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$\sum\limits_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{(-1)^m}{m^2+\lambda n^2}$ AND RELATED SERIES

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ABSTRACT

Let a, b, c be real numbers with $a \neq 0$. The explicit evaluation of the infinite series

$$\sum_{\substack{n=-\infty \\ an^2+bn+c\neq 0}}^{\infty} \frac{1}{an^2+bn+c} \quad \text{and} \quad \sum_{\substack{n=-\infty \\ an^2+bn+c\neq 0}}^{\infty} \frac{(-1)^n}{an^2+bn+c}$$

is carried out and applied to the evaluation of double infinite series of the type specified in the title.

1. Introduction

Recently Li Jian Lin[4] determined the sum $S(a, b)$ of the infinite

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series $\sum\limits_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n+1}}{an^2+bn}$, where a and b are real numbers with $a > b > 0$. Prior to this evaluation a number of authors [1], [3], [5], [7] had found the value of $S(3,1)$, which had occurred originally in the work of Turan [9].

We begin by giving a simpler determination of $S(a,b)$ than that in [4] by making use of the infinite partial fraction expansion of $\csc \pi z$ (z a complex number):

$$\pi \csc \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{z-n} - \frac{1}{z+n} \right) \qquad (z \notin Z) \tag{1}$$

where Z denotes the set of integers.

Theorem. (Li Jian Lin [4]) *Let a and b be real numbers with $a \neq 0$ and $b/a \notin Z$. Then*

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^n}{an^2+bn} = \frac{1}{b}(a/b - \pi \csc b\pi/a).$$

Proof. We have

$$\begin{aligned} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^n}{an^2+bn} &= \frac{1}{a} \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n^2 + \frac{b}{a}n} + \frac{1}{n^2 - \frac{b}{a}n} \right) \\ &= -\frac{1}{b} \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{\frac{b}{a} - n} + \frac{1}{\frac{b}{a} + n} \right) \\ &= -\frac{1}{b}(\pi \csc b\pi/a - a/b). \end{aligned}$$

□

Using the ideas of this proof and the infinite partial fraction expansion of $\cot \pi z$:

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) \qquad (z \notin Z) \tag{2}$$

we obtain the following generalization of Lin's theorem.

Theorem 1. *Let a, b, c be real numbers with $a \neq 0$. Let α and β be the roots of the quadratic equation $az^2 + bz + c = 0$. Then*

$$\sum_{\substack{n=-\infty \\ an^2+bn+c\neq 0}}^{\infty} \frac{1}{an^2+bn+c} = \begin{cases} \frac{-\pi(\cot\alpha\pi - \cot\beta\pi)}{a(\alpha-\beta)}, & \text{if } \alpha \neq \beta,\ \alpha \notin Z,\ \beta \notin Z, \\ \frac{1}{a(\alpha-\beta)^2} + \frac{\pi\cot\beta\pi}{a(\alpha-\beta)}, & \text{if } a \neq \beta,\ \alpha \in Z,\ \beta \notin Z, \\ \frac{2}{a(\alpha-\beta)^2}, & \text{if } \alpha \neq \beta,\ \alpha \in Z,\ \beta \in Z, \\ \frac{\pi^2\csc^2\alpha\pi}{a}, & \text{if } \alpha = \beta \notin Z, \\ \frac{\pi^2}{3a}, & \text{if } \alpha = \beta \in Z, \end{cases}$$

and

$$\sum_{\substack{n=-\infty \\ an^2+bn+c\neq 0}}^{\infty} \frac{(-1)^n}{an^2+bn+c} = \begin{cases} \frac{-\pi(\csc\alpha\pi - \csc\beta\pi)}{a(\alpha-\beta)}, & \text{if } \alpha \neq \beta,\ \alpha \notin Z,\ \beta \notin Z, \\ \frac{(-1)^n}{a(\alpha-\beta)^2} + \frac{\pi\csc\beta\pi}{a(\alpha-\beta)}, & \text{if } a \neq \beta,\ \alpha \in Z,\ \beta \notin Z, \\ \frac{(-1)^\alpha+(-1)^\beta}{a(\alpha-\beta)^2}, & \text{if } \alpha \neq \beta,\ \alpha \in Z,\ \beta \in Z, \\ \frac{\pi^2\csc\alpha\pi\cot\alpha\pi}{a}, & \text{if } \alpha = \beta \notin Z, \\ (-1)^{\alpha+1}\frac{\pi^2}{6a}, & \text{if } \alpha = \beta \in Z. \end{cases}$$

Proof. We just treat $\sum_{\substack{n=-\infty \\ an^2+bn+c\neq 0}}^{\infty} \frac{1}{an^2+bn+c}$ when $\alpha \neq \beta, \alpha \notin Z, \beta \notin Z$, as the remaining cases are very similar. In this case we have

$$\begin{aligned} &\sum_{\substack{n=-\infty \\ an^2+bn+c\neq 0}}^{\infty} \frac{1}{an^2+bn+c} \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{a(n-\alpha)(n-\beta)} \\ &= \frac{1}{a\alpha\beta} + \frac{1}{a}\sum_{n=1}^{\infty}\left(\frac{1}{(n-\alpha)(n-\beta)} + \frac{1}{(n+\alpha)(n+\beta)}\right) \\ &= \frac{1}{a\alpha\beta} - \frac{1}{a(\alpha-\beta)}\sum_{n=1}^{\infty}\left(\frac{1}{\alpha-n} + \frac{1}{\alpha+n} - \frac{1}{\beta-n} - \frac{1}{\beta+n}\right) \end{aligned}$$

$$= \frac{1}{a\alpha\beta} - \frac{1}{a(\alpha-\beta)}\left(\left(\pi\cot\alpha\pi - \frac{1}{\alpha}\right) - \left(\pi\cot\beta\pi - \frac{1}{\beta}\right)\right) \quad \text{(by (2))}$$
$$= \frac{-\pi(\cot\alpha\pi - \cot\beta\pi)}{a(\alpha-\beta)}. \qquad \square$$

As a check on our calculations we note that Theorem 1 is consistent with the relation

$$\sum_{\substack{n=-\infty \\ an^2+bn+c\neq 0}}^{\infty} \frac{1}{an^2+bn+c} + \sum_{\substack{n=-\infty \\ an^2+bn+c\neq 0}}^{\infty} \frac{(-1)^n}{an^2+bn+c} = 2\sum_{\substack{n=-\infty \\ 4an^2+2bn+c\neq 0}}^{\infty} \frac{1}{4an^2+2bn+c}.$$

2. Applications of Theorem 1

As an application of Theorem 1, we make use of it in the evaluation of the series

$$\sigma_1(b) = \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{m^2+b^2n^2},$$
$$\sigma_2(b) = \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^{m+n}}{m^2+b^2n^2},$$
$$\sigma_3(b) = \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^n}{m^2+b^2n^2},$$

where b is a positive real number and the prime ($'$) indicates that $(m,n) = (0,0)$ is omitted. More general series than these have been treated by Zucker and Robertson [11] and Zucker [12] by different techniques. It should be noted that the series $\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{m^2+b^2n^2}$ diverges as

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{m^2+b^2n^2} \geq \frac{1}{(1+b^2)} \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{m^2+n^2} \geq \frac{1}{(1+b^2)} \sum_{p\equiv 1 \pmod 4} \frac{1}{p},$$

since every prime $p \equiv 1 \pmod 4$ is the sum of two integral squares and the series $\sum_{p\equiv 1 \pmod 4} \frac{1}{p}$ is known to be divergent.

The roots of $z^2 + b^2n^2 = 0$ are $\alpha = ibn$ and $\beta = -ibn$, which are distinct and non-integral provided $n \neq 0$. Hence, by Theorem 1, we have for $n \neq 0$

$$\begin{aligned}\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{m^2 + b^2n^2} &= -\frac{\pi}{2ibn}\left(\csc(ib\pi n) - \csc(-ib\pi n)\right) \\ &= \frac{i\pi}{bn}\csc(ib\pi n) \\ &= \frac{\pi}{b}\frac{1}{n\sinh(b\pi n)},\end{aligned}$$

and so

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{m^2 + b^2n^2} = \frac{2\pi}{b}\sum_{n=1}^{\infty} \frac{1}{n\sinh(b\pi n)}.$$

Thus

$$\sigma_1(b) = \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{m^2 + b^2n^2} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{m^2 + b^2n^2} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{(-1)^m}{m^2},$$

that is

$$\sigma_1(b) = \frac{2\pi}{b}\sum_{n=1}^{\infty} \frac{1}{n\sinh(b\pi n)} - \frac{\pi^2}{6}. \tag{3}$$

Similarly we can show

$$\sigma_2(b) = \sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^{m+n}}{m^2 + b^2n^2} = \frac{2\pi}{b}\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sinh(b\pi n)} - \frac{\pi^2}{6}. \tag{4}$$

For $\sigma_3(b)$ it is sufficient to observe that

$$\sigma_3(b) = \frac{1}{b^2}\sigma_1(1/b). \tag{5}$$

Next we evaluate $\sum_{n=1}^{\infty} \frac{1}{n\sinh(b\pi n)}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sinh(b\pi n)}$. We set $\lambda = b^2$, so that $b = \sqrt{\lambda}$, and $q = e^{-b\pi} = e^{-\pi\sqrt{\lambda}}$, so that $0 < q < 1$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n\sinh(b\pi n)} = 2\sum_{n=1}^{\infty} \frac{1}{n}\frac{1}{(e^{b\pi n} - e^{-b\pi n})}$$

$$\begin{aligned}
&= 2\sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{-b\pi n}}{(1-e^{-2b\pi n})}\\
&= 2\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=1}^{\infty}e^{-b\pi n(2m-1)}\\
&= 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}\left(q^{2m-1}\right)^n\\
&= -2\sum_{m=1}^{\infty}\log(1-q^{2m-1}),
\end{aligned}$$

that is

$$\sum_{n=1}^{\infty}\frac{1}{n\sinh(b\pi n)} = -2\log\prod_{m=1}^{\infty}(1-q^{2m-1}). \tag{6}$$

Similarly we find

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n\sinh(b\pi n)} = -2\log\prod_{m=1}^{\infty}(1+q^{2m-1}). \tag{7}$$

Ramanujan [6, eqns. (1), (2)] has defined positive algebraic numbers g_λ and G_λ, where λ is a positive rational number, by

$$\prod_{m=1}^{\infty}(1-q^{2m-1}) = 2^{1/4}e^{-\frac{\pi\sqrt{\lambda}}{24}}g_\lambda,\quad \prod_{m=1}^{\infty}(1+q^{2m-1}) = 2^{1/4}e^{-\frac{\pi\sqrt{\lambda}}{24}}G_\lambda, \tag{8}$$

and noted the properties [6, eqns. (5), (6), (7)]

$$\left\{\begin{array}{l} g_{4\lambda} = 2^{1/4}g_\lambda G_\lambda,\\ G_\lambda = G_{1/\lambda},\ 1/g_\lambda = g_{4/\lambda},\\ (g_\lambda G_\lambda)^8(G_\lambda^8 - g_\lambda^8) = 1/4. \end{array}\right. \tag{9}$$

Thus, from (6), (7) and (8),we obtain

Theorem 2.

$$\sum_{n=1}^{\infty}\frac{1}{n\sinh(b\pi n)} = \frac{b\pi}{12} - \log(\sqrt{2}\,g_{b^2}^2), \tag{10}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n\sinh(b\pi n)} = \frac{b\pi}{12} - \log(\sqrt{2}\,G_{b^2}^2). \tag{11}$$

Then appealing to (3), (4), (5), (9), (10) and (11), we deduce

Theorem 3.

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{m^2+b^2n^2} = -\frac{\pi}{b}\log(2g_{b^2}^4), \tag{12}$$

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^{m+n}}{m^2+b^2n^2} = -\frac{\pi}{b}\log(2G_{b^2}^4), \tag{13}$$

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^n}{m^2+b^2n^2} = \frac{\pi}{b}\log(g_{b^2}^4 G_{b^2}^4). \tag{14}$$

The quantities g_λ and G_λ are given by (see [6, p. 27])

$$g_\lambda = 2^{-1/4} f_1(\sqrt{-\lambda}), \qquad G_\lambda = 2^{-1/4} f(\sqrt{-\lambda}), \tag{15}$$

where the functions $f_1(z)$ and $f(z)$ are defined in terms of the Dedekind eta function

$$\eta(z) = e^{\pi iz/12}\prod_{m=1}^{\infty}(1-e^{2\pi imz}) \qquad (z = x+iy,\ y>0) \tag{16}$$

by

$$f_1(z) = \frac{\eta(z/2)}{\eta(z)}, \qquad f(z) = e^{-\frac{\pi i}{24}}\frac{\eta((z+1)/2)}{\eta(z)}, \tag{17}$$

see for example [10, p. 114]. Tables of $f_1(\sqrt{-\lambda})$ and $f(\sqrt{-\lambda})$ for certain positive integral values of λ are given in [10, Table VI] and values of g_λ and G_λ in [6, Table I]. From the last equation in (9), we see that the quantities g_λ and G_λ are not independent. Given one of them we show how to find the other. Writing the last equation in (9) as

$$4g_\lambda^8 G_\lambda^{16} - 4g_\lambda^{16} G_\lambda^8 - 1 = 0, \tag{18}$$

and solving (18) for G_λ^8, we obtain

$$G_\lambda^8 = \frac{g_\lambda^{12} \pm \sqrt{g_\lambda^{24}+1}}{2g_\lambda^4}.$$

However, G_λ is real so that

$$G_\lambda = \left(\frac{g_\lambda^{12} + \sqrt{g_\lambda^{24}+1}}{2g_\lambda^4}\right)^{1/8}. \tag{19}$$

Further, rewriting (18) as

$$4G_\lambda^8 g_\lambda^{16} - 4G_\lambda^{16} g_\lambda^8 + 1 = 0,$$

and solving for g_λ^8, we deduce

$$g_\lambda^8 = \frac{G_\lambda^{12} \pm \sqrt{G_\lambda^{24}-1}}{2G_\lambda^4}. \tag{20}$$

We now determine the correct sign in (20). As g_λ is real we must have

$$G_\lambda \geq 1. \tag{21}$$

From this point on we assume that $\lambda \geq 1$ since we shall apply our results with λ a positive integer. Clearly, from (8), we see that g_λ is a strictly increasing function of λ so that

$$g_\lambda > g_1 \qquad (\lambda > 1). \tag{22}$$

From Table VI in [10] we have $f(\sqrt{-1}) = 2^{1/4}$ so that $G_1 = 1$ by (15). Hence, by (20), $g_1^8 = 1/2$ so $g_1 = 2^{-1/8}$, and thus (22) becomes

$$g_\lambda > 2^{-1/8} \qquad (\lambda > 1). \tag{23}$$

Assume now that there is a value of $\lambda\,(> 1)$, say $\lambda = \lambda_0$, for which the minus sign holds in (20). Then, appealing to (21), we deduce

$$g_{\lambda_0}^8 = \frac{G_{\lambda_0}^{12} - \sqrt{G_{\lambda_0}^{24}-1}}{2G_{\lambda_0}^4} = \frac{1}{2G_{\lambda_0}^4\left(G_{\lambda_0}^{12} + \sqrt{G_{\lambda_0}^{24}-1}\right)} \leq \frac{1}{2},$$

which contradicts (23). Hence the plus sign must hold in (20) for $\lambda \geq 1$, that is,

$$g_\lambda = \left(\frac{G_\lambda^{12} + \sqrt{G_\lambda^{24}-1}}{2G_\lambda^4}\right)^{1/8}. \tag{24}$$

Appealing to the first six values in Table VI of [10], we have

$$f(\sqrt{-1}) = 2^{1/4},\ f_1(\sqrt{-2}) = 2^{1/4},\ f(\sqrt{-3}) = 2^{1/3},$$
$$f_1(\sqrt{-4}) = 2^{3/8},\ f(\sqrt{-5}) = (1+\sqrt{5})^{1/4},\ f_1(\sqrt{-6}) = (4+2\sqrt{2})^{1/6}$$

where the value of $f_1(\sqrt{-4})$ has been replaced by its correct value. Thus, by (15), we deduce

$$G_1 = 1,\ g_2 = 1,\ G_3 = 2^{1/12},$$
$$G_4 = 2^{1/8},\ G_5 = 2^{-1/4}(1+\sqrt{5})^{1/4},\ g_6 = 2^{-1/4}(4+2\sqrt{2})^{1/6}.$$

Appealing to (19) and (24), we obtain

$$g_1 = 2^{-1/8},\ G_2 = 2^{-1/8}(\sqrt{2}+1)^{1/8},\ g_3 = 2^{-1/6}(2+\sqrt{3})^{1/8},$$
$$G_4 = 2^{-3/16}(1+\sqrt{2})^{1/4},\ g_5 = 2^{-1/4}(3+\sqrt{5}+2\sqrt{2+2\sqrt{5}})^{1/8},$$
$$G_6 = 2^{-1/8}(1+\sqrt{2})^{-1/12}(2+\sqrt{3})^{1/8}(\sqrt{2}+\sqrt{3})^{1/8}.$$

Making use of these values in Theorem 2 and 3, we deduce

$$\sum_{n=1}^{\infty} \frac{1}{n\sinh(\pi n)} = \frac{\pi}{12} - \frac{1}{4}\log 2,$$
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sinh(\pi n)} = \frac{\pi}{12} - \frac{1}{2}\log 2,$$
$$\sum_{n=1}^{\infty} \frac{1}{n\sinh(\sqrt{2}\,\pi n)} = \frac{\sqrt{2}}{12}\pi - \frac{1}{2}\log 2,$$
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sinh(\sqrt{2}\,\pi n)} = \frac{\sqrt{2}}{12}\pi - \frac{1}{4}\log(1+\sqrt{2}) - \frac{1}{4}\log 2,$$
$$\sum_{n=1}^{\infty} \frac{1}{n\sinh(\sqrt{3}\,\pi n)} = \frac{\sqrt{3}}{12}\pi - \frac{1}{4}\log(2+\sqrt{3}) - \frac{1}{6}\log 2,$$
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sinh(\sqrt{3}\,\pi n)} = \frac{\sqrt{3}}{12}\pi - \frac{2}{3}\log 2,$$
$$\sum_{n=1}^{\infty} \frac{1}{n\sinh(2\pi n)} = \frac{\pi}{6} - \frac{3}{4}\log 2,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \sinh(2\pi n)} = \frac{\pi}{6} - \frac{1}{2}\log(1+\sqrt{2}) - \frac{1}{8}\log 2,$$
$$\sum_{n=1}^{\infty} \frac{1}{n \sinh(\sqrt{5}\,\pi n)} = \frac{\sqrt{5}}{12}\pi - \frac{1}{4}\log(3+\sqrt{5}+2\sqrt{2+2\sqrt{5}}),$$
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \sinh(\sqrt{5}\,\pi n)} = \frac{\sqrt{5}}{12}\pi - \frac{1}{2}\log(1+\sqrt{5}),$$
$$\sum_{n=1}^{\infty} \frac{1}{n \sinh(\sqrt{6}\,\pi n)} = \frac{\sqrt{6}}{12}\pi - \frac{1}{3}\log(4+2\sqrt{2}),$$
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \sinh(\sqrt{6}\,\pi n)} = \frac{\sqrt{6}}{12}\pi - \frac{1}{4}\log 2 + \frac{1}{6}\log(1+\sqrt{2}) - \frac{1}{4}\log(2+\sqrt{3}) - \frac{1}{4}\log(\sqrt{2}+\sqrt{3}),$$

and the following values of $\sigma_1(b), \sigma_2(b), \sigma_3(b)$ for $b = \sqrt{\lambda}$, $\lambda = 1, 2, \ldots, 6$:

$$\begin{aligned}
\sigma_1(1) &= -\frac{\pi}{2}\log 2,\\
\sigma_1(\sqrt{2}) &= -\frac{\pi}{\sqrt{2}}\log 2,\\
\sigma_1(\sqrt{3}) &= -\frac{\pi}{3\sqrt{3}}\log 2 - \frac{\pi}{2\sqrt{3}}\log(2+\sqrt{3}),\\
\sigma_1(2) &= -\frac{3\pi}{4}\log 2,\\
\sigma_1(\sqrt{5}) &= -\frac{\pi}{2\sqrt{5}}\log(3+\sqrt{5}+2\sqrt{2+2\sqrt{5}}),\\
\sigma_1(\sqrt{6}) &= -\frac{2\pi}{3\sqrt{6}}\log(1+\sqrt{2}) - \frac{\pi}{\sqrt{6}}\log 2,\\
\sigma_2(1) &= -\pi\log 2,\\
\sigma_2(\sqrt{2}) &= -\frac{\pi}{2\sqrt{2}}\log(2+2\sqrt{2}),\\
\sigma_2(\sqrt{3}) &= -\frac{4\pi}{3\sqrt{3}}\log 2,\\
\sigma_2(2) &= -\frac{\pi}{8}\log 2 - \frac{\pi}{2}\log(1+\sqrt{2}),\\
\sigma_2(\sqrt{5}) &= -\frac{\pi}{\sqrt{5}}\log(1+\sqrt{5}),
\end{aligned}$$

$$\begin{aligned}
\sigma_2(\sqrt{6}) &= -\frac{\pi}{2\sqrt{6}}\log 2+\frac{\pi}{3\sqrt{6}}\log(1+\sqrt{2})-\frac{\pi}{2\sqrt{6}}\log(2+\sqrt{3})\\
&\quad -\frac{\pi}{2\sqrt{6}}\log(\sqrt{2}+\sqrt{3}),\\
\sigma_3(1) &= -\frac{\pi}{2}\log 2,\\
\sigma_3(\sqrt{2}) &= -\frac{\pi}{2\sqrt{2}}\log(-2+2\sqrt{2}),\\
\sigma_3(\sqrt{3}) &= -\frac{\pi}{3\sqrt{3}}\log 2+\frac{\pi}{2\sqrt{3}}\log(2+\sqrt{3}),\\
\sigma_3(2) &= -\frac{\pi}{8}\log 2+\frac{\pi}{2}\log(1+\sqrt{2}),\\
\sigma_3(\sqrt{5}) &= -\frac{2\pi}{\sqrt{5}}\log 2+\frac{\pi}{2\sqrt{5}}\log(3+\sqrt{5}+2\sqrt{2+2\sqrt{5}})\\
&\quad +\frac{\pi}{\sqrt{5}}\log(1+\sqrt{5}),\\
\sigma_3(\sqrt{6}) &= -\frac{\pi}{2\sqrt{6}}\log 2+\frac{\pi}{3\sqrt{6}}\log(1+\sqrt{2})+\frac{\pi}{2\sqrt{6}}\log(2+\sqrt{3})\\
&\quad +\frac{\pi}{2\sqrt{6}}\log(\sqrt{2}+\sqrt{3}).
\end{aligned}$$

The tables in [6] and [10] enable us to determine $\sigma_1(b), \sigma_2(b), \sigma_3(b)$, where $b = \sqrt{\lambda}$, for all positive integers λ in the range $1 \le \lambda \le 100$ (except $\lambda = 53, 54, 59, 61, 74, 79, 83, 86, 87, 89, 95$) as well as for certain values of $\lambda > 100$. For example, as $\sqrt{2}\, f_1(\sqrt{-58})^2 = 5+\sqrt{29}$ [10, p. 723], we have $g_{58}^2 = \frac{5+\sqrt{29}}{2}$ by (15) so that $\log(2g_{58}^4) = \log(27+5\sqrt{29})$ and thus by Theorem 3

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{m^2+58n^2} = -\frac{\pi}{\sqrt{58}}\log(27+5\sqrt{29}),$$

a result which is stated explicitly in [11, p. 1225].

We remark that by applying Theorem 2 to the obvious relation

$$\sum_{n=1}^{\infty}\frac{1}{n\sinh(b\pi n)}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n\sinh(b\pi n)}=\sum_{n=1}^{\infty}\frac{1}{n\sinh(2b\pi n)},$$

we get the first of Ramanujan's properties in (9), namely,

$$g_{4\lambda} = 2^{1/4} g_\lambda G_\lambda. \tag{25}$$

Then, from (19) and (25), we obtain

$$G_{4\lambda} = \left(\frac{8g_\lambda^{12}G_\lambda^{12} + \sqrt{64g_\lambda^{24}G_\lambda^{24}+1}}{4g_\lambda^4 G_\lambda^4} \right)^{1/8}. \qquad (26)$$

Formulae (25) and (26), in conjunction with Theorem 3, allow us to determine $\sum'^{\infty}_{m,n=-\infty} \frac{(-1)^m}{m^2+4\lambda n^2}$, $\sum'^{\infty}_{m,n=-\infty} \frac{(-1)^{m+n}}{m^2+4\lambda n^2}$, and $\sum'^{\infty}_{m,n=-\infty} \frac{(-1)^n}{m^2+4\lambda n^2}$ from g_λ and G_λ.

By subtracting (11) from (10) in Theorem 2, we obtain

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)\sinh(b\pi(2n+1))} = \log(G_{b^2}/g_{b^2}),$$

so that for example

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)\sinh(\pi(2n+1))} = \frac{1}{8}\log 2.$$

Adding (12), (13) and (14) in Theorem 3, we obtain

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m + (-1)^n + (-1)^{m+n}}{m^2+\lambda n^2} = -\frac{2\pi}{\sqrt{\lambda}}\log 2. \qquad (27)$$

This result can also be obtained directly as a simple consequence of Kronecker's limit formula ([2] or [8, p. 14])

$$\lim_{s\to 1^+} \left(\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m^2+\lambda n^2)^s} - \frac{\pi}{\sqrt{\lambda}}\frac{1}{s-1} \right) \text{ exists.} \qquad (28)$$

For $s > 1$ we have

$$\begin{aligned} \sum_{m,n=-\infty}^{\infty}{}' \frac{(1+(-1)^m)(1+(-1)^n)}{(m^2+\lambda n^2)^s} &= \sum_{m,n=-\infty}^{\infty}{}' \frac{4}{((2m)^2+\lambda(2n)^2)^s} \\ &= \frac{1}{2^{2s-2}} \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m^2+\lambda n^2)^s}, \end{aligned}$$

so that

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{((-1)^m + (-1)^n + (-1)^{m+n})}{(m^2+\lambda n^2)^s} = \left(\frac{1}{2^{2s-2}} - 1\right) \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m^2+\lambda n^2)^s}.$$

Now for s close to 1 we have

$$\frac{1}{2^{2s-2}} - 1 = -2(\log 2)(s-1) + O((s-1)^2),$$

and by (28)

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m^2+\lambda n^2)^s} = \frac{\pi}{\sqrt{\lambda}} \cdot \frac{1}{s-1} + O(1),$$

so that

$$\left(\frac{1}{2^{2s-2}} - 1\right) \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m^2+\lambda n^2)^s} = \frac{-2\pi \log 2}{\sqrt{\lambda}} + O((s-1)).$$

Letting $s \to 1^+$ we obtain (27).

Finally we mention that by applying Theorem 1 to positive-definite, binary quadratic forms $am^2 + bmn + cn^2$ other than $m^2 + \lambda n^2$ we can obtain results like

$$\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^m}{m^2+mn+n^2} = -\frac{4\pi}{3\sqrt{3}} \log 2.$$

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