

## A REFINEMENT OF H. C. WILLIAMS' $q$ th ROOT ALGORITHM

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*Dedicated to the memory of D. H. Lehmer*

**ABSTRACT.** Let  $p$  and  $q$  be primes such that  $p \equiv 1 \pmod{q}$ . Let  $a$  be an integer such that  $a^{(p-1)/q} \equiv 1 \pmod{p}$ . In 1972, H. C. Williams gave an algorithm which determines a solution of the congruence  $x^q \equiv a \pmod{p}$  in  $O(q^3 \log p)$  steps, once an integer  $b$  has been found such that  $(b^q - a)^{(p-1)/q} \not\equiv 0, 1 \pmod{p}$ . A step is an arithmetic operation  $\pmod{p}$  or an arithmetic operation on  $q$ -bit integers. We present a refinement of this algorithm which determines a solution in  $O(q^4) + O(q^2 \log p)$  steps, once  $b$  has been determined. Thus the new algorithm is better when  $q$  is small compared with  $p$ .

### 1. INTRODUCTION

Let  $p$  and  $q$  be primes and let  $a$  be an integer not divisible by  $p$ . If  $p \not\equiv 1 \pmod{q}$ , the congruence

$$(1.1) \quad x^q \equiv a \pmod{p}$$

has one solution  $x = a^u$ , where  $u$  and  $v$  are integers such that  $qu - (p-1)v = 1$ . The integer  $u$  is easily found by applying the Euclidean algorithm to  $q$  and  $p-1$ . If  $p \equiv 1 \pmod{q}$  and  $a^{(p-1)/q} \not\equiv 1 \pmod{p}$ , the congruence (1.1) has no solutions. If  $p \equiv 1 \pmod{q}$  and  $a^{(p-1)/q} \equiv 1 \pmod{p}$ , (1.1) has  $q$  solutions. H. C. Williams [14] has given an algorithm for finding a solution  $x$  of (1.1) when  $q$  is odd. Briefly, his algorithm may be described as follows: first determine by trial an integer  $b$  such that  $b^q - a$  is not a  $q$ th power residue of  $p$ ; then use the formula

$$U_{j,m+n} \equiv \sum_{i=0}^j U_{i,n} U_{j-i,m} + (a - b^q) \sum_{i=1}^{q-1-j} U_{j+i,n} U_{q-i,m} \pmod{p}$$

$$(j = 0, 1, \dots, q-1; m = 1, 2, \dots; n = 1, 2, \dots)$$

recursively, starting with the initial values

$$U_{0,1} = b, \quad U_{1,1} = 1, \quad U_{j,1} = 0 \quad (j = 2, \dots, q-1),$$

to compute  $x = U_{0,(p^q-1)/((p-1)q)}$ . Then  $x$  is a solution of (1.1). Once  $b$  has been determined, Williams' algorithm requires  $O(q^3 \log p)$  steps to solve (1.1),

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where by a step we mean an arithmetic operation in  $GF(p)$  or an arithmetic operation on  $q$ -bit integers. All  $q$  solutions of the congruence (1.1) are given by

$$x(b^q - a)^{j(p-1)/q}, \quad j = 0, 1, \dots, q - 1.$$

In this paper we present a refinement of Williams' algorithm which determines a solution of (1.1) in  $O(q^4) + O(q^2 \log p)$  steps, once the integer  $b$  has been found. Thus, our algorithm is better when  $q$  is small compared with  $p$ , roughly when  $q = O_\epsilon((\log p)^{1-\epsilon})$ , where  $0 < \epsilon < \frac{1}{2}$ .

We remark that Williams' algorithm is a  $q$ th power version of an algorithm for computing square roots in  $GF(p)$ , which was published by Cipolla [5] in 1903 (see also [2], [9, pp. 132–134]). Shanks [12] has also given an algorithm for determining  $q$ th roots in  $GF(p)$ . His algorithm is an extension of an algorithm of Tonelli [13]. Adleman, Manders, and Miller [1] have shown, *assuming the extended Riemann hypothesis*, that there is a deterministic algorithm running in time  $O(n \log^c(p + a))$  for some  $c > 0$  such that on inputs  $a, p, n$ , where  $p$  is prime, it outputs the least positive integer  $x$  such that  $x^n \equiv a \pmod{p}$  or "no" if no such  $x$  exists. It is an open problem to find a polynomial-time algorithm—polynomial in  $\log q$  and  $\log p$ —for  $q$ th roots in  $GF(p)$ .

Algorithms for the more general problem of factoring polynomials over finite fields have been given by a number of authors, notably, Berlekamp [3], Moenck [10], Rabin [11], and Cantor and Zassenhaus [4] (see also [8, Chapter 4]). Cantor and Zassenhaus give a heuristic argument to suggest that the expected running time of their algorithm to factor a polynomial of degree  $n$  in  $GF(p^m)$  is  $O(n^3 + n^2 \log(p^m))$ .

## 2. IDEA OF ALGORITHM

Let  $p$  and  $q$  be primes with  $q|p - 1$ . Let  $a$  be a nonzero element of  $k = GF(p)$  which is the  $q$ th power of an element in  $k$ . We wish to determine a  $q$ th root of  $a$ . The algorithm constructs an extension field  $K = k[\theta] \simeq GF(p^q)$  together with an element  $\alpha \in K$  which, when raised to the power  $(p^q - 1)/(p - 1)$ , gives  $a$ . It then follows that  $\alpha^{(p^q - 1)/(q(p - 1))} \in k$  is the desired  $q$ th root of  $a$ . This strategy, in rather disguised form, is used by Williams [14]. The contribution of this paper is a way to compute the high power of  $\alpha$  somewhat more quickly than the usual repeated squaring algorithm does. The idea is to write the exponent in base  $p$  and use automorphisms of  $K/k$  to get the effect of raising elements to  $p^e$ th powers.

## 3. THE ALGORITHM

Let  $p$  and  $q$  be primes satisfying  $p \equiv 1 \pmod{q}$ . Let  $a \in k \setminus \{0\}$  be such that  $a^{(p-1)/q} = 1$ . We first show that there exists  $b \in k$  with  $(b^q - a)^{(p-1)/q} \neq 0, 1$ . Clearly, we can identify  $k$  with the residues  $\{1, 1 - a, 1 - 2a, \dots, 1 - (p - 1)a\}$  modulo  $p$ . As  $k$  contains  $(p - 1)(q - 1)/q \geq q - 1 \geq 1$  elements which are not  $q$ th powers, we can let  $l$  be the smallest nonnegative integer such that  $1 - la$  is not a  $q$ th power of an element of  $k$ . Clearly, we have  $l \geq 1$  and  $1 - (l - 1)a = b^q$  for some  $b \in k$ . Then we have  $b^q - a = 1 - la$ , and so, as  $1 - la$  is not a  $q$ th power, we have  $(b^q - a)^{(p-1)/q} \neq 0, 1$ . We set

$$(3.1) \quad c = (b^q - a)^{(p-1)/q}.$$

Clearly,  $c$  is a primitive  $q$ th root of unity in  $k$ . Since  $b^q - a$  is not a  $q$ th power in  $k$ , we can adjoin a  $q$ th root  $\theta$  of this quantity to  $k$  and obtain an extension field

$$(3.2) \quad K = k[\theta] = GF(p)[\theta] \simeq GF(p^q), \quad \text{where } \theta^q = b^q - a.$$

In  $K$  we have  $\theta^p = (\theta^q)^{(p-1)/q} \theta = (b^q - a)^{(p-1)/q} \theta = c\theta$ , so that

$$(3.3) \quad \theta^{p^n} = c^n \theta, \quad n = 0, 1, 2, \dots$$

Now define  $x \in K$  by

$$(3.4) \quad x = (b - \theta)^{(p^q - 1)/((p-1)q)}.$$

As  $(p^q - 1)/(p - 1) = 1 + p + p^2 + \dots + p^{q-1}$ , we have

$$(3.5) \quad x^q = \prod_{j=0}^{q-1} (b - \theta)^{p^j}.$$

Next we observe that  $(b - \theta)^p = b^p - \theta^p = b - c\theta$ , so that

$$(3.6) \quad (b - \theta)^{p^j} = b - c^j \theta, \quad j = 0, 1, 2, \dots$$

As  $c$  is a primitive  $q$ th root of unity in  $k$ , we have

$$(3.7) \quad \prod_{j=0}^{q-1} (b - c^j \theta) = b^q - \theta^q = a,$$

so that by (3.5), (3.6), and (3.7), we see that  $x^q = a$ . Since the equation  $y^q = c$  has at most  $q$  solutions in the field  $K$ , and since it has exactly  $q$  solutions in the subfield  $k$ , every solution must belong to  $k$ . Thus, in particular, we have  $x \in k$ . We have thus shown that  $x = (b - \theta)^{(p^q - 1)/((p-1)q)}$  is a  $q$ th root of  $a$  in  $k$ . We remark that H. C. Williams' algorithm is equivalent to computing  $\frac{1}{q} \text{tr}_{K/k}((b - \theta)^{(p^q - 1)/((p-1)q)})$ , which is also a  $q$ th root of  $a$ . Note also that  $N_{K/k}(b - \theta) = a$ .

In order to compute  $x$ , we write it in the form

$$(3.8) \quad x = E_1^{(p-1)/q} E_2,$$

where

$$(3.9) \quad E_1 = (b - \theta)^{(p-1)q-2}, \quad E_2 = (b - \theta)^{(p^q - 1)/q(p-1) - (p-1)q-1/q}.$$

First we consider  $E_1$ . Applying the binomial theorem to  $(p - 1)^{q-2}$ , and appealing to (3.6), we obtain

$$(3.10) \quad E_1 = \prod_{i=0}^{q-2} (b - c^i \theta)^{(-1)^{q-i} \binom{q-2}{i}},$$

say

$$(3.11) \quad E_1 = \sum_{i=0}^{q-1} a_i \theta^i,$$

where  $a_i \in k$ ,  $i = 0, 1, \dots, q - 1$ . Now define  $a_i(j) \in k$  for  $i = 0, 1, \dots, q - 1$  and  $j = 1, 2, 3, \dots$  by

$$(3.12) \quad \sum_{i=0}^{q-1} a_i(j)\theta^i = \left( \sum_{i=0}^{q-1} a_i\theta^i \right)^j,$$

so that

$$(3.13) \quad a_i(1) = a_i, \quad i = 0, 1, \dots, q - 1,$$

and

$$(3.14) \quad E_1^{(p-1)/q} = \sum_{i=0}^{q-1} a_i((p-1)/q)\theta^i.$$

Next we consider  $E_2$ . Again, by the binomial theorem and (3.6), we obtain

$$(3.15) \quad E_2 = \prod_{i=1}^{q-1} (b - c^{q-i-1}\theta)^{(1-(-1)^i \binom{q-1}{i})/q}.$$

It is easily proved by induction on  $i$  that the exponent  $(1 - (-1)^i \binom{q-1}{i})/q$  is an integer. Thus we have

$$(3.16) \quad E_2 = \sum_{i=0}^{q-1} b_i\theta^i,$$

where  $b_i \in k$ ,  $i = 0, 1, \dots, q - 1$ . From (3.8), (3.14), and (3.16), we deduce

$$x = \left( \sum_{i=0}^{q-1} a_i((p-1)/q)\theta^i \right) \left( \sum_{j=0}^{q-1} b_j\theta^j \right),$$

that is

$$(3.17) \quad x = a_0((p-1)/q)b_0 + (b^q - a) \sum_{i=1}^{q-1} a_i((p-1)/q)b_{q-i}.$$

Formula (3.17) is the expression we use to calculate  $x$ . We can now give the algorithm.

**Algorithm to determine all solutions  $x$  of the congruence  $x^q \equiv a \pmod{p}$ .**

*Input.*  $p, q$  primes satisfying  $p \equiv 1 \pmod{q}$ .  $a$  an integer not divisible by  $p$ .

*Step 1.* Compute  $a^{(p-1)/q}$  in  $k = GF(p)$ . If  $a^{(p-1)/q} \neq 1$ , then  $x^q = a$  has no solutions in  $k$  and the algorithm terminates. Otherwise,  $x^q = a$  has  $q$  solutions in  $k$  and the algorithm continues with Step 2.

*Step 2.* Try  $b = 1, 2, 3, \dots$  until the first integer  $b$  is found such that  $(b^q - a)^{(p-1)/q} \neq 0, 1$ , and set  $c = (b^q - a)^{(p-1)/q}$ .

*Step 3.* In  $K = k[\theta] = \{c_0 + c_1\theta + \dots + c_{q-1}\theta^{q-1} \mid c_0, c_1, \dots, c_{q-1} \in k\}$ , where  $\theta^q = b^q - a$ , compute the quantities  $X_i = (b - c^i\theta)^{(-1)^{q-i} \binom{q-1}{i}}$  for  $i = 0, 1, \dots, q - 2$  and  $Y_i = (b - c^{q-i-1}\theta)^{(1-(-1)^i \binom{q-1}{i})/q}$  for  $i = 1, \dots, q - 1$ . Then compute the products  $E_1 = \prod_{i=0}^{q-2} X_i = \sum_{i=0}^{q-1} a_i\theta^i$  and  $E_2 = \prod_{i=1}^{q-1} Y_i = \sum_{i=0}^{q-1} b_i\theta^i$  to obtain  $a_0, a_1, \dots, a_{q-1}, b_0, b_1, \dots, b_{q-1} \in k$ .

Step 4. Use the recurrence relation in  $k$ ,

$$(3.18) \quad a_i(m+n) = \sum_{j=0}^i a_j(m)a_{i-j}(n) + (b^q - a) \sum_{j=i+1}^{q-1} a_j(m)a_{q+i-j}(n) \quad (m, n = 1, 2, \dots),$$

subject to the initial conditions

$$(3.19) \quad a_i(1) = a_i \quad (i = 0, 1, \dots, q-1),$$

to calculate  $a_i((p-1)/q)$  ( $i = 0, 1, \dots, q-1$ ).

Output. A solution  $x$  of the congruence  $x^q \equiv a \pmod{p}$  is given by

$$(3.20) \quad x = a_0((p-1)/q)b_0 + (b^q - a) \sum_{i=1}^{q-1} a_i((p-1)/q)b_{q-i}.$$

All solutions are given by  $x_j = c^j x$ ,  $j = 0, 1, \dots, q-1$ .

We conclude this section by determining the running time of the algorithm. Recall that a step is an arithmetic operation in  $k = GF(p)$  or an arithmetic operation on  $q$ -bit integers. Note that arithmetic operations in  $K = GF(p^q)$  take  $O(q^2)$  steps.

Step 1. The calculation of  $a^{(p-1)/q}$  can be carried out in  $O(\log p)$  steps in  $k$  by the repeated squaring technique.

Step 2. Let  $N$  denote the number of  $(x, y) \in k \times k$  with  $x^q - y^q = a$  and  $B$  the number of values of  $b \in k$  for which  $(b^q - a)^{(p-1)/q} = 0$  or 1. Then we have

$$N = \sum_{\substack{x \in k \\ x^q - y^q = a}} \sum_{y \in k} 1 = \sum_{\substack{x \in k \\ x^q = a}} 1 + \sum_{x \in k} \sum_{\substack{y \in k \\ y \neq 0 \\ y^q = x^q - a}} 1 = q + q \sum_{\substack{x \in k \\ (x^q - a)^{(p-1)/q} = 1}} 1 = q + q(B - q).$$

From the work of Davenport and Hasse [6, p. 174] we have

$$|N - p| \leq q - 1 + ((q-1)^2 - (q-1))\sqrt{p},$$

so that

$$|qB - q(q-1) - p| \leq q - 1 + (q-1)(q-2)\sqrt{p}.$$

Hence, we have

$$\begin{aligned} B &\leq \frac{p}{q} + \frac{(q^2 - 1)}{q} + \frac{(q-1)(q-2)}{q} \sqrt{p} \\ &\leq \frac{p}{q} + \frac{(2q^2 - 3q + 1)}{q} \sqrt{p} \leq \frac{p}{q} + 2q\sqrt{p} \end{aligned}$$

and

$$B \geq \frac{p}{q} + \frac{(q-1)^2}{q} - \frac{(q-1)(q-2)}{q} \sqrt{p} \geq \frac{p}{q} - q\sqrt{p},$$

so that

$$\left| \frac{B}{p} - \frac{1}{q} \right| \leq \frac{2q}{\sqrt{p}}.$$

Thus, for  $q$  small compared with  $p$ , say for example  $q \leq p^{1/4}$ , a random value of  $b$  does not satisfy  $(b^q - a)^{(p-1)/q} \neq 0, 1$  with probability

$$\frac{B}{p} = \frac{1}{q} + O\left(\frac{q}{\sqrt{p}}\right) = \frac{1}{q} + O(p^{-1/4}).$$

Thus finding an appropriate value of  $b$  is usually quite fast in practice.

*Step 3.* First we observe that all of the values of  $b^i$  ( $i = 0, 1, \dots, q-1$ ) and  $c^i$  ( $i = 0, 1, \dots, (q-1)^2$ ) can be computed in  $O(q^2)$  arithmetic operations in  $K$ .

Next we remark that as

$$\binom{q-2}{i} \leq 2^{q-2} < 2^q < 10^q$$

for  $i = 0, 1, \dots, q-2$ , each entry in the first  $q-2$  rows of Pascal's triangle can be represented as a  $q$ -bit integer, and so  $O(q^2)$  additions of  $q$ -bit integers are required to compute all the binomial coefficients  $\binom{q-2}{i}$  ( $i = 0, 1, \dots, q-2$ ) from Pascal's triangle.

Knowing the values of  $c^i$  ( $i = 0, 1, \dots, q-2$ ) and  $\binom{q-2}{i}$  ( $i = 0, 1, \dots, q-2$ ), we can, when  $q-i$  is even, compute each quantity  $(b - c^i\theta)^{(-1)^{q-i}\binom{q-2}{i}} = (b - c^i\theta)^{\binom{q-2}{i}}$  by repeated squarings in  $K$  in  $O(q^2 \log \binom{q-2}{i})$  steps. Knowing the values of  $b^i$  ( $i = 0, 1, \dots, q-1$ ),  $(c^i)^j$  ( $i = 0, 1, \dots, q-1; j = 0, 1, \dots, q-1$ ) and  $\binom{q-2}{i}$  ( $i = 0, 1, \dots, q-2$ ), as

$$(3.21) \quad (b - c^i\theta)^{-1} = a^{p-2}(b^{q-1} + b^{q-2}c^i\theta + \dots + (c^i)^{q-1}\theta^{q-1}),$$

we can, when  $q-i$  is odd, compute each quantity

$$\begin{aligned} (b - c^i\theta)^{(-1)^{q-i}\binom{q-2}{i}} &= (b - c^i\theta)^{-\binom{q-2}{i}} \\ &= (a^{p-2}(b^{q-1} + b^{q-2}c^i\theta + \dots + (c^i)^{q-1}\theta^{q-1}))^{\binom{q-2}{i}} \end{aligned}$$

by repeated squarings in  $K$  in

$$O(\log p) + O(q) + O\left(q^2 \log \binom{q-2}{i}\right)$$

steps. Hence, all of

$$X_i = (b - c^i\theta)^{(-1)^{q-2-i}\binom{q-2}{i}} \quad (i = 0, 1, \dots, q-2)$$

can be computed in

$$O(q^2) + \sum_{i=0}^{q-2} \left( O(\log p) + O(q) + O\left(q^2 \log \binom{q-2}{i}\right) \right) = O(q \log p) + O(q^4)$$

steps, as

$$\sum_{i=0}^n \log \binom{n}{i} \sim \frac{1}{2}n^2, \quad \text{as } n \rightarrow \infty,$$

see [7]. Multiplying the  $X_i$  together in  $K$  to obtain  $E_1 = \prod_{i=0}^{q-2} X_i = \sum_{i=0}^{q-1} a_i\theta^i$  takes a further  $O(q)$  multiplications in  $K$ , that is,  $O(q^3)$  steps. Hence,  $a_0, a_1, \dots, a_{q-1}$  can be computed in  $O(q \log p) + O(q^4)$  steps. A similar calculation shows that  $b_0, b_1, \dots, b_{q-1}$  can also be computed in  $O(q \log p) + O(q^4)$  steps.

*Step 4.* The quantities  $a_i((p-1)/q)$  ( $i = 0, 1, \dots, q-1$ ) can be computed from the values of the  $a_i$  ( $i = 0, 1, \dots, q-1$ ) using (3.18) in  $O(q^2 \log p)$  steps, since each use of the recurrence relation (3.18) requires  $O(q)$  operations and each of the  $q$  recurrence relations must be applied  $O(\log(p-1)/q)$  times in the repeated doubling technique.

The calculation of the solution  $x$  of (1.1) from the values of the  $a_i((p-1)/q)$  ( $i = 0, 1, \dots, q-1$ ) and  $b_i$  ( $i = 0, 1, \dots, q-1$ ) using (3.20) takes  $O(q)$  steps, and the calculation of the other solutions  $xc^j$  ( $j = 1, 2, \dots, q-1$ ) can be done in  $O(q)$  steps. Hence the algorithm determines all the solutions of (1.1) in  $O(q^4) + O(q^2 \log p)$  steps, once a suitable  $b$  has been determined in Step 2.

We remark that this algorithm (suitably modified) can be used to compute  $q$ th roots in  $GF(p^n)$ , when  $q$  divides  $p^n - 1$ .

#### 4. EXAMPLE

Following the suggestion of the referee, we present a small example to illustrate our algorithm, which the interested reader can easily check by hand. The algorithm is easily programmed to solve (1.1) for large values of  $p$  and values of  $q$  small compared with  $p$ .

We determine all the solutions  $x$  of the congruence

$$(4.1) \quad x^3 \equiv 2 \pmod{31},$$

using our refinement to the algorithm of H. C. Williams. Here,  $p = 31$ ,  $q = 3$ ,  $a = 2$ ,  $(p-1)/q = 10$ ,  $(p-1)^{q-2} = 30$  and  $(p^q-1)/q(p-1) - (p-1)^{q-1}/q = 31$ . As

$$a^{(p-1)/q} = 2^{10} \equiv 32^2 \equiv 1^2 \equiv 1 \pmod{p},$$

the congruence (4.1) is solvable. We can take  $b = 2$ ,  $c = 25$ , as

$$(b^q - a)^{(p-1)/q} = (2^3 - 2)^{10} = 6^{10} = 36^5 \equiv 5^5 \equiv 3125 \equiv 25 \pmod{p}.$$

Also,  $\theta$  is a root of  $\theta^q = b^q - a$ , that is,  $\theta^3 = 6$ . We perform calculations in  $k = GF(31)$  and  $K = GF(31)[\theta] \simeq GF(31^3)$ .

Appealing to (3.9), (3.10), and (3.21), we have

$$E_1 = (2 - \theta)^{30} = (2 - \theta)^{-1}(2 - 25\theta) = (2 + \theta + 16\theta^2)(2 + 6\theta) = 22 + 14\theta + 7\theta^2,$$

so that  $a_0 = 22$ ,  $a_1 = 14$ ,  $a_2 = 7$ . Making use of the recurrence relations

$$\begin{cases} a_i(m+n) = \sum_{j=0}^i a_j(m)a_{i-j}(n) + 6 \sum_{j=i+1}^2 a_j(m)a_{3+i-j}(n), \\ a_i(1) = a_i, \quad i = 0, 1, 2, 3, \end{cases} \quad m, n = 1, 2, \dots,$$

we obtain the values in Table 1 (next page).

Next, from (3.9) and (3.15), we have  $E_2 = 2 - 25\theta$ , so that  $b_0 = 2$ ,  $b_1 = 6$ ,  $b_2 = 0$ . Finally, appealing to (3.20), we obtain

$$x = a_0(10)b_0 + 6(a_1(10)b_2 + a_2(10)b_1) = 9 \times 2 + 6 \times 19 \times 6 = 20.$$

TABLE 1

$j$	$a_0(j)$	$a_1(j)$	$a_2(j)$
1	22	14	7
2	17	11	8
4	12	14	21
8	14	6	18
10	9	4	19

We note that  $x = 20$  is indeed a solution of (4.1), as  $20^3 \equiv (-11)^3 \equiv (-121)11 \equiv 3 \times 11 = 33 \equiv 2 \pmod{31}$ . All solutions of (4.1) are given by  $x \equiv 20 \cdot 25^j \pmod{31}$ ,  $j = 0, 1, 2$ , that is,  $x \equiv 20, 4, 7 \pmod{31}$ .

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