Canad. Math. Bull. Vol. **44** (1), 2001 pp. 97–104

On the Density of Cyclic Quartic Fields

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Abstract. An asymptotic formula is obtained for the number of cyclic quartic fields over Q with discriminant $\leq x$.

1 Introduction

Let $h(n)$ denote the number of cyclic quartic fields over the rational number field Q with discriminant n. We consider

$$N(x) = \sum_{n \leq x} h(n).$$

In [1, Theorem 9] Baily proved

$$N(x) \sim \frac{3}{\pi^2}\left\{ \frac{25}{24} \prod_{p \equiv 1 \pmod 4} \left(1 + \frac{2}{(p+1)\sqrt{p}}\right) - 1 \right\} x^{1/2}, \tag{1.1}$$

where p runs through primes $p \equiv 1 \pmod 4$. Unfortunately Baily's generating function $f(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$ is given incorrectly, and so the constant in (1.1) is wrong. In giving the Euler product for $f(s)$, Baily [1, p. 209] overlooks that the discriminant is $\frac{1}{2} f_4^3 f_2^2$ in one case rather than $f_4^3 f_2^2$ and so his term $4 \cdot 16^{-3s} = 4 \cdot 2^{-12s}$ should be replaced by $4 \cdot 2^{-11s}$.

In this paper, using the representation of a cyclic quartic field given by Hardy, Hudson, Richman, Williams and Holtz [2], see also [3], and an elementary method, we correct Baily's result and at the same time give an estimate for the error term. We prove

Theorem

$$N(x) = \frac{3}{\pi^2}\left\{ \frac{24 + \sqrt{2}}{24} \prod_{p \equiv 1 \pmod 4} \left(1 + \frac{2}{(p+1)\sqrt{p}}\right) - 1 \right\} x^{1/2} + O(x^{1/3} \log^3 x). \tag{1.2}$$

Received by the editors March 11, 1999.

Research of the first author was supported by the China Scholarship Council. Research of the second author was supported by Natural Sciences and Engineering Research Council of Canada grant A-7233.

AMS subject classification: 11R16, 11R29.

Keywords: cyclic quartic fields, density, discriminant.

2 Representation of a Cyclic Quartic Field

In [2] the authors show that a cyclic quartic extension K of the rational number field Q can be expressed uniquely in the form

$$K = Q\big(\sqrt{A(D + B\sqrt{D})}\big), \tag{2.1}$$

where A, B, D are integers such that

$$\begin{cases} A \text{ is squarefree and odd,} \\ B \geq 1, D \geq 2, \\ D \text{ is squarefree and } D - B^2 \text{ is a square,} \\ (A, D) = 1, \end{cases} \tag{2.2}$$

where (A, D) denotes the gcd of A and D. The discriminant $d(K)$ of K is given by

$$d(K) = \begin{cases} 2^8 A^2 D^3, & \text{if } D \equiv 0 \pmod 2, \\ 2^6 A^2 D^3, & \text{if } D \equiv 1 \pmod 2,\ B \equiv 1 \pmod 2, \\ 2^4 A^2 D^3, & \text{if } D \equiv 1 \pmod 2,\ B \equiv 0 \pmod 2,\ A + B \equiv 3 \pmod 4, \\ A^2 D^3, & \text{if } D \equiv 1 \pmod 2,\ B \equiv 0 \pmod 2,\ A + B \equiv 1 \pmod 4. \end{cases} \tag{2.3}$$

3 Proof of the Theorem

Let K be a cyclic quartic extension of Q. From (2.1)–(2.3) we see that the discriminant $d(K)$ of K is of the form

$$d(K) = 2^{\alpha}(p_1 \cdots p_m)^2 (q_1 \cdots q_r)^3, \tag{3.1}$$

where $\alpha = 0, 4, 6$ or 11 and $p_1, \ldots, p_m, q_1, \ldots, q_r$ are distinct odd primes with $m \geq 0, r \geq 1$ if $\alpha = 0, 4, 6$, $r \geq 0$ if $\alpha = 11$, and $q_j \equiv 1 \pmod 4$, $j = 1, \ldots, r$. We set

$$A = p_1 \cdots p_m, \quad D = q_1 \cdots q_r. \tag{3.2}$$

We note that A and D defined in (3.2) are slightly different from the A and D in Section 2.

If $\alpha = 0$ then $n = d(K) = A^2 D^3$ and $K = Q\big(\sqrt{\varepsilon A(D + B\sqrt{D})}\big)$ for some $\varepsilon = \pm 1$ and some positive integer B such that

$$B \equiv 0 \pmod 2, \quad B \equiv 1 - \varepsilon p_1 \cdots p_m \pmod 4, \quad D - B^2 = \text{square}.$$

Moreover distinct pairs (ε, B) give different fields K. Thus

$$\begin{aligned} h(n) &= \sum_{\varepsilon=-1,+1} \sum_{\substack{B>0,2|B \\ B\equiv 1-\varepsilon p_1\cdots p_m \pmod 4 \\ D-B^2=\square}} 1 = \sum_{\substack{B>0,2|B \\ D-B^2=\square}} 1 \\ &= \sum_{\substack{C>0,2\nmid C \\ D-C^2=\square}} 1 = \frac{1}{2} \sum_{\substack{B>0 \\ D-B^2=\square}} 1 = \frac{1}{2} \sum_{\substack{B<0 \\ D-B^2=\square}} 1 \\ &= \frac{1}{4} \sum_{\substack{B\neq 0 \\ D-B^2=\square}} = \frac{1}{4} \sum_{\substack{B \\ D-B^2=\square}} 1 = \frac{1}{8} \sum_{\substack{B,C \\ D=B^2+C^2}} 1 = \frac{1}{8} r_2(D) \\ &= \frac{1}{8} 2^{r+2} = 2^{r-1} = \frac{1}{2} d(D), \end{aligned}$$

where $r_2(k)$ denotes the number of representations of the positive integer k as the sum of two squares and $d(k)$ denotes the number of positive divisors of k.

If $\alpha = 4$ then $n = d(K) = 2^4 A^2 D^3$ and $K = Q\big(\sqrt{\varepsilon A(D + B\sqrt{D})}\big)$ for some $\varepsilon = \pm 1$ and some positive integer B such that

$$B \equiv 0 \pmod 2,\ B \equiv 3 - \varepsilon p_1 \cdots p_m \pmod 4, D - B^2 = \text{square}.$$

Moreover distinct pairs (ε, B) give different fields K. Thus

$$h(n) = \sum_{\varepsilon=-1,+1} \sum_{\substack{B>0,2|B \\ B\equiv 3-\varepsilon p_1\cdots p_m \pmod 4 \\ D-B^2=\square}} 1 = \sum_{\substack{B>0,2|B \\ D-B^2=\square}} 1 = \frac{1}{2} d(D).$$

If $\alpha = 6$ then $n = d(K) = 2^6 A^2 D^3$ and $K = Q\big(\sqrt{\varepsilon A(D + B\sqrt{D})}\big)$ for some $\varepsilon = \pm 1$ and some positive integer B such that

$$B \equiv 1 \pmod 2, \quad D - B^2 = \text{square}.$$

Moreover distinct pairs (ε, B) give different fields K. Thus

$$h(n) = 2 \sum_{\substack{B>0,2\nmid B \\ D-B^2=\square}} 1 = 2 \sum_{\substack{C>0,2|C \\ D-C^2=\square}} 1 = 2^r = d(D).$$

If $\alpha = 11$ then $n = d(K) = 2^{11} A^2 D^3$ and $K = Q\big(\sqrt{\varepsilon A(2D + B\sqrt{2D})}\big)$ for some $\varepsilon = \pm 1$ and some positive integer B such that

$$2D - B^2 = \text{square}.$$

Moreover distinct pairs (ε, B) give different fields K. Thus

$$h(n) = 2 \sum_{\substack{B>0 \\ 2D-B^2=\square}} 1 = 2 \sum_{\substack{B<0 \\ 2D-B^2=\square}} 1 = \sum_{\substack{B\neq 0 \\ 2D-B^2=\square}} 1$$

$$= \sum_{\substack{B \\ 2D-B^2=\square}} 1 = \frac{1}{2} \sum_{\substack{B,C \\ 2D=B^2+C^2}} 1 = \frac{1}{2} r_2(2D) = \frac{1}{2} 2^{r+2} = 2^{r+1} = 2d(D).$$

Summarizing we have

$$h(n) = \begin{cases} 2d(D), & \text{if } n = 2^{11}A^2D^3, \\ d(D), & \text{if } n = 2^6A^2D^3, \\ \frac{1}{2}d(D), & \text{if } n = 2^4A^2D^3 \text{ or } A^2D^3. \end{cases} \tag{3.3}$$

Recalling that $D = 1$ can only occur when $n = 2^{11}A^2D^3$, we have

$$\sum_{n\leq x} h(n) = 2 \sum_{2^{11}A^2\leq x} 1 + 2 \sum_{2^{11}A^2D^3\leq x} d(D) + \sum_{2^6A^2D^3\leq x} d(D) + \frac{1}{2} \sum_{2^4A^2D^3\leq x} d(D) + \frac{1}{2} \sum_{A^2D^3\leq x} d(D),$$

so that

$$\sum_{n\leq x} h(n) = 2 \sum_{\substack{A\leq (x/2^{11})^{1/2} \\ A \text{ squarefree} \\ A \text{ odd}}} 1 + 2S(2^{-11}x) + S(2^{-6}x) + \frac{1}{2}S(2^{-4}x) + \frac{1}{2}S(x), \tag{3.4}$$

where

$$S(x) = \sum_{A^2D^3\leq x} d(D) \tag{3.5}$$

and the sum is over all positive integers A and D such that

$$A = p_1 \cdots p_m \quad (m \geq 0), \; D = q_1 \cdots q_r \quad (r \geq 1), \tag{3.6}$$

where $p_1, \dots, p_m, q_1, \dots, q_r$ are distinct odd primes with $q_j \equiv 1 \pmod 4$ $(j = 1, \dots, r)$. We set

$$\mathcal{P} = \{D \mid D = q_1 \cdots q_r (r \geq 1), q_1, \dots, q_r \text{ are distinct primes } \equiv 1 \pmod 4\}, \tag{3.7}$$

so that

$$S(x) = \sum_{\substack{D\leq x^{1/3} \\ D\in\mathcal{P}}} d(D) \sum_{\substack{1\leq A\leq \sqrt{xD^{-3}} \\ A \text{ squarefree} \\ (A,2D)=1}} 1. \tag{3.8}$$

Note that $1 \notin \mathcal{P}$.

We first estimate $\sum_{\substack{A \le y \\ A \text{ squarefree} \\ A \text{ odd}}} 1$, where $y = (x/2^{11})^{1/2}$. We have

$$\begin{aligned}
\sum_{\substack{A \le y \\ A \text{ squarefree} \\ A \text{ odd}}} 1 &= \sum_{\substack{A \le y \\ A \text{ odd}}} \sum_{d^2 | A} \mu(d) \\
&= \sum_{\substack{d \le y^{1/2} \\ d \text{ odd}}} \mu(d) \sum_{\substack{a \le y/d^2 \\ a \text{ odd}}} 1 \\
&= \sum_{\substack{d \le y^{1/2} \\ d \text{ odd}}} \mu(d) \left(\frac{y}{2d^2} + O(1) \right) \\
&= \frac{y}{2} \sum_{\substack{d \le y^{1/2} \\ d \text{ odd}}} \frac{\mu(d)}{d^2} + O(y^{1/2}) \\
&= \frac{y}{2} \sum_{\substack{d=1 \\ d \text{ odd}}}^{\infty} \frac{\mu(d)}{d^2} + O(y^{1/2}) \\
&= \frac{y}{2} \prod_{p \ne 2} \left(1 - \frac{1}{p^2} \right) + O(y^{1/2}) \\
&= \frac{y}{2} \left(1 - \frac{1}{2^2} \right)^{-1} \prod_{p} \left(1 - \frac{1}{p^2} \right) + O(y^{1/2}) \\
&= \frac{4}{\pi^2} y + O(y^{1/2}) \\
&= \frac{x^{1/2}}{2^{7/2} \pi^2} + O(x^{1/4}).
\end{aligned}$$

We now turn to the estimation of $S(x)$. The inner sum in (3.8) is

$$\begin{aligned}
\sum_{\substack{A \le \sqrt{xD^{-3}} \\ (A,2D)=1}} \sum_{d^2 | A} \mu(d) &= \sum_{\substack{d \le (xD^{-3})^{1/4} \\ (d,2D)=1}} \mu(d) \sum_{\substack{a \le d^{-2} \sqrt{xD^{-3}} \\ (a,2D)=1}} 1 \\
&= \sum_{\substack{d \le (xD^{-3})^{1/4} \\ (d,2D)=1}} \mu(d) \sum_{e | 2D} \mu(e) \sum_{b \le e^{-1} d^{-2} \sqrt{xD^{-3}}} 1 \\
&= \sum_{e | 2D} \mu(e) \sum_{\substack{d \le (xD^{-3})^{1/4} \\ (d,2D)=1}} \mu(d) \left[\frac{\sqrt{xD^{-3}}}{d^2 e} \right]
\end{aligned}$$

$$= \sqrt{xD^{-3}} \sum_{e|2D} \frac{\mu(e)}{e} \sum_{\substack{d \le (xD^{-3})^{1/4} \\ (d,2D)=1}} \frac{\mu(d)}{d^2} + O\left(d(2D) \left(\frac{x}{D^3} \right)^{1/4} \right)$$

$$= \sqrt{xD^{-3}} \sum_{e|2D} \frac{\mu(e)}{e} \sum_{(d,2D)=1} \frac{\mu(d)}{d^2} + O\left(d(D) \left(\frac{x}{D^3} \right)^{1/4} \right)$$

$$+ O\left(\left(\frac{x}{D^3} \right)^{1/2} \sum_{e|2D} \frac{1}{e} \left(\frac{x}{D^3} \right)^{-1/4} \right)$$

$$= \sqrt{xD^{-3}} \frac{\varphi(2D)}{2D} \frac{6}{\pi^2} \left(\prod_{p|2D} \left(1 - \frac{1}{p^2} \right)^{-1} \right) + O\left(d(D) \left(\frac{x}{D^3} \right)^{1/4} \right),$$

since

$$\sum_{\substack{d=1 \\ (d,2D)=1}}^{\infty} \frac{\mu(d)}{d^2} = \prod_{(p,2D)=1} \left(1 - \frac{1}{p^2} \right) = \prod_{p} \left(1 - \frac{1}{p^2} \right) \Big/ \prod_{p|2D} \left(1 - \frac{1}{p^2} \right)$$

$$= \frac{1}{\zeta(2)} \prod_{p|2D} \left(1 - \frac{1}{p^2} \right)^{-1} = \frac{6}{\pi^2} \prod_{p|2D} \left(1 - \frac{1}{p^2} \right)^{-1}$$

and Euler's phi function $\varphi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$. Thus

$$S(x) = \frac{3}{\pi^2} x^{1/2} \sum_{\substack{D \le x^{1/3} \\ D \in \mathcal{P}}} d(D) \varphi(D) D^{-5/2} \prod_{p|2D} \left(1 - \frac{1}{p^2} \right)^{-1}$$

$$+ O\left(x^{1/4} \sum_{\substack{D \le x^{1/3} \\ D \in \mathcal{P}}} d^2(D) D^{-3/4} \right)$$

$$= \frac{4}{\pi^2} x^{1/2} \sum_{\substack{D=1 \\ D \in \mathcal{P}}}^{\infty} d(D) \varphi(D) D^{-5/2} \prod_{p|D} \left(1 - \frac{1}{p^2} \right)^{-1}$$

$$+ O\left(x^{1/2} \sum_{\substack{D > x^{1/3} \\ D \in \mathcal{P}}} d(D) \varphi(D) D^{-5/2} \right) + O\left(x^{1/4} \sum_{\substack{D \le x^{1/3} \\ D \in \mathcal{P}}} d^2(D) D^{-3/4} \right),$$

as

$$\prod_{p|D} \left(1 - \frac{1}{p^2} \right)^{-1} = \frac{\pi^2}{6} \prod_{p \nmid D} \left(1 - \frac{1}{p^2} \right) < \frac{\pi^2}{6}.$$

Clearly

$$\sum_{\substack{D=1 \\ D \in \mathcal{P}}}^{\infty} d(D) \varphi(D) D^{-5/2} \prod_{p|D} \left(1 - \frac{1}{p^2} \right)^{-1} = \prod_{p \equiv 1 \ (\mathrm{mod}\ 4)} \left(1 + \frac{2}{(p+1)\sqrt{p}} \right) - 1.$$

It remains to estimate $R_1 = \sum_{\substack{D \le x^{1/3} \\ D \in \mathcal{P}}} d^2(D) D^{-3/4}$ and $R_2 = \sum_{\substack{D > x^{1/3} \\ D \in \mathcal{P}}} d(D)\varphi(D) D^{-5/2}$. Firstly

$$\sum_{\substack{D \le x \\ D \in \mathcal{P}}} d^2(D) = \sum_{\substack{D \le x \\ D \in \mathcal{P}}} d(D) \sum_{a|D} 1 = \sum_{\substack{ab \le x \\ a,b \in \mathcal{P} \\ (a,b)=1}} d(ab) + 2 \sum_{\substack{D \le x \\ D \in \mathcal{P}}} d(D)$$

$$\le \sum_{\substack{a \le x \\ a \in \mathcal{P}}} d(a) \sum_{\substack{b \le x/a \\ b \in \mathcal{P}}} d(b) + 2 \sum_{\substack{D \le x \\ D \in \mathcal{P}}} d(D)$$

$$\sum_{\substack{D \le x \\ D \in \mathcal{P}}} d(D) = \sum_{\substack{D \le x \\ D \in \mathcal{P}}} \sum_{a|D} 1 \le \sum_{\substack{a \le x \\ a \in \mathcal{P}}} \Big(2 + \sum_{\substack{b \le x/a \\ b \in \mathcal{P}}} 1 \Big) \ll x \log x,$$

so

$$\sum_{\substack{D \le x \\ D \in \mathcal{P}}} d^2(D) \ll x \log x + \sum_{\substack{a \le x \\ a \in \mathcal{P}}} d(a) \frac{x}{a} \log \frac{x}{a}$$

$$\ll x \log x + x \log x \sum_{\substack{a \le x \\ a \in \mathcal{P}}} \frac{d(a)}{a}$$

$$\ll x \log^3 x.$$

By partial summation we have

$$R_1 = x^{-\frac{1}{4}} \sum_{\substack{D \le x^{1/3} \\ D \in \mathcal{P}}} d^2(D) - \int_1^{x^{1/3}} \Big(\sum_{\substack{D \le y \\ D \in \mathcal{P}}} d^2(D) \Big) d(y^{-3/4})$$

$$= O(x^{1/3 - 1/4} \log^3 x) = O(x^{1/12} \log^3 x)$$

and

$$R_2 \le \sum_{\substack{D > x^{1/3} \\ D \in \mathcal{P}}} d(D) D^{-3/2} = -\int_{x^{1/3}}^{\infty} \Big(\sum_{\substack{D \le y \\ D \in \mathcal{P}}} d(D) \Big) d(y^{-3/2}) = O(x^{-1/6} \log x).$$

Therefore

$$S(x) = \frac{4c_0}{\pi^2} x^{1/2} + O(x^{1/3} \log^3 x),$$

where

$$c_0 = \prod_{p \equiv 1 \ (\mathrm{mod}\ 4)} \left(1 + \frac{2}{(p+1)\sqrt{p}} \right) - 1,$$

and

$$\begin{aligned}
\sum_{n \le x} h(n) &= \frac{x^{1/2}}{2^{5/2}\pi^2} + O(x^{1/4}) \\
&\quad + \frac{4c_0}{\pi^2}\left(2 \cdot 2^{-11/2} + 2^{-3} + \frac{1}{2}2^{-2} + \frac{1}{2}\right) x^{1/2} + O(x^{1/3}\log^3 x) \\
&= \left(\frac{(24+\sqrt{2})}{8\pi^2}c_0 + \frac{\sqrt{2}}{8\pi^2}\right) x^{1/2} + O(x^{1/3}\log^3 x) \\
&= \left(\frac{24+\sqrt{2}}{8\pi^2} \prod_{p \equiv 1 \pmod 4} \left(1 + \frac{2}{(p+1)\sqrt{p}}\right) - \frac{24}{8\pi^2}\right) x^{1/2} + O(x^{1/3}\log^3 x) \\
&= \frac{3}{\pi^2}\left\{\frac{24+\sqrt{2}}{24} \prod_{p \equiv 1 \pmod 4} \left(1 + \frac{2}{(p+1)\sqrt{p}}\right) - 1\right\} x^{1/2} + O(x^{1/3}\log^3 x).
\end{aligned}$$

This completes the proof of (1.2).

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