The proportion of cyclic quartic fields with discriminant divisible by a given prime

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Abstract: An asymptotic formula is given for the number of cyclic quartic fields with discriminant $\leq x$ and divisible by a given prime.

Key words: Discriminant; cyclic quartic field.

1. Introduction. It was shown in [1, Theorem, p. 97] that the number $N(x)$ of cyclic quartic fields K with discriminant $d(K) \leq x$ satisfies

$$(1.1)\qquad N(x) = \frac{3}{\pi^2}\left\{\frac{(24+\sqrt{2})}{24}C - 1\right\}x^{1/2}, + O\big(x^{1/3}\log^3 x\big),$$

as $x \to \infty$, where

$$(1.2)\qquad C = \prod_{p\equiv 1 \pmod 4}\left(1 + \frac{2}{(p+1)\sqrt{p}}\right).$$

Here and throughout this paper p denotes a prime. Let q be a fixed prime. In this paper, which should be viewed as a continuation of [1], we determine an asymptotic formula for the number $N_q(x)$ of cyclic quartic fields K with discriminant $d(K) \leq x$ and $d(K) \equiv 0 \pmod q$. We prove

Theorem. *Let q be a prime. Then*

$$(1.3)\qquad N_q(x) = E_q x^{1/2} + O\big(x^{1/3}\log^3 x\big),$$

as $x \to \infty$, where

$$E_2 = \frac{1}{\pi^2}\left(\frac{(8+\sqrt{2})}{8}C - 1\right),$$

$$E_q = \frac{3}{\pi^2(q+1)}\left(\left(\frac{24+\sqrt{2}}{24}\right)C - 1\right), \quad \text{if } q \equiv 3 \pmod 4,$$

$$E_q = \frac{3}{\pi^2(q+1)} \times \left(\left(\frac{24+\sqrt{2}}{24}\right)\left(\frac{1+\frac{2}{\sqrt{q}}}{1+\frac{2}{(q+1)\sqrt{q}}}\right)C - 1\right), \quad \text{if } q \equiv 1 \pmod 4.$$

This theorem is proved in Section 3 after some preliminary results are given in Section 2.

The proportion d_q of cyclic quartic fields with discriminant divisible by the fixed prime q is

$$d_q = \lim_{x\to\infty}\frac{N_q(x)}{N(x)} = \frac{E_q}{\frac{3}{\pi^2}\left\{\frac{(24+\sqrt{2})}{24}C - 1\right\}}.$$

Appealing to the values of E_q given in the Theorem, the proportion d_q is given by

$$d_q = \frac{(8+\sqrt{2})C - 8}{(24+\sqrt{2})C - 24}, \quad \text{if } q = 2,$$

$$d_q = \frac{1}{q+1}, \quad \text{if } q \equiv 3 \pmod 4,$$

$$d_q = \frac{(24+\sqrt{2})\left(\dfrac{1+\dfrac{2}{\sqrt{q}}}{1+\dfrac{2}{\sqrt{q}(q+1)}}\right)C - 24}{(q+1)((24+\sqrt{2})C - 24)}, \quad \text{if } q \equiv 1 \pmod 4.$$

2. Some Lemmas. The results of this section are used in Section 3. They are either contained in [1] or [2] or are simple extensions of results there. We use 'n sqf' to indicate that the positive integer n is required to be squarefree. As usual, for $n \in \mathbf{N}$, $\phi(n)$ is Euler's totient function and $d(n)$ counts the number of positive divisors of n. The greatest common divisor of the positive integers a and b is

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denoted by (a,b).

Lemma 2.1. *Let $k \in \mathbf{N}$. Then*

$$\sum_{\substack{1\le n\le x\\ n\ sqf\\ (n,k)=1}} 1 = x\frac{6}{\pi^2}\frac{\phi(k)}{k}\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)^{-1} + O(x^{1/2}d(k)),$$

as $x \to \infty$, where the implied constant is absolute.

Proof. See [2, Lemma 3, p. 182]. □

Lemma 2.2. *Let $k \in \mathbf{N}$. Let q be an odd prime. Then*

$$\sum_{\substack{1\le n\le x\\ n\ sqf\\ (n,k)=1\\ q\mid n}} 1 = \begin{cases} 0, & \text{if } q \mid k,\\ \dfrac{x}{q+1}\dfrac{6}{\pi^2}\dfrac{\phi(k)}{k}\displaystyle\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)^{-1} \\ \quad + O(x^{1/2}q^{-1/2}d(k)), & \text{if } q \nmid k,\end{cases}$$

where the implied constant is absolute.

Proof. The result is clear for $q \mid k$. For $q \nmid k$ we have

$$\begin{aligned}\sum_{\substack{1\le n\le x\\ n\ \mathrm{sqf}\\ (n,k)=1\\ q\mid n}} 1 &= \sum_{\substack{1\le n\le x/q\\ n\ \mathrm{sqf}\\ q\nmid n\\ (qn,k)=1}} 1 = \sum_{\substack{1\le n\le x/q\\ n\ \mathrm{sqf}\\ (n,qk)=1}} 1\\ &= \frac{x}{q}\frac{6}{\pi^2}\frac{\phi(qk)}{qk}\prod_{p\mid qk}\left(1-\frac{1}{p^2}\right)^{-1}\\ &\quad + O\left(\left(\frac{x}{q}\right)^{1/2} d(qk)\right)\\ &= \frac{x}{q+1}\frac{6}{\pi^2}\frac{\phi(k)}{k}\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)^{-1}\\ &\quad + O\big(x^{1/2}q^{-1/2}d(k)\big),\end{aligned}$$

by Lemma 2.1. □

Following [1, eq. (3.7), p. 100] we set

$$\wp = \big\{D \mid D = q_1\cdots q_r\ (r\ge 1),\ q_1,\dots,q_r \text{ distinct primes } \equiv 1 \pmod 4\big\}.$$

Note that $1 \notin \wp$. We set (as in [1, eq. (3.8), p. 100])

$$S(x) := \sum_{\substack{D\le x^{1/3}\\ D\in\wp}} d(D) \sum_{\substack{1\le A\le\sqrt{xD^{-3}}\\ A\ \mathrm{sqf}\\ (A,2D)=1}} 1.$$

Lemma 2.3.

$$S(x) = \frac{4}{\pi^2}(C-1)x^{1/2} + O(x^{1/3}\log^3 x),$$

where the implied constant is absolute.

Proof. See [1, p. 103]. Note that we have $c_0 + 1 = C$. □

Let q be an odd prime. We define

$$S_1(x) := \sum_{\substack{D\le x^{1/3}\\ D\in\wp\\ q\mid D}} d(D) \sum_{\substack{1\le A\le\sqrt{xD^{-3}}\\ A\ \mathrm{sqf}\\ (A,2D)=1}} 1$$

and

$$S_2(x) := \sum_{\substack{D\le x^{1/3}\\ D\in\wp}} d(D) \sum_{\substack{1\le A\le\sqrt{xD^{-3}}\\ A\ \mathrm{sqf}\\ (A,2D)=1\\ q\mid A}} 1.$$

We note that

$$S_1(x) = 0, \quad \text{if } q \equiv 3 \pmod 4.$$

Lemma 2.4. *Let q be an odd prime. Then*

$$S_2(x) = \frac{4}{\pi^2(q+1)}(C'-1)x^{1/2} + O(x^{1/3}\log^3 x),$$

where the implied constant depends only on q, and

$$C' = \prod_{\substack{p\equiv 1 \pmod 4\\ p\ne q}}\left(1+\frac{2}{(p+1)\sqrt{p}}\right). \tag{2.1}$$

We note that $C' = C$ if $q \equiv 3 \pmod 4$, whereas

$$C' = \frac{C}{\left(1+\dfrac{2}{(q+1)\sqrt{q}}\right)} \tag{2.2}$$

if $q \equiv 1 \pmod 4$.

Proof. We have by Lemma 2.2

$$\begin{aligned} S_2(x) &= \sum_{\substack{D\le x^{1/3}\\ D\in\wp\\ q\nmid D}} d(D)\Bigg\{\frac{x^{1/2}}{D^{3/2}}\frac{1}{q+1}\\ &\qquad\times\frac{6}{\pi^2}\frac{\phi(2D)}{2D}\prod_{p\mid 2D}\left(1-\frac{1}{p^2}\right)^{-1}\\ &\qquad + O(x^{1/4}D^{-3/4}d(D))\Bigg\}\\ &= \frac{4}{\pi^2}\frac{x^{1/2}}{q+1}\sum_{\substack{D\le x^{1/3}\\ D\in\wp\\ q\nmid D}} d(D)D^{-5/2}\\ &\qquad\times\phi(D)\prod_{p\mid D}\left(1-\frac{1}{p^2}\right)^{-1}\end{aligned}$$

$$+\,O(x^{1/4}\sum_{\substack{D\le x^{1/3}\\ D\in\wp}} d^2(D)D^{-3/4}).$$

It is shown in [1, p. 103] that

$$\sum_{\substack{D\le x^{1/3}\\ D\in\wp}} d^2(D)D^{-3/4} = O(x^{1/12}\log^3 x).$$

Also

$$\begin{aligned}\sum_{\substack{D\le x^{1/3}\\ D\in\wp\\ q\nmid D}} & d(D)D^{-5/2}\phi(D)\prod_{p\mid D}\left(1-\frac{1}{p^2}\right)^{-1}\\ &= \sum_{\substack{D=1\\ D\in\wp\\ q\nmid D}}^{\infty} d(D)D^{-5/2}\phi(D)\prod_{p\mid D}\left(1-\frac{1}{p^2}\right)^{-1}\\ &\qquad + O\Bigg(\sum_{\substack{D>x^{1/3}\\ D\in\wp}} d(D)D^{-5/2}\phi(D)\Bigg),\end{aligned}$$

as

$$\prod_{p\mid D}\left(1-\frac{1}{p^2}\right)^{-1} < \frac{\pi^2}{6}.$$

Clearly

$$\begin{aligned}\sum_{\substack{D=1\\ D\in\wp\\ q\nmid D}}^{\infty} & d(D)D^{-5/2}\phi(D)\prod_{p\mid D}\left(1-\frac{1}{p^2}\right)^{-1}\\ &= \prod_{\substack{p\equiv 1 \pmod 4\\ p\neq q}}\left(1+\frac{2}{(p+1)\sqrt{p}}\right) - 1 = C'-1.\end{aligned}$$

Also

$$\sum_{\substack{D>x^{1/3}\\ D\in\wp}} d(D)D^{-5/2}\phi(D) = O(x^{-1/6}\log x),$$

see [1, p. 103]. Thus

$$\begin{aligned}S(x) &= \frac{4}{\pi^2}\frac{x^{1/2}}{q+1}(C'-1) + O(x^{1/2-1/6}\log x)\\ &\quad + O\big(x^{1/4+1/12}\log^3 x\big),\end{aligned}$$

which gives the asserted result. □

Lemma 2.5. *Let q be a prime $\equiv 1 \pmod 4$. Then*

$$S_1(x) = \frac{8}{\pi^2}\frac{x^{1/2}}{(q+1)\sqrt{q}}C' + O(x^{1/3}\log^3 x).$$

Proof. We have by Lemma 2.1

$$\begin{aligned}S_1(x) &= \sum_{\substack{D\le x^{1/3}\\ D\in\wp\\ q\mid D}} d(D)\Bigg\{\left(\frac{x}{D^3}\right)^{1/2}\frac{6}{\pi^2}\\ &\qquad\times\frac{\phi(2D)}{2D}\prod_{p\mid 2D}\left(1-\frac{1}{p^2}\right)^{-1}\\ &\qquad\qquad + O\left(\left(\frac{x}{D^3}\right)^{1/4} d(D)\right)\Bigg\}\\ &= \frac{4}{\pi^2}x^{1/2}\sum_{\substack{D\le x^{1/3}\\ D\in\wp\\ q\mid D}} d(D)D^{-5/2}\\ &\qquad\times\phi(D)\prod_{p\mid D}\left(1-\frac{1}{p^2}\right)^{-1}\\ &\qquad\quad + O\Bigg(x^{1/4}\sum_{\substack{D\le x^{1/3}\\ D\in\wp}} d^2(D)D^{-3/4}\Bigg).\end{aligned}$$

As in Lemma 2.4 the error term is

$$O\big(x^{1/4+1/12}\log^3 x\big) = O\big(x^{1/3}\log^3 x\big).$$

Also

$$\begin{aligned}\sum_{\substack{D\le x^{1/3}\\ D\in\wp\\ q\mid D}} & d(D)D^{-5/2}\phi(D)\prod_{p\mid D}\left(1-\frac{1}{p^2}\right)^{-1}\\ &= \sum_{\substack{D=1\\ D\in\wp\\ q\mid D}}^{\infty} d(D)D^{-5/2}\phi(D)\prod_{p\mid D}\left(1-\frac{1}{p^2}\right)^{-1}\\ &\qquad + O\Bigg(\sum_{\substack{D>x^{1/3}\\ D\in\wp}} d(D)D^{-5/2}\phi(D)\Bigg)\\ &= d(q)q^{-5/2}\phi(q)\left(1-\frac{1}{q^2}\right)^{-1}\\ &\quad\times\prod_{\substack{p\equiv 1 \pmod 4\\ p\neq q}}\left(1+\frac{2}{(p+1)\sqrt{p}}\right)\\ &\qquad + O\big(x^{-1/6}\log x\big)\\ &= \frac{2}{(q+1)\sqrt{q}}C' + O\big(x^{-1/6}\log x\big).\end{aligned}$$

Finally

$$S_1(x) = \frac{4x^{1/2}}{\pi^2}\left(\frac{2}{(q+1)\sqrt{q}}C' + O\big(x^{-1/6}\log x\big)\right) + O\big(x^{1/3}\log^3 x\big)$$
$$= \frac{8}{\pi^2}\frac{x^{1/2}}{(q+1)\sqrt{q}}C' + O\big(x^{1/3}\log^3 x\big),$$

which is the assertion of Lemma 2.5. □

3. Proof of Theorem. From [1, (3.3) and (3.4), p. 100] we see that if $q = 2$

$$N_q(x) = 2\sum_{\substack{A\le(x/2^{11})^{1/2}\\ A \text{ sqf}\\ A \text{ odd}}} 1 + 2S(2^{-11}x) + S(2^{-6}x) + \frac{1}{2}S(2^{-4}x);$$

if $q \equiv 3 \pmod 4$

$$N_q(x) = 2\sum_{\substack{A\le(x/2^{11})^{1/2}\\ A \text{ sqf}\\ A \text{ odd}\\ q\mid A}} 1 + 2S_2(2^{-11}x) + S_2(2^{-6}x) + \frac{1}{2}S_2(2^{-4}x) + \frac{1}{2}S_2(x);$$

and if $q \equiv 1 \pmod 4$

$$N_q(x) = 2\sum_{\substack{A\le(x/2^{11})^{1/2}\\ A \text{ sqf}\\ A \text{ odd}\\ q\mid A}} 1 + 2S_2(2^{-11}x) + S_2(2^{-6}x) + \frac{1}{2}S_2(2^{-4}x) + \frac{1}{2}S_2(x) + 2S_1(2^{-11}x) + S_1(2^{-6}x) + \frac{1}{2}S_1(2^{-4}x) + \frac{1}{2}S_1(x).$$

For $q = 2$ we have by Lemmas 2.1 and 2.3

$$N_q(x) = 2\left(\left(\frac{x}{2^{11}}\right)^{1/2}\frac{6}{\pi^2}\frac{\phi(2)}{2} \times \prod_{p\mid 2}\left(1-\frac{1}{p^2}\right)^{-1} + O\big(x^{1/4}\big)\right) + \frac{8}{\pi^2}(C-1)\left(\frac{x}{2^{11}}\right)^{1/2} + \frac{4}{\pi^2}(C-1)\left(\frac{x}{2^{6}}\right)^{1/2} + \frac{2}{\pi^2}(C-1)\left(\frac{x}{2^{4}}\right)^{1/2} + O\big(x^{1/3}\log^3 x\big)$$
$$= \frac{1}{2^{5/2}}\frac{x^{1/2}}{\pi^2} + \frac{2(C-1)}{\pi^2}x^{1/2}\left(\frac{4}{2^{11/2}} + \frac{2}{2^3} + \frac{1}{2^2}\right) + O\big(x^{1/3}\log^3 x\big)$$
$$= \frac{x^{1/2}}{8\pi^2}\big(\sqrt{2} + (C-1)(\sqrt{2}+8)\big) + O\big(x^{1/3}\log^3 x\big)$$
$$= \frac{x^{1/2}}{8\pi^2}\big((8+\sqrt{2})C - 8\big) + O\big(x^{1/3}\log^3 x\big)$$
$$= \frac{1}{\pi^2}\left(\frac{(8+\sqrt{2})}{8}C - 1\right)x^{1/2} + O\big(x^{1/3}\log^3 x\big).$$

For $q \equiv 3 \pmod 4$ we have by Lemmas 2.2 and 2.4

$$N_q(x) = 2\left(\left(\frac{x}{2^{11}}\right)^{1/2}\frac{1}{q+1}\frac{6}{\pi^2}\frac{\phi(2)}{2} \times \prod_{p\mid 2}\left(1-\frac{1}{p^2}\right)^{-1} + O\big(x^{1/4}\big)\right) + \frac{8}{\pi^2}\frac{(C-1)}{(q+1)}\left(\frac{x}{2^{11}}\right)^{1/2} + \frac{4(C-1)}{\pi^2(q+1)}\left(\frac{x}{2^{6}}\right)^{1/2} + \frac{2(C-1)}{\pi^2(q+1)}\left(\frac{x}{2^{4}}\right)^{1/2} + \frac{2(C-1)}{\pi^2(q+1)}x^{1/2} + O\big(x^{1/3}\log^3 x\big)$$
$$= \frac{x^{1/2}}{(q+1)\pi^2}\left(\frac{2^3}{2^{11/2}}\right) + \frac{(C-1)}{(q+1)\pi^2}x^{1/2} \times \left(\frac{8}{2^{11/2}} + \frac{4}{2^3} + \frac{2}{2^2} + 2\right) + O\big(x^{1/3}\log^3 x\big)$$
$$= \frac{3}{\pi^2}\frac{x^{1/2}}{(q+1)}\left(\left(\frac{24+\sqrt{2}}{24}\right)C - 1\right) + O\big(x^{1/3}\log^3 x\big).$$

For $q \equiv 1 \pmod 4$ we have by Lemmas 2.2, 2.4 and 2.5

$$N_q(x) = \frac{3}{\pi^2}\frac{x^{1/2}}{q+1}\left(\left(\frac{24+\sqrt{2}}{24}\right)C' - 1\right) + \frac{8C'x^{1/2}}{\pi^2(q+1)\sqrt{q}}\left\{\frac{2}{2^{11/2}} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2}\right\} + O(x^{1/3}\log^3 x)$$
$$= \frac{3}{\pi^2}\frac{x^{1/2}}{(q+1)} \times \left(\left(\frac{24+\sqrt{2}}{24}\right)\left(1+\frac{2}{\sqrt{q}}\right)C' - 1\right) + O(x^{1/3}\log^3 x).$$

This completes the proof of the Theorem. □

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References

[1] Ou, Z. M., and Williams, K. S.: On the density of cyclic quartic fields. Canad. Math. Bull., **44**, 97–104 (2001).

[2] Spearman, B. K., and Williams, K. S.: Integers which are discriminants of bicyclic or cyclic quartic fields. JP J. Algebra Number Theory Appl., **1**, 179–194 (2001).