{\substack{0\le b_2<2n_2 \\ b_2^2\equiv d \pmod{4n_2}}} \sum_{\substack{0\le B<2n_1n_2 \\ B^2\equiv d \pmod{4n_1n_2} \\ B\equiv b_1 \pmod{2n_1} \\ B\equiv b_2 \pmod{2n_2} \\ [n_1,\, B,\, (B^2-d)/4n_1]=K_1 \\ [n_2,\, B,\, (B^2-d)/4n_2]=K_2}} 1$$

$$= \sum_{K_1K_2=K} \sum_{\substack{0\le b_1<2n_1 \\ b_1^2\equiv d \pmod{4n_1} \\ [n_1,\, b_1,\, (b_1^2-d)/4n_1]=K_1}} \sum_{\substack{0\le b_2<2n_2 \\ b_2^2\equiv d \pmod{4n_2} \\ [n_2,\, b_2,\, (b_2^2-d)/4n_2]=K_2}} \sum_{\substack{0\le B<2n_1n_2 \\ B^2\equiv d \pmod{4n_1n_2} \\ B\equiv b_1 \pmod{2n_1} \\ B\equiv b_2 \pmod{2n_2}}} 1$$

$$= \sum_{K_1K_2=K} \sum_{\substack{0\le b_1<2n_1 \\ b_1^2\equiv d \pmod{4n_1} \\ [n_1,\, b_1,\, (b_1^2-d)/4n_1]=K_1}} \sum_{\substack{0\le b_2<2n_2 \\ b_2^2\equiv d \pmod{4n_2} \\ [n_2,\, b_2,\, (b_2^2-d)/4n_2]=K_2}} 1 \qquad \text{(by Lemma 2.1)}$$

$$= \sum_{K_1K_2=K} H_{K_1}(n_1)H_{K_2}(n_2),$$

as asserted.

6. The Quantities $[K, L]$ and $\chi(K, L)$

As $H(d)$ is a finite abelian group, there exist positive integers $h_1, h_2, \ldots, h_\nu$ and $A_1, A_2, \ldots, A_\nu \in H(d)$ such that

$$h_1h_2\cdots h_\nu = h(d),\ \ 1 < h_1 \mid h_2 \mid \cdots \mid h_\nu,\ \ \mathrm{ord}(A_i) = h_i \quad (i = 1, \ldots, \nu),$$

and, for each $K \in H(d)$, there exist unique integers $k_1, \ldots, k_\nu$ with

$$K = A_1^{k_1} \cdots A_\nu^{k_\nu}\ (0 \le k_j < h_j,\ 1 \le j \le \nu).$$

We fix once and for all the generators $A_1, \ldots, A_\nu$. With this notation we make the following definition.

Definition 6.1. For $j = 1, \ldots, \nu$ we set

$$\mathrm{ind}_{A_j}(K) = k_j,$$

and for $K, L \in H(d)$, we set

$$[K, L] = \sum_{j=1}^{v} \frac{\mathrm{ind}_{A_j}(K)\mathrm{ind}_{A_j}(L)}{h_j}.$$

The following is immediate.

Lemma 6.1. *Let $K, L, M \in H(d)$. Then*

$$[K, L] = [L, K], [K, I] = 0, [KL, M] \equiv [K, M] + [L, M] (\mathrm{mod}\, 1),$$

and

$$[K^r, L^s] \equiv rs[K, L] (\mathrm{mod}\, 1) \text{ for integers } r, s.$$

Definition 6.2. For $K, L \in H(d)$, we set

$$\chi(K, L) = e^{2\pi i[K, L]}.$$

The next lemma follows immediately from Lemma 6.1 and Definition 6.2.

Lemma 6.2. *Let $K, L, M \in H(d)$. Let r and s be integers. Then*

$$\chi(K, L) = \chi(L, K), \chi(K, I) = 1, \chi(KL, M) = \chi(K, M)\chi(L, M),$$

and

$$\chi(K^r, L^s) = \chi(K, L)^{rs}.$$

Moreover

$$\sum_{U \in H(d)} \chi(K, U) = \begin{cases} h(d), & \text{if } K = I, \\ 0, & \text{if } K \neq I, \end{cases}$$

and

$$\sum_{U \in H(d)} \chi(K, U)\chi(L, U)^{-1} = \begin{cases} h(d), & \text{if } K = L, \\ 0, & \text{if } K \neq L. \end{cases}$$

7. The Quantities $Y_K(n)$, $j(K, d)$ and $\sum_{n=1}^{\infty} \frac{Y_K(n)}{n^s}$

It is convenient to make the following definition.

Definition 7.1. Let $K \in H(d)$. We set

$$Y_K(n) = \sum_{L \in H(d)} \chi(K, L) H_L(n).$$

If p is a prime such that $p \mid f$, we have by (26),

$$0 \leq H(p^j, d) \leq 2f.$$

Thus for $K \in H(d)$, we have

$$|Y_K(p^j)| = \left| \sum_{L \in H(d)} \chi(K, L) H_L(p^j) \right| \leq \sum_{L \in H(d)} H_L(p^j) = H(p^j, d) \leq 2f, \quad (30)$$

by (28). We next develop the properties of $Y_K(n)$.

Lemma 7.1. $Y_K(1) = 1$ *for* $K \in H(d)$.

Proof. We have

$$Y_K(1) = \sum_{L \in H(d)} \chi(K, L) H_L(1) = \chi(K, I) = 1$$

by Lemmas 5.2 and 6.2.

Lemma 7.2. *Let* $K \in H(d)$ *and let* n_1 *and* n_2 *be relatively prime positive integers. Then*

$$Y_K(n_1 n_2) = Y_K(n_1) Y_K(n_2).$$

Proof. We have

$$\begin{aligned} Y_K(n_1 n_2) &= \sum_{L \in H(d)} \chi(K, L) H_L(n_1 n_2) \\ &= \sum_{L \in H(d)} \chi(K, L) \sum_{L_1 L_2 = L} H_{L_1}(n_1) H_{L_2}(n_2) \quad \text{(by Lemma 5.5)} \\ &= \sum_{L \in H(d)} \sum_{L_1 L_2 = L} \chi(K, L_1 L_2) H_{L_1}(n_1) H_{L_2}(n_2) \\ &= \sum_{L_1 \in H(d)} \sum_{L_2 \in H(d)} \chi(K, L_1) \chi(K, L_2) H_{L_1}(n_1) H_{L_2}(n_2) \\ &\qquad \text{(by Lemma 6.2)} \end{aligned}$$

$$= \sum_{L_1 \in H(d)} \chi(K, L_1) H_{L_1}(n_1) \sum_{L_2 \in H(d)} \chi(K, L_2) H_{L_2}(n_2)$$

$$= Y_K(n_1) Y_K(n_2),$$

as asserted.

Lemma 7.3. *Let* $K \in H(d)$ *and let* p *be a prime.*

(a) *If* $\left(\frac{d}{p}\right) = -1$, *then* $Y_K(p^\alpha) = 0$ *for* $\alpha \geq 1$.

(b) *If* $\left(\frac{d}{p}\right) = 1$, *then* $Y_K(p^\alpha) = \chi(K, K_p)^\alpha + \chi(K, K_p)^{-\alpha}$ *for* $\alpha \geq 1$.

(c) *If* $p \mid d$, $p \nmid f$, *then*

$$Y_K(p^\alpha) = \begin{cases} \chi(K, K_p), & \text{if } \alpha = 1, \\ 0, & \text{if } \alpha \geq 2. \end{cases}$$

(d) *For all primes* p *and all integers* $\alpha \geq 0$, *we have*

$$|Y_K(p^\alpha)| \leq 2f.$$

Proof. If $p \nmid d$, we have by Lemma 5.4,

$$Y_K(p^\alpha) = \sum_{L \in H(d)} \chi(K, L) H_L(p^\alpha) = \sum_{L \in H(d)} \chi(K, L) \sum_{M^\alpha = L} H_M(p). \quad (31)$$

(a) This follows from (31) and Lemma 5.3(a).

(b) Let $\left(\frac{d}{p}\right) = 1$. If $K_p^\alpha \neq K_p^{-\alpha}$, then we have, by (31) and Lemma 5.3(b),

$$Y_K(p^\alpha) = \sum_{L \neq K_p^\alpha, K_p^{-\alpha}} \chi(K, L) \sum_{M^\alpha = L} H_M(p) + \chi(K, K_p^\alpha) \sum_{M^\alpha = K_p^\alpha} H_M(p)$$

$$+ \chi(K, K_p^{-\alpha}) \sum_{M^\alpha = K_p^{-\alpha}} H_M(p)$$

$$= \chi(K, K_p^\alpha) H_{K_p}(p) + \chi(K, K_p^{-\alpha}) H_{K_p^{-1}}(p)$$

$$= \chi(K, K_p^\alpha) + \chi(K, K_p^{-\alpha}).$$

Similarly, if $K_p^\alpha = K_p^{-\alpha}$, we have

$$Y_K(p^\alpha) = \chi(K, K_p^\alpha) \sum_{M^\alpha = K_p^\alpha} H_M(p)$$

$$= \begin{cases} \chi(K, K_p^\alpha)(H_{K_p}(p) + H_{K_p^{-1}}(p)), & \text{if } K_p \neq K_p^{-1}, \\ \chi(K, K_p^\alpha) H_{K_p}(p), & \text{if } K_p = K_p^{-1}, \end{cases}$$

$$= 2\chi(K, K_p^\alpha)$$

$$= \chi(K, K_p^\alpha) + \chi(K, K_p^{-\alpha}).$$

Thus, in both cases, we have

$$Y_K(p^\alpha) = \chi(K, K_p^\alpha) + \chi(K, K_p^{-\alpha}) = \chi(K, K_p)^\alpha + \chi(K, K_p)^{-\alpha},$$

by Lemma 6.2.

(c) Let $p \mid d$, $p \nmid f$. By Lemma 5.4, we have

$$Y_K(p^\alpha) = \sum_{L \in H(d)} \chi(K, L) H_L(p^\alpha) = 0 \text{ if } \alpha \geq 2.$$

Also, by Lemma 5.3(d), we have

$$Y_K(p) = \sum_{L \in H(d)} \chi(K, L) H_L(p) = \chi(K, K_p).$$

(d) The asserted inequality follows from Lemma 7.1, (30), and parts (a), (b), (c) of this lemma.

We next investigate the series $\displaystyle\sum_{n=1}^{\infty} \frac{Y_K(n)}{n^s}$.

Lemma 7.4. *Let p be a prime and let $K \in H(d)$. Then the series*

$$\sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}}$$

converges absolutely and uniformly for $s \geq 1$. *Moreover, for* $s \geq 1$, *we have*

$$\sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}} = \begin{cases} 1, & \textit{if } \left(\frac{d}{p}\right) = -1, \\ \dfrac{1 - \frac{1}{p^{2s}}}{\left(1 - \frac{\chi(K, K_p)}{p^s}\right)\left(1 - \frac{\chi(K, K_p)^{-1}}{p^s}\right)}, & \textit{if } \left(\frac{d}{p}\right) = 1, \\ 1 + \dfrac{\chi(K, K_p)}{p^s}, & \textit{if } p \mid d,\ p \nmid f. \end{cases}$$

Proof. By Lemma 7.3(d) the series $\sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}}$ converges absolutely and uniformly for $s \geq 1$. The values of $\sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}}$ for the cases given in the statement of the lemma follow immediately on using Lemmas 7.1 and 7.3.

Lemma 7.5. *Let* $K \in H(d)$. *Then for* $s > 1$, *we have*

$$\sum_{n=1}^{\infty} \frac{Y_K(n)}{n^s} = \prod_{\left(\frac{d}{p}\right)=1} \frac{1 - \frac{1}{p^{2s}}}{\left(1 - \frac{\chi(K, K_p)}{p^s}\right)\left(1 - \frac{\chi(K, K_p)^{-1}}{p^s}\right)}$$

$$\times \prod_{\substack{p \mid d \\ p \nmid f}} \left(1 + \frac{\chi(K, K_p)}{p^s}\right) \prod_{p \mid f} \sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}}.$$

Proof. Let $s > 1$. For any prime p, Lemma 7.3(d) gives

$$\sum_{j=1}^{\infty} \left| \frac{Y_K(p^j)}{p^{js}} \right| \leq \sum_{j=1}^{\infty} \frac{2f}{p^{js}} = \frac{2f}{p^s - 1}.$$

Hence $\sum_p \sum_{j=1}^{\infty} \left| \frac{Y_K(p^j)}{p^{js}} \right|$ converges so that

$$\prod_p \left(1 + \sum_{j=1}^{\infty} \left| \frac{Y_K(p^j)}{p^{js}} \right| \right) = \prod_p \sum_{j=0}^{\infty} \left| \frac{Y_K(p^j)}{p^{js}} \right|$$

converges. Since $Y_K(n)$ is multiplicative (Lemma 7.2), it follows that

$$\sum_{n=1}^{\infty} \frac{Y_K(n)}{n^s} = \prod_p \sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}}.$$

The result now follows on using Lemma 7.4.

Lemma 7.6. *Let* $K(\neq I) \in H(d)$. *Then*

$$j(K, d) = \lim_{s \to 1^+} \prod_{\substack{p \\ \left(\frac{d}{p}\right)=1}} \left(1 - \frac{\chi(K, K_p)}{p^s} \right) \left(1 - \frac{\chi(K^{-1}, K_p)}{p^s} \right) \tag{32}$$

exists and is a nonzero real number such that $j(K, d) = j(K^{-1}, d)$.

Proof. The existence of the above limit and the fact that it is nonzero has been proved by Bernays [1, Teil I, Section 3, Section 4, pp. 36-68], The fact that it is real follows easily since

$$\prod_{\substack{p \\ \left(\frac{d}{p}\right)=1}} \left(1 - \frac{\chi(K, K_p)}{p^s} \right) \left(1 - \frac{\chi(K^{-1}, K_p)}{p^s} \right)$$

is real for $s > 1$. The equality $j(K, d) = j(K^{-1}, d)$ is clear.

Lemma 7.7. *Let* $K(\neq I) \in H(d)$. *Set*

$$t_1(d) = \prod_{\left(\frac{d}{p}\right)=1} \left(1 - \frac{1}{p^2} \right), \tag{33}$$

$$A(K, d, p) = \sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^j}, \tag{34}$$

$$l(K, d) = \prod_{\substack{p \mid d \\ p \nmid f}} \left(1 + \frac{\chi(K, K_p)}{p}\right) \prod_{p \mid f} A(K, d, p). \tag{35}$$

Then

$$\sum_{n=1}^{\infty} \frac{Y_K(n)}{n^s} = \frac{t_1(d)}{j(K, d)} l(K, d)(1 + o(1))$$

as $s \to 1^+$.

Proof. By the uniform convergence of $\sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}}$ for $s \geq 1$ (Lemma 7.4), we have

$$\sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}} = A(K, d, p)(1 + o(1)), \tag{36}$$

as $s \to 1^+$. Also

$$\prod_{\left(\frac{d}{p}\right)=1} \left(1 - \frac{1}{p^{2s}}\right) = t_1(d)(1 + o(1)), \tag{37}$$

$$1 + \frac{\chi(K, K_p)}{p^s} = \left(1 + \frac{\chi(K, K_p)}{p}\right)(1 + o(1)), \tag{38}$$

as $s \to 1^+$. The required result follows from (36), (37), (38), Lemmas 7.5 and 7.6.

Lemma 7.8. *Let* $s > 1$. *Then*

$$\zeta(2s) \sum_{n=1}^{\infty} \frac{Y_I(n)}{n^s} = \frac{1}{w(d)} \sum_{n=1}^{\infty} \frac{N(n, d)}{n^s}.$$

Proof. We have

$$\zeta(2s)\sum_{n=1}^{\infty}\frac{Y_I(n)}{n^s} = \sum_{L\in H(d)}\sum_{m=1}^{\infty}\frac{1}{m^{2s}}\sum_{n=1}^{\infty}\frac{H_L(n)}{n^s}$$

$$= \sum_{L\in H(d)}\sum_{l=1}^{\infty}\frac{1}{l^s}\sum_{m^2\mid l}H_L(l/m^2)$$

$$= \frac{1}{w(d)}\sum_{L\in H(d)}\sum_{l=1}^{\infty}\frac{R_L(l,d)}{l^s} \quad \text{(by (29))}$$

$$= \frac{1}{w(d)}\sum_{l=1}^{\infty}\frac{N(l,d)}{l^s} \quad \text{(by (19))},$$

completing the proof.

8. The Series $\sum_{n=1}^{\infty}\frac{R_K(n,d)}{n^s}$

We now turn our attention to the series $\sum_{n=1}^{\infty}\frac{R_K(n,d)}{n^s}$.

Lemma 8.1. *Let $K\in H(d)$. For $s>1$, we have*

$$\sum_{n=1}^{\infty}\frac{R_K(n,d)}{n^s} = \frac{w(d)}{h(d)}\sum_{\substack{L\in H(d)\\ L\neq I}}\chi(L,K)^{-1}\left(\zeta(2s)\sum_{n=1}^{\infty}\frac{Y_L(n)}{n^s}\right)+\frac{1}{h(d)}\sum_{n=1}^{\infty}\frac{N(n,d)}{n^s}.$$

Proof. We have

$$\sum_{L\in H(d)}\chi(L,K)^{-1}Y_L(n)$$

$$= \sum_{L\in H(d)}\chi(L,K)^{-1}\sum_{M\in H(d)}\chi(L,M)H_M(n)$$

$$= \sum_{M\in H(d)}H_M(n)\sum_{L\in H(d)}\chi(L,M)\chi(L,K)^{-1}$$

$$= H_K(n)h(d),$$

by Lemma 6.2. Thus

$$H_K(n) = \frac{1}{h(d)} \sum_{L \in H(d)} \chi(L, K)^{-1} Y_L(n).$$

Hence, for $s > 1$, we have

$$\sum_{n=1}^{\infty} \frac{R_K(n, d)}{n^s} = w(d) \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{m^2 \mid n} H_K(n/m^2)$$

$$= w(d)\zeta(2s) \sum_{n=1}^{\infty} \frac{H_K(n)}{n^s}$$

$$= \frac{w(d)\zeta(2s)}{h(d)} \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{L \in H(d)} \chi(L, K)^{-1} Y_L(n)$$

$$= \frac{w(d)}{h(d)} \sum_{L \in H(d)} \chi(L, K)^{-1} \left(\zeta(2s) \sum_{n=1}^{\infty} \frac{Y_L(n)}{n^s} \right)$$

$$= \frac{w(d)}{h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K)^{-1} \left(\zeta(2s) \sum_{n=1}^{\infty} \frac{Y_L(n)}{n^s} \right)$$

$$+ \frac{w(d)}{h(d)} \zeta(2s) \sum_{n=1}^{\infty} \frac{Y_I(n)}{n^s}.$$

The result follows on using Lemma 7.8.

We now consider the behavior of the series $\sum_{n=1}^{\infty} \frac{N(n, d)}{n^s}$ as $s \to 1^+$. We denote Euler's constant by γ.

Lemma 8.2. *As* $s \to 1^+$ *we have*

$$\sum_{n=1}^{\infty} \frac{N(n, d)}{n^s} = \frac{2\pi h(d)}{\sqrt{|d|}} \frac{1}{s-1} + B(d) + O(s-1),$$

where

$$B(d) = \frac{2\pi h(d)}{\sqrt{|d|}} \log(2\pi) + \frac{4\pi\gamma h(d)}{\sqrt{|d|}}$$

$$- \frac{2\pi h(d)}{\sqrt{|d|}} \sum_{p \mid f} \alpha_p(\Delta, f) \log p$$

$$- \frac{\pi h(d) w(\Delta)}{\sqrt{|d|} h(\Delta)} \sum_{m=1}^{|\Delta|} \left(\frac{\Delta}{m}\right) \log \Gamma\left(\frac{m}{|\Delta|}\right), \tag{39}$$

and

$$\alpha_p(\Delta, f) = \frac{(p^{v_p(f)} - 1)(1 - (\Delta/p))}{p^{v_p(f)-1}(p-1)(p - (\Delta/p))}. \tag{40}$$

Proof. Let $G \in G(d)$. Recall from (16) that $|G(d)| = 2^{t(d)}$. By [9, Theorem 10.2], we have as $s \to 1^+$,

$$\sum_{n=1}^{\infty} \frac{R_G(n, d)}{n^s} = \frac{\pi h(d)}{2^{t(d)-1}\sqrt{|d|}} \frac{1}{s-1} + B_G(d) + O(s-1), \tag{41}$$

where

$$B_G(d) = \frac{1}{2^{t(d)}} B(d) - \frac{8\pi}{\sqrt{|d|}} \sum_{\substack{d_1 \in F(d) \\ d_1 > 1}} \beta(d_1, d, G) \log(\eta_{d_1}), \tag{42}$$

$$\beta(d_1, d, G) = \frac{-w(d)\gamma_{d_1}(G) f(d/d_1) h^*(d_1) h(\Delta(d/d_1))}{w(\Delta(d/d_1)) 2^{t(d)+1}}$$

$$\times \sum_{m \mid f(d/d_1)} \frac{1}{m} \prod_{p \mid f/m} \left(1 - \frac{\left(\frac{d_1}{p}\right)}{p}\right)\left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p}\right)$$

$$\times \prod_{\substack{p \mid m \\ p \nmid f/m}} \left(1 - \frac{\left(\frac{\Delta}{p}\right)}{p}\right), \tag{43}$$

η_{d_1} and $h^*(d_1)$ are the fundamental unit and classnumber of the real quadratic field $\mathbb{Q}(\sqrt{d_1})$, respectively, $F(d) = \{d_1 : d_1$ is a fundamental discriminant, $d_1 \mid d$ and $d/d_1 \equiv 0, 1 \pmod 4\}$, and each γ_{d_1} is a group character of $G(d)$, see [9, pp. 277-279]. Hence, as $s \to 1^+$, we have

$$\sum_{n=1}^{\infty} \frac{N(n, d)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{G \in G(d)} R_G(n, d) \quad \text{(by (19))}$$

$$= \sum_{G \in G(d)} \sum_{n=1}^{\infty} \frac{R_G(n, d)}{n^s}$$

$$= \sum_{G \in G(d)} \left(\frac{\pi h(d)}{2^{t(d)-1}\sqrt{|d|}} \frac{1}{s-1} + B_G(d) + O(s-1) \right) \quad \text{(by (41))}$$

$$= \frac{2\pi h(d)}{\sqrt{|d|}} \frac{1}{s-1} + \sum_{G \in G(d)} B_G(d) + O(s-1).$$

But

$$\sum_{G \in G(d)} B_G(d) = B(d) - \frac{8\pi}{\sqrt{|d|}} \sum_{G \in G(d)} \sum_{\substack{d_1 \in F(d) \\ d_1 > 1}} \beta(d_1, d, G) \log(\eta_{d_1}) = B(d),$$

since

$$\sum_{G \in G(d)} \gamma_{d_1}(G) = 0 \quad \text{for } d_1 > 1,$$

see [9, p. 279]. This completes the proof of the lemma.

Lemma 8.3. *Let $K \in H(d)$. Then*

$$\sum_{n=1}^{\infty} \frac{R_K(n, d)}{n^s} = \frac{2\pi}{\sqrt{|d|}} \frac{1}{s-1} + A(K, d) + o(1),$$

as $s \to 1^+$, where

$$A(K, d) = \frac{B(d)}{h(d)} + \frac{\pi^2 w(d)}{6h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K)^{-1} \frac{t_1(d)}{j(L, d)} l(L, d). \tag{44}$$

Proof. The asserted result follows on using Lemmas 7.7, 8.1 and 8.2.

Lemma 8.4. *Let* $K = [a, b, c] \in H(d)$. *Then*

$$\sum_{n=1}^{\infty} \frac{R_K(n, d)}{n^s} = \frac{2\pi}{\sqrt{|d|}} \frac{1}{s-1} + B(a, b, c) + o(1),$$

as $s \to 1^+$, *where*

$$B(a, b, c) = \frac{4\pi\gamma}{\sqrt{|d|}} - \frac{2\pi \log(|d|)}{\sqrt{|d|}} - \frac{8\pi}{\sqrt{|d|}} \log\left(a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|\right). \tag{45}$$

Proof. This is Kronecker's limit formula, see for example [18, Theorem 1, p. 14] or [9, p. 300].

9. Evaluation of $|\eta((b+\sqrt{d})/2a)|$

We are now in a position to prove our main result. It is convenient to define for $K \in H(d)$,

$$E(K, d) = \frac{\pi\sqrt{|d|}w(d)}{48h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K)^{-1} \frac{t_1(d)}{j(L, d)} l(L, d). \tag{46}$$

Theorem 1. *Let* $K = [a, b, c] \in H(d)$. *Then*

$$a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right| = (2\pi|d|)^{-1/4} \prod_{p \mid f} p^{\alpha_p(\Delta, f)/4}$$

$$\times \left(\prod_{m=1}^{|\Delta|} \Gamma\left(\frac{m}{|\Delta|}\right)^{(\Delta/m)}\right)^{\frac{w(\Delta)}{8h(\Delta)}} e^{-E(K,d)}.$$

Proof. By Lemmas 8.3 and 8.4, we have

$$A(K, d) = B(a, b, c). \tag{47}$$

Using (44) and (45) in (47) and after simplifying and exponentiating the resulting equality, we obtain the asserted result.

We remark that the product $\prod_{p \mid f} p^{\alpha_p(\Delta, f)/4}$ in Theorem 1 can be replaced by $\prod_{p} p^{\alpha_p(\Delta, f)/4}$, as $\alpha_p(\Delta, f) = 0$ for $p \nmid f$, see (40).

Next, we use Theorem 1 to deduce the following result of Kaneko [10], Nakkajima and Taguchi [13], and Kaplan and Williams [11], which is the extension of the Chowla-Selberg formula [5], [17] to arbitrary discriminants.

Corollary 1.

$$\prod_{[a,b,c] \in H(d)} a^{-1/4} \left| \eta\left(\frac{b + \sqrt{d}}{2a} \right) \right| = (2\pi |d|)^{-h(d)/4} \left(\prod_{p \mid f} p^{\alpha_p(\Delta, f)} \right)^{h(d)/4} \times \left(\prod_{m=1}^{|\Delta|} \Gamma\left(\frac{m}{|\Delta|} \right)^{(\Delta/m)} \right)^{\frac{w(\Delta) h(d)}{8h(\Delta)}}$$

Proof. The result follows on multiplying the result of Theorem 1 over all the $h(d)$ classes of $H(d)$ and the observation that

$$\sum_{K \in H(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K)^{-1} \frac{t_1(d)}{j(L, d)} l(L, d)$$

$$= \sum_{\substack{L \in H(d) \\ L \neq I}} \frac{t_1(d) l(L, d)}{j(L, d)} \sum_{K \in H(d)} \chi(L^{-1}, K)$$

$$= 0,$$

by Lemma 6.2.

Next we deduce the following result of van der Poorten and Williams [16] from Theorem 1.

Corollary 2. *Let* d *be a fundamental discriminant with* $d < 0$. *Let* $K = [a, b, c] \in H(d)$. *Then*

$$a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right| = (2\pi|d|)^{-1/4}\left(\prod_{m=1}^{|d|}\Gamma\left(\frac{m}{|d|}\right)^{(d/m)}\right)^{\frac{w(d)}{8h(d)}}$$

$$\times \exp\left(-\frac{\pi w(d)\sqrt{|d|}}{48h(d)}\sum_{\substack{L\in H(d)\\ L\neq I}}\chi(L, K)\frac{t_1(d)}{j(L, d)}l(L, d)\right).$$

Proof. Since d is fundamental, we have $\Delta = d$ and $f = 1$. From (35) we obtain

$$l(L, d) = \prod_{p|d}\left(1+\frac{\chi(L, K_p)}{p}\right).$$

Thus

$$l(L^{-1}, d) = \prod_{p|d}\left(1+\frac{\chi(L^{-1}, K_p)}{p}\right) = \prod_{p|d}\left(1+\frac{\chi(L, K_p^{-1})}{p}\right) = l(L, d)$$

since $\chi(L^{-1}, K_p) = \chi(L, K_p^{-1})$ by Lemma 6.2 and $K_p = K_p^{-1}$ if $p|d$ and $p\nmid f$, see Definition 5.2. Hence

$$\sum_{\substack{L\in H(d)\\ L\neq I}}\chi(L, K)^{-1}\frac{t_1(d)}{j(L, d)}l(L, d)$$

$$= \sum_{\substack{L\in H(d)\\ L\neq I}}\chi(L^{-1}, K)\frac{t_1(d)}{j(L, d)}l(L, d)$$

$$= \sum_{\substack{L\in H(d)\\ L\neq I}}\chi(L, K)\frac{t_1(d)}{j(L^{-1}, d)}l(L^{-1}, d)$$

$$= \sum_{\substack{L\in H(d)\\ L\neq I}}\chi(L, K)\frac{t_1(d)}{j(L, d)}l(L, d),$$

since $j(L, d) = j(L^{-1}, d)$ by Lemma 7.6. The result now follows on using Theorem 1.

10. Evaluation of Weber's Functions at Quadratic Irrationalities

We prove

Theorem 2. *Let* $K = [a, b, c] \in H(d)$. *Set*

$$q_0 = a + b + c, \ q_1 = c, \ q_2 = a,$$

$$\lambda_i = \begin{cases} 1, & \text{if } q_i \equiv 2 \pmod 4, \\ 1, & \text{if } q_i \equiv 0 \pmod 4, \ b \equiv 1 \pmod 2, \\ 1/2, & \text{if } q_i \equiv 0 \pmod 4, \ b \equiv 0 \pmod 2, \\ 2, & \text{if } q_i \equiv 1 \pmod 2, \end{cases}$$

for $i = 0, 1, 2$,

$$M_0 = \left[2a\lambda_0, \ \lambda_0(2a+b), \ \frac{\lambda_0}{2}(a+b+c)\right] \in H(\lambda_0^2 d),$$

$$M_1 = \left[2a\lambda_1, \ \lambda_1 b, \ \frac{\lambda_1}{2}c\right] \in H(\lambda_1^2 d),$$

$$M_2 = \left[\frac{\lambda_2}{2}a, \ \lambda_2 b, \ 2\lambda_2 c\right] \in H(\lambda_2^2 d),$$

$$m_i = 2 - 2^{1-\nu_2(\lambda_i)} = \begin{cases} 0, & \text{if } \lambda_i = 1, \\ 1, & \text{if } \lambda_i = 2, \\ -2, & \text{if } \lambda_i = 1/2, \end{cases}$$

for $i = 0, 1, 2$. *Then*

$$\left| f_i\left(\frac{b+\sqrt{d}}{2a}\right) \right| = \left(\frac{2}{\lambda_i}\right)^{1/4} 2^{m_i \frac{1-(\Delta/2)}{2-(\Delta/2)} 2^{-\dot{2}-v_2(f)}} e^{E(K,d)-E(M_i,\lambda_i^2 d)}$$

for $i = 0, 1, 2$.

Proof. We note that $\Delta(\lambda_i^2 d) = \Delta(d) = \Delta$ and $f(\lambda_i^2 d) = \lambda_i f(d) = \lambda_i f$. Applying Theorem 1 to the classes K, M_0, M_1 and M_2, we obtain expressions for

$$\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|, \quad \left|\eta\left(\frac{1+\dfrac{b+\sqrt{d}}{2a}}{2}\right)\right|, \quad \left|\eta\left(\frac{1}{2}\,\frac{b+\sqrt{d}}{2a}\right)\right| \text{ and } \left|\eta\left(2\,\frac{b+\sqrt{d}}{2a}\right)\right|.$$

Using these expressions in (4), (5) and (6), we obtain

$$\left| f_i\left(\frac{b+\sqrt{d}}{2a}\right)\right| = \left(\frac{2}{\lambda_i}\right)^{1/4}\left(\prod_p p^{(\alpha_p(\Delta,\lambda_i f)-\alpha_p(\Delta,f))/4}\right)e^{E(K,d)-E(M_i,\lambda_i^2 d)}$$

for $i = 0, 1, 2$.

If p is odd, we have $\alpha_p(\Delta, \lambda_i f) = \alpha_p(\Delta, f)$ as $v_p(\lambda_i f) = v_p(f)$. Thus

$$\prod_p p^{(\alpha_p(\Delta,\lambda_i f)-\alpha_p(\Delta,f))/4} = 2^{(\alpha_2(\Delta,\lambda_i f)-\alpha_2(\Delta,f))/4} = 2^{m_i\frac{1-(\Delta/2)}{2-(\Delta/2)}2^{-2-v_2(f)}}$$

as required.

The following theorem follows easily from Theorem 2 as $f_i(\sqrt{-n}) \in \mathbb{R}^+$ for $i = 0, 1, 2$.

Theorem 3. *Let n be a positive integer and let $d = -4n$. Let $K = [1, 0, n] \in H(d)$.*

(a) $n \equiv 0 \pmod 4$. *Set*

$$M_0 = [4, 4, n+1] \in H(4d),$$

$$M_1 = \left[1, 0, \frac{n}{4}\right] \in H\left(\frac{d}{4}\right),$$

$$M_2 = [1, 0, 4n] \in H(4d).$$

Let $n = 4^\alpha \mu$, where α is a positive integer and $\mu \equiv 1, 2$ or $3 \pmod 4$.

(i) $\mu \equiv 1$ *or* $2 \pmod 4$ *(so that Δ is even and $v_2(f) = \alpha$). We have*

$$f_0(\sqrt{-n}) = 2^{\frac{1}{2^{\alpha+3}}}e^{E(K,d)-E(M_0,4d)},$$

$$f_1(\sqrt{-n}) = 2^{\frac{2^{\alpha+1}-1}{2^{\alpha+2}}}e^{E(K,d)-E(M_1,d/4)},$$

$$f_2(\sqrt{-n}) = 2^{\frac{1}{2^{\alpha+3}}}e^{E(K,d)-E(M_2,4d)}.$$

(ii) $\mu \equiv 3 \pmod 4$ *(so that* $\Delta \equiv -\mu \pmod 8$ *and* $v_2(f) = \alpha + 1$*). If* $\mu \equiv 3 \pmod 8$*, we have*

$$f_0(\sqrt{-n}) = 2^{\frac{1}{3\cdot 2^{\alpha+2}}} e^{E(K,d)-E(M_0,4d)},$$

$$f_1(\sqrt{-n}) = 2^{\frac{3\cdot 2^{\alpha}-1}{3\cdot 2^{\alpha+1}}} e^{E(K,d)-E(M_1,d/4)},$$

$$f_2(\sqrt{-n}) = 2^{\frac{1}{3\cdot 2^{\alpha+2}}} e^{E(K,d)-E(M_2,4d)}.$$

If $\mu \equiv 7 \pmod 8$*, then we have*

$$f_0(\sqrt{-n}) = e^{E(K,d)-E(M_0,4d)},$$

$$f_1(\sqrt{-n}) = \sqrt{2}\, e^{E(K,d)-E(M_1,d/4)},$$

$$f_2(\sqrt{-n}) = e^{E(K,d)-E(M_2,4d)}.$$

(b) $n \equiv 1 \pmod 4$ *(so that* Δ *is even and* f *is odd). Set*

$$M_0 = \left[2,\ 2,\ \frac{n+1}{2}\right] \in H(d),$$

$$M_1 = [4,\ 0,\ n] \in H(4d),$$

$$M_2 = [1,\ 0,\ 4n] \in H(4d).$$

Then

$$f_0(\sqrt{-n}) = 2^{1/4} e^{E(K,d)-E(M_0,d)},$$

$$f_1(\sqrt{-n}) = 2^{1/8} e^{E(K,d)-E(M_1,4d)},$$

$$f_2(\sqrt{-n}) = 2^{1/8} e^{E(K,d)-E(M_2,4d)}.$$

(c) $n \equiv 2 \pmod 4$ *(so that* Δ *is even and* f *is odd). Set*

$$M_0 = [4,\ 4,\ n+1] \in H(4d),$$

$$M_1 = \left[2,\ 0,\ \frac{n}{2}\right] \in H(d),$$

$$M_2 = [1,\ 0,\ 4n] \in H(4d).$$

Then

$$f_0(\sqrt{-n}) = 2^{1/8}e^{E(K,d)-E(M_0,4d)},$$

$$f_1(\sqrt{-n}) = 2^{1/4}e^{E(K,d)-E(M_1,d)},$$

$$f_2(\sqrt{-n}) = 2^{1/8}e^{E(K,d)-E(M_2,4d)}.$$

(d) $n \equiv 3 \pmod 4$ (*so that* $n \equiv -\Delta \pmod 8$ *and* $f \equiv 2 \pmod 4$). *Set*

$$M_0 = \left[1,\ 1,\ \frac{n+1}{4}\right] \in H\left(\frac{d}{4}\right),$$

$$M_1 = [4,\ 0,\ n] \in H(4d),$$

$$M_2 = [1,\ 0,\ 4n] \in H(4d).$$

Then, for $n \equiv 3 \pmod 8$, *we have*

$$f_0(\sqrt{-n}) = 2^{1/3}e^{E(K,d)-E(M_0,d/4)},$$

$$f_1(\sqrt{-n}) = 2^{1/12}e^{E(K,d)-E(M_1,4d)},$$

$$f_2(\sqrt{-n}) = 2^{1/12}e^{E(K,d)-E(M_2,4d)},$$

and, for $n \equiv 7 \pmod 8$, *we have*

$$f_0(\sqrt{-n}) = \sqrt{2}e^{E(K,d)-E(M_0,d/4)},$$

$$f_1(\sqrt{-n}) = e^{E(K,d)-E(M_1,4d)},$$

$$f_2(\sqrt{-n}) = e^{E(K,d)-E(M_2,4d)}.$$

11. Evaluation of $f(\sqrt{-19})$

We illustrate Theorem 3(d) by using it to determine the (known) value of $f(\sqrt{-19})$, see for example [2], [20]. In another paper we plan to use our results to determine other values of Weber's functions.

We take $n = 19$ so that $d = -76$, $\Delta = -19$, $f = 2$, $K = [1, 0, 19] = I \in H(-76)$ and $M_0 = [1, 1, 5] \in H(-19)$. By Theorem 3(d) we have

$$f(\sqrt{-19}) = 2^{1/3} e^{E(K,-76)-E(M_0,-19)}, \tag{48}$$

where $E(K, d)$ is defined in (46). Since $h(-19) = 1$, we have $E(M_0, -19) = 0$, so that (48) becomes

$$f(\sqrt{-19}) = 2^{1/3} e^{E(K,-76)}. \tag{49}$$

Also $H(-76) = \{I, A, A^{-1}\}$, where $A = [4, 2, 5]$. We have $\nu = 1$, $h_1 = 3$, $\chi(A, A) = e^{2\pi i/3}$, $\chi(A, A^{-1}) = e^{-2\pi i/3}$ and $\chi(A^{-1}, A^{-1}) = e^{2\pi i/3}$. By (46) we have

$$E(K, -76) = \frac{\pi\sqrt{19}}{36} \frac{t_1(-76)}{j(A, -76)} (l(A, -76) + l(A^{-1}, -76)),$$

since $j(A, -76) = j(A^{-1}, -76)$ by Lemma 7.6.

For $L = A$ and A^{-1}, we have by (35),

$$l(L, -76) = \left(1 + \frac{\chi(L, K_{19})}{19}\right) A(L, -76, 2) = \frac{20}{19} A(L, -76, 2).$$

Next, by (34), we have

$$A(L, -76, 2) = 1 + \sum_{j=1}^{\infty} \frac{Y_L(2^j)}{2^j}.$$

By Lemma 4.3 we have

$$H(2^j, -76) = 0 \text{ for } j = 1 \text{ and } j > 2.$$

For any class $M \in H(-76)$ we have by (27),

$$0 \le H_M(2^j) \le H(2^j, -76)$$

so that

$$H_M(2^j) = 0 \text{ for } j = 1 \text{ and } j > 2.$$

Thus, by Definition 7.1, we obtain

$$Y_L(2^j) = \sum_{M \in H(-76)} \chi(L, M) H_M(2^j) = 0 \text{ for } j = 1 \text{ and } j > 2.$$

Hence

$$A(L, -76, 2) = 1 + \frac{Y_L(4)}{4}.$$

From Definition 5.1 we deduce that

$$H_I(4) = 0, \ H_A(4) = 1, \ H_{A^{-1}}(4) = 1,$$

so that, for $L = A$ and A^{-1}, we have

$$Y_L(4) = \sum_{M \in H(-76)} \chi(L, M) H_M(4) = e^{2\pi i/3} + e^{-2\pi i/3} = -1.$$

Hence

$$A(L, -76, 2) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Thus

$$l(L, -76) = 15/19 \text{ for } L = A \text{ and } A^{-1}$$

so that

$$E(K, -76) = \frac{5\pi}{6\sqrt{19}} \frac{t_1(-76)}{j(A, -76)}. \tag{50}$$

By (32) we have

$$j(A, -76) = \lim_{s \to 1^+} \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ K_p = K}} \left(1 - \frac{1}{p^s}\right)^2$$

$$\times \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ K_p \neq K}} \left(1 - \frac{\chi(A, K_p)}{p^s}\right)\left(1 - \frac{\chi(A^{-1}, K_p)}{p^s}\right)$$

$$= \lim_{s\to 1^+} \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ K_p=K}} \left(1-\frac{1}{p^s}\right)^2 \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ K_p\neq K}} \left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}\right).$$

Applying a result of Spearman and Williams [19] to the irreducible polynomial $x^3 - 2x - 2$, we deduce that if $\left(\frac{-76}{p}\right) = 1$, then

$$x^3 - 2x - 2 \equiv 0 \pmod p \text{ is solvable} \Leftrightarrow p \text{ is represented by } K = [1, 0, 19]$$

$$\Leftrightarrow K_p = K.$$

Hence

$$j(A, -76) = \lim_{s\to 1^+} \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2\equiv 0 \pmod p \\ \text{solvable}}} \left(1-\frac{1}{p^s}\right)^2 \times \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2\equiv 0 \pmod p \\ \text{insolvable}}} \left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}\right). \tag{51}$$

As $\operatorname{disc}(x^3 - 2x - 2) = -76 < 0$, $x^3 - 2x - 2$ has one real root and two non-real roots. Let θ be the unique real root of $c(x) = x^3 - 2x - 2$. As $c(1) = -3 < 0$ and $c(2) = 2 > 0$ we have $1 < \theta < 2$, so that θ is positive. We set $F = \mathbb{Q}(\theta)$. Next we link $E(K, -76)$ to the Dedekind zeta function $\zeta_F(s)$ of the cubic field F. By a theorem of Llorente and Nart [12], we find that rational primes decompose into prime ideals in the field F as follows:

$$2 = P^3, \quad N(P) = 2,$$

$$19 = PQ^2, \quad N(P) = N(Q) = 19.$$

If $\left(\frac{-76}{p}\right) = 1$ and $x^3 - 2x - 2 \equiv 0 \pmod p$ is solvable, then

$$p = PQR, \quad N(P) = N(Q) = N(R) = p.$$

If $\left(\frac{-76}{p}\right) = 1$ and $x^3 - 2x - 2 \equiv 0 \pmod p$ is insolvable, then

$$p = P, \quad N(P) = p^3.$$

If $\left(\frac{-76}{p}\right) = -1$, then

$$p = PQ, \quad N(P) = p, \quad N(Q) = p^2.$$

Thus

$$\zeta_F(s) = \prod_P (1 - (N(P))^{-s})^{-1}$$

$$= (1-2^{-s})^{-1}(1-19^{-s})^{-2} \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2\equiv 0 \pmod p \\ \text{solvable}}} (1-p^{-s})^{-3}$$

$$\times \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2\equiv 0 \pmod p \\ \text{insolvable}}} (1-p^{-3s})^{-1} \prod_{\left(\frac{-76}{p}\right)=-1} (1-p^{-s})^{-1}(1-p^{-2s})^{-1}$$

$$= \zeta(s)(1-19^{-s})^{-1} \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2\equiv 0 \pmod p \\ \text{solvable}}} (1-p^{-s})^{-2}$$

$$\times \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2\equiv 0 \pmod p \\ \text{insolvable}}} (1+p^{-s}+p^{-2s})^{-1} \prod_{\left(\frac{-76}{p}\right)=-1} (1-p^{-2s})^{-1}$$

$$= \zeta(s)\zeta(2s)(1-2^{-2s})(1-19^{-s})^{-1}(1-19^{-2s})$$

$$\prod_{\left(\frac{-76}{p}\right)=1} (1-p^{-2s}) \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2\equiv 0 (\bmod p) \\ \text{solvable}}} (1-p^{-s})^{-2}$$

$$\times \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2\equiv 0 (\bmod p) \\ \text{insolvable}}} (1+p^{-s}+p^{-2s})^{-1}.$$

Hence

$$\lim_{s\to 1^+}(s-1)\zeta_F(s) = \frac{\pi^2}{6}\left(1-\frac{1}{4}\right)\left(1+\frac{1}{19}\right)\frac{t_1(-76)}{j(A,\,-76)} = \frac{3\pi}{\sqrt{19}}E(K,\,-76), \quad (52)$$

by (50), (51) and (33). It is well-known that [14, p. 326]

$$\lim_{s\to 1^+}(s-1)\zeta_F(s) = \frac{2^{s+t}\pi^t R_F h_F}{W_F\,|\,d_F\,|^{1/2}},$$

where

s = number of real embeddings of F,

$2t$ = number of imaginary embeddings of F,

h_F = class number of F,

W_F = number of roots of unity in F,

d_F = discriminant of F,

R_F = regulator of F.

Here we have

$$s = 1,\ t = 1,\ h_F = 1,\ W_F = 2,\ d_F = -76,\ R_F = \log(1+\theta),$$

as $1+\theta$ is the fundamental unit of F, see [6, p. 519]. Hence

$$\lim_{s\to 1^+}(s-1)\zeta_F(s) = \frac{\pi}{\sqrt{19}}\log(1+\theta). \tag{53}$$

By (52) and (53), we obtain

$$E(K, -76) = \frac{1}{3}\log(1+\theta) = \log\left(\frac{\theta}{2^{1/3}}\right)$$

since $\theta^3 = 2\theta + 2$. Thus, by (49), we have

$$f(\sqrt{-19}) = 2^{1/3}e^{E(K, -76)} = \theta.$$

We have reproved the following result of Weber [20].

Theorem 4. *Let* θ *be the unique real root of*

$$x^3 - 2x - 2 = 0.$$

Then

$$f(\sqrt{-19}) = \theta.$$

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