Remark on the infinite series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)\cdots(k+m)}$

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Let m be a positive integer and set

$$S(m) \;=\; \sum_{k=1}^{\infty} \frac{1}{k(k+1)\cdots(k+m)}\,.$$

In a recent article in ***Crux with Mayhem***, it was shown [1, Corollary 5] that

$$S(m) \;=\; \sum_{k=1}^{m} \frac{(-1)^{m-k}\sum_{l=m-k+1}^{m}\frac{1}{l}}{(m-k)!k!}\,.$$

We note that

$$S(m) \;=\; \frac{1}{m\,m!}\,.$$

This is easily seen by summing the identity

$$\frac{m}{k(k+1)\cdots(k+m)} \;=\; \frac{1}{k(k+1)\cdots(k+m-1)} - \frac{1}{(k+1)(k+2)\cdots(k+m)}$$

over all positive integers k to obtain

$$mS(m) \;=\; S(m-1) - \left(S(m-1) - \frac{1}{m!}\right)\,.$$

References

1. Z. Mashreghi and J. Mashreghi, *On the closed form of power series*, Crux Math. 27 (2001), 436–439.

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