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Some Lambert Series Expansions of Products of Theta Functions*

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Abstract. Let q be a complex number satisfying $|q| < 1$. The theta function $\phi(q)$ is defined by $\phi(q) = \sum_{x=-\infty}^{\infty} q^{x^2}$. Ramanujan has given a number of Lambert series expansions such as

$$\phi(q)\phi(q^2) = 1 - 2\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2} q^{2n-1}}{1 - q^{2n-1}}.$$

A formula is proved which includes this and other expansions as special cases.

Key words: theta functions, Lambert series, binary quadratic forms

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Let q be a complex number satisfying $|q| < 1$. Following Ramanujan we define the theta functions $\phi(q)$ and $\psi(q)$ by

$$\phi(q) = \sum_{x=-\infty}^{\infty} q^{x^2} \tag{1}$$

and

$$\psi(q) = \sum_{x=0}^{\infty} q^{x(x+1)/2}, \tag{2}$$

see for example [1, Entry 22, p. 36], [3, 10]. The transformation $x \to -x - 1$ in (2) shows that

$$\sum_{x=-\infty}^{\infty} q^{x(x+1)/2} = 2\psi(q). \tag{3}$$

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A series of the type

$$\sum_{n=1}^{\infty} a_n \frac{q^n}{1-q^n},$$

where the $a_n (n = 1, 2, \ldots)$ are real numbers is called a Lambert series [7, p. 257]. In his second notebook Ramanujan [10] gave the Lambert series expansion of many functions involving $\phi(q)$ and $\psi(q)$. For example [1, Entry 17(i), p. 302]

$$q\psi(q)\psi(q^7) = \frac{q}{1-q} - \frac{q^3}{1-q^3} - \frac{q^5}{1-q^5} + \frac{q^9}{1-q^9} + \frac{q^{11}}{1-q^{11}} - \frac{q^{13}}{1-q^{13}} + \cdots, \tag{4}$$

where the cycle of coefficients is of length 14. Since the Kronecker symbol (see for example [6, p. 77])

$$\left(\frac{-28}{n}\right) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2 \text{ or } n \equiv 0 \pmod 7, \\ +1, & \text{if } n \equiv 1, 9, 11 \pmod{14}, \\ -1, & \text{if } n \equiv 3, 5, 13 \pmod{14}, \end{cases}$$

formula (4) can be written as

$$q\psi(q)\psi(q^7) = \sum_{n=1}^{\infty} \left(\frac{-28}{n}\right) \frac{q^n}{1-q^n}. \tag{5}$$

Berndt remarks [1, p. 303] that it appears very difficult to prove this formula without the addition theorem for elliptic integrals of the second kind. In addition Berndt's proof makes use of the Jacobi triple product identity. For the proof of the corresponding Lambert series for $\phi(q)\phi(q^7)$, namely,

$$\begin{aligned} \phi(q)\phi(q^7) = 1 + 2\bigg(& \frac{q}{1-q} - \frac{q^2}{1-q^2} - \frac{q^3}{1-q^3} + \frac{q^4}{1-q^4} - \frac{q^5}{1-q^5} + \frac{q^6}{1-q^6} + \frac{q^8}{1-q^8} \\ & + \frac{q^9}{1-q^9} + \frac{q^{10}}{1-q^{10}} + \frac{q^{11}}{1-q^{11}} - \frac{q^{12}}{1-q^{12}} - \frac{q^{13}}{1-q^{13}} + \frac{q^{15}}{1-q^{15}} \\ & + \frac{q^{16}}{1-q^{16}} - \frac{q^{17}}{1-q^{17}} - \frac{q^{18}}{1-q^{18}} - \frac{q^{19}}{1-q^{19}} - \frac{q^{20}}{1-q^{20}} - \frac{q^{22}}{1-q^{22}} \\ & + \frac{q^{23}}{1-q^{23}} - \frac{q^{24}}{1-q^{24}} + \frac{q^{25}}{1-q^{25}} + \frac{q^{26}}{1-q^{26}} - \frac{q^{27}}{1-q^{27}} + \cdots \bigg), \end{aligned} \tag{6}$$

where the cycle of coefficients is of length 28, Berndt [1, Entry 17(ii), p. 302] makes use of a modular equation of the seventh order.

In this paper we present a systematic approach to the problem of finding Lambert series expansions of products of theta functions which yields (4), (6) and many new Lambert series.

Our approach is based on the classical theory of binary quadratic forms due to Gauss, see for example [6, Chap. 5]. The main tool is the recent formula of Huard, Kaplan and Williams [8, Theorem 9.1] which counts the number of representations of a positive integer by a representative system of inequivalent primitive positive-definite integral binary quadratic forms of given discriminant. This formula generalizes a classical formula of Dirichlet (see for example [6, Theorem 64, p. 78], [9]). Before stating and proving our main result we give the basic facts about binary quadratic forms that we shall need.

We consider only integral binary quadratic forms $ax^2 + bxy + cy^2$ (a, b, c integers), which are primitive (that is $\gcd(a, b, c) = 1$) and positive-definite (that is $f(x, y) > 0$ for all integers x, y with $(x, y) \neq (0, 0)$, equivalently, $a > 0$ and $b^2 - 4ac < 0$). We call a primitive positive-definite integral binary quadratic form a form for short, and we denote the form $ax^2 + bxy + cy^2$ by (a, b, c). The integer $d = b^2 - 4ac$ is called the discriminant of the form (a, b, c). We note that $d < 0$ and $d \equiv 0$ or $1 \pmod 4$. If d is a negative integer with $d \equiv 0$ or $1 \pmod 4$ then the forms

$$\begin{cases} (1, 0, -d/4), & \text{if } d \equiv 0 \pmod 4, \\ (1, 1, (1-d)/4), & \text{if } d \equiv 1 \pmod 4, \end{cases} \tag{7}$$

have discriminant d. Two forms (a, b, c) and (a', b', c') of discriminant d are said to be equivalent if there exist integers p, q, r, s with $ps - qr = 1$ such that

$$ax^2 + bxy + cy^2 = a'(px + qy)^2 + b'(px + qy)(rx + sy) + c'(rx + sy)^2. \tag{8}$$

If (a, b, c) and (a', b', c') are equivalent we write $(a, b, c) \sim (a', b', c')$. Clearly $\sim$ is an equivalence relation on the set of forms of discriminant d. It is a classical result of Gauss that there are only finitely many such equivalence classes [6, Chap. 5]. The set of these equivalence classes is denoted by $H(d)$ and the number of classes is called the (form) class number and is denoted by $h(d)$. The class containing the form (a, b, c) is denoted by $[a, b, c]$. Each class in $H(d)$ contains one and only one reduced form [6, p. 68], that is, a form (a, b, c) satisfying

$$-a < b \le a \le c \quad \text{with} \quad b \ge 0 \ \text{ if } a = c. \tag{9}$$

The class number $h(d)$ is just the number of reduced forms of discriminant d. We denote by $l(d)$ the number of reduced forms (a, b, c) with $b = 0$. Clearly $0 \le l(d) \le h(d)$ and

$$\begin{cases} l(d) = 0, & \text{if } d \equiv 1 \pmod 4, \\ l(d) \ge 1, & \text{if } d \equiv 0 \pmod 4. \end{cases} \tag{10}$$

Gauss showed how to compose two classes to obtain a third class, see for example [4, Chap. 7], [6, Chap. 9]. Under Gaussian composition $H(d)$ is an abelian group called the (form) class group. The identity of the class group $H(d)$ is the class I containing the form $(1, 0, -d/4)$ if $d \equiv 0 \pmod 4$ and the form $(1, 1, (1-d)/4)$ if $d \equiv 1 \pmod 4$. The inverse of the class $[a, b, c]$ is the class $[a, -b, c]$. We note that if $K = [a, 0, c] \in H(d)$ then $K = K^{-1}$ so that $K^2 = I$.

The conductor of the discriminant d is the largest positive integer $f(d)$ such that $d/f(d)^2$ is a discriminant. The discriminant $\Delta(d) = d/f(d)^2$ is called the fundamental discriminant of d. The discriminant d is fundamental if and only if $f(d) = 1$. We also set

$$w(d) = \begin{cases} 6, & \text{if } d = -3, \\ 4, & \text{if } d = -4, \\ 2, & \text{if } d < -4. \end{cases} \tag{11}$$

From this point on we consider only discriminants d for which

$$H(d) \simeq Z_2 \times \cdots \times Z_2, \tag{12}$$

where Z_2 denotes the cyclic group of order 2. Thus $K^2 = I$ for all $K \in H(d)$, and so each reduced form of discriminant d is of the form

$$(a, 0, c), \quad 1 \le a \le c,$$

or

$$(a, a, c), \quad 1 \le a < c,$$

or

$$(a, b, a), \quad 1 \le b \le a.$$

Since $[a, b, a] = [2a - b, 2a - b, a]$ we may take the $h(d)$ classes of $H(d)$ as

$$\begin{cases} [a_i, 0, c_i], & i = 1, 2, \ldots, l(d), \\ [a_i, a_i, c_i], & i = l(d) + 1, \ldots, h(d). \end{cases} \tag{13}$$

For positive integers n and k we set

$$\delta(n, k) = \begin{cases} 1, & \text{if } k \mid n, \\ 0, & \text{if } k \nmid n, \end{cases} \tag{14}$$

with the understanding that $\delta(n, k)F(n, k)$ is 0 for $k \nmid n$ even if the function $F(n, k)$ is not defined for $k \nmid n$. As usual $\mu(n)$ is the Möbius function [7, p. 234] and $(\frac{d}{n})$ is the Kronecker symbol, see for example [6, p. 77], [8, p. 278]. We prove

Theorem. *Let d be a discriminant for which* (12) *holds. Let the $h(d)$ classes of $H(d)$ be as given in* (13). *Then*

$$\sum_{i=1}^{l(d)} \phi(q^{a_i})\phi(q^{c_i}) + \sum_{i=l(d)+1}^{h(d)} \phi(q^{a_i})\phi(q^{4c_i - a_i}) + 4 \sum_{i=l(d)+1}^{h(d)} q^{c_i}\psi(q^{2a_i})\psi(q^{8c_i - 2a_i})$$
$$= h(d) + \sum_{n=1}^{\infty} a_n(d) \frac{q^n}{1 - q^n}, \tag{15}$$

where

$$a_n(d) = \sum_{\substack{m,\,e \\ m|f(d),\, e|f(d)/m}} \mu(e) w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)}\delta(n, m^2 e)\left(\frac{d/m^2}{n/m^2 e}\right). \tag{16}$$

Remark 1. We note that when d is a fundamental discriminant, that is $f(d) = 1$, formula (16) becomes

$$a_n(d) = w(d)\left(\frac{d}{n}\right). \tag{17}$$

Remark 2. When $f(d) = 2$ formula (16) becomes

$$a_n(d) = w(d)\left(\frac{d}{n}\right) - w(d)\delta(n,2)\left(\frac{d}{n/2}\right) + w\left(\frac{d}{4}\right)\frac{h(d)}{h(d/4)}\delta(n,4)\left(\frac{d/4}{n/4}\right). \tag{18}$$

Proof: We consider the sums

$$S_1 = \sum_{i=1}^{l(d)} \sum_{x,y=-\infty}^{\infty} q^{a_i x^2 + c_i y^2}$$

and

$$S_2 = \sum_{i=l(d)+1}^{h(d)} \sum_{x,y=-\infty}^{\infty} q^{a_i x^2 + a_i xy + c_i y^2}.$$

Clearly in view of (1) we have

$$S_1 = \sum_{i=1}^{l(d)} \phi(q^{a_i})\phi(q^{c_i}).$$

For $i = l(d)+1, \ldots, h(d)$ we have

$$\begin{aligned}
\sum_{x,y=-\infty}^{\infty} q^{a_i x^2 + a_i xy + c_i y^2} &= \sum_{\substack{x,\,y=-\infty \\ y \text{ even}}}^{\infty} q^{a_i x^2 + a_i xy + c_i y^2} + \sum_{\substack{x,\,y=-\infty \\ y \text{ odd}}}^{\infty} q^{a_i x^2 + a_i xy + c_i y^2} \\
&= \sum_{x,y=-\infty}^{\infty} q^{a_i x^2 + 2a_i xy + 4c_i y^2} + \sum_{x,y=-\infty}^{\infty} q^{a_i x^2 + a_i x(2y+1) + c_i (2y+1)^2} \\
&= \sum_{x,y=-\infty}^{\infty} q^{a_i (x+y)^2 + (4c_i - a_i)y^2} + \sum_{x,y=-\infty}^{\infty} q^{a_i (x+y)(x+y+1) + (4c_i - a_i)y(y+1) + c_i} \\
&= \sum_{x,y=-\infty}^{\infty} q^{a_i x^2 + (4c_i - a_i)y^2} + q^{c_i} \sum_{x,y=-\infty}^{\infty} q^{a_i x(x+1) + (4c_i - a_i)y(y+1)}
\end{aligned}$$

$$= \phi(q^{a_i})\phi(q^{4c_i-a_i}) + q^{c_i}2\psi(q^{2a_i})2\psi\left(q^{2(4c_i-a_i)}\right)$$
$$= \phi(q^{a_i})\phi(q^{4c_i-a_i}) + 4q^{c_i}\psi(q^{2a_i})\psi(q^{8c_i-2a_i})$$

so that

$$S_2 = \sum_{i=l(d)+1}^{h(d)} \phi(q^{a_i})\phi(q^{4c_i-a_i}) + 4\sum_{i=l(d)+1}^{h(d)} q^{c_i}\psi(q^{2a_i})\psi(q^{8c_i-2a_i}).$$

Thus

$$S_1 + S_2 = \sum_{i=1}^{l(d)} \phi(q^{a_i})\phi(q^{c_i}) + \sum_{i=l(d)+1}^{h(d)} \phi(q^{a_i})\phi(q^{4c_i-a_i}) + 4\sum_{i=l(d)+1}^{h(d)} q^{c_i}\psi(q^{2a_i})\psi(q^{8c_i-2a_i}).$$

On the other hand, we have

$$S_1 + S_2 = \sum_{x,y=-\infty}^{\infty} \left(\sum_{i=1}^{l(d)} q^{a_i x^2 + c_i y^2} + \sum_{i=l(d)+1}^{h(d)} q^{a_i x^2 + a_i xy + c_i y^2} \right)$$
$$= h(d) + \sum_{n=1}^{\infty} q^n N(n,d),$$

where $N(n,d)$ is the number of representations of the positive integer n by the forms $a_i x^2 + c_i y^2$ $(i = 1, 2, \ldots, l(d))$ and $a_i x^2 + a_i xy + c_i y^2$ $(i = l(d)+1, \ldots, h(d))$, that is, by the $h(d)$ classes of $H(d)$. A prime p is said to be a null prime [8, p. 283] with respect to n and d if

$$v_p(n) \equiv 1 \pmod 2, \quad v_p(n) < 2v_p(f(d)),$$

where $p^{v_p(n)} \mid n$, $p^{v_p(n)+1} \nmid n$. The set of all such null primes is denoted by $\text{Null}(n,d)$. It is known that if $\text{Null}(n,d) \neq \emptyset$ then $N(n,d) = 0$, see [8, Proposition 4.1]. Moreover Huard, Kaplan and Williams [8, Theorem 9.1] have shown that if $\text{Null}(n,d) = \emptyset$ then

$$N(n,d) = w\left(\frac{d}{M(n,d)^2}\right) \frac{h(d)}{h(d/M(n,d)^2)} \sum_{v \mid n/M(n,d)^2} \left(\frac{\Delta(d)}{v}\right),$$

where $M(n,d)$ is the largest integer such that $M(n,d)^2 \mid n$, $M(n,d) \mid f(d)$, and $w(d)$ was defined in (11). Hence

$$\sum_{n=1}^{\infty} q^n N(n,d) = \sum_{\substack{n=1 \\ \text{Null}(n,d)=\emptyset}}^{\infty} q^n w\left(\frac{d}{M(n,d)^2}\right) \frac{h(d)}{h(d/M(n,d)^2)} \sum_{v \mid n/M(n,d)^2} \left(\frac{\Delta(d)}{v}\right)$$
$$= \sum_{m \mid f(d)} \sum_{\substack{n=1 \\ \text{Null}(n,d)=\emptyset \\ M(n,d)=m}}^{\infty} q^n w\left(\frac{d}{m^2}\right) \frac{h(d)}{h(d/m^2)} \sum_{v \mid n/m^2} \left(\frac{\Delta(d)}{v}\right)$$

$$= \sum_{m|f(d)} w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \sum_{\substack{n=1\\ \mathrm{Null}(n,d)=\emptyset\\ M(n,d)=m}}^{\infty} q^n \sum_{v|n/m^2} \left(\frac{\Delta(d)}{v}\right)$$

$$= \sum_{m|f(d)} w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \sum_{\substack{n=1\\ m^2|n\\ \mathrm{Null}(n/m^2,d/m^2)=\emptyset\\ M(n/m^2,d/m^2)=1}}^{\infty} q^n \sum_{v|n/m^2} \left(\frac{\Delta(d)}{v}\right)$$

$$= \sum_{m|f(d)} w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \sum_{\substack{n=1\\ m^2|n\\ (n/m^2,\ f(d)/m)=1}}^{\infty} q^n \sum_{v|n/m^2} \left(\frac{\Delta(d)}{v}\right)$$

(by [8, Lemma 4.1 (b)])

$$= \sum_{m|f(d)} w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \sum_{\substack{N=1\\ (N,\ f(d)/m)=1}}^{\infty} q^{Nm^2} \sum_{v|N} \left(\frac{\Delta(d)}{v}\right)$$

$$= \sum_{m|f(d)} w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \sum_{\substack{v,s=1\\ (vs,\ f(d)/m)=1}}^{\infty} q^{vsm^2} \left(\frac{\Delta(d)}{v}\right)$$

$$= \sum_{m|f(d)} w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \sum_{\substack{v=1\\ (v,\ f(d)/m)=1}}^{\infty} \left(\frac{\Delta(d)}{v}\right) \sum_{\substack{s=1\\ (s,\ f(d)/m)=1}}^{\infty} q^{vsm^2}$$

$$= \sum_{m|f(d)} w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \sum_{\substack{v=1\\ (v,\ f(d)/m)=1}}^{\infty} \left(\frac{\Delta(d)}{v}\right) \sum_{s=1}^{\infty} q^{vsm^2} \sum_{e|(s,\ f(d)/m)} \mu(e)$$

$$= \sum_{m|f(d)} w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \sum_{\substack{v=1\\ (v,\ f(d)/m)=1}}^{\infty} \left(\frac{\Delta(d)}{v}\right) \sum_{e|f(d)/m} \mu(e) \sum_{\substack{s=1\\ e|s}}^{\infty} q^{vm^2 s}$$

$$= \sum_{m|f(d)} w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \sum_{\substack{v=1\\ (v,\ f(d)/m)=1}}^{\infty} \left(\frac{\Delta(d)}{v}\right) \sum_{e|f(d)/m} \mu(e) \sum_{t=1}^{\infty} q^{vm^2 et}$$

$$= \sum_{m|f(d)} w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \sum_{\substack{v=1 \\ (v,\ f(d)/m)=1}}^{\infty} \left(\frac{\Delta(d)}{v}\right) \sum_{e|f(d)/m} \mu(e)\frac{q^{vm^2e}}{1-q^{vm^2e}}$$

$$= \sum_{m|f(d)} w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \sum_{v=1}^{\infty} \left(\frac{d/m^2}{v}\right) \sum_{e|f(d)/m} \mu(e)\frac{q^{vm^2e}}{1-q^{vm^2e}}$$

$$= \sum_{m|f(d)} \sum_{e|f(d)/m} \sum_{v=1}^{\infty} w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \left(\frac{d/m^2}{v}\right)\mu(e)\frac{q^{vm^2e}}{1-q^{vm^2e}}$$

$$= \sum_{n=1}^{\infty} \left\{ \sum_{\substack{m,e \\ m|f(d),\ e|f(d)/m,\ m^2e|n}} w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \left(\frac{d/m^2}{n/m^2e}\right)\mu(e) \right\} \frac{q^n}{1-q^n}$$

$$= \sum_{n=1}^{\infty} \left\{ \sum_{\substack{m,e \\ m|f(d),\ e|f(d)/m}} \mu(e) w\left(\frac{d}{m^2}\right)\frac{h(d)}{h(d/m^2)} \delta(n, m^2e) \left(\frac{d/m^2}{n/m^2e}\right) \right\}$$

$$\times \frac{q^n}{1-q^n}.$$

This completes the proof of the theorem. □

It follows from a theorem of Chowla (see for example [5]) that (12) holds for only finitely many $d < 0$. It is conjectured that the complete list of such d consists of the following 101 discriminants:

$-d =$ 3, 4, 7, 8, 11, 12, 15, 16, 19, 20, 24, 27, 28, 32, 35, 36, 40, 43, 48, 51, 52, 60, 64, 67, 72, 75, 84, 88, 91, 96, 99, 100, 112, 115, 120, 123, 132, 147, 148, 160, 163, 168, 180, 187, 192, 195, 228, 232, 235, 240, 267, 280, 288, 312, 315, 340, 352, 372, 403, 408, 420, 427, 435, 448, 480, 483, 520, 532, 555, 595, 627, 660, 672, 708, 715, 760, 795, 840, 928, 960, 1012, 1092, 1120, 1155, 1248, 1320, 1380, 1428, 1435, 1540, 1632, 1848, 1995, 2080, 3003, 3040, 3315, 3360, 5280, 5460, 7392.

Chowla and Briggs [5] have shown that there is at most one further such fundamental discriminant d with $d < -10^{60}$. In view of the theorem, for all of these values of d, we have a Lambert series expansion of the type (15). We illustrate this with the following values of d.

Example	d
1	-3
2	-12
3	-4
4	-7
5	-28
6	-11
7	-8
8	-16
9	-15
10	-60
11	-7392

Example 1. $d = -3$. Here $f(-3) = 1$, $\Delta(-3) = -3$ and

$$H(-3) = \{[1, 1, 1]\},\ h(-3) = 1,\ l(-3) = 0,\ w(-3) = 6,$$

so the theorem gives

$$\phi(q)\phi(q^3) + 4q\psi(q^2)\psi(q^6) = 1 + \sum_{n=1}^{\infty} a_n(-3)\frac{q^n}{1-q^n}, \tag{19}$$

where

$$a_n(-3) = 6\left(\frac{-3}{n}\right). \tag{20}$$

Formula (19) is given in [1, eq. (3.6), p. 462], see also [1, Entries 3(i), (ii), p. 223], [2, Entry 37, p. 189]. We examine $\phi(q)\phi(q^3)$ and $q\psi(q^2)\psi(q^6)$ separately after treating the case $d = -12$ in the next example.

Example 2. $d = -12$. Here $f(-12) = 2$, $\Delta(-12) = -3$ and

$$H(-12) = \{[1, 0, 3]\},\ h(-12) = 1,\ l(-12) = 1,\ w(-12) = 2,$$

so the theorem gives

$$\phi(q)\phi(q^3) = 1 + \sum_{n=1}^{\infty} a_n(-12)\frac{q^n}{1-q^n}, \tag{21}$$

where

$$a_n(-12) = 2\left(\frac{-12}{n}\right) - 2\delta(n, 2)\left(\frac{-12}{n/2}\right) + 6\delta(n, 4)\left(\frac{-3}{n/4}\right)$$

$$= \begin{cases} 2\left(\frac{-12}{n}\right), & \text{if } n \equiv 1 \pmod 2, \\ 2\left(\frac{-12}{n}\right) - 2\left(\frac{-12}{n/2}\right), & \text{if } n \equiv 2 \pmod 4, \\ 2\left(\frac{-12}{n}\right) - 2\left(\frac{-12}{n/2}\right) + 6\left(\frac{-3}{n/4}\right), & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

$$= \begin{cases} 2\left(\frac{-3}{n}\right), & \text{if } n \equiv 1 \pmod 2, \\ -2\left(\frac{-3}{n/2}\right), & \text{if } n \equiv 2 \pmod 4, \\ 6\left(\frac{-3}{n/4}\right), & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

that is

$$a_n(-12) = \begin{cases} 2\left(\frac{-3}{n}\right), & \text{if } n \equiv 1 \pmod 2, \\ 2\left(\frac{-3}{n}\right), & \text{if } n \equiv 2 \pmod 4, \\ 6\left(\frac{-3}{n}\right), & \text{if } n \equiv 0 \pmod 4, \end{cases} \tag{22}$$

as $(\frac{-3}{2}) = -1$. Hence

$$\phi(q)\phi(q^3) = 1 + 6\sum_{n=1}^{\infty}\left(\frac{-3}{n}\right)\frac{q^n}{1-q^n} - 4\sum_{\substack{n=1 \\ 4\nmid n}}^{\infty}\left(\frac{-3}{n}\right)\frac{q^n}{1-q^n}. \tag{23}$$

From (19), (20) and (23) we obtain

$$q\psi(q^2)\psi(q^6) = \sum_{\substack{n=1 \\ 4\nmid n}}^{\infty}\left(\frac{-3}{n}\right)\frac{q^n}{1-q^n}. \tag{24}$$

We now put (23) and (24) in the forms given by Ramanujan. We set

$$A = \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty}\left(\frac{-3}{n}\right)\frac{q^n}{1-q^n}, \tag{25}$$

$$B = \sum_{\substack{n=1 \\ n\equiv 2 \pmod 4}}^{\infty} \left(\frac{-3}{n}\right) \frac{q^n}{1-q^n}, \tag{26}$$

$$C = \sum_{\substack{n=1 \\ n\equiv 0 \pmod 4}}^{\infty} \left(\frac{-3}{n}\right) \frac{q^n}{1-q^n}, \tag{27}$$

so that (23) and (24) become

$$\phi(q)\phi(q^3) = 1 + 6(A+B+C) - 4(A+B) = 1 + 2A + 2B + 6C \tag{28}$$

and

$$q\psi(q^2)\psi(q^6) = A + B. \tag{29}$$

Now

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n}{1+(-q)^n} &= \sum_{n=1}^{\infty} \left(\frac{-3}{n}\right) \frac{q^n}{1+(-q)^n} \\
&= \sum_{\substack{n=1 \\ n\equiv 0 \pmod 2}}^{\infty} \left(\frac{-3}{n}\right) \frac{q^n}{1+q^n} + \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \left(\frac{-3}{n}\right) \frac{q^n}{1-q^n} \\
&= \sum_{\substack{n=1 \\ n\equiv 0 \pmod 2}}^{\infty} \left(\frac{-3}{n}\right) \left(\frac{q^n}{1-q^n} - \frac{2q^{2n}}{1-q^{2n}}\right) + A \\
&= (B+C) + 2C + A \quad \left(\text{as } \left(\frac{-3}{2}\right) = -1\right) \\
&= A + B + 3C
\end{aligned}$$

so that (28) becomes

$$\phi(q)\phi(q^3) = 1 + 2\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n}{1+(-q)^n} \tag{30}$$

as asserted by Ramanujan [1, Entry 8 (iv), p. 114]. Also

$$\begin{aligned}
\sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \left(\frac{-3}{n}\right) \frac{q^n}{1-q^{2n}} &= \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \left(\frac{-3}{n}\right) \left(\frac{q^n}{1-q^n} - \frac{q^{2n}}{1-q^{2n}}\right) \\
&= \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \left(\frac{-3}{n}\right) \frac{q^n}{1-q^n} + \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \left(\frac{-3}{2n}\right) \frac{q^{2n}}{1-q^{2n}} \\
&= A + B,
\end{aligned}$$

so that by (29) we have

$$q\psi(q^2)\psi(q^6) = \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \left(\frac{-3}{n}\right) \frac{q^n}{1-q^{2n}} \tag{31}$$

as asserted by Ramanujan [1, Entry 3 (i), p. 223].

Example 3. $d = -4$. Here $f(-4) = 1$, $\Delta(-4) = -4$ and

$$H(-4) = \{[1,0,1]\},\ h(-4) = 1,\ l(-4) = 1,\ w(-4) = 4,$$

so the theorem gives

$$\phi(q)^2 = 1 + \sum_{n=1}^{\infty} a_n(-4)\frac{q^n}{1-q^n}, \tag{32}$$

where

$$a_n(-4) = 4\left(\frac{-4}{n}\right). \tag{33}$$

This agrees with [1, Entry 8 (i), p. 114] as

$$\left(\frac{-4}{n}\right) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2, \\ (-1)^k, & \text{if } n \equiv 1 \pmod 2, \text{ where } n = 2k+1. \end{cases}$$

Example 4. $d = -7$. Here $f(-7) = 1$, $\Delta(-7) = -7$ and

$$H(-7) = \{[1,1,2]\},\ h(-7) = 1,\ l(-7) = 0,\ w(-7) = 2,$$

so the theorem gives

$$\phi(q)\phi(q^7) + 4q^2\psi(q^2)\psi(q^{14}) = 1 + \sum_{n=1}^{\infty} a_n(-7)\frac{q^n}{1-q^n}, \tag{34}$$

where

$$a_n(-7) = 2\left(\frac{-7}{n}\right). \tag{35}$$

We examine $\phi(q)\phi(q^7)$ and $q\psi(q)\psi(q^7)$ individually in the next example, where the case $d = -28$ is treated.

Example 5. $d = -28$. Here $f(-28) = 2$, $\Delta(-28) = -7$ and

$$H(-28) = \{[1,0,7]\},\ h(-28) = 1,\ l(-28) = 1,\ w(-28) = 2,$$

so the theorem gives

$$\phi(q)\phi(q^7) = 1 + \sum_{n=1}^{\infty} a_n(-28)\frac{q^n}{1-q^n}, \tag{36}$$

where

$$a_n(-28) = 2\left(\frac{-28}{n}\right) - 2\delta(n,2)\left(\frac{-28}{n/2}\right) + 2\delta(n,4)\left(\frac{-7}{n/4}\right) \tag{37}$$

so that

$$a_n(-28) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 7, \\ 2, & \text{if } n \equiv 1, 4, 6, 8, 9, 10, 11, 15, 16, 23, 25, 26 \pmod{28}, \\ -2, & \text{if } n \equiv 2, 3, 5, 12, 13, 17, 18, 19, 20, 22, 24, 27 \pmod{28}, \end{cases} \tag{38}$$

which is in agreement with [1, Entry 17 (ii), p. 302]. From Example 4 we deduce that

$$4q^2\psi(q^2)\psi(q^{14}) = \sum_{n=1}^{\infty}(a_n(-7) - a_n(-28))\frac{q^n}{1-q^n}. \tag{39}$$

From (35) and (37) we obtain

$$\begin{aligned} a_n(-7) - a_n(-28) &= 2\left(\frac{-7}{n}\right) - 2\left(\frac{-28}{n}\right) + 2\delta(n,2)\left(\frac{-28}{n/2}\right) - 2\delta(n,4)\left(\frac{-7}{n/4}\right) \\ &= \begin{cases} 0, & \text{if } n \equiv 1 \pmod 2, \\ 4\left(\frac{-7}{n}\right), & \text{if } n \equiv 2 \pmod 4, \\ 0, & \text{if } n \equiv 0 \pmod 4, \end{cases} \end{aligned}$$

so that

$$\begin{aligned} 4q^2\psi(q^2)\psi(q^{14}) &= 4 \sum_{\substack{n=1 \\ n\equiv 2 \pmod 4}}^{\infty} \left(\frac{-7}{n}\right)\frac{q^n}{1-q^n} \\ &= 4\sum_{m=1}^{\infty}\left(\frac{-7}{4m-2}\right)\frac{q^{4m-2}}{1-q^{4m-2}} \\ &= 4\sum_{m=1}^{\infty}\left(\frac{-7}{2m-1}\right)\frac{q^{4m-2}}{1-q^{4m-2}} \\ &= 4\sum_{m=1}^{\infty}\left(\frac{2m-1}{7}\right)\frac{q^{4m-2}}{1-q^{4m-2}}. \end{aligned}$$

Replacing q by $q^{1/2}$, we obtain

$$q\psi(q)\psi(q^7) = \sum_{m=1}^{\infty} \left(\frac{2m-1}{7}\right) \frac{q^{2m-1}}{1-q^{2m-1}}, \tag{40}$$

which is in agreement with [1, Entry 17 (i), p. 302].

Example 6. $d = -11$. Here $f(-11) = 1$, $\Delta(-11) = -11$ and

$$H(-11) = \{[1, 1, 3]\},\ h(-11) = 1,\ l(-11) = 0,\ w(-11) = 2,$$

so the theorem gives

$$\phi(q)\phi(q^{11}) + 4q^3\psi(q^2)\psi(q^{22}) = 1 + \sum_{n=1}^{\infty} a_n(-11)\frac{q^n}{1-q^n}, \tag{41}$$

where

$$a_n(-11) = 2\left(\frac{-11}{n}\right). \tag{42}$$

I could not find (41) in [1]. Since $H(-44) \simeq Z_3$ it is not possible by our methods to determine the Lambert series of each of $\phi(q)\phi(q^{11})$ and $q^3\psi(q^2)\psi(q^{22})$ separately.

Example 7. $d = -8$. Here $f(-8) = 1$, $\Delta(-8) = -8$ and

$$H(-8) = \{[1, 0, 2]\},\ h(-8) = 1,\ l(-8) = 1,\ w(-8) = 2,$$

so the theorem gives

$$\phi(q)\phi(q^2) = 1 + \sum_{n=1}^{\infty} a_n(-8)\frac{q^n}{1-q^n}, \tag{43}$$

where

$$a_n(-8) = 2\left(\frac{-8}{n}\right). \tag{44}$$

This agrees with [1, Entry 8 (iii), p. 114] as

$$\left(\frac{-8}{n}\right) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2, \\ -(-1)^{k(k+1)/2}, & \text{if } n \equiv 1 \pmod 2, \text{ where } n = 2k-1. \end{cases}$$

Example 8. $d = -16$. Here $f(-16) = 1$, $\Delta(-16) = -4$ and

$$H(-16) = \{[1, 0, 4]\},\ h(-16) = 1,\ l(-16) = 1,\ w(-16) = 2,$$

so the theorem gives

$$\phi(q)\phi(q^4) = 1 + \sum_{n=1}^{\infty} a_n(-16)\frac{q^n}{1-q^n}, \tag{45}$$

where

$$a_n(-16) = 2\left(\frac{-16}{n}\right) - 2\delta(n,2)\left(\frac{-16}{n/2}\right) + 4\delta(n,4)\left(\frac{-4}{n/4}\right), \tag{46}$$

that is

$$a_n(-16) = \begin{cases} 2, & \text{if } n \equiv 1 \pmod 4 \text{ or } n \equiv 6 \pmod 8, \\ -2, & \text{if } n \equiv 3 \pmod 4 \text{ or } n \equiv 2 \pmod 8, \\ 4, & \text{if } n \equiv 4 \pmod{16}, \\ -4, & \text{if } n \equiv 12 \pmod{16}, \\ 0, & \text{otherwise.} \end{cases} \tag{47}$$

Formula (45) does not seem to appear in [1, 2]. However an elegant companion formula for $q\psi(q)\psi(q^4)$ was derived by Ramanujan, see [2, eq. (22.1), p. 153].

Example 9. $d = -15$. Here $f(-15) = 1$, $\Delta(-15) = -15$ and

$$H(-15) = \{[1,1,4],[5,5,2]\} \simeq Z_2$$
$$h(-15) = 2,\ l(-15) = 0,\ w(-15) = 2,$$

so the theorem gives

$$\begin{aligned} &\phi(q)\phi(q^{15}) + \phi(q^5)\phi(q^3) + 4q^4\psi(q^2)\psi(q^{30}) + 4q^2\psi(q^{10})\psi(q^6) \\ &\quad = 2 + \sum_{n=1}^{\infty} a_n(-15)\frac{q^n}{1-q^n}, \end{aligned} \tag{48}$$

where

$$a_n(-15) = 2\left(\frac{-15}{n}\right). \tag{49}$$

We determine $\phi(q)\phi(q^{15}) + \phi(q^3)\phi(q^5)$ and $q^2\psi(q)\psi(q^{15}) + q\psi(q^3)\psi(q^5)$ individually at the end of Example 10 where we consider $d = -60$.

Example 10. $d = -60$. Here $f(-60) = 2$, $\Delta(-60) = -15$ and

$$H(-60) = \{[1,0,15],[3,0,5]\} \simeq Z_2$$
$$h(-60) = 2,\ l(-60) = 2,\ w(-60) = 2,$$

so the theorem gives

$$\phi(q)\phi(q^{15}) + \phi(q^3)\phi(q^5) = 2 + \sum_{n=1}^{\infty} a_n(-60)\frac{q^n}{1-q^n}, \tag{50}$$

where

$$a_n(-60) = 2\left(\frac{-60}{n}\right) - 2\delta(n,2)\left(\frac{-60}{n/2}\right) + 2\delta(n,4)\left(\frac{-15}{n/4}\right). \tag{51}$$

Since

$$a_n(-60) = \begin{cases} 0, & \text{if } n \equiv 0, 3, 5, 6, 9, 10, 12, 15, 18, 20, 21, \\ & \quad 24, 25, 27, 30, 33, 35, 36, 39, 40, 42, \\ & \quad 45, 48, 50, 51, 54, 55, 57 \pmod{60}, \\ 2, & \text{if } n \equiv 1, 4, 8, 14, 16, 17, 19, 22, 23, 26, \\ & \quad 31, 32, 47, 49, 53, 58 \pmod{60}, \\ -2, & \text{if } n \equiv 2, 7, 11, 13, 28, 29, 34, 37, 38, \\ & \quad 41, 43, 44, 46, 52, 56, 59 \pmod{60}, \end{cases} \tag{52}$$

formula (50) agrees with [1, Entry 10 (vi), p. 379]. From (48) and (50) we obtain

$$4q^4\psi(q^2)\psi(q^{30}) + 4q^2\psi(q^6)\psi(q^{10}) = \sum_{n=1}^{\infty}(a_n(-15) - a_n(-60))\frac{q^n}{1-q^n}.$$

Now

$$\begin{aligned} a_n(-15) - a_n(-60) &= 2\left(\frac{-15}{n}\right) - 2\left(\frac{-60}{n}\right) + 2\delta(n,2)\left(\frac{-60}{n/2}\right) - 2\delta(n,4)\left(\frac{-15}{n/4}\right) \\ &= \begin{cases} 0, & \text{if } n \equiv 1 \pmod 2, \\ 4\left(\frac{-15}{n}\right), & \text{if } n \equiv 2 \pmod 4, \\ 0, & \text{if } n \equiv 0 \pmod 4, \end{cases} \end{aligned}$$

so that

$$\begin{aligned} q^4\psi(q^2)\psi(q^{30}) + q^2\psi(q^6)\psi(q^{10}) &= \sum_{\substack{n=1 \\ n\equiv 2 \pmod 4}}^{\infty} \left(\frac{-15}{n}\right)\frac{q^n}{1-q^n} \\ &= \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \left(\frac{-15}{n}\right)\frac{q^{2n}}{1-q^{2n}}. \end{aligned}$$

Replacing q by $q^{1/2}$, we obtain

$$q^2\psi(q)\psi(q^{15}) + q\psi(q^3)\psi(q^5) = \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \left(\frac{-15}{n}\right)\frac{q^n}{1-q^n}, \tag{53}$$

in agreement with [1, Entry 10 (v), p. 379]. As a final example we take $d = -7392$.

Example 11. $d = -7392$. Here $f(-7392) = 2$, $\Delta(-7392) = -1848$, and

$$\begin{aligned} H(-7392) = \{&[1, 0, 1848], [3, 0, 616], [7, 0, 264], [8, 0, 231], [11, 0, 168], [21, 0, 88],\\ &[24, 0, 77], [33, 0, 56], [4, 4, 463], [8, 8, 233], [12, 12, 157], [24, 24, 83],\\ &[28, 28, 73], [44, 44, 53], [88, 88, 43], [132, 132, 47]\} \simeq Z_2 \times Z_2 \times Z_2 \times Z_2,\end{aligned}$$
$h(-7392) = 16,\ l(-7392) = 8,\ w(-7392) = 2.$

By the theorem we obtain

$$\begin{aligned} &\phi(q)\phi(q^{1848}) + \phi(q^3)\phi(q^{616}) + \phi(q^7)\phi(q^{264}) + \phi(q^8)\,\phi(q^{231})\\ &\quad + \phi(q^{11})\phi(q^{168}) + \phi(q^{21})\phi(q^{88}) + \phi(q^{24})\phi(q^{77}) + \phi(q^{33})\phi(q^{56})\\ &\quad + \phi(q^4)\phi(q^{1848}) + \phi(q^8)\phi(q^{924}) + \phi(q^{12})\phi(q^{616}) + \phi(q^{24})\phi(q^{308})\\ &\quad + \phi(q^{28})\phi(q^{264}) + \phi(q^{44})\phi(q^{168}) + \phi(q^{88})\phi(q^{84}) + \phi(q^{132})\phi(q^{56})\\ &\quad + 4q^{463}\phi(q^8)\phi(q^{3696}) + 4q^{233}\phi(q^{16})\phi(q^{1848}) + 4q^{157}\phi(q^{24})\phi(q^{1232})\\ &\quad + 4q^{83}\phi(q^{48})\phi(q^{616}) + 4q^{73}\phi(q^{56})\phi(q^{528}) + 4q^{53}\phi(q^{88})\phi(q^{336})\\ &\quad + 4q^{43}\phi(q^{176})\phi(q^{168}) + 4q^{47}\phi(q^{264})\phi(q^{112})\\ &\quad = 16 + \sum_{n=1}^{\infty} a_n(-7392)\frac{q^n}{1-q^n}, \end{aligned} \tag{54}$$

where

$$a_n(-7392) = 2\left(\frac{-7392}{n}\right) - 2\delta(n, 2)\left(\frac{-7392}{n/2}\right) + \delta(n, 4)\left(\frac{-1848}{n/4}\right).$$

In conclusion we offer as a friendly challenge the problem of proving (54) using modular equations!

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