JP Jour. Algebra, Number Theory & Appl. 3(1) (2003), 121-144

THE INDEX OF A DIHEDRAL QUARTIC FIELD

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Abstract

Let $c \neq 1$ be a squarefree integer. The set of all possible field indices for non-pure dihedral quartic fields containing the quadratic field $\mathbb{Q}(\sqrt{c})$ is determined by means of congruences on c modulo 24.

2000 Mathematics Subject Classification: 11R16.

Key words and phrases: dihedral quartic field, index.

The first author was supported by an undergraduate student research award at Okanagan University College from the Natural Sciences and Engineering Research Council of Canada.

The second and third authors were supported by research grants from the Natural Sciences and Engineering Research Council of Canada.

Received October 18, 2002

1. Introduction

Let $c \neq 1$ be a squarefree integer, so that $\mathbb{Q}(\sqrt{c})$ is a quadratic field. Let

$$F_c = \text{set of all non-pure dihedral quartic fields } F \supseteq \mathbb{Q}(\sqrt{c}) \tag{1.1}$$

and

$$I_c = \{i(F) \mid F \in F_c\}, \tag{1.2}$$

where $i(F)$ denotes the index of the field F. From the work of Engstrom [1, p. 234], it is known that

$$I_c \subseteq \{1, 2, 3, 4, 6, 12\}. \tag{1.3}$$

Funakura [2, p. 36] has shown that $i(F) = 1$ or 2 for a pure quartic field. For $F \in F_c$, by consideration of the factorization of 2 and 3 in $\mathbb{Q}(\sqrt{c})$, it follows from [1, p. 234] that

$$2 \nmid i(F), \quad \text{if} \quad c \equiv 2, 3 \pmod 4, \tag{1.4}$$

$$2^2 \nmid i(F), \quad \text{if} \quad c \equiv 5 \pmod 8, \tag{1.5}$$

$$3 \nmid i(F); \quad \text{if} \quad c \equiv 0, 2 \pmod 3. \tag{1.6}$$

Further, if $F \in F_c$ and $2 \mid d(F)$ (the discriminant of F), then 2 ramifies in F and so by [1, p. 234] we see that

$$4 \nmid i(F), \quad \text{if} \quad 2 \mid d(F),\ F \in F_c. \tag{1.7}$$

From (1.4), (1.5) and (1.6), we see that

$$I_c = \{1\}, \quad \text{if} \quad c \equiv 2, 3, 6, 11 \pmod{12}, \tag{1.8}$$

$$I_c \subseteq \{1, 2\}, \quad \text{if} \quad c \equiv 5, 21 \pmod{24}, \tag{1.9}$$

$$I_c \subseteq \{1, 3\}, \quad \text{if} \quad c \equiv 7, 10 \pmod{12}, \tag{1.10}$$

$$I_c \subseteq \{1, 2, 4\}, \quad \text{if} \quad c \equiv 9, 17 \pmod{24}, \tag{1.11}$$

$$I_c \subseteq \{1, 2, 3, 6\}, \quad \text{if} \quad c \equiv 13 \pmod{24}. \tag{1.12}$$

The objective of this paper is to prove the following theorem.

Theorem. *Let $c \neq 1$ be a squarefree integer. Then*

$$I_c = \{1, 2, 3, 4, 6, 12\}, \quad \text{if} \quad c \equiv 1 \pmod{24},$$

$$I_c = \{1, 2, 4\}, \quad \text{if} \quad c \equiv 9, 17 \pmod{24},$$

$$I_c = \{1, 2, 3, 6\}, \quad \text{if} \quad c \equiv 13 \pmod{24},$$

$$I_c = \{1, 2\}, \quad \text{if} \quad c \equiv 5, 21 \pmod{24},$$

$$I_c = \{1, 3\}, \quad \text{if} \quad c \equiv 7, 10 \pmod{12},$$

$$I_c = \{1\}, \quad \text{if} \quad c \equiv 2, 3, 6, 11 \pmod{12}.$$

The last assertion of the Theorem is (1.8), so we can suppose that $c \not\equiv 2, 3, 6, 11 \pmod{12}$. For each of the remaining congruence classes of c modulo 12 or 24, we exhibit infinitely many non-pure, dihedral, quartic fields $\supseteq \mathbb{Q}(\sqrt{c})$ having all the possible indices permitted by (1.9)-(1.12). These fields are given in Proposition 2 below. Their indices are determined in the proof of the Theorem in Section 2.

In order to construct these fields we make use of a simple extension of Nagel's theorem for quadratic polynomials [5], see also [6, pp. 1103-1104]. First we state Nagel's theorem for quadratic polynomials.

Nagel's theorem. *Let $f(x) \in \mathbb{Z}[x]$ be a quadratic polynomial, which is primitive and has a nonzero discriminant. Then there exist infinitely many $x \in \mathbb{N}$ such that $f(x)$ is squarefree.*

We need the following extension of this theorem.

Proposition 1. *Let $d \neq 0, e, f \in \mathbb{Z}$ be such that $(d, e, f) = 1$ and $e^2 - 4df \neq 0$. Let m be a positive squarefree integer. Let r be an integer such that $dr^2 + er + f \neq 0$ and for every prime p satisfying $p \mid m$, $p^2 \mid dr^2 + er + f$ we have $p \nmid 2dr + e$. Then there exist infinitely many positive integers $x \equiv r \pmod{m}$ such that $dx^2 + ex + f$ is squarefree.*

Proof. We define the positive squarefree integers G and H by

$$G = \prod_{\substack{p \mid m \\ p \parallel dr^2+er+f}} p, \quad H = \prod_{\substack{p \mid m \\ p^2 \mid dr^2+er+f}} p. \tag{1.13}$$

Next we set

$$A = dm^2GH, \tag{1.14}$$

$$B = m(2d(mG + r) + e), \tag{1.15}$$

$$C = \frac{d(mG + r)^2 + e(mG + r) + f}{GH}. \tag{1.16}$$

We note that

$$C = \frac{dm^2G^2 + mG(2dr + e) + (dr^2 + er + f)}{GH}. \tag{1.17}$$

Clearly $A(\neq 0) \in \mathbb{Z}$ and $B \in \mathbb{Z}$. As $GH \mid m$ and $GH \mid dr^2 + er + f$ we see that $C \in \mathbb{Z}$. Next we show that $(A, B, C) = 1$. Suppose that q is a prime such that $q \mid A$, $q \mid B$, $q \mid C$. Then

$$q \mid dm^2GH, \tag{1.18}$$

$$q \mid m(2d(mG + r) + e), \tag{1.19}$$

$$q \,\Big|\, \frac{dm^2G^2 + mG(2dr + e) + (dr^2 + er + f)}{GH}. \tag{1.20}$$

If $q \mid m$, then from (1.20) we see that $q \mid dr^2 + er + f$. If $q \parallel dr^2 + er + f$, then by (1.13) $q \parallel G$, $q \nmid H$ and so by (1.20) $q^2 \mid dr^2 + er + f$, a contradiction. If $q^2 \mid dr^2 + er + f$, then by (1.13) $q \nmid G$, $q \parallel H$ and by assumption $q \nmid 2dr + e$. Then, by (1.20), we have $q^2 \mid m$, contradicting m squarefree. Thus $q \nmid m$. Hence, by (1.13), $q \nmid G$, $q \nmid H$. Thus, by (1.18)-(1.20), we deduce that $q \mid d$, $q \mid e$, $q \mid f$, contradicting $(d, e, f) = 1$. This completes the proof that

$$(A, B, C) = 1. \tag{1.21}$$

Next we show that the discriminant of $Ax^2 + Bx + C$ is nonzero. This is clear as

$$B^2 - 4AC = m^2(e^2 - 4df) \neq 0.$$

Further, for any integer y, we have $(Ay^2 + By + C, m) = (C, m) = 1$, as, by (1.14), (1.15), and (1.21), $m \mid A$, $m \mid B$ and $(A, B, C) = 1$.

Thus, by Nagel's theorem, there exist infinitely many positive integers y such that $Ay^2 + By + C$ is squarefree and coprime with GH (as $GH \mid m$). These infinitely many positive integers y are such that

$$GH(Ay^2 + By + C) = d(MGHy + (mG + r))^2 + e(mGHy + (mG + r)) + f$$

is squarefree. Hence there exist infinitely many positive integers $x \equiv r \pmod{m}$ with $dx^2 + ex + f$ squarefree.

Applying Proposition 1 to the quadratic polynomials $-16cx^2 + 1$ and $16x^2 - 9c$, where $c(\neq 1)$ is a squarefree integer, we obtain the following result.

Corollary. *Let $c(\neq 1)$ be a squarefree integer.*

(a) *There exist infinitely many positive integers k in each residue class modulo 6 such that $1 - 16ck^2$ is squarefree.*

(b) *If $c \not\equiv 2 \pmod 4$, then there exist infinitely many positive integers k in both residue classes modulo 2 such that $16k^2 - 9c$ is squarefree.*

(c) *If $c \not\equiv 2 \pmod 4$, then there exist infinitely many positive integers k in each of the residue classes 1, 2, 4, 5 modulo 6 such that $16k^2 - 9c$ is squarefree.*

Proof. (a) We take $d = -16c$, $e = 0$, $f = 1$, $m = 6$, $r = 0, 1, 2, 3, 4, 5$ in Proposition 1. All the conditions of the Proposition are satisfied. The only one which is not immediately obvious is

$$9 \mid 1 - 16cr^2 \Rightarrow 3 \nmid 16cr^2 \Rightarrow 3 \nmid cr \Rightarrow 3 \nmid -32cr.$$

(b) In this case $c \equiv 1 \pmod 2$. We take $d = 16$, $e = 0$, $f = -9c$, $m = 2$, $r = 0, 1$. We have only to note that $4 \nmid 16r^2 - 9c$.

(c) In this case $c \equiv 1 \pmod 2$. We take $d = 16$, $e = 0$, $f = -9c$, $m = 6$, $r = 1, 2, 4, 5$. We have only to note that $4 \nmid 16r^2 - 9c$ and $9 \nmid 16r^2 - 9c$.

Next for any squarefree integer $c (\neq 1)$, any positive integer m, and any integer $i \in \{0, 1, \ldots, m-1\}$, we define the sets

$$U_{i,m}(c) = \{k \in \mathbb{N} \mid k \equiv i \pmod m,\ 1 - 16ck^2 \text{ is squarefree}\} \tag{1.22}$$

and

$$V_{i,m}(c) = \{k \in \mathbb{N} \mid k \equiv i \pmod m,\ 16k^2 - 9c \text{ is squarefree}\}. \tag{1.23}$$

By the Corollary the sets $U_{i,6}(c)(i = 0, 1, 2, 3, 4, 5)$, $V_{i,2}(c)(c \not\equiv 2 \pmod 4$, $i = 0, 1)$ and $V_{i,6}(c)(c \not\equiv 2 \pmod 4$, $i = 1, 2, 4, 5)$ are infinite sets.

Further, for any squarefree integer $c \neq 1$ and any integer k, we define the fields

$$K(k, c) = \mathbb{Q}(\sqrt{\mu}), \quad \text{where } \mu = 1 + 4k\sqrt{c}, \tag{1.24}$$

and

$$L(k, c) = \mathbb{Q}(\sqrt{\lambda}), \quad \text{where } \lambda = 4k - 3\sqrt{c}. \tag{1.25}$$

We prove

Proposition 2. (a) *For each* $i = 0, 1, 2, 3, 4, 5$ *and each squarefree integer* $c \neq 1$, *the set*

$$\{K(k, c) \mid k \in U_{i,6}(c)\}$$

consists of infinitely many distinct non-pure, dihedral, quartic fields containing $\mathbb{Q}(\sqrt{c})$ *with*

$$d(K(k, c)) = \begin{cases} (1 - 16ck^2)c^2, & \text{if } c \equiv 1 \pmod 4, \\ 2^4(1 - 16ck^2)c^2, & \text{if } c \equiv 2, 3 \pmod 4. \end{cases} \tag{1.26}$$

(b) *For each* $i = 0, 1$ *and each squarefree integer* $c \neq 1$ *with* $c \equiv 1 \pmod 4$, *the set*

$$\{L(k, c) \mid k \in V_{i,2}(c)\}$$

consists of infinitely many distinct non-pure, dihedral, quartic fields containing $\mathbb{Q}(\sqrt{c})$ *with*

$$d(L(k, c)) = \begin{cases} 2^2(16k^2 - 9c)c^2, & \text{if } c \equiv 1 \pmod 8, \\ 2^4(16k^2 - 9c)c^2, & \text{if } c \equiv 5 \pmod 8). \end{cases} \tag{1.27}$$

(c) *For each* $i = 1, 2, 4, 5$ *and each squarefree integer* $c \neq 1$ *with* $c \equiv 1 \pmod 4$, *the set*

$$\{L(k, c) \mid k \in V_{i,6}(c)\}$$

consists of infinitely many distinct non-pure, dihedral, quartic fields containing $\mathbb{Q}(\sqrt{c})$ *with*

$$d(L(k, c)) = \begin{cases} 2^2(16k^2 - 9c)c^2, & \text{if } c \equiv 1 \pmod 8, \\ 2^4(16k^2 - 9c)c^2, & \text{if } c \equiv 5 \pmod 8). \end{cases} \tag{1.28}$$

Proof. We just treat (a) as (b) and (c) can be proved similarly. Let $i \in \{0, 1, 2, 3, 4, 5\}$, c squarefree $\neq 1$, and $k \in U_{i,6}(c)$. Clearly from (1.24) $K(k, c) \supseteq \mathbb{Q}(\sqrt{c})$. The norm of $1 + 4k\sqrt{c}$ is $1 - 16ck^2$, which is a squarefree integer $\neq 1$, so that $[K(k, c) : \mathbb{Q}] = 4$. The formula for $d(K(k, c))$ in (1.26) follows from [4, Theorem 1]. For $k_1, k_2 \in U_{i,6}(c)$ with $k_1 \neq k_2$ we see from (1.26) that $d(K(k_1, c)) \neq d(K(k_2, c))$ so that $\{K(k, c) \mid k \in U_{i,6}(c)\}$ is an infinite collection of distinct quartic fields for each $i \in \{0, 1, 2, 3, 4, 5\}$. As $1 - 16ck^2$ is squarefree and not equal to 1, it follows that none of $1 - 16ck^2$, $\frac{1 - 16ck^2}{c}$, $\frac{1 - 16ck^2}{-c}$ are perfect squares, and so by [4, equation (1) and Proposition 2] $K(k, c)$ is a non-pure, dihedral, quartic field.

2. Proof of Theorem

We just give the proof in the case $c \equiv 1 \pmod{24}$. Details are summarized in Table 1. The remaining cases $c \equiv 5, 9, 13, 17, 21 \pmod{24}$ and $c \equiv 7, 10 \pmod{12}$ can be treated in a similar fashion and the needed information is provided in Tables 2-8.

Let c be a squarefree integer $\neq 1$ with $c = 24v + 1$. Let k be any positive integer such that $1 - 16ck^2$ is squarefree. By [4, Table D'] an integral basis for $K(k, c)$ is

$$\left\{1, \frac{1+\sqrt{\mu}}{2}, \frac{1+\sqrt{c}}{2}, \frac{(1-\sqrt{c})(1+\sqrt{\mu})}{4}\right\}.$$

The index form $i(X, Y, Z)$ corresponding to this integral basis is given by

$$i(X, Y, Z) = \sqrt{\frac{D\left(X\left(\frac{1+\sqrt{\mu}}{2}\right) + Y\left(\frac{1+\sqrt{c}}{2}\right) + Z\left(\frac{(1-\sqrt{c})(1+\sqrt{\mu})}{4}\right)\right)}{d(K(k, c))}},$$

where $D(\alpha)$ denotes the discriminant of the element α. The index form of a dihedral quartic field has been discussed by Gaál, Pethö and Pohst [3]. Using MAPLE we find $i(X, Y, Z)$ as given in Table 1. Then, with multiplicities omitted, we obtain

$$\{i(X, Y, Z) \pmod 2 \mid X, Y, Z \pmod 2\} = \{0\},$$

$$\{i(X, Y, Z) \pmod 4 \mid X, Y, Z \pmod 4\} = \{0, 2vk^2, 2vk^2 + 2k^2\},$$

$$\{i(X, Y, Z) \pmod 3 \mid X, Y, Z \pmod 3\} = \{0, k^2, 2k^2\}.$$

Thus the index

$$i(K(k, c)) = \gcd\{i(X, Y, Z) \mid X, Y, Z \in \mathbb{Z}\}$$

satisfies

$$i(K(k, c)) \equiv 0 \pmod 2,$$

$$i(K(k, c)) \equiv 0 \pmod 4 \Leftrightarrow k \equiv 0 \pmod 2,$$

$$i(K(k, c)) \equiv 0 \pmod 3 \Leftrightarrow k \equiv 0 \pmod 3.$$

Hence

$$i(K(k, c)) = 2, 4, 6, 12 \text{ according as } k \equiv \pm 1, \pm 2, 3, 0 \pmod 6.$$

Thus

$$i(K(k, c)) = 2 \text{ for the infinitely many } k \in U_{1,6}(c),$$

$$i(K(k, c)) = 4 \text{ for the infinitely many } k \in U_{2,6}(c),$$

$$i(K(k, c)) = 6 \text{ for the infinitely many } k \in U_{3,6}(c),$$

$$i(K(k, c)) = 12 \text{ for the infinitely many } k \in U_{0,6}(c).$$

Now let k be any positive integer such that $16k^2 - 9c$ is squarefree and not equal to 1. By [4, Table D'] an integral basis for $K(k, c)$ is

$$\left\{1, \sqrt{\lambda}, \frac{1+\sqrt{c}}{2}, \frac{(1+\sqrt{c})(1+\sqrt{\lambda})}{4}\right\}.$$

The index form $i(X, Y, Z)$ corresponding to this integral basis was found using MAPLE and is given in Table 1. Then, with multiplicities omitted, we obtain

$$\{i(X, Y, Z) \pmod 2 \mid X, Y, Z \pmod 2\} = \left\{0, k^2 + \frac{c+23}{24}\right\}$$

and

$$\{i(X, Y, Z) \pmod 3 \mid X, Y, Z \pmod 3\} = \{0, k+2, 1+2k\}.$$

Hence

$$i(L(k, c)) \equiv 0 \pmod 2 \Leftrightarrow k \equiv \frac{c+23}{24} \pmod 2,$$

$$i(L(k, c)) \equiv 0 \pmod 3 \Leftrightarrow k \equiv 1 \pmod 3.$$

From (1.7) and (1.27) we deduce that $i(L(k, c)) \not\equiv 0 \pmod 4$. Thus

$i(L(k, c)) = 1, 2, 3, 6$ according as

$$k \equiv \frac{c-1}{8} \text{ or } \frac{c+15}{8}, \frac{c+23}{8} \text{ or } \frac{c+39}{8}, \frac{c+31}{8}, \frac{c+7}{8} \pmod 6.$$

Finally

$$i(L(k, c)) = 1 \text{ for the infinitely many } k \in V_{\frac{c+15}{8},6}(c),$$

$$i(L(k, c)) = 3 \text{ for the infinitely many } k \in V_{\frac{c+31}{8},6}(c).$$

We note that $\frac{c+15}{8} \equiv 2$ or $5 \pmod 6$ and $\frac{c+31}{8} \equiv 1$ or $4 \pmod 6$. Thus for $c \equiv 1 \pmod{24}$ we have shown that $I_c = \{1, 2, 3, 4, 6, 12\}$.

For the remaining congruence classes for c, we find (see Tables 2-8):

$c \equiv 5 \pmod{24}$	$i(K(k, c)) = 2 \quad k \in U_{0,1}(c)$ $i(L(k, c)) = 1 \quad k \in V_{0,1}(c)$
$c \equiv 9 \pmod{24}$	$i(K(k, c)) = 2 \quad k \in U_{1,2}(c)$ $i(K(k, c)) = 4 \quad k \in U_{0,2}(c)$ $i(L(k, c)) = 1 \quad k \in V_{\frac{c+15}{24},2}(c)$
$c \equiv 13 \pmod{24}$	$i(K(k, c)) = 2 \quad k \in U_{1,3}(c)$ $i(K(k, c)) = 6 \quad k \in U_{0,3}(c)$ $i(L(k, c)) = 1 \quad k \in V_{2,6}(c)$ $i(L(k, c)) = 3 \quad k \in V_{1,6}(c)$
$c \equiv 17 \pmod{24}$	$i(K(k, c)) = 2 \quad k \in U_{1,2}(c)$ $i(K(k, c)) = 4 \quad k \in U_{0,2}(c)$ $i(L(k, c)) = 1 \quad k \in V_{\frac{c-17}{24},2}(c)$
$c \equiv 21 \pmod{24}$	$i(K(k, c)) = 2 \quad k \in U_{0,1}(c)$ $i(L(k, c)) = 1 \quad k \in V_{1,6}(c)$
$c \equiv 7, 10 \pmod{12}$	$i(K(k, c)) = 1 \quad k \in U_{1,3}(c)$ $i(K(k, c)) = 3 \quad k \in U_{0,3}(c)$

Proposition 2 guarantees that in each case there are infinitely many such fields $K(k, c)$ and $L(k, c)$.

Table 1

$c = 24v + 1$
$d(K(k, c)) = (1 - 16ck^2)c^2$
$O_{K(k,c)} = \left[1, \frac{1+\sqrt{\mu}}{2}, \frac{1+\sqrt{c}}{2}, \frac{(1-\sqrt{c})(1+\sqrt{\mu})}{4}\right]$
$i(X, Y, Z) = (Y^2 + YZ - 6vZ^2)(24vX^4 + X^4 - 2X^3Z - 48vX^3Z - X^2Y^2 - X^2YZ + 96kvX^2YZ + 4kX^2YZ + 30vX^2Z^2 + 48kvX^2Z^2 + 2kX^2Z^2 + X^2Z^2 + XY^2Z + XYZ^2 - 4kXYZ^2 - 96kvXYZ^2 - 48kvXZ^3 - 2kXZ^3 - 6vXZ^3 + 4k^2Y^4 - 2kY^3Z + 8k^2Y^3Z + 8k^2Y^2Z^2 + 48k^2vY^2Z^2 - 3kY^2Z^2 + 48k^2vYZ^3 + 12kvYZ^3 + 4k^2YZ^3 - kYZ^3 + 6kvZ^4 + k^2Z^4 + 144k^2v^2Z^4 + 24k^2vZ^4)$
$i(K(k, c)) \equiv 0 \pmod 2$ $i(K(k, c)) \equiv 0 \pmod 4 \Leftrightarrow 2 \mid k$ $i(K(k, c)) \equiv 0 \pmod 3 \Leftrightarrow 3 \mid k$
$i(K(k, c)) = 2, 4, 6, 12$ according as $k \equiv \pm 1, \pm 2, 3, 0 \pmod 6$

Table 1 (continued)

$c = 24v + 1$
$d(L(k, c)) = 2^2(16k^2 - 9c)c^2$
$O_{L(k,c)} = \left[1, \sqrt{\lambda}, \frac{1+\sqrt{c}}{2}, \frac{(1+\sqrt{c})(1+\sqrt{\lambda})}{4}\right]$
$i(X, Y, Z) = (2Y^2 + YZ - 3vZ^2)(24vX^4 + X^4 + 2X^3Z + 48vX^3Z - 16kX^2Y^2 + 6X^2YZ + 144vX^2YZ - 8kX^2YZ - 24kvX^2Z^2 - 2kX^2Z^2 + 72vX^2Z^2 + 3X^2Z^2 - 16kXY^2Z + 6XYZ^2 - 8kXYZ^2 + 144vXYZ^2 + 48vXZ^3 + 2XZ^3 - 2kXZ^3 - 24kvXZ^3 + 36Y^4 - 48kY^3Z + 36Y^3Z + 108vY^2Z^2 + 16k^2Y^2Z^2 - 40kY^2Z^2 + 18Y^2Z^2 + 8k^2YZ^3 + 90vYZ^3 - 72kvYZ^3 - 14kYZ^3 + 6YZ^3 - 2kZ^4 + 81v^2Z^4 + 24vZ^4 + k^2Z^4 - 24kvZ^4 + Z^4)$
$i(L(k, c)) \equiv 0 \pmod 2 \Leftrightarrow k \equiv \frac{c+23}{24} \pmod 2$ $i(L(k, c)) \equiv 0 \pmod 3 \Leftrightarrow k \equiv 1 \pmod 3$ $i(L(k, c)) \not\equiv 0 \pmod 4$ (by (1.7))
$i(L(k, c)) = 1, 2, 3, 6$ according as $k \equiv \frac{c-1}{8}$ or $\frac{c+15}{8}, \frac{c+23}{8}$ or $\frac{c+39}{8}, \frac{c+31}{8}, \frac{c+7}{8} \pmod 6$

Table 2

$c = 24v + 5$
$d(K(k, c)) = (1 - 16ck^2)c^2$
$O_{K(k,c)} = \left[1, \dfrac{1+\sqrt{\mu}}{2}, \dfrac{1+\sqrt{c}}{2}, \dfrac{(1-\sqrt{c})(1+\sqrt{\mu})}{4}\right]$
$i(X, Y, Z) = (Y^2 + YZ - Z^2 - 6vZ^2)(24vX^4 + 5X^4 - 10X^3Z - 48vX^3Z - X^2Y^2 - X^2YZ + 96kvX^2YZ + 20kX^2YZ + 30vX^2Z^2 + 48kvX^2Z^2 + 10kX^2Z^2 + 6X^2Z^2 + XY^2Z + XYZ^2 - 20kXYZ^2 - 96kvXYZ^2 - XZ^3 - 10kXZ^3 - 48kvXZ^3 - 6vXZ^3 + 4k^2Y^4 - 2kY^3Z + 8k^2Y^3Z + 48k^2vY^2Z^2 - 3kY^2Z^2 + 16k^2Y^2Z^2 + 48k^2vYZ^3 + 12kvYZ^3 + 12k^2YZ^3 + kYZ^3 + 6kvZ^4 + 144k^2v^2Z^4 + 72k^2vZ^4 + 9k^2Z^4 + kZ^4)$
$i(K(k, c)) \equiv 0 \pmod 2$ $i(K(k, c)) \not\equiv 0 \pmod 3$ (by (1.6)) $i(K(k, c)) \not\equiv 0 \pmod 4$ (by (1.5))
$i(K(k, c)) = 2$

Table 2 (continued)

$c = 24v + 5$
$d(L(k, c)) = 2^4(16k^2 - 9c)c^2$
$O_{L(k,c)} = \left[1, \sqrt{\lambda}, \frac{1+\sqrt{c}}{2}, \frac{(1-\sqrt{c})\sqrt{\lambda}}{2}\right]$
$i(X, Y, Z) = (Y^2 + YZ - Z^2 - 6vZ^2)(5X^4 + 24vX^4 - 16kX^2Y^2 - 60X^2YZ - 16kX^2YZ - 288vX^2YZ - 24kX^2Z^2 - 144vX^2Z^2 - 30X^2Z^2 - 96kvX^2Z^2 + 36Y^4 + 96kY^3Z + 72Y^3Z + 144kY^2Z^2 + 432vY^2Z^2 + 144Y^2Z^2 + 64k^2Y^2Z^2 + 108YZ^3 + 192kYZ^3 + 432vYZ^3 + 576kvYZ^3 + 64k^2YZ^3 + 16k^2Z^4 + 288kvZ^4 + 81Z^4 + 1296v^2Z^4 + 648vZ^4 + 72kZ^4)$
$i(L(k, c)) \not\equiv 0 \pmod 2$ $i(L(k, c)) \not\equiv 0 \pmod 3$ (by (1.6))
$i(L(k, c)) = 1$

Table 3

$c = 24v + 9$
$d(K(k, c)) = (1 - 16ck^2)c^2$
$O_{K(k,c)} = \left[1, \frac{1+\sqrt{\mu}}{2}, \frac{1+\sqrt{c}}{2}, \frac{(1-\sqrt{c})(1+\sqrt{\mu})}{4}\right]$
$i(X, Y, Z) = (Y^2 + YZ - 2Z^2 - 6vZ^2)(24vX^4 + 9X^4 - 18X^3Z - 48vX^3Z - X^2Y^2 - X^2YZ + 96kvX^2YZ + 36kX^2YZ + 30vX^2Z^2 + 48kvX^2Z^2 + 18kX^2Z^2 + 11X^2Z^2 + XY^2Z + XYZ^2 - 36kXYZ^2 - 96kvXYZ^2 - 2XZ^3 - 18kXZ^3 - 48kvXZ^3 - 6vXZ^3 + 4k^2Y^4 - 2kY^3Z + 8k^2Y^3Z + 48k^2vY^2Z^2 - 3kY^2Z^2 + 24k^2Y^2Z^2 + 48k^2vYZ^3 + 12kvYZ^3 + 20k^2YZ^3 + 3kYZ^3 + 6kvZ^4 + 144k^2v^2Z^4 + 120k^2vZ^4 + 25k^2Z^4 + 2kZ^4)$
$i(K(k, c)) \equiv 0 \pmod 2$ $i(K(k, c)) \equiv 0 \pmod 4 \Leftrightarrow 2 \mid k$ $i(K(k, c)) \not\equiv 0 \pmod 3$ (by (1.6))
$i(K(k, c)) = 2$ or 4 according as $k \equiv 1$ or $0 \pmod 2$

Table 3 (continued)

$$c = 24v + 9$$

$$d(L(k, c)) = 2^2(16k^2 - 9c)c^2$$

$$O_{L(k,c)} = \left[1, \sqrt{\lambda}, \frac{1+\sqrt{c}}{2}, \frac{(1+\sqrt{c})(1+\sqrt{\lambda})}{4}\right]$$

$$\begin{aligned} i(X, Y, Z) = {} & (2Y^2 + YZ - Z^2 - 3vZ^2)(9X^4 + 24vX^4 + 48vX^3Z \\ & + 18X^3Z - 16kX^2Y^2 + 144vX^2YZ + 54X^2YZ - 8kX^2YZ \\ & + 72vX^2Z^2 - 24kvX^2Z^2 - 10kX^2Z^2 + 27X^2Z^2 - 16kXY^2Z \\ & + 54XYZ^2 - 8kXYZ^2 + 144vXYZ^2 - 10kXZ^3 - 24kvXZ^3 \\ & + 48vXZ^3 + 18XZ^3 + 36Y^4 - 48kY^3Z + 36Y^3Z - 40kY^2Z^2 \\ & + 108vY^2Z^2 + 54Y^2Z^2 + 16k^2Y^2Z^2 + 8k^2YZ^3 + 90vYZ^3 \\ & - 72kvYZ^3 - 38kYZ^3 + 36YZ^3 + k^2Z^4 + 18Z^4 - 24kvZ^4 \\ & + 81v^2Z^4 + 78vZ^4 - 10kZ^4) \end{aligned}$$

$$i(L(k, c)) \equiv 0 \pmod 2 \Leftrightarrow k \equiv \frac{c-9}{24} \pmod 2$$

$$i(L(k, c)) \not\equiv 0 \pmod 3 \quad \text{(by (1.6))}$$

$$i(L(k, c)) \not\equiv 0 \pmod 4 \quad \text{(by (1.7))}$$

$$i(L(k, c)) = 1 \text{ or } 2 \text{ according as } k \equiv \frac{c+15}{24} \text{ or } \frac{c-9}{24} \pmod 2$$

Table 4

$c = 24v + 13$
$d(K(k, c)) = (1 - 16ck^2)c^2$
$O_{K(k,c)} = \left[1, \frac{1+\sqrt{\mu}}{2}, \frac{1+\sqrt{c}}{2}, \frac{(1-\sqrt{c})(1+\sqrt{\mu})}{4}\right]$
$i(X, Y, Z) = (Y^2 + YZ - 3Z^2 - 6vZ^2)(24vX^4 + 13X^4 - 26X^3Z - 48vX^3Z - X^2Y^2 - X^2YZ + 96kvX^2YZ + 52kX^2YZ + 30vX^2Z^2 + 48kvX^2Z^2 + 26kX^2Z^2 + 16X^2Z^2 + XY^2Z + XYZ^2 - 52kXYZ^2 - 96kvXYZ^2 - 3XZ^3 - 26kXZ^3 - 48kvXZ^3 - 6vXZ^3 + 4k^2Y^4 - 2kY^3Z + 8k^2Y^3Z + 48k^2vY^2Z^2 - 3kY^2Z^2 + 32k^2Y^2Z^2 + 48k^2vYZ^3 + 12kvYZ^3 + 28k^2YZ^3 + 5kYZ^3 + 6kvZ^4 + 144k^2v^2Z^4 + 168k^2vZ^4 + 49k^2Z^4 + 3kZ^4)$
$i(K(k, c)) \equiv 0 \pmod 2$ $i(K(k, c)) \equiv 0 \pmod 3 \Leftrightarrow k \equiv 0 \pmod 3$ $i(K(k, c)) \not\equiv 0 \pmod 4$ (by (1.5))
$i(K(k, c)) = 2, 6$ according as $k \equiv \pm 1, 0 \pmod 3$

Table 4 (continued)

$c = 24v + 13$
$d(L(k, c)) = 2^4(16k^2 - 9c)c^2$
$O_{L(k,c)} = \left[1, \sqrt{\lambda}, \frac{1+\sqrt{c}}{2}, \frac{(1-\sqrt{c})\sqrt{\lambda}}{2}\right]$
$i(X, Y, Z) = (Y^2 + YZ - 3Z^2 - 6vZ^2)(13X^4 + 24vX^4 - 16kX^2Y^2 - 156X^2YZ - 16kX^2YZ - 288vX^2YZ - 56kX^2Z^2 - 144vX^2Z^2 - 78X^2Z^2 - 96kvX^2Z^2 + 36Y^4 + 96kY^3Z + 72Y^3Z + 144kY^2Z^2 + 432vY^2Z^2 + 288Y^2Z^2 + 64k^2Y^2Z^2 + 252YZ^3 + 384kYZ^3 + 432vYZ^3 + 576kvYZ^3 + 64k^2YZ^3 + 16k^2Z^4 + 288kvZ^4 + 441Z^4 + 1296v^2Z^4 + 1512vZ^4 + 168kZ^4)$
$i(L(k, c)) \not\equiv 0 \pmod 2$ $i(L(k, c)) \equiv 0 \pmod 3 \Leftrightarrow k \equiv 1 \pmod 3$
$i(L(k, c)) = 1, 3$ according as $k \equiv 0$ or $2, 1 \pmod 3$

Table 5

$c = 24v + 17$
$d(K(k, c)) = (1 - 16ck^2)c^2$
$O_{K(k,c)} = \left[1, \frac{1+\sqrt{\mu}}{2}, \frac{1+\sqrt{c}}{2}, \frac{(1-\sqrt{c})(1+\sqrt{\mu})}{4}\right]$
$i(X, Y, Z) = (Y^2 + YZ - 4Z^2 - 6vZ^2)(24vX^4 + 17X^4 - 34X^3Z - 48vX^3Z - X^2Y^2 - X^2YZ + 96kvX^2YZ + 68kX^2YZ + 30vX^2Z^2 + 48kvX^2Z^2 + 34kX^2Z^2 + 21X^2Z^2 + XY^2Z + XYZ^2 - 68kXYZ^2 - 96kvXYZ^2 - 4XZ^3 - 34kXZ^3 - 48kvXZ^3 - 6vXZ^3 + 4k^2Y^4 - 2kY^3Z + 8k^2Y^3Z + 48k^2vY^2Z^2 - 3kY^2Z^2 + 40k^2Y^2Z^2 + 48k^2vYZ^3 + 12kvYZ^3 + 36k^2YZ^3 + 7kYZ^3 + 6kvZ^4 + 144k^2v^2Z^4 + 216k^2vZ^4 + 81k^2Z^4 + 4kZ^4)$
$i(K(k, c)) \equiv 0 \pmod 2$ $i(K(k, c)) \equiv 0 \pmod 4 \Leftrightarrow k \equiv 0 \pmod 2$ $i(K(k, c)) \not\equiv 0 \pmod 3$ (by (1.6))
$i(K(k, c)) = 2$ or 4 according as $k \equiv 1$ or $0 \pmod 2$

Table 5 (continued)

$c = 24v + 17$
$d(L(k, c)) = 2^2(16k^2 - 9c)c^2$
$O_{L(k,c)} = \left[1, \sqrt{\lambda}, \frac{1+\sqrt{c}}{2}, \frac{(1+\sqrt{c})(1+\sqrt{\lambda})}{4}\right]$
$i(X, Y, Z) = (2Y^2 + YZ - 2Z^2 - 3vZ^2)(17X^4 + 24vX^4 + 48vX^3Z + 34X^3Z - 16kX^2Y^2 + 102X^2YZ + 144vX^2YZ - 8kX^2YZ + 51X^2Z^2 + 72vX^2Z^2 - 24kvX^2Z^2 - 18kX^2Z^2 - 16kXY^2Z + 144vXYZ^2 - 8kXYZ^2 + 102XYZ^2 + 48vXZ^3 - 24kvXZ^3 + 34XZ^3 - 18kXZ^3 + 36Y^4 - 48kY^3Z + 36Y^3Z + 108vY^2Z^2 + 16k^2Y^2Z^2 + 90Y^2Z^2 - 40kY^2Z^2 - 72kvYZ^3 + 90vYZ^3 - 62kYZ^3 + 66YZ^3 + 8k^2YZ^3 - 24kvZ^4 - 18kZ^4 + 81v^2Z^4 + 53Z^4 + 132vZ^4 + k^2Z^4)$
$i(L(k, c)) \equiv 0 \pmod 2 \Leftrightarrow k \equiv \frac{c+7}{24} \pmod 2$ $i(L(k, c)) \not\equiv 0 \pmod 3$ (by (1.6)) $i(L(k, c)) \not\equiv 0 \pmod 4$ (by (1.7))
$i(L(k, c)) = 1$ or 2 according as $k \equiv \frac{c-17}{24}$ or $\frac{c+7}{24} \pmod 2$

Table 6

$c = 24v + 21$
$d(K(k, c)) = (1 - 16ck^2)c^2$
$O_{K(k,c)} = \left[1, \frac{1+\sqrt{\mu}}{2}, \frac{1+\sqrt{c}}{2}, \frac{(1-\sqrt{c})(1+\sqrt{\mu})}{4}\right]$
$i(X, Y, Z) = (Y^2 + YZ - 5Z^2 - 6vZ^2)(24vX^4 + 21X^4 - 42X^3Z - 48vX^3Z - X^2Y^2 - X^2YZ + 96kvX^2YZ + 84kX^2YZ + 30vX^2Z^2 + 48kvX^2Z^2 + 42kX^2Z^2 + 26X^2Z^2 + XY^2Z + XYZ^2 - 84kXYZ^2 - 96kvXYZ^2 - 5XZ^3 - 42kXZ^3 - 48kvXZ^3 - 6vXZ^3 + 4k^2Y^4 - 2kY^3Z + 8k^2Y^3Z + 48k^2vY^2Z^2 - 3kY^2Z^2 + 48k^2Y^2Z^2 + 48k^2vYZ^3 + 12kvYZ^3 + 44k^2YZ^3 + 9kYZ^3 + 6kvZ^4 + 144k^2v^2Z^4 + 264k^2vZ^4 + 121k^2Z^4 + 5kZ^4)$
$i(K(k, c)) \equiv 0 \pmod 2$ $i(K(k, c)) \not\equiv 0 \pmod 3$ (by (1.6)) $i(K(k, c)) \not\equiv 0 \pmod 4$ (by 1.5))
$i(K(k, c)) = 2$

Table 6 (continued)

$c = 24v + 21$
$d(L(k, c)) = 2^4(16k^2 - 9c)c^2$
$O_{L(k,c)} = \left[1, \sqrt{\lambda}, \frac{1+\sqrt{c}}{2}, \frac{(1-\sqrt{c})\sqrt{\lambda}}{2}\right]$
$i(X, Y, Z) = (Y^2 + YZ - 5Z^2 - 6vZ^2)(21X^4 + 24vX^4 - 16kX^2Y^2 - 252X^2YZ - 16kX^2YZ - 288vX^2YZ - 88kX^2Z^2 - 144vX^2Z^2 - 126X^2Z^2 - 96kvX^2Z^2 + 36Y^4 + 96kY^3Z + 72Y^3Z + 144kY^2Z^2 + 432vY^2Z^2 + 432Y^2Z^2 + 64k^2Y^2Z^2 + 396YZ^3 + 576kYZ^3 + 432vYZ^3 + 576kvYZ^3 + 64k^2YZ^3 + 16k^2Z^4 + 288kvZ^4 + 1089Z^4 + 1296v^2Z^4 + 2376vZ^4 + 264kZ^4)$
$i(L(k, c)) \not\equiv 0 \pmod 2$ $i(L(k, c)) \not\equiv 0 \pmod 3$ (by (1.6))
$i(L(k, c)) = 1$

Table 7

$c = 12v + 7$
$d(K(k, c)) = 2^4(1 - 16ck^2)c^2$
$O_{K(k,c)} = \left[1, \frac{1+\sqrt{\mu}}{2}, \sqrt{c}, \frac{(1-\sqrt{c})(1+\sqrt{\mu})}{2}\right]$
$i(X, Y, Z) = (Y^2 + 2YZ - 6Z^2 - 12vZ^2)(48vX^4 + 28X^4 - 56X^3Z - 96vX^3Z - X^2Y^2 - 2X^2YZ + 96kvX^2YZ + 56kX^2YZ + 60vX^2Z^2 + 96kvX^2Z^2 + 56kX^2Z^2 + 34X^2Z^2 + XY^2Z + 2XYZ^2 - 56kXYZ^2 - 96kvXYZ^2 - 6XZ^3 - 56kXZ^3 - 96kvXZ^3 - 12vXZ^3 + k^2Y^4 - kY^3Z + 4k^2Y^3Z + 24k^2vY^2Z^2 - 3kY^2Z^2 + 20k^2Y^2Z^2 + 48k^2vYZ^3 + 12kvYZ^3 + 32k^2YZ^3 + 4kYZ^3 + 12kvZ^4 + 144k^2v^2Z^4 + 192k^2vZ^4 + 64k^2Z^4 + 6kZ^4)$
$i(K(k, c)) \equiv 0 \pmod 3 \Leftrightarrow k \equiv 0 \pmod 3$ $i(K(k, c)) \not\equiv 0 \pmod 2$ (by (1.4))
$i(K(k, c)) = 1$ or 3 according as $k \equiv \pm 1$ or $0 \pmod 3$

Table 8

$c = 12v + 10$
$d(K(k, c)) = 2^4(1 - 16ck^2)c^2$
$O_{K(k,c)} = \left[1, \frac{1+\sqrt{\mu}}{2}, \sqrt{c}, \frac{\sqrt{c}(1+\sqrt{\mu})}{2}\right]$
$i(X, Y, Z) = (Y^2 - 12vZ^2 - 10Z^2)(48vX^4 + 40X^4 + 96vX^3Z + 80X^3Z - X^2Y^2 - 80kX^2YZ - 96kvX^2YZ + 60vX^2Z^2 + 50X^2Z^2 - XY^2Z - 80kXYZ^2 - 96kvXYZ^2 + 12vXZ^3 + 10XZ^3 + k^2Y^4 + kY^3Z + 20k^2Y^2Z^2 + 24k^2vY^2Z^2 - 12kvYZ^3 - 10kYZ^3 + 240k^2vZ^4 + 100k^2Z^4 + 144k^2v^2Z^4)$
$i(K(k, c)) \equiv 0 \pmod 3 \Leftrightarrow k \equiv 0 \pmod 3$ $i(K(k, c)) \not\equiv 0 \pmod 2$ (by (1.4))
$i(K(k, c)) = 1$ or 3 according as $k \equiv \pm 1$ or $0 \pmod 3$

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