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APPLICATION OF THE HURWITZ ZETA FUNCTION TO THE EVALUATION OF CERTAIN INTEGRALS

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ABSTRACT. The Hurwitz zeta function $\zeta(s,a)$ is defined by the series

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s},$$

for $0<a\le 1$ and $\sigma=\mathrm{Re}(s)>1$, and can be continued analytically to the whole complex plane except for a simple pole at $s=1$ with residue 1. The integral functions $C(s,a)$ and $S(s,a)$ are defined in terms of the Hurwitz zeta function as follows:

$$C(s,a)=\frac{(2\pi)^s}{4}\frac{(\zeta(1-s,a)+\zeta(1-s,1-a))}{\Gamma(s)\cos\frac{\pi}{2}s},$$

$$S(s,a)=\frac{(2\pi)^s}{4}\frac{(\zeta(1-s,a)-\zeta(1-s,1-a))}{\Gamma(s)\sin\frac{\pi}{2}s}.$$

Using integral representations of $C(s,a)$ and $S(s,a)$, we evaluate explicitly a class of improper integrals. For example if $0<a<1$ we show that

$$\int_0^\infty \frac{e^{-x}\log x}{e^{-2x}-2e^{-x}\cos 2\pi a+1}\,dx=\frac{\pi}{2}\frac{1}{\sin 2\pi a}\log\left((2\pi)^{1-2a}\frac{\Gamma(1-a)}{\Gamma(a)}\right).$$

1. **Introduction.** The Hurwitz zeta function $\zeta(s,a)$ is defined by the series

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \tag{1.1}$$

for $0<a\le 1$ and $\sigma=\mathrm{Re}(s)>1$. The reader will find the basic properties of $\zeta(s,a)$ in [3, Chapter 12]. When $a=1$ $\zeta(s,a)$ reduces to the Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Following [17, §2.17], an integral representation of $\zeta(s,a)$ is

$$\zeta(s,a)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{e^{(1-a)x}}{e^x-1}x^{s-1}\,dx,\quad \sigma>1. \tag{1.2}$$

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Using this integral representation $\zeta(s,a)$ can be continued analytically to the whole complex plane except for a simple pole at $s=1$ with residue 1 by means of the integral

$$\zeta(s,a)=\frac{e^{-\pi i s}\Gamma(1-s)}{2\pi i}\int_C \frac{e^{(1-a)z}}{e^z-1}z^{s-1}\,dz, \tag{1.3}$$

where C is the contour consisting of the real axis from ∞ to ϵ $(0<\epsilon)$, the circle $|z|=\epsilon$, and the real axis from ϵ to ∞. We remark that relations and values for the Hurwitz zeta function and its derivatives have been given by many authors, see for example [1], [2], [4], [5], [6], [7], [9], [10], [14], [15].

For $\sigma<0$ we deduce from (1.3) that $\zeta(s,a)$ can be expressed in the form

$$\zeta(s,a)=\frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\Big(\sin\frac{\pi s}{2}C(1-s,a)+\cos\frac{\pi s}{2}S(1-s,a)\Big), \tag{1.4}$$

where $C(s,a)$ and $S(s,a)$ are the functions defined by

$$C(s,a)=\sum_{n=1}^{\infty}\frac{\cos 2n\pi a}{n^s},\quad S(s,a)=\sum_{n=1}^{\infty}\frac{\sin 2n\pi a}{n^s},\quad 0<a<1,\quad \sigma>0. \tag{1.5}$$

The functions $C(s,a)$ and $S(s,a)$ can be continued analytically to the whole complex plane. In terms of the Hurwitz zeta function, we define the functions

$$\lambda(s,a)=\zeta(s,a)+\zeta(s,1-a)=\frac{4}{(2\pi)^{1-s}}\Gamma(1-s)\sin\frac{\pi s}{2}C(1-s,a), \tag{1.6}$$

$$\mu(s,a)=\zeta(s,a)-\zeta(s,1-a)=\frac{4}{(2\pi)^{1-s}}\Gamma(1-s)\cos\frac{\pi s}{2}S(1-s,a). \tag{1.7}$$

In §2 we determine explicitly the value of $S'(1,a)$, $0<a<1$ (see Proposition). We also obtain integral representations of $C(s,a)$ and $S(s,a)$ (see (2.16) and (2.17)).

In §3 we use the integral representations for $C(s,a)$ and $S(s,a)$ to evaluate a class of improper integrals. One of the results obtained is the following: for $0<a<1$

$$\int_0^\infty \frac{e^{-x}\log x}{e^{-2x}-2e^{-x}\cos 2\pi a+1}\,dx=\frac{\pi}{2}\frac{1}{\sin 2\pi a}\log\Big((2\pi)^{1-2a}\frac{\Gamma(1-a)}{\Gamma(a)}\Big).$$

This integral can be found in [13, p. 572]. Special cases of this integral are discussed in [18]. In addition the integral

$$\int_0^\infty \frac{(e^{-x}\cos 2\pi a-e^{-2x})\log x}{e^{-2x}-2e^{-x}\cos 2\pi a+1}\,dx$$

is evaluated for certain values of a, namely, $a=1/2, 1/3, 1/4, 1/6$. The values of the integrals obtained when $a=1/2, 1/4$ appear in [13, p. 572] but those for $a=1/3, 1/6$ appear to be new.

Finally in §4 we use the integral representations of $S(s,1/4)$ (resp. $C(s,0)$ and $C(s,1/2)$) to obtain the following integral representation of $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}$ (resp. $\sum_{n=1}^{\infty}\frac{1}{n^s}$):

$$S(s)=S(s,1/4)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}=\frac{1}{\Gamma(s)}\int_0^\infty\frac{e^x}{e^{2x}+1}x^{s-1}\,dx,\quad \sigma>0,$$

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{(1-\frac{1}{2^s})\Gamma(s)}\int_0^\infty\frac{e^x}{e^{2x}-1}x^{s-1}\,dx,\quad \sigma>1.$$

Using the first of these representations with $s = 2k + 1$, we obtain a new recurrence relation for $S(2k + 1)$. The second of the two representations with $s = 2k$ yields the recurrence relation for $\zeta(2k)$ given by G. Stoica in [16].

2. **Evaluation of $S'(1, a)$.** From the theory of Fourier series, for $0 < a < 1$, we have

$$S(1,a) = \sum_{n=1}^{\infty} \frac{\sin 2n\pi a}{n} = \pi\left(\frac{1}{2} - a\right), \tag{2.1}$$

$$C(1,a) = \sum_{n=1}^{\infty} \frac{\cos 2n\pi a}{n} = -\log(2\sin \pi a). \tag{2.2}$$

From (1.4) and (2.1), we obtain

$$\zeta(0,a) = \sum_{n=1}^{\infty} \frac{\sin 2n\pi a}{n\pi} = \frac{1}{2} - a, \tag{2.3}$$

and hence by (1.6) and (1.7), we have

$$\lambda(0,a) = \zeta(0,a) + \zeta(0,1-a) = 0, \tag{2.4}$$

$$\mu(0,a) = \zeta(0,a) - \zeta(0,1-a) = 1 - 2a. \tag{2.5}$$

PROPOSITION. *For $0 < a < 1$, we have*

$$S'(1,a) = \frac{\pi}{2}\left\{\log \frac{\Gamma(1-a)}{\Gamma(a)} + (1-2a)(\gamma + \log 2\pi)\right\}. \tag{2.6}$$

PROOF. Differentiating both sides of (1.7), putting $s = 0$, and appealing to (2.1) and (2.5), we obtain

$$\mu'(0,a) = (\gamma + \log 2\pi)(1-2a) - \frac{2}{\pi} S'(1,a). \tag{2.7}$$

However, from Hermite's formula for the Hurwitz zeta function

$$\zeta(s,a) = \frac{1}{2}a^{-s} + \frac{a^{1-s}}{s-1} + 2\int_0^{\infty} (a^2 + y^2)^{-\frac{s}{2}} \left\{\sin\left(s \arctan \frac{y}{a}\right)\right\} \frac{dy}{e^{2\pi y} - 1}, \tag{2.8}$$

it is easy to see ([19, p. 271]) that

$$\zeta'(0,a) = \log \Gamma(a) - \frac{1}{2}\log 2\pi \tag{2.9}$$

and from (1.7)

$$\mu'(0,a) = \log\big(\Gamma(a)/\Gamma(1-a)\big). \tag{2.10}$$

From (2.7) and (2.9), we deduce (2.6). ■

REMARK 1. Since

$$S'(1,a) = -\sum_{n=1}^{\infty} \frac{\sin 2n\pi a}{n} \log n,$$

we have from (2.6), making use of $\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}$,

$$(2.11) \quad \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin 2n\pi a}{n} \log n = \log \Gamma(a) - (\gamma + \log 2\pi)\left(\frac{1}{2} - a\right) - \frac{1}{2}\log \pi + \frac{1}{2}\log(\sin \pi a),$$

which is a famous formula due to Kummer [11] (see also [19, p. 210]).

Differentiating (1.6) and putting $s = 0$ we have, using (2.4) and (2.2),

$$(2.12) \qquad \lambda'(0,a) = C(1,a) = -\log(2\sin \pi a).$$

If we differentiate both sides of (1.6) twice and take $s = 0$, we see that

$$(2.13) \qquad \lambda''(0,a) = 2\gamma C(1,a) - 2C'(1,a) + 2(\log 2\pi)C(1,a),$$

or equivalently

$$(2.13)' \qquad \lambda''(0,a) = -2(\gamma + \log 2\pi)\log(2\sin \pi a) + 2\sum_{n=1}^{\infty} \frac{\cos 2n\pi a}{n} \log n.$$

So far we have the expressions (1.2), (1.6), and (1.7) for $\zeta(s,a)$, $C(s,a)$ and $S(s,a)$ respectively. Now we obtain other integral representations of these functions. Taking the real and imaginary parts of the identity

$$\frac{re^{2\pi ai}}{1 - re^{2\pi ai}} = \sum_{n=1}^{\infty} r^n e^{2\pi nai}, \quad |r| < 1,$$

we have

$$(2.14) \qquad \frac{r \sin 2\pi a}{r^2 - 2r\cos 2\pi a + 1} = \sum_{n=1}^{\infty} r^n \sin 2\pi na, \quad |r| < 1,$$

$$(2.15) \qquad \frac{r \cos 2\pi a - r^2}{r^2 - 2r\cos 2\pi a + 1} = \sum_{n=1}^{\infty} r^n \cos 2\pi na, \quad |r| < 1.$$

For $\sigma > 0$, we have

$$\int_0^{\infty} e^{-nx} x^{s-1}\, dx = \frac{\Gamma(s)}{n^s}.$$

Multiplying this equality by $\sin nt$, summing over n, interchanging the order of summation and integration, and appealing to (2.14), we obtain

$$\Gamma(s)S(s,a) = \sin 2\pi a \int_0^{\infty} \frac{e^{-x}x^{s-1}}{e^{-2x} - 2e^{-x}\cos 2\pi a + 1}\, dx.$$

Hence we have

$$(2.16) \quad \sin 2\pi a \int_0^{\infty} \frac{e^{-x}x^{s-1}}{e^{-2x} - 2e^{-x}\cos 2\pi a + 1}\, dx = \Gamma(s)S(s,a), \quad 0 \le a \le 1, \quad \sigma > 0.$$

Similarly, from (2.15), we have

$$\int_0^\infty \frac{(e^{-x}\cos 2\pi a - e^{-2x})}{e^{-2x} - 2e^{-x}\cos 2\pi a + 1} x^{s-1}\,dx = \Gamma(s)C(s,a), \quad 0 < a < 1, \quad \sigma > 0; \quad \text{or } a = 0, 1, \quad \sigma > 1. \tag{2.17}$$

The formulae (2.16) and (2.17) give integral representations of $S(s,a)$ and $C(s,a)$ respectively. Then, from (1.4), (2.16) and (2.17), we obtain the integral representation of $\zeta(s,a)$:

$$\zeta(1-s,a) = 2(2\pi)^{-s}\int_0^\infty \frac{e^x\cos(\frac{\pi s}{2} - 2\pi a) - \cos\frac{\pi s}{2}}{e^{2x} - 2e^x\cos 2\pi a + 1} x^{s-1}\,dx, \quad 0 < a < 1, \quad \sigma > 0; \quad \text{or } a = 1, \quad \sigma > 1,$$

or

$$\zeta(s,a) = 2(2\pi)^{s-1}\int_0^\infty \frac{e^x\sin(\frac{\pi s}{2} + 2\pi a) - \sin\frac{\pi s}{2}}{e^{2x} - 2e^x\cos 2\pi a + 1} x^{-s}\,dx, \quad 0 < a < 1, \quad \sigma < 1; \quad \text{or } a = 1, \quad \sigma < 0. \tag{2.18}$$

These expressions will be used in the following sections.

3. **Evaluation of certain integrals.** By differentiating (2.16) and (2.17) and using the values of $S(1,a)$, $S'(1,a)$, $C(1,a)$, and $C'(1,a)$ obtained in §2, we are able to evaluate certain improper integrals.

THEOREM 1. *For $0 < a < 1$ we have*

$$\int_0^\infty \frac{e^x \log x}{e^{2x} - 2e^x\cos 2\pi a + 1}\,dx = \frac{\pi}{2}\frac{1}{\sin 2\pi a}\log\left((2\pi)^{1-2a}\frac{\Gamma(1-a)}{\Gamma(a)}\right). \tag{3.1}$$

In particular, for $a = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ the integrals in (3.1) become

$$\int_0^\infty \frac{e^x \log x}{e^{2x} - e^x + 1}\,dx = \frac{2\pi}{\sqrt{3}}\left\{\frac{5}{6}\log 2\pi - \log\Gamma(1/6)\right\}, \tag{3.2}$$

$$\int_0^\infty \frac{e^x \log x}{e^{2x} + 1}\,dx = \frac{\pi}{2}\log\frac{\sqrt{2\pi}\,\Gamma(3/4)}{\Gamma(1/4)}, \tag{3.3}$$

$$\int_0^\infty \frac{e^x \log x}{e^{2x} + e^x + 1}\,dx = \frac{\pi}{\sqrt{3}}\log\frac{(2\pi)^{1/3}\Gamma(2/3)}{\Gamma(1/3)}, \tag{3.4}$$

$$\int_0^\infty \frac{e^x \log x}{(e^x + 1)^2}\,dx = \frac{1}{2}\left(\log\frac{\pi}{2} - \gamma\right). \tag{3.5}$$

PROOF. From (2.16), we obtain

$$\begin{aligned}\int_0^\infty \frac{e^{-x}\log x}{e^{-2x} - 2e^{-x}\cos 2\pi a + 1}\,dx &= \frac{1}{\sin 2\pi a}\Big(\Gamma(s)S(s,a)\Big)'_{s=1} \\ &= \frac{1}{\sin 2\pi a}\{\Gamma(1)S'(1,a) + \Gamma'(1)S(1,a)\}.\end{aligned}$$

In view of (2.1) and (2.6), we have

$$\begin{aligned}
&\int_0^\infty \frac{e^{-x}\log x}{e^{-2x}-2e^{-x}\cos 2\pi a+1}\,dx \\
&= \frac{1}{\sin 2\pi a}\left\{\frac{\pi}{2}\left[\log\frac{\Gamma(1-a)}{\Gamma(a)}+(1-2a)(\gamma+\log 2\pi)\right]-\gamma\pi\left(\frac{1}{2}-a\right)\right\} \\
&= \frac{\pi}{2\sin 2\pi a}\left\{\log\frac{\Gamma(1-a)}{\Gamma(a)}+(1-2a)\log 2\pi\right\},
\end{aligned}$$

which is (3.1).

For $a = 1/2$, the value of the integral on the right side of (3.1) should be considered as the limiting value as $a \longrightarrow 1/2$:

$$\begin{aligned}
\int_0^\infty \frac{e^x \log x}{(e^x+1)^2}\,dx &= \frac{\pi}{2}\lim_{a\to 1/2}\frac{1}{\sin 2\pi a}\left\{\log\frac{\Gamma(1-a)}{\Gamma(a)}+(1-2a)\log 2\pi\right\} \\
&= \frac{1}{2}\left\{\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})}+\log 2\pi\right\}.
\end{aligned}$$

Taking $s = 1$ in the well-known formula [12, p. 320]

$$\frac{\Gamma'(s)}{\Gamma(s)} = \int_0^\infty \left(\frac{e^{-x}}{x}-\frac{e^{-sx}}{1-e^{-x}}\right)dx$$

we obtain

$$\gamma = -\frac{\Gamma'(1)}{\Gamma(1)} = \int_0^\infty \left(\frac{e^{-x}}{1-e^{-x}}-\frac{e^{-x}}{x}\right)dx,$$

and taking $s = 1/2$ we obtain

$$\begin{aligned}
\frac{\Gamma'(1/2)}{\Gamma(1/2)} &= \int_0^\infty \left(\frac{e^{-x}}{x}-\frac{e^{-\frac{1}{2}x}}{1-e^{-x}}\right)dx \\
&= -\gamma+\int_0^\infty \frac{e^{-x}-e^{-\frac{1}{2}x}}{1-e^{-x}}\,dx \\
&= -\gamma+\int_0^1 \frac{1-t^{-\frac{1}{2}}}{1-t}\,dt \\
&= -\gamma-2\log 2.
\end{aligned}$$

Hence we have

$$\int_0^\infty \frac{e^x\log x}{(e^x+1)^2}\,dx = \frac{1}{2}\left(\log\frac{\pi}{2}-\gamma\right),$$

which proves (3.5). ∎

REMARK 2. The integral in (3.1) can be expressed in the following equivalent forms:

$$\frac{1}{2}\int_0^\infty \frac{\log x}{\cosh x-\cos 2\pi a}\,dx = \int_0^1 \frac{\log\log\frac{1}{x}}{x^2-2x\cos 2\pi a+1}\,dx = \int_1^\infty \frac{\log\log x}{x^2-2x\cos 2\pi a+1}\,dx.$$

Similarly, from (2.12), (2.13) and (2.17), we have

$$\int_0^\infty \frac{(e^{-x}\cos 2\pi a - e^{-2x})}{e^{-2x} - 2e^{-x}\cos 2\pi a + 1}\log x\,dx = \big(\Gamma(s)C(s,a)\big)'_{s=1}$$
$$= \Gamma(1)C'(1,a) + \Gamma'(1)C(1,a),$$

that is by (2.2)

$$\int_0^\infty \frac{(e^{-x}\cos 2\pi a - e^{-2x})}{e^{-2x} - 2e^{-x}\cos 2\pi a + 1}\log x\,dx = C'(1,a) + \gamma\log(2\sin\pi a), \tag{3.6}$$

or by (2.13)

$$\int_0^\infty \frac{(e^{-x}\cos 2\pi a - e^{-2x})}{e^{-2x} - 2e^{-x}\cos 2\pi a + 1}\log x\,dx = -(\log 2\pi)\log(2\sin\pi a) - \frac{1}{2}\lambda''(0,a). \tag{3.6$'$}$$

It appears to be difficult to determine $C'(1,a)$ explicitly for general a, so we just evaluate $C'(1,a)$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$. For these values of a, $C(s,a)$ can be expressed in terms of $\zeta(s)$.

THE CASE $a = 1/2$. We have

$$C\Big(s,\frac{1}{2}\Big) = (2^{1-s} - 1)\zeta(s)$$
$$= \Big\{ -(\log 2)(s-1) + \frac{1}{2}(\log^2 2)(s-1)^2 + \cdots\Big\}\Big\{\frac{1}{s-1} + \gamma + \cdots\Big\}$$
$$= -\log 2 + \Big(\frac{1}{2}\log^2 2 - \gamma\log 2\Big)(s-1) + \cdots$$

so that

$$C'\Big(1,\frac{1}{2}\Big) = \frac{1}{2}\log^2 2 - \gamma\log 2. \tag{3.7}$$

From (3.6) with $a = 1/2$ and (3.7) we obtain

$$\int_0^\infty \frac{\log x}{e^x + 1}dx = -\frac{1}{2}(\log 2)^2. \tag{3.8}$$

THE CASE $a = 1/3$. We have

$$C\Big(s,\frac{1}{3}\Big) = \frac{1}{2}(3^{1-s} - 1)\zeta(s)$$
$$= \frac{1}{2}\Big\{ -(\log 3)(s-1) + \frac{1}{2}(\log^2 3)(s-1)^2 + \cdots\Big\}\Big\{\frac{1}{s-1} + \gamma + \cdots\Big\}$$
$$= -\frac{1}{2}\log 3 + \Big(\frac{1}{4}\log^2 3 - \frac{\gamma}{2}\log 3\Big)(s-1) + \cdots$$

so that

$$C'\Big(1,\frac{1}{3}\Big) = \frac{1}{4}\log^2 3 - \frac{1}{2}\gamma\log 3. \tag{3.9}$$

From (3.6) with $a = 1/3$ and (3.9) we obtain

$$\int_0^\infty \frac{(e^x + 2)\log x}{e^{2x} + e^x + 1}\,dx = -\frac{1}{2}(\log 3)^2. \tag{3.10}$$

THE CASE $a = 1/4$. We have

$$\begin{aligned}
C\Big(s, \frac{1}{4}\Big) &= 2^{-s}(2^{1-s} - 1)\zeta(s)\\
&= \Big(\frac{1}{2} - \frac{(s-1)\log 2}{2} + \frac{(s-1)^2\log^2 2}{4} + \cdots\Big)\\
&\qquad \Big(-(s-1)\log 2 + \frac{(s-1)^2}{2}\log^2 2 + \cdots\Big)\zeta(s)\\
&= \Big(-\frac{(s-1)\log 2}{2} + \frac{3}{4}(s-1)^2\log^2 2 + \cdots\Big)\Big(\frac{1}{s-1} + \gamma + \cdots\Big)\\
&= -\frac{1}{2}\log 2 + \Big(\frac{3}{4}\log^2 2 - \frac{\gamma}{2}\log 2\Big)(s-1) + \cdots
\end{aligned}$$

so that

$$C'\Big(1, \frac{1}{4}\Big) = \frac{3}{4}\log^2 2 - \frac{\gamma}{2}\log 2. \tag{3.11}$$

From (3.6) with $a = 1/4$ and (3.11) we obtain

$$\int_0^\infty \frac{\log x}{e^{2x} + 1}dx = -\frac{3}{4}\log^2 2. \tag{3.12}$$

Replacing x by $x/2$ in (3.12), as

$$\int_0^\infty \frac{dx}{e^x + 1} = \log 2,$$

we recover (3.8).

THE CASE $a = 1/6$. We have

$$\begin{aligned}
C\Big(s, \frac{1}{6}\Big) &= \frac{1}{2}(1 - 2^{1-s})(1 - 3^{1-s})\zeta(s)\\
&= \frac{1}{2}\Big((s-1)\log 2 - \frac{(s-1)^2}{2}\log^2 2 + \cdots\Big)\\
&\qquad \Big((s-1)\log 3 - \frac{(s-1)^2}{2}\log^2 3 + \cdots\Big)\zeta(s)\\
&= \Big(\frac{1}{2}(s-1)^2(\log 2)(\log 3) + \cdots\Big)\Big(\frac{1}{s-1} + \gamma + \cdots\Big)\\
&= \frac{1}{2}(\log 2)(\log 3)(s-1) + \cdots
\end{aligned}$$

so that

$$C'\Big(1, \frac{1}{6}\Big) = \frac{1}{2}(\log 2)(\log 3). \tag{3.13}$$

From (3.6) with $a = 1/6$ and (3.13) we obtain

$$\int_0^\infty \frac{(e^x-2)\log x}{e^{2x}-e^x+1}\,dx = (\log 2)(\log 3). \tag{3.14}$$

REMARK 3. Since

$$C'(1,a) = -\sum_{n=1}^\infty \frac{\cos 2n\pi a}{n}\log n$$

we deduce respectively from (3.7) (or (3.11)), (3.9), (3.13)

$$\sum_{n=1}^\infty \frac{(-1)^n \log n}{n} = \gamma \log 2 - \frac{1}{2}\log^2 2 \tag{3.15}$$

$$\sum_{n=1}^\infty \frac{\cos\frac{2n\pi}{3}\log n}{n} = \frac{1}{2}\gamma\log 3 - \frac{1}{4}\log^2 3, \tag{3.16}$$

$$\sum_{n=1}^\infty \frac{\cos\frac{n\pi}{3}\log n}{n} = -\frac{1}{2}(\log 2)(\log 3). \tag{3.17}$$

4. **A recurrence relation for** $\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^{2k+1}}$. Taking $a = \frac{1}{4}$ in (2.16) and defining

$$S(s) = S\left(s, \frac{1}{4}\right) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^s}, \quad \sigma > 0, \tag{4.1}$$

we have

$$\Gamma(s)S(s) = \int_0^\infty \frac{e^x}{e^{2x}+1}x^{s-1}\,dx, \quad \sigma > 0. \tag{4.2}$$

It is very easy to see that $C(s,0) = \zeta(s)$, and (2.17) with $a = 0$ becomes the well-known formula:

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx, \quad \sigma > 1. \tag{4.3}$$

Also

$$C\left(s,\frac{1}{2}\right) = \sum_{n=1}^\infty \frac{(-1)^n}{n^s} = -(1-2^{1-s})\zeta(s),$$

and (2.17) with $a = 1/2$ reduces to

$$(1-2^{1-s})\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x+1}\,dx, \quad \sigma > 0. \tag{4.4}$$

Adding (4.3) and (4.4), we obtain

$$(2-2^{1-s})\Gamma(s)\zeta(s) = 2\int_0^\infty \frac{e^x}{e^{2x}-1}x^{s-1}\,dx, \quad \sigma > 1. \tag{4.5}$$

We are now ready to prove the following theorem.

THEOREM 2. *For nonnegative integers k, we have*

$$\sum_{j=0}^{k}(-1)^j(2j)!\, C_{2j}^{2k}\pi^{2k-2j}S(2j+1)+(-1)^k(2k)!\, S(2k+1)=(\pi/2)^{2k+1}. \tag{4.6}$$

PROOF. Taking $s=2k+1$ in (4.2), we have

$$(2k)!\, S(2k+1)=\int_0^\infty \frac{e^x x^{2k}}{e^{2x}+1}\,dx=\int_1^\infty \frac{(\log t)^{2k}}{t^2+1}\,dt.$$

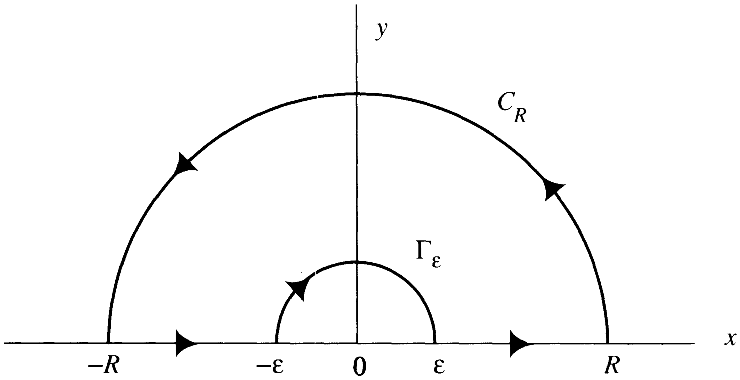
But in view of

$$\int_1^\infty \frac{(\log t)^{2k}}{t^2+1}\,dt=\int_0^1 \frac{(\log t)^{2k}}{t^2+1}\,dt,$$

we have

$$2(2k)!\, S(2k+1)=\int_0^\infty \frac{(\log t)^{2k}}{t^2+1}\,dt. \tag{4.7}$$

Considering the integral of the complex function $F(z)=\frac{(\log z)^{2k}}{1+z^2}$ along the contour shown in the figure below, we obtain by Cauchy's residue theorem



$$\int_{C_R} F(z)dz+\int_{\Gamma_\varepsilon} F(z)\,dz+\int_{-R}^{-\varepsilon} F(x)\,dx+\int_{\varepsilon}^{R} F(x)\,dx=2\pi i\,\mathrm{Res}\big(F(z),i\big). \tag{4.8}$$

Now we evaluate the residue on the right side of (4.8). We have

$$\mathrm{Res}\big(F(z),i\big)=\frac{1}{z+i}(\log z)^{2k}|_{z=i}=\frac{1}{2i}(\log i)^{2k}=\frac{(-1)^k}{2i}(\pi/2)^{2k}.$$

On the semicircle C_R, we have

$$F(z)=O\left(\frac{(\log R)^{2k}}{R^2}\right),\int_{C_R} F(z)\,dz=O\left(\frac{(\log R)^{2k}}{R}\right)\longrightarrow 0,\quad \text{as } R\longrightarrow\infty$$

and on the semicircle Γ_ε, we have

$$F(z)=O\big((\log\varepsilon)^{2k}\big),\int_{\Gamma_\varepsilon} F(z)\,dz=O\big(\varepsilon(\log\varepsilon)^{2k}\big)\longrightarrow 0,\quad \text{as } \varepsilon\longrightarrow 0.$$

In addition we have

$$\int_{-R}^{-\varepsilon} F(x)\,dx = \int_{\varepsilon}^{R} \frac{(\log t + \pi i)^{2k}}{1+t^2}\,dt.$$

Hence letting $\varepsilon \longrightarrow 0$ and $R \longrightarrow \infty$ in (4.8) we obtain

$$\int_0^\infty \frac{(\log t)^{2k} + (\log t + \pi i)^{2k}}{1+t^2}\,dt = (-1)^k \frac{\pi^{2k+1}}{2^{2k}}.$$

Taking the real part of the above equation, we deduce (4.6). ■

In particular, taking $k = 0, 1, 2, 3, 4$ in (4.6), we have $S(1) = \frac{\pi}{4}$, $S(3) = \frac{\pi^3}{2.2^4}$, $S(5) = \frac{5\pi^5}{3.2^9}$, $S(7) = \frac{61\pi^7}{5.9.2^{12}}$, $S(9) = \frac{277\pi^9}{7.9.2^{17}}$.

Similarly, making the substitution $t = e^x$ in (4.5), we obtain

$$(2 - 2^{1-s})\Gamma(s)\zeta(s) = 2\int_1^\infty \frac{(\log t)^{s-1}}{t^2-1}\,dt, \quad \sigma > 1,$$

and with $s = 2k$

$$(2 - 2^{1-2k})(2k-1)!\,\zeta(2k) = 2\int_1^\infty \frac{(\log t)^{2k-1}}{t^2-1}\,dt.$$

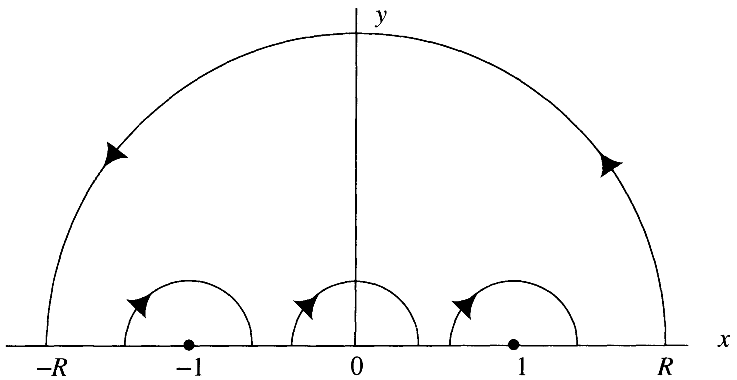
Since

$$\int_1^\infty \frac{(\log t)^{2k-1}}{t^2-1}\,dt = \int_0^1 \frac{(\log t)^{2k-1}}{t^2-1}\,dt,$$

we have

(4.9) $$2\left(1 - \frac{1}{2^{2k}}\right)(2k-1)!\,\zeta(2k) = \int_0^\infty \frac{(\log t)^{2k-1}}{t^2-1}\,dt.$$

Considering the integral of $\frac{(\log z)^{2k-1}}{t^2-1}$ along the contour shown in the figure below



and applying Cauchy's residue theorem, we obtain

(4.10)
$$\sum_{j=1}^{k}(-1)^j(2j-1)!\,C_{2j-1}^{2k-1}\pi^{2k-2j}\left(1 - \frac{1}{2^{2j}}\right)\zeta(2j) + (-1)^k(2k-1)!\left(1 - \frac{1}{2^{2k}}\right)\zeta(2k)$$
$$= -\pi^{2k}/4, \quad k \geq 1.$$

The recurrence relation (4.10) was obtained in [16] by a longer argument. In particular,

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}, \quad \zeta(10) = \frac{\pi^{10}}{93555}.$$

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