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Pascal's Triangle (mod 8)

JAMES G. HUARD, BLAIR K. SPEARMAN AND KENNETH S. WILLIAMS

Lucas' theorem gives a congruence for a binomial coefficient modulo a prime. Davis and Webb (*Europ. J. Combinatorics*, **11** (1990), 229–233) extended Lucas' theorem to a prime power modulus. Making use of their result, we count the number of times each residue class occurs in the nth row of Pascal's triangle (mod 8). Our results correct and extend those of Granville (*Amer. Math. Monthly*, **99** (1992), 318–331).

1. INTRODUCTION

Let n denote a nonnegative integer. The nth row of Pascal's triangle consists of the $n+1$ binomial coefficients $\binom{n}{r}$ $(r = 0, 1, \ldots, n)$. For integers t and m with $0 \le t < m$, we denote by $N_n(t, m)$ the number of integers in the nth row of Pascal's triangle which are congruent to t modulo m. Clearly if d is a positive integer dividing m then

$$\sum_{j=0}^{(m/d)-1} N_n(jd + t, m) = N_n(t, d), \qquad 0 \le t < d. \tag{1.1}$$

When $d = 1$ the right-hand side of (1.1) is $N_n(0, 1) = n + 1$. In 1899 Glaisher [3] showed that $N_n(1, 2)$ is always a power of 2, see Theorem A. In 1991 Davis and Webb [2] determined $N_n(t, 4)$ for $t = 0, 1, 2, 3$, see Theorem B. Their results show that $N_n(1, 4)$ is always a power of 2 and that $N_n(3, 4)$ is either 0 or a power of 2. These facts were also observed by Granville [4] in 1992. In addition Granville found for odd t that $N_n(t, 8)$ is either 0 or a power of 2. Unfortunately some of Granville's results on the distribution of the odd binomial coefficients modulo 8 in the nth row of Pascal's triangle are incorrect. We label Granville's five assertions following Figure 12 on p. 324 of [4] as (α), (β), (γ), (δ), and (ε). (Note that in the wording describing Figure 12 the assertion 'each $u_j \ge 2$' is not correct as the block of 0's in $(n)_2$ furthermost to the right may contain just one zero, for example, $n = 78$ has $(n)_2 = 1001110$.) We comment on each of (α), (γ), (δ), and (ε).

- (α) This assertion is false. Take $n = 3$ so that $(n)_2 = 11$. Thus $t_1 = 2$ and there are no other t_j's. Hence $n = 3$ falls under (α). However, the third row of Pascal's triangle is $1, 3, 3, 1$ contradicting the assertion of (α).
- (γ) This assertion is false. Take $n = 19$ so that $(n)_2 = 10011$. Thus $t_1 = 1$, $u_1 = 2$, and $t_2 = 2$. Hence $n = 19$ falls under (γ). However, the first half of the 19th row of Pascal's triangle modulo 8 is $1, 3, 3, 1, 4, 4, 4, 4, 6, 2$ so that $N_{19}(1, 8) = 4$, $N_{19}(7, 8) = 0$, contradicting Granville's claim that $N_{19}(1, 8) = N_{19}(7, 8)$.
- (δ) This assertion does not tell the full story. Take $n = 39$ so that $(n)_2 = 100111$. Thus $t_1 = 1$, $u_1 = 2$, $t_2 = 3$, and $n = 39$ falls under (δ). Here $N_{39}(t, 8) = 4$ $(t = 1, 3, 5, 7)$. On the other hand if $n = 156$ then $(n)_2 = 10011100$ so that $t_1 = 1$, $u_1 = 2$, $t_2 = 3$, $u_2 = 2$, and $n = 156$ falls under (δ). Here $N_{156}(1, 8) = N_{156}(3, 8) = 8$, $N_{156}(5, 8) = N_{156}(7, 8) = 0$.
- (ε) This assertion is false. If $n = 3699$ then $(n)_2 = 111001110011$ so that $t_1 = 3$, $u_1 = 2$, $t_2 = 3$, $u_2 = 2$, $t_3 = 2$, and $n = 3699$ falls under (ε). However $N_{3699}(1, 8) = N_{3699}(3, 8) = 128$, $N_{3699}(5, 8) = N_{3699}(7, 8) = 0$, contradicting Granville's claim that $N_{3699}(1, 8) = N_{3699}(3, 8) = N_{3699}(5, 8) = N_{3699}(7, 8)$.

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This article was motivated by the desire to find the correct evaluation of $N_n(t, 8)$ when t is odd (see Theorem C (third part)). In addition our method enables us to determine the value of $N_n(t, 8)$ when t is even, a problem not considered by Granville, see Theorem C (first and second parts). Our evaluation of $N_n(t, 8)$ involves the binary representation of n, namely

$$n = a_0 + a_1 2 + a_2 2^2 + \cdots + a_\ell 2^\ell,$$

where $\ell \geq 0$, each $a_i = 0$ or 1, and $a_\ell = 1$ unless $n = 0$ in which case $\ell = 0$ and $a_0 = 0$.

For brevity we write $a_0 a_1 \ldots a_\ell$ for the binary representation of n. Note that our notation is the reverse of Granville's notation [4]. On occasion it is more convenient to consider $a_0 a_1 \ldots a_\ell$ as a string of 0's and 1's. The context will make it clear which interpretation is being used. The length of the ith block of 0's (respectively 1's) in $a_0 a_1 \ldots a_\ell$ is denoted by v_i (respectively s_i). We consider n to begin with a block of 0's and to finish with a block of 1's. Thus, the binary representation of $n = 389743$ is 1111011001001111101 and $v_1 = 0$, $s_1 = 4$, $v_2 = 1$, $s_2 = 2$, $v_3 = 2$, $s_3 = 1$, $v_4 = 2$, $s_4 = 5$, $v_5 = 1$, $s_5 = 1$.

Throughout this article r denotes an arbitrary integer between 0 and n inclusive. The binary representation of r is (with additional zeros at the right-hand end if necessary) $r = b_0 b_1 \ldots b_\ell$. The exact power of 2 dividing the binomial coefficient $\binom{n}{r}$ is given by a special case of Kummer's theorem [5].

PROPOSITION 1 (KUMMER [5]). *Let $c(n, r)$ denote the number of carries when adding the binary representations of r and $n - r$. Then*

$$2^{c(n,r)} \,\|\, \binom{n}{r}.$$

Consider now the addition of the binary representation $b_0 b_1 \ldots b_\ell$ of r to that of $n - r$ to obtain the binary representation $a_0 a_1 \ldots a_\ell$ of n. If no carry occurs in the $(i - 1)$th position then there is no carry in the ith position if $b_i \leq a_i$, whereas there is a carry in the ith position if $b_i > a_i$. This simple observation enables us to say when $c(n, r) = 0$, 1 or 2.

PROPOSITION 2.

(a) $c(n, r) = 0 \Leftrightarrow b_i \leq a_i$ $(i = 0, 1, \ldots, \ell)$.
(b) $c(n, r) = 1$ *and the carry occurs in the* f*th position* $(0 \leq f \leq \ell - 1) \Leftrightarrow a_f a_{f+1} = 01$, $b_f b_{f+1} = 10$, *and* $b_i \leq a_i$ $(i \neq f, f + 1)$.
(c) $c(n, r) = 2$ *and the carries occur in the* f*th and* g*th positions* $(0 \leq f < g \leq \ell - 1)$

$$\begin{aligned} \Leftrightarrow\ & a_f a_{f+1} = 01, \quad b_f b_{f+1} = 10, \quad a_g a_{g+1} = 01, \quad b_g b_{g+1} = 10, \qquad \text{if } g \neq f+1, \\ & a_f a_{f+1} a_{f+2} = 011, \quad b_f b_{f+1} b_{f+2} = 110, \qquad \text{or} \\ & a_f a_{f+1} a_{f+2} = 001, \quad b_f b_{f+1} b_{f+2} = 1{*}0, \qquad \text{if } g = f+1, \end{aligned}$$

and

$$b_i \leq a_i \qquad (i \neq f, f+1, g, g+1).$$

($*$ denotes 0 *or* 1.*)*

If S denotes a nonempty string of 0's and 1's, we denote by n_S the number of occurrences of S in the string $a_0 a_1 \ldots a_\ell$. For example, if $n = 1496 = 00011011101$ then $n_0 = 5$, $n_1 = 6$, $n_{00} = 2$, $n_{01} = 3$, $n_{10} = 2$, $n_{11} = 3$, $n_{000} = 1$, $n_{001} = 1$.

Propositions 1 and 2(a) enable us to prove Glaisher's formulae [3].

THEOREM A (GLAISHER [3]). $N_n(0,2) = n + 1 - 2^{n_1}$, $N_n(1,2) = 2^{n_1}$.

PROOF. We have

$$N_n(1,2) = \sum_{\substack{r=0 \\ \binom{n}{r}\equiv 1 \pmod 2}}^{n} 1 = \sum_{\substack{r=0 \\ c(n,r)=0}}^{n} 1 = \sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell} 1 = (a_0+1)\cdots(a_\ell+1) = 2^{n_1}.$$

The formula for $N_n(0,2)$ now follows from (1.1) with $d = 1$ and $m = 2$. □

Similarly we can prove Davis and Webb's formulae [2].

THEOREM B (FIRST PART, DAVIS AND WEBB [2]).

$$N_n(0,4) = n + 1 - 2^{n_1} - n_{01}2^{n_1-1}, \quad N_n(2,4) = n_{01}2^{n_1-1}.$$

PROOF. Appealing to Propositions 1 and 2(b), we have

$$\begin{aligned} N_n(2,4) &= \sum_{\substack{r=0 \\ \binom{n}{r}\equiv 2 \pmod 4}}^{n} 1 = \sum_{f=0}^{\ell-1} \sum_{\substack{r=0 \\ c(n,r)=1 \\ \text{carry in } f\text{th place}}}^{n} 1 = \sum_{\substack{f=0 \\ a_f a_{f+1}=01}}^{\ell-1} \sum_{\substack{b_0,\dots,b_{f-1},b_{f+2},\dots,b_\ell=0 \\ b_f b_{f+1}=10}}^{a_0,\dots,a_{f-1},a_{f+2},\dots,a_\ell} 1 \\ &= \sum_{\substack{f=0 \\ a_f a_{f+1}=01}}^{\ell-1} \prod_{\substack{j=0 \\ j\neq f,f+1}}^{\ell} (a_j+1) = \sum_{\substack{f=0 \\ a_f a_{f+1}=01}}^{\ell-1} 2^{n_1-1} = n_{01}2^{n_1-1}. \end{aligned}$$

From (1.1) with $m = 4$, $d = 2$, and $t = 0$, we have $N_n(0,4) + N_n(2,4) = N_n(0,2)$, from which the value of $N_n(0,4)$ follows. □

Likewise we can use Propositions 1 and 2(c) to determine $N_n(0,8)$ and $N_n(4,8)$.

THEOREM C (FIRST PART).

$$\begin{aligned} &N_n(0,8) = n + 1 - (n_{001}+1)2^{n_1} - n_{011}2^{n_1-2} - n_{01}(n_{01}+3)2^{n_1-3}, \\ &N_n(4,8) = n_{001}2^{n_1} + n_{011}2^{n_1-2} + n_{01}(n_{01}-1)2^{n_1-3}. \end{aligned}$$

PROOF. We have

$$
\begin{aligned}
N_n(4,8) &= \sum_{\substack{r=0\\ \binom{n}{r}\equiv 4 \pmod 8}}^{n} 1 = \sum_{\substack{r=0\\ c(n,r)=2}}^{n} 1\\
&= \sum_{\substack{f=0\\ a_fa_{f+1}a_{f+2}=011}}^{\ell-2} \sum_{\substack{b_0,\dots,b_{f-1},b_{f+3},\dots,b_\ell=0\\ b_fb_{f+1}b_{f+2}=110}}^{a_0,\dots,a_{f-1},a_{f+3},\dots,a_\ell} 1 + \sum_{\substack{f=0\\ a_fa_{f+1}a_{f+2}=001}}^{\ell-2} \sum_{\substack{b_0,\dots,b_{f-1},b_{f+3},\dots,b_\ell=0\\ b_fb_{f+1}b_{f+2}=1*0}}^{a_0,\dots,a_{f-1},a_{f+3},\dots,a_\ell} 1\\
&\quad + \sum_{\substack{f=0\\ a_fa_{f+1}=01}}^{\ell-3} \sum_{\substack{g=f+2\\ a_ga_{g+1}=01}}^{\ell-1} \sum_{\substack{b_0,\dots,b_{f-1},b_{f+2},\dots,b_{g-1},b_{g+2},\dots,b_\ell=0\\ b_fb_{f+1}=b_gb_{g+1}=10}}^{a_0,\dots,a_{f-1},a_{f+2},\dots,a_{g-1},a_{g+2},\dots,a_\ell} 1\\
&= \sum_{\substack{f=0\\ a_fa_{f+1}a_{f+2}=011}}^{\ell-2} \prod_{\substack{j=0\\ j\neq f,f+1,f+2}}^{\ell} (a_j+1) + 2 \sum_{\substack{f=0\\ a_fa_{f+1}a_{f+2}=001}}^{\ell-2} \prod_{\substack{j=0\\ j\neq f,f+1,f+2}}^{\ell} (a_j+1)\\
&\quad + \sum_{\substack{f=0\\ a_fa_{f+1}=01}}^{\ell-3} \sum_{\substack{g=f+2\\ a_ga_{g+1}=01}}^{\ell-1} \prod_{\substack{j=0\\ j\neq f,f+1,g,g+1}}^{\ell} (a_j+1)\\
&= \sum_{\substack{f=0\\ a_fa_{f+1}a_{f+2}=011}}^{\ell-2} 2^{n_1-2} + 2 \sum_{\substack{f=0\\ a_fa_{f+1}a_{f+2}=001}}^{\ell-2} 2^{n_1-1} + \sum_{\substack{f=0\\ a_fa_{f+1}=01}}^{\ell-3} \sum_{\substack{g=f+2\\ a_ga_{g+1}=01}}^{\ell-1} 2^{n_1-2}\\
&= n_{011}2^{n_1-2} + n_{001}2^{n_1} + \frac{n_{01}(n_{01}-1)}{2}2^{n_1-2}.
\end{aligned}
$$

The value of $N_n(0,8)$ follows from (1.1) with $m = 8$, $d = 4$, and $t = 0$. □

Although Kummer's result (Proposition 1) enabled us to determine $N_n(1,2)$, $N_n(2,4)$, and $N_n(4,8)$, it is clear that we need a more precise congruence for $\binom{n}{r}$ to be able to determine $N_n(t,4)$ for $t = 1,3$ and $N_n(t,8)$ for $t = 1,2,3,5,6,7$. The required congruences for $\binom{n}{r}$ modulo 4 and modulo 8 are provided by the Davis–Webb congruence, which is the subject of the next section.

It is understood throughout that an empty sum has the value 0, an empty product the value 1, and

$$0^n = \begin{cases} 1, & \text{if } n = 0,\\ 0, & \text{if } n \geq 1.\end{cases}$$

2. THE DAVIS–WEBB CONGRUENCE

In order to state the Davis–Webb congruence for $\binom{n}{r}$ modulo 2^h, we need the binary version of the symbol $\left\langle {c \atop d} \right\rangle$ defined by Davis and Webb [1] for arbitrary nonnegative integers $c = c_0c_1\dots c_s$ and $d = d_0d_1\dots d_s$ (where additional zeros have been included at the right-hand of either c or d, if necessary, to make their binary representations the same length). If $c_0c_1\dots c_i < d_0d_1\dots d_i$ for $i = 0,1,\dots,s$, we set

$$\left\langle {c \atop d} \right\rangle = 2^{s+1}.$$

Otherwise we let u denote the largest integer between 0 and s inclusive for which $c_0c_1\dots c_u \geq d_0d_1\dots d_u$, and set

$$\left\langle {c \atop d} \right\rangle = 2^{s-u}\binom{c_0c_1\dots c_u}{d_0d_1\dots d_u}.$$

TABLE 1.
Values of $\left[\begin{smallmatrix} c_0 \\ d_0 \end{smallmatrix}\right]$.

	c_0	
d_0	0	1
0	1	1
1	1	1

TABLE 2.
Values of $\left[\begin{smallmatrix} c_0 & c_1 \\ d_0 & d_1 \end{smallmatrix}\right]$.

	c_0c_1			
d_0d_1	00	10	01	11
00	1	1	1	1
10	1	1	1	3
01	1	1	1	3
11	1	1	1	1

Thus, for example, if $c = 26 = 010110$ and $d = 39 = 111001$, we have $s = 5$ and $u = 4$ so that

$$\left\langle \begin{matrix} 26 \\ 39 \end{matrix} \right\rangle = 2^{5-4} \binom{01011}{11100} = 2\binom{26}{7}.$$

The symbol $\left\langle\begin{smallmatrix} c \\ d \end{smallmatrix}\right\rangle$ is an extension of the ordinary binomial coefficient since for $0 \le d \le c$ we have $u = s$ and so $\left\langle\begin{smallmatrix} c \\ d \end{smallmatrix}\right\rangle = \binom{c}{d}$. The odd part of $\left\langle\begin{smallmatrix} c \\ d \end{smallmatrix}\right\rangle$ is denoted by $\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$. The values of $\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ for $s = 0$, 1, and 2 are given in Tables 1–3 respectively.

For our purposes it is also convenient to set for $s \ge 1$

$$\begin{bmatrix} c \\ d \end{bmatrix}' = \text{odd part of } \frac{\left\langle \begin{matrix} c_0c_1 \ldots c_s \\ d_0d_1 \ldots d_s \end{matrix} \right\rangle}{\left\langle \begin{matrix} c_0c_1 \ldots c_{s-1} \\ d_0d_1 \ldots d_{s-1} \end{matrix} \right\rangle}.$$

The values of $\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]'$ for $s = 1$ and 2 are given in Tables 4 and 5 respectively. From Tables 1–5 we obtain the assertions of Lemma 1.

LEMMA 1.

(a) $\begin{bmatrix} c_0 \\ d_0 \end{bmatrix} = 1.$

(b) $\begin{bmatrix} c_0 & c_1 \\ d_0 & d_1 \end{bmatrix} \equiv \begin{cases} 1 \pmod 4, & \textit{if } c_0c_1 \ne 11, \\ (-1)^{d_0+d_1} \pmod 4, & \textit{if } c_0c_1 = 11. \end{cases}$

(c) $\begin{bmatrix} c_0 & c_1 \\ d_0 & d_1 \end{bmatrix} \equiv \begin{cases} 1 \pmod 8, & \textit{if } c_0c_1 \ne 11, \\ (-1)^{d_0+d_1}5^{d_0+d_1} \pmod 8, & \textit{if } c_0c_1 = 11. \end{cases}$

(d) $\begin{bmatrix} c_0 & c_1 \\ d_0 & d_1 \end{bmatrix}' \equiv \begin{cases} 1 \pmod 4, & \textit{if } c_0c_1 \ne 11, \\ (-1)^{d_0+d_1} \pmod 4, & \textit{if } c_0c_1 = 11. \end{cases}$

(e) $\begin{bmatrix} c_0 & c_1 & 0 \\ d_0 & d_1 & d_2 \end{bmatrix}' = 1.$

TABLE 3.
Values of $\begin{bmatrix} c_0c_1c_2 \\ d_0d_1d_2 \end{bmatrix}$.

$d_0d_1d_2$	$c_0c_1c_2$: 000	100	010	110	001	101	011	111
000	1	1	1	1	1	1	1	1
100	1	1	1	3	1	5	3	7
010	1	1	1	3	3	5	15	21
110	1	1	1	1	1	5	5	35
001	1	1	1	1	1	5	15	35
101	1	1	1	3	1	1	3	21
011	1	1	1	3	1	1	1	7
111	1	1	1	1	1	1	1	1

TABLE 4.
Values of $\begin{bmatrix} c_0c_1 \\ d_0d_1 \end{bmatrix}'$.

d_0d_1	c_0c_1: 00	10	01	11
00	1	1	1	1
10	1	1	1	3
01	1	1	1	3
11	1	1	1	1

(f) $\begin{bmatrix} c_0 & 0 & 1 \\ d_0 & 1 & 0 \end{bmatrix}' \equiv (-1)^{1+c_0} \pmod 4$, *if* $c_0 \geq d_0$.

(g) $\begin{bmatrix} 0 & 1 & c_2 \\ 1 & 0 & d_2 \end{bmatrix}' \equiv (-1)^{c_2} \pmod 4$.

(h) $\begin{bmatrix} c_0 & 1 & 1 \\ d_0 & d_1 & d_2 \end{bmatrix}' \equiv (-1)^{d_1+d_2} \pmod 4$, *if* $c_0 \geq d_0$.

(i) $\begin{bmatrix} c_0 & c_1 & c_2 \\ d_0 & d_1 & d_2 \end{bmatrix}' \equiv 1 \pmod 4$, *if* $c_1c_2 \neq 11$ *and* $c_1 \geq d_1$.

(j) $\begin{bmatrix} 1 & 0 & 1 \\ d_0 & 0 & d_2 \end{bmatrix}' \equiv 5^{d_0+d_2} \pmod 8$.

(k) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & d_1 & d_2 \end{bmatrix}' \equiv (-1)^{d_1+d_2} \pmod 8$.

(l) $\begin{bmatrix} 1 & 1 & 1 \\ d_0 & d_1 & d_2 \end{bmatrix}' \equiv (-1)^{d_1+d_2}5^{d_0+d_2} \pmod 8$.

(m) $\begin{bmatrix} c_0 & c_1 & c_2 \\ d_0 & d_1 & d_2 \end{bmatrix}' \equiv 1 \pmod 8$, *if* $c_0c_1c_1 \neq 101, 011, 111$, *and* $c_i \geq d_i$ $(i = 0, 1, 2)$.

Let h be an integer with $h \geq 2$. When $\ell \geq h - 1$, Davis and Webb [1] have given a congruence for $\binom{n}{r} \pmod{p^h}$ for any prime p. (Granville's Proposition 2 in [4] is the special case of Davis and Webb's congruence when $p \nmid \binom{n}{r}$.) When $p = 2$ their congruence can be expressed using Proposition 1 in the form:

DAVIS–WEBB CONGRUENCE $\pmod{2^h}$. *For* $2 \leq h \leq \ell + 1$

$$\binom{n}{r} \equiv 2^{c(n,r)} \begin{bmatrix} a_0a_1 \dots a_{h-2} \\ b_0b_1 \dots b_{h-2} \end{bmatrix} \prod_{i=0}^{\ell-h+1} \begin{bmatrix} a_ia_{i+1} \dots a_{i+h-1} \\ b_ib_{i+1} \dots b_{i+h-1} \end{bmatrix}' \pmod{2^h}. \quad (2.1)$$

TABLE 5.
Values of $\begin{bmatrix} c_0c_1c_2 \\ d_0d_1d_2 \end{bmatrix}'$.

	$c_0c_1c_2$							
$d_0d_1d_2$	000	100	010	110	001	101	011	111
000	1	1	1	1	1	1	1	1
100	1	1	1	1	1	5	3	7/3
010	1	1	1	1	3	5	15	7
110	1	1	1	1	1	5	5	35
001	1	1	1	1	1	5	15	35
101	1	1	1	1	1	1	3	7
011	1	1	1	1	1	1	1	7/3
111	1	1	1	1	1	1	1	1

Our next task is to make the Davis–Webb congruence (mod 2^h) explicit in certain cases when $h = 2$ and $h = 3$ by means of Lemma 1. It is convenient to set

$$E_1 = \sum_{\substack{i=0 \\ a_i a_{i+1}=11}}^{\ell-1} (b_i + b_{i+1}), \quad E_2 = \sum_{\substack{i=0 \\ a_i a_{i+2}=11}}^{\ell-2} (b_i + b_{i+2}).$$

For an integer f with $0 \le f \le \ell - 1$ we also set

$$H_f = \sum_{\substack{i=0 \\ i \ne f-1,\, f,\, f+1 \\ a_i a_{i+1}=11}}^{\ell-1} (b_i + b_{i+1}).$$

DAVIS–WEBB CONGRUENCE (mod 4). *For $\ell \ge 1$ and $c(n, r) = 0$, we have*

$$\binom{n}{r} \equiv (-1)^{E_1} \pmod 4.$$

PROOF. Taking $h = 2$ and $c(n, r) = 0$ in (2.1), we obtain for $\ell \ge 1$

$$\binom{n}{r} \equiv \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \prod_{i=0}^{\ell-1} \begin{bmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{bmatrix}' \pmod 4.$$

Appealing to Lemma 1(a)(d), we obtain

$$\binom{n}{r} \equiv \prod_{\substack{i=0 \\ a_i a_{i+1}=11}}^{\ell-1} (-1)^{b_i + b_{i+1}} \equiv (-1)^{\sum_{\substack{i=0 \\ a_i a_{i+1}=11}}^{\ell-1} (b_i + b_{i+1})} \equiv (-1)^{E_1} \pmod 4.$$

□

DAVIS–WEBB CONGRUENCE (mod 8). *(a) For $\ell \ge 2$ and $c(n, r) = 0$, we have*

$$\binom{n}{r} \equiv (-1)^{E_1} 5^{E_2} \pmod 8, \qquad \text{if } a_0 a_1 \ne 11,$$
$$\binom{n}{r} \equiv (-1)^{E_1} 5^{b_0 + b_1 + E_2} \pmod 8, \qquad \text{if } a_0 a_1 = 11.$$

(b) For $\ell \geq 2$ and $c(n, r) = 1$, we have

$$\binom{n}{r} \equiv 2(-1)^{1+a_{f-1}+a_{f+2}+H_f} \pmod 8,$$

where f $(0 \leq f \leq \ell - 1)$ is the position of the carry when adding the binary representations of r and $n - r$, and

$$a_{-1} = -1, \qquad a_{\ell+1} = 0.$$

PROOF. (a) Taking $h = 3$ and $c(n, r) = 0$ in (2.1), we obtain for $\ell \geq 2$

$$\binom{n}{r} \equiv \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix} \prod_{i=0}^{\ell-2} \begin{bmatrix} a_i & a_{i+1} & a_{i+2} \\ b_i & b_{i+1} & b_{i+2} \end{bmatrix}' \pmod 8.$$

By Proposition 2(a) we have $b_i \leq a_i$ for $i = 0, \ldots, \ell$. Appealing to Lemma 1(j)(k)(l)(m), we obtain mod 8:

$$\begin{aligned} \binom{n}{r} &\equiv \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix} \prod_{\substack{i=0 \\ a_i a_{i+1} a_{i+2} = 101}}^{\ell-2} 5^{b_i+b_{i+2}} \prod_{\substack{i=0 \\ a_i a_{i+1} a_{i+2} = 011}}^{\ell-2} (-1)^{b_{i+1}+b_{i+2}} \\ &\quad \times \prod_{\substack{i=0 \\ a_i a_{i+1} a_{i+2} = 111}}^{\ell-2} (-1)^{b_{i+1}+b_{i+2}} 5^{b_i+b_{i+2}} \\ &\equiv \begin{bmatrix} a_0 & a_1 \\ b_0 & b_i \end{bmatrix} \prod_{\substack{i=0 \\ a_i a_{i+2} = 11}}^{\ell-2} 5^{b_i+b_{i+2}} \prod_{\substack{i=0 \\ a_{i+1} a_{i+2} = 11}}^{\ell-2} (-1)^{b_{i+1}+b_{i+2}} \\ &\equiv \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix} \prod_{\substack{i=0 \\ a_i a_{i+2} = 11}}^{\ell-2} 5^{b_i+b_{i+2}} \prod_{\substack{i=1 \\ a_i a_{i+1} = 11}}^{\ell-1} (-1)^{b_i+b_{i+1}} \\ &\equiv \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix} 5^{\sum\limits_{\substack{i=0 \\ a_i a_{i+2} = 11}}^{\ell-2} (b_i+b_{i+2})} (-1)^{\sum\limits_{\substack{i=1 \\ a_i a_{i+1} = 11}}^{\ell-1} (b_i+b_{i+1})}. \end{aligned}$$

If $a_0 a_1 \neq 11$, then, by Lemma 1(c), we have $\binom{n}{r} \equiv (-1)^{E_1} 5^{E_2} \pmod 8$. If $a_0 a_1 = 11$ then, by Lemma 1(c), we have

$$\binom{n}{r} \equiv (-1)^{b_0+b_1} 5^{b_0+b_1} 5^{E_2} (-1)^{E_1-(b_0+b_1)} \equiv (-1)^{E_1} 5^{b_0+b_1+E_2} \pmod 8.$$

(b) Taking $h = 3$ and $c(n, r) = 1$ in (2.1), we obtain for $\ell \geq 2$

$$\binom{n}{r} \equiv 2 \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix} \prod_{i=0}^{\ell-2} \begin{bmatrix} a_i & a_{i+1} & a_{i+2} \\ b_i & b_{i+1} & b_{i+2} \end{bmatrix}' \pmod 8.$$

We let f $(0 \leq f \leq \ell - 1)$ be the position of the carry so that by Proposition 2(b)

$a_f a_{f+1} = 01$, $b_f b_{f+1} = 10$, and $a_k \geq b_k$ for $k \neq f,\ f+1$. We have, by Lemma 1(h)(i)(l),

$$\begin{aligned}
&\begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix} \prod_{\substack{i=0 \\ i\neq f-2,\, f-1,\, f}}^{\ell-2} \begin{bmatrix} a_i & a_{i+1} & a_{i+2} \\ b_i & b_{i+1} & b_{i+2} \end{bmatrix}' \\
&\equiv \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix} \prod_{\substack{i=1 \\ i\neq f-1,\, f,\, f+1}}^{\ell-1} \begin{bmatrix} a_{i-1} & a_i & a_{i+1} \\ b_{i-1} & b_i & b_{i+1} \end{bmatrix}' \\
&\equiv \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix} (-1)^{\sum_{\substack{i=1 \\ i\neq f-1,\, f,\, f+1 \\ a_i a_{i+1}=11}}^{\ell-1} (b_i + b_{i+1})} \\
&\equiv (-1)^{\sum_{\substack{i=0 \\ i\neq f-1,\, f,\, f+1 \\ a_i a_{i+1}=11}}^{\ell-1} (b_i + b_{i+1})} \equiv (-1)^{H_f} \pmod 4.
\end{aligned}$$

We now consider four cases: (i) $f = 0$; (ii) $f = 1$; (iii) $2 \leq f \leq \ell - 2$; and (iv) $f = \ell - 1$. In each case we must determine

$$P = \prod_{\substack{i=0 \\ i=f-2,\, f-1,\, f}}^{\ell-2} \begin{bmatrix} a_i & a_{i+1} & a_{i+2} \\ b_i & b_{i+1} & b_{i+2} \end{bmatrix}' \pmod 4.$$

Case (i). $f = 0$. In this case we have

$$P = \begin{bmatrix} 0 & 1 & a_2 \\ 1 & 0 & b_2 \end{bmatrix}' \equiv (-1)^{a_2} \pmod 4,$$

by Lemma 1(g), so that

$$\binom{n}{r} \equiv 2(-1)^{H_f}(-1)^{a_2} \equiv 2(-1)^{1+a_{f-1}+a_{f+2}+H_f} \pmod 8.$$

Case (ii). $f = 1$. Here

$$P = \begin{bmatrix} a_0 & 0 & 1 \\ b_0 & 1 & 0 \end{bmatrix}' \begin{bmatrix} 0 & 1 & a_3 \\ 1 & 0 & b_3 \end{bmatrix}' \equiv (-1)^{1+a_0}(-1)^{a_3} \equiv (-1)^{1+a_0+a_3} \pmod 4,$$

by Lemma 1(f)(g), so that

$$\binom{n}{r} \equiv 2(-1)^{H_f}(-1)^{1+a_0+a_3} \equiv 2(-1)^{1+a_{f-1}+a_{f+2}+H_f} \pmod 8.$$

Case (iii). $2 \leq f \leq \ell - 2$. Here

$$P = \begin{bmatrix} a_{f-2} & a_{f-1} & 0 \\ b_{f-2} & b_{f-1} & 1 \end{bmatrix}' \begin{bmatrix} a_{f-1} & 0 & 1 \\ b_{f-1} & 1 & 0 \end{bmatrix}' \begin{bmatrix} 0 & 1 & a_{f+2} \\ 1 & 0 & b_{f+2} \end{bmatrix}' \equiv (-1)^{1+a_{f-1}+a_{f+2}} \pmod 4,$$

by Lemma 1(e)(f)(g), so that

$$\binom{n}{r} \equiv 2(-1)^{H_f}(-1)^{1+a_{f-1}+a_{f+2}} \equiv 2(-1)^{1+a_{f-1}+a_{f+2}+H_f} \pmod 8.$$

Case (iv). $f = \ell - 1$. Here

$$P = \begin{bmatrix} a_{\ell-3} & a_{\ell-2} & 0 \\ b_{\ell-3} & b_{\ell-2} & 1 \end{bmatrix}' \begin{bmatrix} a_{\ell-2} & 0 & 1 \\ b_{\ell-2} & 1 & 0 \end{bmatrix}' \equiv (-1)^{1+a_{\ell-2}} \pmod 4,$$

by Lemma 1(e)(f), so that

$$\binom{n}{r} \equiv 2(-1)^{H_f}(-1)^{1+a_{\ell-2}} \equiv 2(-1)^{1+a_{f-1}+a_{f+2}+H_f} \pmod 8.$$

□

Our final task in this section is to give the mechanism whereby we can count the number of integers r $(0 \le r \le n)$ for which $\binom{n}{r}$ is in a particular residue class (mod 4) or (mod 8). This mechanism is provided by the next lemma.

LEMMA 2. *Let $c_0, \ldots, c_\ell$ be integers. Then*

$$\sum_{b_0,\ldots,b_\ell=0}^{a_0,\ldots,a_\ell} (-1)^{\sum_{i=0}^{\ell} c_i b_i} = \begin{cases} 2^{n_1}, & \text{if } c_i \equiv 0 \pmod 2 \text{ for each } i = 0, 1, \ldots, \ell \text{ with } a_i = 1, \\ 0, & \text{if } c_i \equiv 1 \pmod 2 \text{ for some } i\ (0 \le i \le \ell) \text{ with } a_i = 1. \end{cases}$$

PROOF. We have

$$\sum_{b_0,\ldots,b_\ell=0}^{a_0,\ldots,a_\ell} (-1)^{\sum_{i=0}^{\ell} c_i b_i} = \prod_{i=0}^{\ell}\left(\sum_{b_i=0}^{a_i}(-1)^{c_i b_i}\right) = \prod_{\substack{i=0 \\ a_i=1}}^{\ell}\left(\sum_{b_i=0}^{1}(-1)^{c_i b_i}\right) = \prod_{\substack{i=0 \\ a_i=1}}^{\ell}(1+(-1)^{c_i})$$

$$= \begin{cases} \displaystyle\prod_{\substack{i=0 \\ a_i=1}}^{\ell} 2, & \text{if } c_i \equiv 0 \pmod 2 \text{ for each } i \text{ with } a_i = 1, \\ 0, & \text{if } c_i \equiv 1 \pmod 2 \text{ for some } i \text{ with } a_i = 1. \end{cases}$$

□

In applying Lemma 2 in the evaluation of $N_n(t, 4)$ $(t = 1, 3)$ and $N_n(t, 8)$ $(t = 1, 2, 3, 5, 6, 7)$ a number of finite sums involving E_1 and E_2 arise. These sums are evaluated in Lemmas 3–7.

LEMMA 3.

$$\sum_{b_0,\ldots,b_\ell=0}^{a_0,\ldots,a_\ell} (-1)^{E_1} = 0^{n_{11}} 2^{n_1}.$$

PROOF. Set $S = \sum_{b_0,\ldots,b_\ell=0}^{a_0,\ldots,a_\ell} (-1)^{E_1}$. For $j = 0, 1, \ldots, \ell$ let c_j denote the number of occurrences of b_j in

$$E_1 = \sum_{\substack{i=0 \\ a_i a_{i+1}=11}}^{\ell-1} (b_i + b_{i+1}),$$

so that $S = \sum_{b_0,\ldots,b_\ell=0}^{a_0,\ldots,a_\ell} (-1)^{\sum_{i=0}^{\ell} c_i b_i}$. If $n_{11} = 0$ then $c_j = 0$ $(0 \le j \le \ell)$ so $S = 2^{n_1}$ by Lemma 2. If $n_{11} > 0$ let u be the least integer $(0 \le u \le \ell - 1)$ such that $a_u a_{u+1} = 11$. Then $c_u = 1$ and $S = 0$ by Lemma 2. □

LEMMA 4.

$$\sum_{b_0,\ldots,b_\ell=0}^{a_0,\ldots,a_\ell} (-1)^{E_2} = \begin{cases} 2^{n_1}, & \text{if } n_{101} = n_{111} = 0, \\ 0, & \text{if } n_{101} > 0 \text{ or } n_{111} > 0. \end{cases}$$

PROOF. Set $S = \sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell}(-1)^{E_2}$. For $j = 0, 1, \dots, \ell$ let c_j denote the number of occurrences of b_j in

$$E_2 = \sum_{\substack{i=0 \\ a_i a_{i+2}=11}}^{\ell-2} (b_i + b_{i+2}).$$

If $n_{101} = n_{111} = 0$ then $c_j = 0$ $(0 \le j \le \ell)$ so, by Lemma 2, we have $S = 2^{n_1}$. If $n_{101} > 0$ or $n_{111} > 0$ let s be the least integer such that $a_s a_{s+2} = 11$ $(0 \le s \le \ell - 2)$. Then $c_s = 1$ and $S = 0$ by Lemma 2. □

Before stating the next lemma, we remind the reader that the length of the ith block of 0's in $a_0 a_1 \dots a_\ell$ is denoted by v_i and the length of the ith block of 1's by s_i. We consider $a_0 a_1 \dots a_\ell$ to start with a block of 0's and finish with a block of 1's.

LEMMA 5. *For* $n \ge 1$

$$\sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell} (-1)^{E_1+E_2} = \begin{cases} 0, & \text{if } n_{101} > 0 \text{ or } n_{1111} > 0, \\ 0, & \text{if } n_{101} = n_{1111} = 0 \text{ and some } s_i = 2, \\ 2^{n_1}, & \text{if } n_{101} = n_{1111} = 0 \text{ and each } s_i = 1 \text{ or } 3. \end{cases}$$

PROOF. The lemma is easily checked for $\ell = 0, 1, 2$ so we may suppose that $\ell \ge 3$. Let $S = \sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell}(-1)^{E_1+E_2}$. For $j = 0, 1, \dots, \ell$ let c_j denote the number of occurrences of b_j in

$$E_1 + E_2 = \sum_{\substack{i=0 \\ a_i a_{i+1}=11}}^{\ell-1} (b_i + b_{i+1}) + \sum_{\substack{i=0 \\ a_i a_{i+1} a_{i+2}=101}}^{\ell-2} (b_i + b_{i+2}) + \sum_{\substack{i=0 \\ a_i a_{i+1} a_{i+2}=111}}^{\ell-2} (b_i + b_{i+2}).$$

Suppose first that $n_{101} = n_{1111} = 0$ and some $s_i = 2$, where $i \ge 1$. Hence there exists an integer u $(0 \le u \le \ell - 1)$ such that $a_u a_{u+1} = 11$, $a_{u-1} = 0$ if $u \ge 1$, and $a_{u+2} = 0$ if $u \le \ell - 2$. Let u be the least such integer. Then $c_u = 1$ and $S = 0$ by Lemma 2.

Suppose next that $n_{101} = n_{1111} = 0$ and each $s_i = 1$ or 3. Let j $(0 \le j \le \ell)$ be an integer such that $a_j = 1$. If $j = 0$ and $a_1 = 0$, then $a_2 = 0$ and $c_j = 0$. If $j = 0$ and $a_1 = 1$, then $a_2 = 1$ and $c_j = 2$. If $j = \ell$ and $a_{\ell-1} = 0$, then $a_{\ell-2} = 0$ and $c_j = 0$. If $j = \ell$ and $a_{\ell-1} = 1$, then $a_{\ell-2} = 1$ and $c_j = 2$. Now suppose $1 \le j \le \ell - 1$. If $a_{j-1} a_j a_{j+1} = 010$ then $c_j = 0$. If $a_{j-1} a_j a_{j+1} = 110$ then $j \ge 2$ and $a_{j-2} a_{j-1} a_j a_{j+1} = 1110$ so that $c_j = 2$. If $a_{j-1} a_j a_{j+1} = 011$ then $j \le \ell - 2$ and $a_{j-1} a_j a_{j+1} a_{j+2} = 0111$ so that $c_j = 2$. If $a_{j-1} a_j a_{j+1} = 111$ then $c_j = 2$. Hence c_j is even for every j with $a_j = 1$. Thus, by Lemma 2, we have $S = 2^{n_1}$.

Now suppose that $n_{101} > 0$. Let s be the least integer such that $a_s a_{s+1} a_{s+2} = 101$. If $s = 0$ then $c_s = 1$. If $s \ge 1$ and $a_{s-1} = 0$ then $c_s = 1$. If $s = 1$ and $a_0 = 1$ then $c_0 = 1$. If $s \ge 2$, $a_{s-1} = 1$, and $a_{s-2} = 0$ then $c_{s-1} = 1$. If $s \ge 2$, $a_{s-1} = 1$, and $a_{s-2} = 1$ then $c_s = 3$. Hence, by Lemma 2, we have $S = 0$.

Finally suppose that $n_{1111} > 0$. Let w be the least integer such that $a_w a_{w+1} a_{w+2} a_{w+3} = 1111$. Then $c_{w+1} = 3$ and, by Lemma 2, we have $S = 0$. □

LEMMA 6. *If* $a_0 a_1 = 1$ *then*

$$\sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell} (-1)^{b_0+b_1+E_2} = 0.$$

PROOF. Set $S = \sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell}(-1)^{b_0+b_1+E_2}$. Let c_j $(0 \le j \le \ell)$ denote the number of occurrences of b_j in

$$b_0 + b_1 + E_2 = b_0 + b_1 + \sum_{\substack{i=0 \\ a_i a_{i+2}=11}}^{\ell-2} (b_i + b_{i+2}).$$

Let $k (1 \le k \le \ell)$ be the largest integer such that $a_0 a_1 \dots a_k = 11 \dots 1$. Then $c_{k-1} = 1$. Hence, by Lemma 2, $S = 0$. □

LEMMA 7. *If* $a_0 a_1 = 11$ *then*

$$\sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell} (-1)^{b_0+b_1+E_1+E_2} = \begin{cases} 0, & \textit{if } n_{101} > 0 \textit{ or } n_{1111} > 0, \\ 0, & \textit{if } n_{101} = n_{1111} = 0 \textit{ and some } s_i = 2 \textit{ with } i \ge 2, \\ 0, & \textit{if } n_{101} = n_{1111} = 0 \textit{ and each } s_i = 1 \textit{ or } 3, \\ 2^{n_1}, & \textit{if } n_{101} = n_{1111} = 0, s_1 = 2, \\ & \textit{and each } s_i = 1 \textit{ or } 3 \textit{ with } i \ge 2. \end{cases}$$

PROOF. Set $S = \sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell}(-1)^{b_0+b_1+E_1+E_2}$. For $j = 0, 1, \dots, \ell$, let c_j denote the number of occurrences of b_j in

$$b_0 + b_1 + \sum_{\substack{i=0 \\ a_i a_{i+1}=11}}^{\ell-1} (b_i + b_{i+1}) + \sum_{\substack{i=0 \\ a_i a_{i+2}=11}}^{\ell-2} (b_i + b_{i+2}).$$

Suppose first that $n_{101} > 0$. Let s $(0 \le s \le \ell-2)$ be the least integer such that $a_s a_{s+1} a_{s+2} = 101$. As $a_0 a_1 = 11$ we have $s \ge 1$. If $s = 1$ then $a_{s-1} = a_0 = 1$ and $c_1 = 3$. If $s \ge 2$ and $a_{s-1} = 0$, then $a_{s-2} = 0$ and $s \ge 4$, so that $c_s = 1$. If $s \ge 2$ and $a_{s-2} = a_{s-1} = 1$ then $c_s = 3$. If $s \ge 2$ and $a_{s-2} = 0$, $a_{s-1} = 1$, then $s \ge 3$ and $a_{s-3} = 0$, so that $c_{s-1} = 1$. Hence, by Lemma 2, $S = 0$.

Suppose next that $n_{1111} > 0$. Let s $(0 \le s \le \ell - 3)$ be the least integer such that $a_s a_{s+1} a_{s+2} a_{s+3} = 1111$. If $s = 0$ then $c_0 = 3$. If $s \ge 1$ then $a_{s-1} = 0$ and $c_{s+1} = 3$. Hence, by Lemma 2, $S = 0$.

Now suppose that $n_{101} = n_{1111} = 0$ and some $s_i = 2$ with $i \ge 2$, say $a_s a_{s+1} = 11$. As $a_0 a_1 = 11$ and $n_{101} = 0$ we have $s \ge 4$. Clearly $a_{s-2} a_{s-1} = 00$. Hence $c_s = 1$, and, by Lemma 2, we have $S = 0$.

Next suppose that $n_{101} = n_{1111} = 0$ and each $s_i = 1$ or 3. Then, as $a_0 a_1 = 11$, we must have $a_2 = 1$. Thus $c_0 = 3$ and, by Lemma 2, we have $S = 0$.

Finally suppose that $n_{101} = n_{1111} = 0$, $s_1 = 2$, and each $s_i (i \ge 2) = 1$ or 3. Clearly $c_0 = c_1 = 2$, $c_i = 0$ if $a_{i-1} a_i a_{i+1} = 010$ $(4 \le i \le \ell - 1)$, $c_\ell = 0$ if $a_{\ell-1} a_\ell = 01$, and $c_{i-1} = c_i = c_{i+1} = 2$ if $a_{i-1} a_i a_{i+1} = 111$ $(5 \le i \le \ell - 1)$. Thus c_i is even for all i with $a_i = 1$ so that, by Lemma 2, we have $S = 2^{n_1}$. □

In Section 3 we use the Davis–Webb congruence (mod 4) to determine $N_n(1, 4)$ and $N_n(3, 4)$, thereby reproving the formulae due to Davis and Webb [2] (see Theorem B (second part)). In Sections 4 and 5 we employ the Davis–Webb congruence (mod 8) to determine $N_n(t, 8)$ $(t = 2, 6)$ (see Theorem C (second part) in Section 4) and $N_n(t, 8)$ $(t = 1, 3, 5, 7)$ (see Theorem C (third part) in Section 5).

3. EVALUATION OF $N_n(1, 4)$ AND $N_n(3, 4)$

In this section we illustrate our methods by re-establishing the formulae for $N_n(1, 4)$ and $N_n(3, 4)$ due to Davis and Webb [2].

THEOREM B (SECOND PART, DAVIS AND WEBB [2]).

$$N_n(1,4) = \begin{cases} 2^{n_1}, & \textit{if } n_{11} = 0, \\ 2^{n_1-1}, & \textit{if } n_{11} > 0, \end{cases} \qquad N_n(3,4) = \begin{cases} 0, & \textit{if } n_{11} = 0, \\ 2^{n_1-1}, & \textit{if } n_{11} > 0. \end{cases}$$

PROOF. It is easily checked that the formulae hold for $n = 0, 1$ so that we may take $n \geq 2$. Thus $\ell \geq 1$. For $t = 1$ and 3, we have

$$N_n(t,4) = \sum_{\substack{r=0 \\ \binom{n}{r} \equiv t \,(\mathrm{mod} 4)}}^{n} 1 = \sum_{\substack{b_0,\dots,b_\ell=0 \\ (-1)^{E_1} \equiv t \,(\mathrm{mod} 4)}}^{a_0,\dots,a_\ell} 1,$$

by Proposition 1, Proposition 2(a), and the Davis–Webb congruence (mod 4). Hence

$$\begin{aligned} N_n(t,4) &= \sum_{\substack{b_0,\dots,b_\ell=0 \\ E_1 \equiv \frac{1}{2}(t-1) \pmod 2}}^{a_0,\dots,a_\ell} 1 = \frac{1}{2} \sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell} (1 + (-1)^{\frac{1}{2}(t-1)+E_1}) \\ &= 2^{n_1-1} + \frac{1}{2}(-1)^{\frac{1}{2}(t-1)} \sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell} (-1)^{E_1} \\ &= \begin{cases} 2^{n_1-1} + (-1)^{\frac{1}{2}(t-1)} 2^{n_1-1}, & \text{if } n_{11} = 0, \\ 2^{n_1-1}, & \text{if } n_{11} > 0, \end{cases} \end{aligned}$$

by Lemma 3. □

4. EVALUATION OF $N_n(2,8)$ AND $N_n(6,8)$

In this section we evaluate $N_n(2,8)$ and $N_n(6,8)$.

THEOREM C (SECOND PART).

$$N_n(2,8) = \begin{cases} n_{01} 2^{n_1-1} - n_{001} 2^{n_1-1}, & \textit{if } n_{11} = 0, \\ n_{01} 2^{n_1-2} - n_{011} 2^{n_1-2} + n_{0011} 2^{n_1-1}, & \textit{if } n_{11} = 1, \\ n_{01} 2^{n_1-2}, & \textit{if } n_{11} \geq 2. \end{cases}$$

$$N_n(6,8) = \begin{cases} n_{001} 2^{n_1-1}, & \textit{if } n_{11} = 0, \\ n_{01} 2^{n_1-2} + n_{011} 2^{n_1-2} - n_{0011} 2^{n_1-1}, & \textit{if } n_{11} = 1, \\ n_{01} 2^{n_1-2}, & \textit{if } n_{11} \geq 2. \end{cases}$$

PROOF. It is easily checked that the theorem holds for $n = 0, 1, 2, 3$ (equivalently $\ell = 0, 1$). Hence we may assume that $\ell \geq 2$. For $t = 2$ and 6 we have, by Proposition 1,

$$N_n(t,8) = \sum_{\substack{r=0 \\ \binom{n}{r} \equiv t \pmod 8}}^{n} 1 = \sum_{\substack{r=0 \\ c(n,r)=1 \\ \binom{n}{r} \equiv t \pmod 8}}^{n} 1 = \sum_{f=0}^{\ell-1} \sum_{\substack{r=0 \\ c(n,r)=1 \\ \text{carry in } f\text{th place} \\ \binom{n}{r} \equiv t \pmod 8}}^{n} 1.$$

Before continuing it is convenient to introduce some notation. Let S be a string of 0's and 1's of length k. For $0 \leq i \leq i + k - 1 \leq \ell$ we set

$$\left(\!\!\binom{a_i a_{i+1} \dots a_{i+k-1}}{S}\!\!\right) = \begin{cases} 1, & \text{if } a_i a_{i+1} \dots a_{i+k-1} = S, \\ 0, & \text{if } a_i a_{i+1} \dots a_{i+k-1} \neq S. \end{cases}$$

Now, by the Davis–Webb congruence (mod 8), we have

$$\binom{n}{r} \equiv t \pmod 8 \Leftrightarrow \frac{1}{4}(t+2) + a_{f-1} + a_{f+2} + H_f \equiv 0 \pmod 2.$$

Next, let

$$E_1' = \sum_{\substack{i=0 \\ a_i a_{i+1}=11}}^{f-2} (b_i + b_{i+1}), \quad E_1'' = \sum_{\substack{i=f+2 \\ a_i a_{i+1}=11}}^{\ell-1} (b_i + b_{i+1}),$$

so that $E_1' + E_1'' = H_f$. Hence the Davis–Webb congruence (mod 8) becomes

$$\binom{n}{r} \equiv t \pmod 8 \Leftrightarrow \frac{1}{4}(t+2) + a_{f-1} + a_{f+2} + E_1' + E_1'' \equiv 0 \pmod 2.$$

Hence

$$\begin{aligned} N_n(t,8) &= \sum_{\substack{f=0 \\ a_f a_{f+1}=01}}^{\ell-1} \sum_{\substack{b_0,\dots,b_{f-1},b_{f+2},\dots,b_\ell=0 \\ b_f b_{f+1}=10}}^{a_0,\dots,a_{f-1},a_{f+2},\dots,a_\ell} \frac{1}{2}(1 + (-1)^{\frac{1}{4}(t+2)+a_{f-1}+a_{f+2}+E_1'+E_1''}) \\ &= n_{01}2^{n_1-2} + \frac{1}{2}(-1)^{\frac{1}{4}(t+2)} \sum_{\substack{f=0 \\ a_f a_{f+1}=01}}^{\ell-1} (-1)^{a_{f-1}+a_{f+2}} \sum_{b_0,\dots,b_{f-1}=0}^{a_0,\dots,a_{f-1}} (-1)^{E_1'} \\ &\quad \times \sum_{b_{f+2},\dots,b_\ell=0}^{a_{f+2},\dots,a_\ell} (-1)^{E_1''}. \end{aligned}$$

Now, by Lemma 3,we have

$$\sum_{b_0,\dots,b_{f-1}=0}^{a_0,\dots,a_{f-1}} (-1)^{E_1'} = 0^{n_{11}'} 2^{n_1'}, \qquad \sum_{b_{f+2},\dots,b_\ell=0}^{a_{f+2},\dots,a_\ell} (-1)^{E_1''} = 0^{n_{11}''} 2^{n_1''},$$

where n_1' is the number of 1's in $a_0 a_1 \dots a_{f-1}$, n_1'' is the number of 1's in $a_{f+2} \dots a_\ell$, n_{11}' is the number of occurrences of 11 in $a_0 a_1 \dots a_{f-1}$, and n_{11}'' is the number of occurrences of 11 in $a_{f+2} \dots a_\ell$. Hence

$$\sum_{b_0,\dots,b_{f-1}=0}^{a_0,\dots,a_{f-1}} (-1)^{E_1'} \sum_{b_{f+2},\dots,b_\ell=0}^{a_{f+2},\dots,a_\ell} (-1)^{E_1''} = 0^{n_{11}'+n_{11}''} 2^{n_1'+n_1''} = \begin{cases} 0^{n_{11}} 2^{n_1-1}, & \text{if } a_{f+2}=0, \\ 0^{n_{11}-1} 2^{n_1-1}, & \text{if } a_{f+2}=1. \end{cases}$$

Thus for $n_{11} > 1$ we have $N_n(t,8) = n_{01}2^{n_1-2}$. Next for $n_{11} = 1$ we have

$$\begin{aligned} N_n(t,8) &= n_{01}2^{n_1-2} + \frac{1}{2}(-1)^{\frac{1}{4}(t+2)} \sum_{\substack{f=0 \\ a_f a_{f+1} a_{f+2}=011}}^{\ell-1} (-1)^{a_{f-1}+a_{f+2}} 2^{n_1-1} \\ &= n_{01}2^{n_1-2} - (-1)^{\frac{1}{4}(t+2)} 2^{n_1-2} \sum_{\substack{f=0 \\ a_f a_{f+1} a_{f+2}=011}}^{\ell-1} (-1)^{a_{f-1}} \\ &= n_{01}2^{n_1-2} + (-1)^{\frac{1}{4}(t-2)} 2^{n_1-2} \left(-\left(\left(\begin{matrix} a_0 & a_1 & a_2 \\ 0 & 1 & 1 \end{matrix} \right) \right) + n_{0011} - n_{1011} \right) \\ &= n_{01}2^{n_1-2} + (-1)^{\frac{1}{4}(t-2)} 2^{n_1-2} (2n_{0011} - n_{011}), \end{aligned}$$

as

$$\left(\!\!\left(\begin{array}{ccc} a_0 & a_1 & a_2 \\ 0 & 1 & 1 \end{array}\right)\!\!\right) = n_{011} - n_{0011} - n_{1011}.$$

Hence

$$N_n(2,8) = n_{01}2^{n_1-2} + 2^{n_1-1}n_{0011} - 2^{n_1-2}n_{011}$$

and

$$N_n(6,8) = n_{01}2^{n_1-2} - 2^{n_1-1}n_{0011} + 2^{n_1-2}n_{011}.$$

Finally for $n_{11} = 0$ we have

$$\begin{aligned} N_n(t,8) &= n_{01}2^{n_1-2} + \frac{1}{2}(-1)^{\frac{1}{4}(t+2)} \sum_{\substack{f=0 \\ a_f a_{f+1} a_{f+2} = 010}}^{\ell-1} (-1)^{a_{f-1}+a_{f+2}} 2^{n_1-1} \\ &= n_{01}2^{n_1-2} + (-1)^{\frac{1}{4}(t+2)} 2^{n_1-2} \left\{ -\left(\!\!\left(\begin{array}{ccc} a_0 & a_1 & a_2 \\ 0 & 1 & 0 \end{array}\right)\!\!\right) + \left(\!\!\left(\begin{array}{ccc} a_{\ell-2} & a_{\ell-1} & a_\ell \\ 0 & 0 & 1 \end{array}\right)\!\!\right) \right. \\ &\quad \left. - \left(\!\!\left(\begin{array}{ccc} a_{\ell-2} & a_{\ell-1} & a_\ell \\ 1 & 0 & 1 \end{array}\right)\!\!\right) + \sum_{\substack{f=1 \\ a_f a_{f+1} a_{f+2} = 010}}^{\ell-2} (-1)^{a_{f-1}} \right\}. \end{aligned}$$

Now as $n_{11} = 0$ we have

$$\begin{aligned} \left(\!\!\left(\begin{array}{ccc} a_0 & a_1 & a_2 \\ 0 & 1 & 0 \end{array}\right)\!\!\right) &= \left(\!\!\left(\begin{array}{cc} a_0 & a_1 \\ 0 & 1 \end{array}\right)\!\!\right) = n_{01} - n_{001} - n_{101}, \\ \left(\!\!\left(\begin{array}{ccc} a_{\ell-2} & a_{\ell-1} & a_\ell \\ 0 & 0 & 1 \end{array}\right)\!\!\right) &= n_{001} - n_{0010} - n_{0011} = n_{001} - n_{0010}, \\ \left(\!\!\left(\begin{array}{ccc} a_{\ell-2} & a_{\ell-1} & a_\ell \\ 1 & 0 & 1 \end{array}\right)\!\!\right) &= n_{101} - n_{1010} - n_{1011} = n_{101} - n_{1010}, \\ \sum_{\substack{f=1 \\ a_f a_{f+1} a_{f+2} = 010}}^{\ell-2} (-1)^{a_{f-1}} &= \sum_{\substack{f=1 \\ a_{f-1} a_f a_{f+1} a_{f+2} = 0010}}^{\ell-2} 1 - \sum_{\substack{f=1 \\ a_{f-1} a_f a_{f+1} a_{f+2} = 1010}}^{\ell-2} 1 = n_{0010} - n_{1010}, \end{aligned}$$

so that

$$\begin{aligned} &-\left(\!\!\left(\begin{array}{ccc} a_0 & a_1 & a_2 \\ 0 & 1 & 0 \end{array}\right)\!\!\right) + \left(\!\!\left(\begin{array}{ccc} a_{\ell-2} & a_{\ell-1} & a_\ell \\ 0 & 0 & 1 \end{array}\right)\!\!\right) - \left(\!\!\left(\begin{array}{ccc} a_{\ell-2} & a_{\ell-1} & a_\ell \\ 1 & 0 & 1 \end{array}\right)\!\!\right) \\ &+ \sum_{\substack{f=1 \\ a_f a_{f+1} a_{f+2} = 010}}^{\ell-2} (-1)^{a_{f-1}} = -n_{01} + 2n_{001}. \end{aligned}$$

Hence

$$\begin{aligned} N_n(t,8) &= n_{01}2^{n_1-2} + (-1)^{\frac{1}{4}(t+2)}(-n_{01} + 2n_{001})2^{n_1-2} \\ &= \begin{cases} n_{01}2^{n_1-1} - n_{001}2^{n_1-1}, & \text{if } t = 2, \\ n_{001}2^{n_1-1}, & \text{if } t = 6. \end{cases} \end{aligned}$$

□

5. Evaluation of $N_n(t,8)$, $t=1,3,5,7$

In this section we carry out the evaluation of $N_n(t,8)$ for $t=1,3,5,7$ using the Davis–Webb congruence $\pmod 8$.

THEOREM C (THIRD PART).

Case No.	n_{1111}	n_{11}	n_{101}	n_{111}	v_1	s_1	$s_i\ (i\geq 2)$	$N_n(1,8)$	$N_n(3,8)$	$N_n(5,8)$	$N_n(7,8)$
(i)	$\neq 0$							2^{n_1-2}	2^{n_1-2}	2^{n_1-2}	2^{n_1-2}
(ii)	0	0	0					2^{n_1}	0	0	0
(iii)			$\neq 0$					2^{n_1-1}	0	2^{n_1-1}	0
(iv)		$\neq 0$	$\neq 0$					2^{n_1-2}	2^{n_1-2}	2^{n_1-2}	2^{n_1-2}
(v)			0	0	$\neq 0$			2^{n_1-1}	0	0	2^{n_1-1}
(vi)					0	1		2^{n_1-1}	0	0	2^{n_1-1}
(vii)						2	*some* $=2$	2^{n_1-2}	2^{n_1-2}	2^{n_1-2}	2^{n_1-2}
(viii)						2	*none* $=2$	2^{n_1-1}	2^{n_1-1}	0	0
(ix)				$\neq 0$	0	3		2^{n_1-2}	2^{n_1-2}	2^{n_1-2}	2^{n_1-2}
(x)						1, 2	*some* $=2$	2^{n_1-2}	2^{n_1-2}	2^{n_1-2}	2^{n_1-2}
(xi)						1, 2	*none* $=2$	2^{n_1-1}	2^{n_1-1}	0	0
(xii)					$\neq 0$	*some* $=2$		2^{n_1-2}	2^{n_1-2}	2^{n_1-2}	2^{n_1-2}
(xiii)						*none* $=2$		2^{n_1-1}	2^{n_1-1}	0	0

PROOF. It is easily checked that the theorem holds for $n=0,1,2,3$ (equivalently $\ell=0,1$). Hence we may assume that $\ell\geq 2$. For $t=1,3,5,7$ we have

$$N_n(t,8)=\sum_{\substack{r=0\\ \binom{n}{r}\equiv t \pmod 8}}^{n} 1=\sum_{\substack{r=0\\ c(n,r)=0\\ \binom{n}{r}\equiv t \pmod 8}}^{n} 1=\sum_{\substack{b_0,\dots,b_\ell=0\\ (-1)^{E_1}5^{(b_0+b_1)\left(\left(\begin{smallmatrix}a_0 & a_1\\ 1 & 1\end{smallmatrix}\right)\right)+E_2}\equiv t \pmod 8}}^{a_0,\dots,a_\ell} 1,$$

by Propositions 1 and 2(a), and part (a) of the Davis–Webb congruence (mod 8). Set

$$\alpha(t)=(t-1)/2. \qquad \beta(t)=(t^2-1)/8,$$

so that $t\equiv(-1)^{\alpha(t)}5^{\beta(t)} \pmod 8$. Hence

$$\begin{aligned}&(-1)^{E_1}5^{(b_0+b_1)\left(\left(\begin{smallmatrix}a_0 & a_1\\ 1 & 1\end{smallmatrix}\right)\right)+E_2}\equiv t \pmod 8\\ &\Leftrightarrow E_1\equiv\alpha(t) \pmod 2,\ (b_0+b_1)\left(\left(\begin{matrix}a_0 & a_1\\ 1 & 1\end{matrix}\right)\right)+E_2\equiv\beta(t) \pmod 2.\end{aligned}$$

Thus

$$\begin{aligned}N_n(t,8)&=\frac{1}{4}\sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell}\left(1+(-1)^{\alpha(t)}(-1)^{E_1}\right)\left(1+(-1)^{\beta(t)}(-1)^{(b_0+b_1)\left(\left(\begin{smallmatrix}a_0 & a_1\\ 1 & 1\end{smallmatrix}\right)\right)+E_2}\right)\\ &=2^{n_1-2}+\frac{(-1)^{\alpha(t)}}{4}\sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell}(-1)^{E_1}\\ &\quad+\frac{(-1)^{\beta(t)}}{4}\sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell}(-1)^{(b_0+b_1)\left(\left(\begin{smallmatrix}a_0 & a_1\\ 1 & 1\end{smallmatrix}\right)\right)+E_2}\\ &\quad+\frac{(-1)^{\alpha(t)+\beta(t)}}{4}\sum_{b_0,\dots,b_\ell=0}^{a_0,\dots,a_\ell}(-1)^{(b_0+b_1)\left(\left(\begin{smallmatrix}a_0 & a_1\\ 1 & 1\end{smallmatrix}\right)\right)+E_1+E_2}.\end{aligned}$$

We treat the two cases $a_0a_1 \neq 11$ and $a_0a_1 = 11$ separately.

If $a_0a_1 \neq 11$ then $\left(\binom{a_0\ a_1}{1\ 1}\right) = 0$ and appealing to Lemmas 3, 4 and 5, we obtain

$$N_n(t,8) = 2^{n_1-2} + \begin{Bmatrix} (-1)^{\alpha(t)}2^{n_1-2}, & \text{if } n_{11}=0 \\ 0, & \text{if } n_{11}>0 \end{Bmatrix} + \begin{Bmatrix} (-1)^{\beta(t)}2^{n_1-2}, & \text{if } n_{101}=n_{111}=0 \\ 0, & \text{if } n_{101} \text{ or } n_{111}>0 \end{Bmatrix}$$
$$+ \begin{cases} (-1)^{\alpha(t)+\beta(t)}2^{n_1-2}, & \text{if } n_{101}=n_{1111}=0 \text{ and each } s_i=1 \text{ or } 3 \\ 0, & \text{if } n_{101}=n_{1111}=0 \\ & \text{ and some } s_i=2; \text{ or } n_{101}>0; \text{ or } n_{1111}>0. \end{cases}$$

Appealing to the case definitions given in the statement of the theorem we obtain the value of $N_n(t,8)$.

Cases (i), (iv), (xii). $N_n(t,8) = 2^{n_1-2} + 0 + 0 + 0 = 2^{n_1-2}$.

Case (ii). Here $n_{11} = 0$ so that each $s_i = 1$.

$$\begin{aligned} N_n(t,8) &= 2^{n_1-2} + (-1)^{\alpha(t)}2^{n_1-2} + (-1)^{\beta(t)}2^{n_1-2} + (-1)^{\alpha(t)+\beta(t)}2^{n_1-2} \\ &= \begin{cases} 2^{n_1}, & \text{if } t=1, \\ 0, & \text{if } t=3,5,7. \end{cases} \end{aligned}$$

Case (iii). $N_n(t,8) = 2^{n_1-2} + (-1)^{\alpha(t)}2^{n_1-2} = \begin{cases} 2^{n_1-1}, & \text{if } t=1,5, \\ 0, & \text{if } t=3,7. \end{cases}$

Cases (v), (vi). Here $n_{11} > 0$, $n_{111} = 0$ implies that some $s_i = 2$.

$$N_n(t,8) = 2^{n_1-2} + 0 + (-1)^{\beta(t)}2^{n_1-2} + 0 = \begin{cases} 2^{n_1-1}, & \text{if } t=1,7, \\ 0, & \text{if } t=3,5. \end{cases}$$

Cases (vii), (viii), (ix). Here $v_1 = 0$ and $s_1 = 2$ or 3 so that $a_0a_1 = 11$, contradicting $a_0a_1 \neq 11$. These cases cannot occur.

Case (x). Here $v_1 = 0$ and $s_1 = 1$ or 2. As $a_0a_1 \neq 11$ we have $s_1 = 1$.

$$N_n(t,8) = 2^{n_1-2} + 0 + 0 + 0 = 2^{n_1-2}.$$

Case (xi) (Here $v_1 = 0$ and $s_1 = 1$ or 2: as $a_0a_1 \neq 11$ we have $s_1 = 1$.) and *Case (xiii).*

$$N_n(t,8) = 2^{n_1-2} + 0 + 0 + (-1)^{\alpha(t)+\beta(t)}2^{n_1-2} = \begin{cases} 2^{n_1-1}, & \text{if } t=1,3, \\ 0, & \text{if } t=5,7. \end{cases}$$

If $a_0a_1 = 11$, appealing to Lemmas 3, 6, and 7, we obtain

$$N_n(t,8) = 2^{n_1-2} + \begin{cases} (-1)^{\alpha(t)+\beta(t)}2^{n_1-2}, & \text{if } n_{101}=n_{1111}=0, s_1=2, \\ & \text{each } s_i(i \geq 2)=1 \text{ or } 3, \\ 0, & \text{otherwise.} \end{cases}$$

Cases (i), (iv), (ix), (x). $N_n(t,8) = 2^{n_1-2} + 0 = 2^{n_1-2}$.

Cases (ii), (iii). Here $a_0a_1 = 11$ implies $n_{11} > 0$ so these cases cannot occur.

Case (v). Here $a_0a_1 = 11$ implies $v_1 = 0$ so this case cannot occur.

Case (vi). Here $a_0a_1 = 11$ implies $v_1 = 0$ and $s_1 \geq 2$ so this case cannot occur.

Case (vii). $N_n(t, 8) = 2^{n_1-2} + 0 = 2^{n_1-2}$.

Case (xi) (Here $a_0a_1 = 11$ implies $v_1 = 0$ and $s_1 \geq 2$.) and *Case (viii).*

$$N_n(t, 8) = 2^{n_1-2} + (-1)^{\alpha(t)+\beta(t)}2^{n_1-2} = \begin{cases} 2^{n_1-1}, & \text{if } t = 1, 3, \\ 0, & \text{if } t = 5, 7. \end{cases}$$

Cases (xii), (xiii). Here $v_1 > 0$ contradicting $a_0a_1 = 11$. These cases cannot occur. □

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James G. Huard
Department of Mathematics,
Canisius College,
Buffalo,
NY 14208,
USA
E-mail: huard@canisius.edu

Blair K. Spearman
Department of Mathematics and Statistics,
Okanagan University College,
Kelowna,
B.C. V1V 1V7,
Canada
E-mail: bkspearm@okanagan.bc.ca

Kenneth S. Williams
Department of Mathematics and Statistics,
Carleton University,
Ottawa,
Ontario K1S 5B6,
Canada
E-mail: williams@math.carleton.ca